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Abstract Near horizon phenomena, like blank strips in the setting sun or Fata Morganas, can be largely understood in terms of geometric optics. Here the existence of a warm layer above the observer in which light rays can be either refracted or reflected, already suffices to explain many of the phenomena; a reasoning which goes back to Alfred Wegener in the 1920's. We review this theory, where first the atmosphere is divided into discrete, optically homogeneous layers, and where later the atmosphere is considered as a continuous, isotropic medium. Thus the theory gets embedded in differential geometry, where light rays are geodesics. A simplifying assumption is that the refraction index only depends on the elevation above the surface of the earth. Here a connection is made with the geodesic problem on a surface of revolution. In the background of this we touch the variational calculus as this runs from Bernoulli to Hamilton, all the way maintaining a completely anachronistic viewpoint.

Keywords Geometric optics · Differential geometry · Nonlinear dynamics

1 Introduction

This paper concerns geometric optics in an atmospheric context. In fact, we study how light rays are being refracted and reflected through layers in the atmosphere, thus leading to various optical phenomena that sometimes occur around the horizon. An example of this is formed by blank strips in the disk of the setting sun, as described by Wegener [43, 44] and later by Minnaert [31]. For a schematic impression see Fig. 1.

Usually temperature decreases monotonically with the distance to the earth surface, but sometimes a *temperature inversion* occurs, in which one or more warmer layers come into existence. In this way discontinuities may arise in the refraction index profile. In Fig. 1 the situation is depicted with one warm layer bounded from below by a circle at height H that resides above the observer W . In this circle reflection can occur, often indicated by the term

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Fig. 1 Minnaert [31], Fig. 60. Qualitative explanation for a blank strip (Dutch: *blinde strook*) in the sun in a model with one warm layer. The lower boundary of the layer is formed by a circle at height H . The observer W roughly looks in the horizontal direction. Transitions between refraction and reflection yield a so-called *Wegener-sector*, within which W only receives light rays from below the horizon. This may lead to a blank strip in the setting sun. For the sake of clarity, here and elsewhere, the figures are depicted far beyond their realistic proportions

Fig. 2 Theoretical sunsets with blank strip (left) and blank zone (right). The atmosphere is colored dark and the earth light gray. In the latter case it appears as if the sun already sets above the horizon, which is a quite common phenomenon. Also note the deformation of the solar disk

upper reflection. In interaction with refraction a *Wegener-sector* can arise, within which W only receives light rays from below the horizon. When the sun is at a suitable height above the horizon, W observes a blank strip or zone in the solar disk, also see Fig. 2.

Furthermore W within, or near, such a *Wegener-sector* by these same refractions and reflections (sometimes called ‘mirages’) can observe *Fata Morganas*: these are images of objects below the horizon. As an example see Fig. 3.

It is to be noted that these descriptions are schematic and qualitative: they serve for understanding rather than for strict quantitative purposes. This is a tradition that goes back at least to Kepler [25]. Here it is good to realize that precise data of refraction index profiles over the entire height of the atmosphere are not available. For the considerably awkward physical background of refraction indices of a medium like air, we refer to, for instance, the

Fig. 3 Theoretical Fata Morganas of a sailing ship disappearing behind the horizon. Here both upper and lower reflections occur

Feynman Lectures [14]. In respect of all this, we are dealing with a so-called *inverse* problem, namely to what extent certain phenomena in the atmosphere can be given a geometric optical ‘explanation’; also compare, for instance, [27, 39, 40].

Remarks

- We give some relevant data. The diameter of the sun is approximately $32' \approx 0.5^\circ$, expressed in minutes and degrees of arc. The radius of the earth is $R = 6400$ km, while the height of the atmosphere approximately equals 11 km. Realistic values for the height of the observer W are 10 to 15 m, say that he observes from the top of a dune or from the deck of a ship. The temperature in a warm layer can vary from a few degrees to more than 20° Celsius.
- In a model with one warm layer as in Fig. 1, a value of $H \approx 80$ m as the height of the lower circular boundary of the warm layer is often appropriate for ‘saving the appearances’. The latter expression goes back to Plato and Aristotle [11].

2 Towards a Mathematical Description

The mathematical description of the phenomena described above in our approach, rests on the Fermat Principle, which says that light rays follow paths of shortest time [1, 37, 41]. This forms the basis of geometric optics. For an explanation how the Fermat Principle is connected to wave optics we refer to Feynman et al. [14] or to Arnold [2].

Below we shall regularly use the terms *isotropic* and *homogenous* as possible properties of an optical medium. Isotropy means that in each point of a medium the speed of light is equal in all directions, say with magnitude v . This justifies a pointwise definition of the

Fig. 4 The Fermat Principle leads to reflection in accordance with Hero (*left*) and refraction in accordance with Snell (*right*). In both cases the broken line ACB forms the path of shortest time to travel from A to B via the boundary $y = 0$. Notice that B' (*left*) is the mirror image of B in this boundary

refraction index by $n = c/v$. Here c is the speed of light in vacuo. In this paper we take $c = 1$, which is possible by an appropriate choice of units. Homogeneity moreover means that v , and therefore n , is independent of the position in the medium. The Fermat Principle implies that in a homogeneous medium the light rays form straight lines that are traversed with a constant velocity.

Fermat Implies Snell's Law If the medium consists of two homogeneous layers bounded by a flat plane, then in that plane reflection takes place according to Hero of Alexandria or refraction according to Snell. In Fig. 4 these laws amount to

$$\beta = \alpha \quad \text{refraction (left)}$$

$$n_2 \sin \beta = n_1 \sin \alpha \quad \text{reflection (right)}$$

We note that Snell's law was already known in ancient times, and certainly also to Descartes and Huygens [24].

Remarks

- A high school proof can be given by considering the time t_{AC} needed to travel from A to C and t_{CB} to travel from C to B . Since $t_{AC} = n_1 |A - C| = n_1 \sqrt{x^2 + b^2}$ as a function of x , it follows that

$$\frac{d}{dx} t_{AC}(x) = \frac{n_1 x}{\sqrt{x^2 + b^2}} = n_1 \sin \alpha,$$

etc. Using this we see that stationarity of the time

$$\frac{d}{dx} (t_{AC} + t_{CB}) = 0$$

is equivalent to (1). In this case the extremum of $t_{AC} + t_{CB}$ also turns out to be a minimum.

- A first proof that these laws follow from the Fermat Principle was published by Leibniz [28]. If the boundary consists of a smooth surface, then the above assertions are also valid, but then the plane mentioned is the tangent plane to this boundary and we have to take angles with respect to its normal.

Fig. 5 Increasing the inclination α , first gives refraction (left) and later reflection (right), in a circular boundary; both cases are separated by the limit angle (middle). In this figure also $n_1 > n_2$ and the rays come from below

Limit Angle We assume that $n_1 > n_2$ and consider Fig. 4 from a somewhat different viewpoint, namely where the point C on the boundary is kept fixed, while the light ray enters from the upper halfplane $y > 0$, where the angle α with the y -direction varies from 0 to $\frac{1}{2}\pi$. Our interest then is how β depends on α . To begin with, we have $\alpha = 0$, where the broken light ray passes vertically through the boundary $y = 0$, compare Fig. 4, right. Initially also the exit angle β increases with α . However, for a certain value $\alpha = \alpha_g < \frac{1}{2}\pi$ the angle β already has reached the value $\beta = \frac{1}{2}\pi$. To be precise, this happens at

$$\alpha_g = \arcsin\left(\frac{n_2}{n_1}\right); \quad (1)$$

this is the so-called *limit angle*, a concept that goes back to Wegener [43, 44]. At that moment the outgoing light ray is tangent to the x -axis. When α increases further the light ray bounces back into the upper halfplane $y > 0$, as in Fig. 4, left.

We conclude that at $\alpha = \alpha_g$ the exit angle β has a discontinuity and jumps from $\beta = \frac{1}{2}\pi$ to $\beta = \pi - \alpha_g$. From that moment on β again changes continuously with α and decreases till $\beta = \frac{1}{2}\pi$, which occurs for $\alpha = \frac{1}{2}\pi$.

See Fig. 5 for how this works for a curved boundary. In this case the light ray comes from below, a reversion that is made clear in the sequel.

2.1 Theoretical Assumptions

Below we shall apply geometric optics to the atmosphere. Our approach mainly is restricted to one vertical plane passing through the eye of the observer and that moreover is determined by one other point, say, the center of the solar disk, the moon disk, by a planet, a ship or mountain, etc.

In the vertical plane we consider, two natural coordinate systems are being used, depending on whether we are dealing with a *flat* or with a *round* earth. For the flat earth this system is Cartesian, where x is the horizontal and y the vertical coordinate. For the round earth we use polar coordinates φ and r with respect to the center of the earth. Here r is the distance to the center and φ the angle with respect to a fixed direction.

Symmetry The layers of different refraction index in the case of a flat earth are assumed to be horizontal, i.e., invariant under horizontal translations. In case of a round earth, we analogously assume that these layers are invariant under rotations about the center of the earth. In other words this means that the refraction index n does not depend on either x or φ . In the former case the corresponding profile therefore reads $n = n(y)$ and in the latter $n = n(r)$.

Fig. 6 Conserved quantities $n(y) \sin \alpha = S$ and $rn(r) \sin \alpha = C$ for light rays in the flat and round earth atmosphere, respectively

Remarks

- That the refraction index is independent of x or φ expresses a symmetry property of the atmosphere. We note that for our considerations this symmetry only has to be valid for a relevant sector of the atmosphere, usually a small neighbourhood of the horizon.
- Evidently rotational symmetry is not always realistic. For a brief discussion of a landscape dependent refraction index profile, see below. In the sequel we present the mathematical theory needed for the computation of general light rays in three-dimensional space. And the end of this paper we also give a few comments on optics beyond the scope of the present treatment.

Conservation Laws Often it is convenient to work with *discrete* homogeneous layers of different refraction index. At the boundaries between layers the refraction and reflection laws are valid as mentioned before and the light ray follows a broken line in the atmosphere. Observe that both for refraction and for reflection the formula

$$n_2 \sin \alpha_2 = n_1 \sin \alpha_1$$

is valid, where the two bounding layers are numbered 1 and 2 and where α is the angle between the light ray and the normal on the boundary. In the atmosphere α therefore is the angle with the vertical direction, from now on called *inclination*. The inclination α thus becomes a parameter along the light ray.

Both in the discrete and in the continuous case, where the refraction index profile $n = n(y)$ or $n = n(r)$ depends continuously or even smoothly on y or r , the refraction and reflection laws lead to a *conservation law*. In the flat case the quantity

$$S = n(y) \sin \alpha \quad (2)$$

is conserved along the entire light ray, while in the round case this holds for

$$C = rn(r) \sin \alpha. \quad (3)$$

The conservation laws corresponding to the expressions (2) and (3) are deduced from the laws of Hero and Snell, using a direct application of planar Euclidean geometry, which in the round earth case amounts to the rule of sines. As a consequence of these conservation laws, the refraction index profiles $n = n(y)$ or $n = n(r)$ completely determine the evolution

Fig. 7 *Left:* Illustration of the brachistochrone problem [6], that asks for a wire profile that minimizes the time for a bead K to slide from A to B under constant gravitation. For the light ray solution we take $n(y) = 1/\sqrt{-2gy}$, $y < 0$, which is Johann Bernoulli's own solution in disguise. *Right:* Geodesics on the Poincaré upper halfplane. In the optical approach we take $n(y) = 1/y$, $y > 0$

of the inclination $\alpha = \alpha(y)$ or $\alpha = \alpha(r)$ and therefore the position of the light ray in the (x, y) -, or the (φ, r) -plane.

As an example we consider the round earth case, where for a given constant C

$$\sin \alpha = \frac{C}{rn(r)}, \quad \text{while also} \quad \tan \alpha = \pm \frac{rd\varphi}{dr},$$

depending on whether the ray is pointing up- or down-ward. This leads to a parametrization

$$\varphi(r) = \pm C \int_{r_0}^r \frac{d\varrho}{\varrho \sqrt{\varrho^2 n^2(\varrho) - C^2}} \quad (4)$$

of the light ray. Note that here the velocity is determined as $v(r) = 1/n(r)$.

2.2 Bernoulli's Brachistochrone and the Poincaré Upper Halfplane

Two classical problems, namely the Bernoulli brachistochrone and the Poincaré upper half-plane fit in the optical approach of the flat earth atmosphere, see Fig. 7.

The brachistochrone problem asks for the curve of swiftest descent between two points A and B in a vertical plane. The motion along the curve is only subject to constant gravitation with acceleration g . Mindful of the Fermat Principle, Johann Bernoulli considered the curve as a light ray with an appropriate refraction index profile. In terms of our flat earth atmosphere this amounts to taking $n(y) = 1/\sqrt{-2gy}$, $y < 0$, obtained by a direct application of energy conservation for motion under constant gravitation. Using formula (2), together with some calculation [8, 9], leads to the cycloid solution

$$\begin{aligned} x(\alpha) &= \frac{1}{4S^2g} (2\alpha - \sin(2\alpha)) \\ y(\alpha) &= \frac{1}{4S^2g} (-1 + \cos(2\alpha)), \end{aligned}$$

parametrized by the inclination α . Here it is assumed that for $\alpha = 0$ we are at rest in A . The constant S determines the scale of the curve such that it passes through B . We note that this problem and its aftermath forms one of the corner stones of the Variational Principle and the ensuing Variational Calculus [6, 18].

Fig. 8 Limit angle behavior in the discrete and the continuous case. As in Fig. 5 also here the ray comes from below

For the geodesic problem on the Poincaré upper halfplane we take $n(y) = 1/y$, $y > 0$. Another application of (2) then immediately leads to the light rays as half circles

$$(x - x_0)^2 + y^2 = \frac{1}{S^2},$$

perpendicular to the x -axis, or as vertical half lines. We recall that this upper halfplane is one of the first models of hyperbolic geometry, of which these light rays exactly are the geodesics.

3 Atmospheric Optics

Following Wegener [43, 44] and Minnaert [31] we start with a rotation symmetric atmosphere that consists of two layers as depicted in Fig. 1. It turns out to be convenient to use the notion of *sight ray*. Consider a light ray that ends in the eye of the observer W . When we reverse its direction this ray becomes a sight ray. Note that, according to the Fermat Principle, such a sight ray also is a light ray.

3.1 Discrete Layers

If M is the center of the earth, then the surface of the earth in the vertical plane of observation gives rise to a circle around M of radius R . The circular lower boundary of the warm layer has height H and thus forms a circle around M of radius $R + H$. Let n_1 be the refraction index of the lower layer and n_2 that of the warm upper one, then $n_1 > n_2$: a discontinuity appears. See Fig. 10, right. This means that the setting exactly is that of Figs. 5 and 8.

Critical Height We consider all sight rays that at the lower boundary circle of the warm layer (so with radius $R + H$) have an incoming limit angle inclination

$$\alpha_g = \arcsin\left(\frac{n_2}{n_1}\right), \quad (5)$$

compare Fig. 9. This means that all exit sight rays are tangent to this boundary circle. In this way two line bundles come into being, that both *envelope* a circle; a circle that by definition is tangent to all these rays. The corresponding height k is called *critical height* and the corresponding circle has center M and radius $R + k$. By construction $k < H$. The observer W moreover has a certain positive height with respect to the earth surface, to be indicated by ε . We always assume that $k < \varepsilon < H$.

Fig. 9 Critical circle with radius $R + k$ as envelope of the light rays with limit angle inclination α_g

Discrete Wegener-Sector Now follows a qualitative description of the Wegener-sector as depicted in Fig. 10, left. The observer looks to the right (say, to the west) and lets his glance wander from above downwards, the sight ray thus varies from WA to WE , with A, B, C, D and E on the lower boundary of the warm layer. In this way the exit inclination into space first increases. When at the boundary point B the limit angle $\alpha = \alpha_g$ (5) has been reached, refraction of the sight ray changes into reflection in the boundary circle. In Fig. 10 reflection is still the case when the observer W , along the ray WC , looks exactly in the horizontal direction. When we come to the ray WD the limit angle inclination $\alpha = \pi - \alpha_g$ is reached, where reflection again changes into refraction, see for instance the ray WE .

The Wegener-sector $\angle BWD$ in the α -direction corresponds to an interval $\alpha_g < \alpha < \pi - \alpha_g$, the boundary points of which have the same sine-value. Apparently all sight rays within the Wegener sector $\angle BWD$ reach below W 's horizon. The corresponding light rays are reflected in the layer at height H (upper reflection) before they reach W .

Theorem 1 (Discrete Wegener-sector [43, 44]) *Assuming the above circumstances, a Wegener-sector occurs for $k < \varepsilon < H$; in other words, when the observer W is in between the critical circle and the lower boundary of the warm layer. In that case we have*

$$\angle CWD = \angle CWB.$$

In other words, the Wegener-sector stretches out equally far above as below the horizontal observation direction WC .

Proof The straight lines WB and WD both are tangent to the critical circle. Since both pass through W we see at once that the horizontal observation direction is the bisectrix of $\angle BWD$. Indeed, from symmetry considerations it follows that MW is the outer-bisectrix of $\angle BWD$, and since this is perpendicular to WC , the latter must be the bisectrix. $\square$

It turns out that, with help of this Wegener-sector many atmospheric phenomena can be given a qualitative explanation. In Fig. 2 we witnessed examples of this in the form of a blank strip or zone in the setting sun, depending on whether $k > 0$ (left) or $k < 0$ (right). In fact the former of these two cases exactly corresponds to the setting of Fig. 10. Also Fata Morgana mirages are within reach of this theory, see Fig. 3, where a further complication occurs in the form of a lower reflection. The latter can be caused by an extra warm layer at the surface of the earth or a smooth sheet of ice or a mirroring lake. Many more examples can be found in [8], including cases with more layers.

Fig. 10 *Top*: Wegener-sector for $k > 0$, also compare Fig. 1 (Minnaert [31], Fig. 60). For a qualitative description see the text. *Bottom*: Corresponding refraction index profile. The unit of the horizontal axis is the hectometer

Remarks

- Sight rays within a Wegener-sector necessarily stay close to the earth, which can only lead to terrestrial Fata Morganas. However, outside but even close to such a sector, also celestial Fata Morganas are possible.
- An example of the latter is the Nova Zembla phenomenon, as observed by Dutch sailors during the winter of 1596–1597. This concerns a conjunction of Jupiter and the moon and a premature observation of the solar disk. Both phenomena, according to the tables, at the time of observation happened behind the horizon. For a nice historic description and good graphics see Van der Werf et al. [39, 40]. Their explanation involves a landscape dependent refraction index profile that is no longer rotation symmetric.

3.2 Continuous Theory

We now turn to an atmosphere where the refraction index profile depends continuously, even smoothly on the position. As before also here we maintain rotation symmetry and restrict to a vertical plane.

Wegener-Sight Ray A *Wegener-sight ray* is defined as a sight ray that cannot escape directly from the atmosphere, but is reflected in a warm layer. After this the ray can land on the earth surface or disappear into space, where in both cases it may give rise to optical illusions. The union of all Wegener-sight rays forms the Wegener-sector.

Fig. 11 Wegener-sight ray in the continuous case

The sight ray can be presented as a parametrized curve

$$t \mapsto (\varphi(t), r(t)),$$

where t denotes time. It is convenient to introduce the notion of *modified refraction index profile*, defined as

$$n_M(r) = rn(r). \quad (6)$$

Clearly n_M is a function that only takes on positive values. Mindful of the conserved quantity (3), we define

$$C(t) = n_M(r(t)) \sin \alpha(t),$$

then the function $C(t) \equiv C(t_0)$ is constant and therefore completely determined by its value at $t = t_0$, associated with $r_0 = r(t_0)$ and $\alpha_0 = \alpha(t_0)$. Here the observer W is situated at height r_0 'emitting' a sight ray that at the instant t_0 has the inclination α_0 . As before we let W look to the right and thus $0 \leq \alpha_0 \leq \pi$, see Fig. 11. Then the following result holds true.

Theorem 2 (Continuous Wegener-sector) *Assume the above circumstances with a smooth refraction index profile $n = n(r)$. If the modified refraction index profile n_M , see (6), has an absolute minimum in $r_m \geq r_0$, then for*

$$\alpha_g = \arcsin\left(\frac{n_M(r_m)}{n_M(r_0)}\right) \quad (7)$$

the interval $[\alpha_g, \pi - \alpha_g]$ corresponds to a Wegener-sector. This sector is symmetric with respect to reflection in the observation direction $\alpha_0 = \frac{1}{2}\pi$.

Proof The sight ray is non-Wegener whenever it can escape directly from the atmosphere, which means that for all $t \geq t_0$ one has that $\alpha(t) \neq \frac{1}{2}\pi$. This is equivalent to saying

$$n_M(r_0) \sin \alpha_0 \equiv n_M(r(t)) \sin \alpha(t) < n_M(r(t)),$$

or to

$$n_M(r_0) \sin \alpha_0 < n_M(r)$$

for any $r > r_0$. In other words, the Wegener-sight rays have an initial inclination α_0 , such that for some $r > r_0$

$$n_M(r) \leq n_M(r_0) \sin \alpha_0.$$

Fig. 12 Continuous Wegener-sector $\alpha_g \leq \alpha_0 \leq \pi - \alpha_g$ in terms of the graph of the function $\alpha_0 \mapsto \sin \alpha_0$

Fig. 13 Left: Continuous sight ray bundle with Wegener-sector. Right: Corresponding modified refraction index profile $r \mapsto n_M(r)$ and transition diagram. Compare Example 1

Since r_m is the absolute minimum of the function n_M it follows that

$$0 < \frac{n_M(r_m)}{n_M(r_0)} \leq 1$$

and a brief observance of the graph of $\alpha_0 \mapsto \sin \alpha_0$, in particular the symmetry property $\sin \alpha_0 = \sin(\pi - \alpha_0)$, teaches us that precisely for all α_0 with

$$\alpha_g \leq \alpha_0 \leq \pi - \alpha_g$$

a Wegener-sight ray occurs, see Fig. 12. □

The interval $[\alpha_g, \pi - \alpha_g]$ indeed is symmetric with respect to the midpoint $\alpha_0 = \frac{1}{2}\pi$. We note that in the exceptional case where the function increases monotonously from the initial value $n_M(r_0)$ on, then $r_m = r_0$ and we have $\alpha_g = \frac{1}{2}\pi$ which reduces the interval $[\alpha_g, \pi - \alpha_g]$ to the singleton set $\{\frac{1}{2}\pi\}$ and thus the Wegener-sector to the horizontal halfline. The inclination α_g as defined by (7), with some good will could be called the limit angle, that in the case of Theorem 1 exactly corresponds to $\angle CWB$.

The entire set-up of Theorem 2, including its proof, is also applicable to Theorem 1, including cases with more than two discrete layers. This follows from formula (4). The existence of a global minimum of the function n_M surely holds in the atmospheric situation where $\lim_{r \rightarrow \infty} n(r) = 1$, compare Fig. 13, right.

Example 1 (Continuous Wegener-sector) The simplest continuous example of a Wegener-sector is given in a qualitative way for a modified refraction index profile n_M as in Fig. 13, right. This function has exactly one maximum and gives rise to a sight ray bundle as in the same figure, left. This example is playing a central role in our story.

The Principles of Fermat and Hamilton We now turn to a dynamical set-up, where next to the positions $q = (x, y)$ or (φ, r) , also the velocities $\dot{q} = (\dot{x}, \dot{y})$ and $(\dot{\varphi}, \dot{r})$ come into play. Thereby the Fermat Principle is extended to a more general Variational Principle, based on methods as developed in the 18th and 19th century by, among others, Euler [13], Lagrange [26] and Hamilton [20]. Meanwhile also elementary differential geometry is introduced: light rays turn out to exactly follow the geodesics in an appropriated Riemannian metric [36]. In this set-up calculus of variations is applied with help of the Euler–Lagrange equations and a translation of these that leads to the Hamilton’s Principle for geometrical optics [2, 20].

By the Chain Rule for a given curve $\tau \mapsto q(\tau)$, that acts as a potential light ray, we can express the (infinitesimal) time as

$$dt = n(q(\tau)) \|\dot{q}(\tau)\| d\tau, \quad (8)$$

where $\dot{q}(\tau) = dq(\tau)/d\tau$ as a tangent vector attached to the point $q(\tau)$. Here $\| - \|$ indicate the Euclidean length.

According to the Fermat Principle we have to minimize

$$\int_{\tau_1}^{\tau_2} dt = \int_{\tau_1}^{\tau_2} n(q(\tau)) \|\dot{q}(\tau)\| d\tau$$

for any fixed end points $q(\tau_1)$ and $q(\tau_2)$. However, it is preferable to minimize

$$\mathcal{I}(q) = \int_{\tau_1}^{\tau_2} \frac{1}{2} n^2(q(\tau)) \|\dot{q}(\tau)\|^2 d\tau \quad (9)$$

as a function of the curve q : locally this is equivalent. Indeed, on a small interval the integrands

$$n(q(\tau)) \|\dot{q}(\tau)\| \quad \text{and} \quad \frac{1}{2} n^2(q(\tau)) \|\dot{q}(\tau)\|^2$$

are almost constant and whenever, for a given path q , one of these is minimal, so must be the other.

The integrand of (9)

$$L(q, \dot{q}) = \frac{1}{2} n^2(q) \|\dot{q}\|^2 \quad (10)$$

is known as the *Lagrangian* of this minimization problem; also to be interpreted as the (kinetic) energy of the present motion. The reformulation of the Fermat Principle then says that light rays extremize the integral (9). This is precisely what is known as Hamilton’s Principle [20].

The Riemannian metric then takes the form

$$G_q(\dot{q}_1, \dot{q}_2) = n^2(q) \langle \dot{q}_1, \dot{q}_2 \rangle, \quad (11)$$

where $\langle \dot{q}_1, \dot{q}_2 \rangle$ denotes the standard (Euclidean) inner product of the tangent vectors $\dot{q}_1$ and $\dot{q}_2$ attached to the point q . From this it follows that

$$L(q, \dot{q}) = \frac{1}{2} G_q \langle \dot{q}, \dot{q} \rangle, \quad (12)$$

compare (10), and we conclude that the light rays, as obtained by the Hamilton Principle, are exactly the geodesics of the Riemannian metric G . See [12, 36, 38] for general references on differential geometry. This provides us with a more modern entrance to geometric optics [4, 19].

Euler–Lagrange Equations Versus the Canonical Formalism We briefly summarize a few central elements from Variational Calculus, for details referring to [2, 17] and the corresponding bibliographies. First we formulate the following results without proof.

Theorem 3 (Euler–Lagrange [13, 26]) *Assume that $n = n(q)$ is a smooth function. Then a curve $\tau \in \mathbb{R} \mapsto q(\tau) \in \mathbb{R}^2$ satisfies the Euler–Lagrange equations*

$$\begin{aligned} \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_1} &= \frac{\partial L}{\partial q_1} \\ \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}_2} &= \frac{\partial L}{\partial q_2} \end{aligned} \quad (13)$$

if and only if the integral $\mathcal{I}$ of (9) is extremal under small variations of the curve q , where the end points $q(\tau_1)$ and $q(\tau_2)$ are kept fixed.

As before, the extremum locally is a minimum. Globally caustics can appear. By the Legendre-transformation an elegant transfer can be made from Euler–Lagrange to the *canonical* formalism of Hamilton.

Theorem 4 (Hamilton [2, 17]) *Consider the Legendre transformation $\mathcal{L} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$*

$$\begin{aligned} \mathcal{L} : (q_1, q_2, \dot{q}_1, \dot{q}_2) &\mapsto (q_1, q_2, p_1, p_2), \quad \text{where} \\ p_j &= \frac{\partial L}{\partial \dot{q}_j} = n^2(q) \dot{q}_j, \end{aligned} \quad (14)$$

for $j = 1, 2$.

The transformation $\mathcal{L}$ is invertible and the Hamiltonian

$$H : \mathbb{R}^4 \longrightarrow \mathbb{R}, \quad \text{where } H(p, q) = L \circ \mathcal{L}^{-1}(q, p) = L\left(q, \frac{p}{n^2(q)}\right) \quad (15)$$

has the form

$$H(p, q) = \frac{1}{2n^2(q)} \|p\|^2. \quad (16)$$

The solutions of the Euler–Lagrange equations (13) correspond to those of the canonical equations

$$\begin{aligned}\dot{q}_j &= \frac{\partial H}{\partial p_j} \\ \dot{p}_j &= -\frac{\partial H}{\partial q_j},\end{aligned}\tag{17}$$

for $j = 1, 2$.

Remarks

- In the atmosphere of the round earth, the Riemannian metric G reads

$$G_q(\dot{q}, \dot{q}) = n^2(\varphi, r)(r^2 d\varphi^2 + dr^2),\tag{18}$$

classically one writes $G_q(\dot{q}, \dot{q}) = ds^2$. So we have that $(q_1, q_2) = (\varphi, r)$, which leads to the momenta

$$p_1 = n^2(q)r\dot{\varphi}, \quad p_2 = n^2(q)\dot{r}.\tag{19}$$

- The canonical equations (17) are also known as *Hamilton–Jacobi* equations. Both systems (13) and (17) in the present optical context are known as the *eikonal* equations of geometrical optics. Note that presently no rotation symmetry is assumed, which makes our set-up generally applicable. Also it is straightforward to extend the theory to a full 3-dimensional position vector q .
- It is well-known that the present theory extends far beyond geometric optics, as was first observed by Hamilton himself [21, 22]. For a wider overview with classical mechanics see Arnold [2], but the Variational Principle as such, extends all over physics and chemistry, as well as Hamiltonian variations do. Compare with, e.g., [14].

Noether’s Principle The rotation symmetry assumption amounts to φ -independence of the refraction index profile: we have $n = n(r)$. This gives rise to the fact that neither L nor H depends on $q_1 = \varphi$, which classically is called a *cyclic* variable. From (13) it immediately follows that

$$\dot{p}_1 = \frac{\partial L}{\partial q_1} \equiv 0,$$

i.e. to conservation of the angular momentum p_1 . If for given constants E and M we write $H(p, q) = E$ and $p_1 = M$, a brief calculation reveals that

$$C = \frac{M}{\sqrt{E}},\tag{20}$$

also compare with Fig. 6 and formula (3). Similar considerations hold for the flat earth atmosphere. For a discussion on Bernoulli’s brachistochrone in these terms, see [8, 9].

Remarks

- Compare this with the conservation of angular momentum for the motion of a particle in a central force field, (or Kepler’s second law) [2, 17]. It seems that the term ‘cyclic variable’ is derived from this setting.

Fig. 14 *The Bottle*: A surface of revolution, the geodesics of which correspond to sight rays of Fig. 13 (left). The *tightest parallel circle* acts as the boundary of the warm layer in which rays can be either refracted (left) or reflected (bottom). The right figure gives limit angle behavior, compare Fig. 8 (right). The warm layer stretches out upwardly, while the observer is positioned somewhere near the widest parallel

And, exactly as in the central force field setting, we here also can reduce the symmetry to a one degree of freedom system on the (q_2, p_2) -plane, where $q_2 = r$ and $p_2 = n^2(r)\dot{r}$.

- There exists a grand connection between symmetries and conservation laws, here coined as the Noether Principle [32]; for an elaborate treatment compare with [2]. In our story this principle is applied to both translation and rotation symmetry.

Surfaces of Revolution The question to be addressed here is under what conditions the geometric optics of the round earth atmosphere can be considered as the geodesic problem on a surface of revolution as embedded in the 3-dimensional space.

We start by generally considering a surface of revolution $S \subset \mathbb{R}^3$, that comes into existence by revolving a planar curve around a disjoint line in the same plane. A natural elaboration runs as follows [2, 12]. We take Cartesian coordinates (x, y, z) , in which the given line is the z -axis (to be considered as ‘vertical’) and assume the curve to lie in the (x, z) -plane, parametrized as

$$x = f(u), \quad z = g(u), \quad (21)$$

where u runs over an open interval. Here always $f(u) > 0$ and for simplicity we also assume that

$$(f'(u))^2 + (g'(u))^2 \equiv 1, \quad (22)$$

which expresses that u is an arclength parameter. If we first introduce cylinder coordinates (φ, r, z) by $x = r \cos \varphi$ and $y = r \sin \varphi$, then the surface S is parametrized by φ and u as follows:

$$x = f(u) \cos \varphi, \quad y = f(u) \sin \varphi, \quad z = g(u). \quad (23)$$

Note that for a point of S with coordinates (φ, u) the distance to the z -axis is given by $f(u)$.

The corresponding Riemannian metric $G = G_q(\dot{q}, \dot{q})$, as inherited from the ambient space $\mathbb{R}^3$, is easily seen to have the form

$$G_q(\dot{q}, \dot{q}) = \langle \dot{q}, \dot{q} \rangle_{G_q} = f^2(u)\dot{\varphi}^2 + \dot{u}^2 \quad (24)$$

with corresponding Lagrangian (12)

$$L(q, \dot{q}) = \frac{1}{2} G_q(\dot{q}, \dot{q}),$$

where now $q = (\varphi, u)$. We recall that also here a conservation law (3) applies, where α is the inclination with the vertical direction. This law is known as Clairaut's Theorem [2, 12]. Also the relation (20) is valid here.

Remarks

- For completeness we also give the Hamiltonian formulation. First of all the momenta are given by $p_1 = f^2(u)\dot{\varphi}$ and $p_2 = \dot{u}$. For the Hamiltonian we then have

$$H(p, q) = \frac{1}{2} \left(\frac{p_1^2}{f^2(q_2)} + p_2^2 \right),$$

from which we see that $q_1 = \varphi$ is a cyclic variable and thus p_1 a conserved quantity.

- Fixing $p_1 = M$, the reduced Hamiltonian reads

$$H_M(p_2, q_2) = \frac{1}{2} p_2^2 + V_M(q_2), \quad \text{where } V_M(q_2) = \frac{M^2}{2f^2(q_2)} \quad (25)$$

is the effective potential. From this we see that the reduced system has the oscillator format $\ddot{u} = -V'_M(u)$. See Figs. 15 and 16 for a reduced planar phase portrait (including effective potential) and a 3-dimensional reconstruction of this in a hypersurface $p_1 = M$.

On the one hand we consider the surface of revolution (23) with the metric G as in (24), and on the other hand the optical configuration plane $\mathbb{R}^2 \setminus \{0\} = \{\varphi, r\}$, endowed with the Riemannian metric

$$G_{\varphi, r}^{\text{opt}}((\dot{\varphi}, \dot{r}), (\dot{\varphi}, \dot{r})) = n^2(r)(r^2\dot{\varphi}^2 + \dot{r}^2), \quad (26)$$

asking when this can be isometrically embedded as a surface of revolution in $\mathbb{R}^3$.

Comparison with the metric (24) teaches us that if we want to identify the two, the curve (21) needs to have the general format

$$x = f(r), \quad z = g(r), \quad (27)$$

i.e., parametrized by the variable r , where necessarily $f(r) = rn(r) = n_M(r)$, which is exactly the *modified refraction index profile* (6). After this it remains to specify the function g .

Instead of (φ, u) we now use (φ, r) to parametrize $\mathcal{S}$. It is straightforward to express both metric and Lagrangian in these new variables. Indeed, the Riemannian metric (24) on $\mathcal{S}$ obtains the form

$$G_{\varphi, r}^{\text{omw}}((\dot{\varphi}, \dot{r}), (\dot{\varphi}, \dot{r})) = f^2(r)\dot{\varphi}^2 + ((f')^2(r) + (g')^2(r))\dot{r}^2, \quad (28)$$

Fig. 15 Effective potential (25) and corresponding reduced phase portrait concerning the *Bottle* of Example 1 and Fig. 14. As mentioned earlier, this set-up fits with the light ray bundle from Fig. 13. See there for a modified refraction index profile

whence the Lagrangian $L^{\text{omw}}(q, \dot{q}) = \frac{1}{2} G^{\text{omw}}(\dot{q}, \dot{q})$ now reads

$$L^{\text{omw}}(\varphi, r, \dot{\varphi}, \dot{r}) = \frac{1}{2} f^2(r) \dot{\varphi}^2 + \frac{1}{2} ((f')^2(r) + (g')^2(r)) \dot{r}^2.$$

So, given $n = n(r)$ we have to find $f = f(r)$ and $g = g(r)$, compare (27), such that the following equations are satisfied:

$$\begin{aligned} f^2(r) &= r^2 n^2(r) \quad (\text{see above}), \\ (f')^2(r) + (g')^2(r) &= n^2(r). \end{aligned} \quad (29)$$

We have the following result.

Theorem 5 (Isometric embeddability as a surface of revolution) *The Riemannian manifold $\mathbb{R}^2 \setminus \{0\} = \{\varphi, r\}$ with metric G^{opt} (26) determined by the refraction index profile $n = n(r)$ can be embedded isometrically as a surface of revolution in $\mathbb{R}^3$*

$$x = f(r) \cos \varphi, \quad y = f(r) \sin \varphi, \quad z = g(r),$$

according to (29), if and only if for each r one has

$$\frac{-2n(r)}{r} \leq n'(r) \leq 0. \quad (30)$$

Fig. 16 Sketch of the invariant surfaces in the dynamics of the *Bottle* in Fig. 14. In the entire Hamiltonian system (17) corresponding to (24), we consider an invariant 3-dimensional hypersurface $p_1 = M$, see (19). Qualitatively this figure comes into existence by revolving the reduced phase portrait in Fig. 15 around the axis $r = 0$

Proof According to (29), given the choice $f(r) = rn(r)$, also the equation

$$(f')^2(r) + (g')^2(r) = n^2(r) \quad (31)$$

has to be solved for g . From the Leibniz formula it follows that $f'(r) = n(r) + rn'(r)$. Hence (31) turns into

$$n^2(r) + 2rn(r)n'(r) + r^2(n')^2(r) + (g')^2(r) = n^2(r),$$

or

$$(g')^2(r) = -(2rn(r)n'(r) + r^2(n')^2(r)).$$

The latter equation can be solved if and only if

$$2n(r)n'(r) + r(n')^2(r) \leq 0,$$

or, equivalently, if

$$n'(r)(2n(r) + rn'(r)) \leq 0.$$

In principle we have either

$$n'(r) \leq 0 \quad \text{and} \quad 2n(r) + rn'(r) \geq 0$$

or

$$n'(r) \geq 0 \quad \text{and} \quad 2n(r) + rn'(r) \leq 0.$$

A priori we know that $n(r) = f(r)/r > 0$, which cancels the latter possibility. So we conclude that only

$$n'(r) \leq 0 \quad \text{and} \quad 2n(r) + rn'(r) \geq 0$$

remains, or equivalently,

$$\frac{-2n(r)}{r} \leq n'(r) \leq 0. \quad \square$$

It thus turns out that this embeddability holds true for a large amount of relevant refraction index profiles $n = n(r)$.

Fig. 17 *Left:* Cuspoidal caustic in geometric optics. *Right and bottom:* Wave optical refinements [35] p. 266

4 Final Remarks

This paper is concluded by a number of miscellaneous remarks on related phenomena and literature.

4.1 Related Phenomena

A first comment concerns acoustics: indeed the entire theory of this paper also applies when ‘light’ is replaced by ‘sound’ [34]. One interesting complication then is that the medium, the air, may change considerably with time. There is also quite a number of interesting optical phenomena in the atmosphere that we did not touch in this paper.

A *caustic* arises when light rays accumulate on each other, for instance as the envelope of other light rays. In Fig. 17, left, we depict a cusp as this occurs as the envelope of a ray bundle generated by a circular light source. A cuspoidal caustic, under suitable incidence of light, can already be observed on the bottom of an almost empty coffee cup. In Catastrophe Theory [3, 35], and more generally in Singularity Theory [4, 5, 19], geometric shapes of caustics are given that have a certain universality and robustness. It is interesting to study refined versions of caustics within the theories of wave optics. For the cusp see Fig. 17, center and right. Compare Poston and Stewart [35] p. 266. For more mathematical background see [4, 5, 7, 19]. Among others, the fold catastrophe occurs in the rainbow, where it can be observed in a frequency dependent form [31, 35]. Here it is important that the refraction index depends on the frequency: roughly speaking red light is bent stronger than green light. The latter is also responsible for the occurrence of the *green flash*, a phenomenon that may occur at sunset: in fact the red sun already has set and the green one—that normally overlap to give the familiar white light—still is briefly visible [33, 42].

Fig. 18 Iso-energetic Poincaré map of the periodically perturbed *Bottle*, compare Fig. 15. The orbits of this map have come into existence by iterations that follow the integral curves of the canonical equations (17) with Hamiltonian (16), while they go around the earth. As in many nonlinear systems, we see the co-existence of order and *chaos*

4.2 Nonlinearity

By the symmetry assumption the optical system (17) with Hamiltonian (16) is completely integrable. If we perturb our *Bottle* model in a periodic way, say by

$$n(\varphi, r) = n_0(r) + \varepsilon n_{\text{pert}}(\varphi, r), \quad (32)$$

we destroy the integrability. In Fig. 18 a few numerically obtained orbits of an iso-energetic Poincaré map are shown for small $|\varepsilon|$, which display the familiar mixture of quasi-periodicity and chaos.

4.3 Background Reading

In the area of geometric optics there is a lot more to read. We surely should mention the work of Carathéodory [10]. More recently we have, e.g., Leonhard and Philbin [29]. Also worth reading are the classical Feynman Lectures on Physics [14], in particular part I, § 26, and, somewhat later, Guillemin and Sternberg [19], Chap. III, or Poston and Stewart [35], Chap. 12. For more physical background of atmospheric optics see, for instance, Humphries [23]. There is a large amount of physical-meteorological literature on the fascinating phenomenology around the setting sun. In this respect I like to mention Floor [15, 16], in recent years increasingly illustrated by fantastic photo and film material. For the latter the reader may consult the internet as well.

4.4 Mathematics and Physics

The contributions of Minnaert [31], of Van der Werf [39], but also for instance of Levi [30], evidently expose the connection and harmony of mathematics and the natural sciences. The present article attempts to contribute to this insight as well.

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