Implicit Hamiltonian formulation of bond graphs

G. Golo[1,2], A.J. van der Schaft[2,1], P.C. Breedveld[1,3], B.M. Maschke[4]

[1]Cornelis J. Drebbel Institute for Mechatronics, Enschede, The Netherlands
[2]University of Twente, Faculty of Mathematical Sciences, The Netherlands
[3]University of Twente, Faculty of Electrical Engineering
[4]Université Claude Bernard Lyon-1,Laboratoire d'Automatique et de Génie des Procédés, France

Abstract

This paper deals with mathematical formulation of bond graphs. It is proven that the power continuous part of bond graphs, the junction structure, can be associated with a Dirac structure and that equations describing a bond graph model correspond to an implicit port-controlled Hamiltonian system with dissipation. The condition for well-posedness of the modelled system are given and representations suitable for numerical simulation are derived. The index of the representations is analysed and sufficient conditions for computational efficiency are given. The results are applied to some models arising in automotive applications.

1 Introduction

In most of the current modelling and simulation approaches of physical systems, some sort of *network representation* is used to make explicit that the physical system model under consideration is an interrelated set of elementary conceptss. This way of modelling has several advantages. From a physical point of view it is usually natural to regard the system as composed of functional components, possibly from different domains (mechanical, electrical, and so on). The knowledge about subsystems can be stored in libraries, and is reusable for later occasions. Due to this modularity, the modelling process can be performed in an *iterative* way, gradually refining -if necessary- the model by adding other subsystems. Furthermore, the approach is suited to general control design where the overall behaviour of the system is sought to be improved by the addition of the other subsystems or controlling devices. From a systematic-theoretic point of view this modular approach naturally emphasises the need for the models of systems with external variables.

In this paper we concentrate on the mathematical description of a network representation of *energy-conserving physical systems* called bond graphs. The bond graph approach is graphically oriented and its outcome, the bond graph model, represents a multiport system based on energy flows [1], [2], [3]. An important feature of this technique is that the cumbersome job of writing down equations describing a physical system is replaced by drawning a picture of the same by using a finite set of symbols. Once the bond graph is obtained, we can start with its analysis and with the derivation of equations suitable for the numerical simulation. These tasks can be fully automated [4], [5].

The purpose of this paper is to develop a theory of bond graphs in a mathematical language. The mathematical results that have been obtained so far are sporadic. The approaches reported in [6], [7], [8], [9], [10], [11], [12] represent combinatorial theory of bond graphs. Here, a geometric approach is developed, elaborating on previous work on the structure of such dynamical systems [13], [14], [15], [16]. Firstly, we consider the power continuous part of a bond graph model, called junction structure, and prove that every junction structure can be associated to the geometric notion of a Dirac structure [17], [18], which represents a general power conserving interconnection structure of a physical system. Secondly, we prove that equations describing a bond graph model corresponds to implicit port-controlled Hamiltonian system with dissipation [17], [18], [19]. Also, the condition for well-posedness of the modelled system are given and representations suitable for the numerical simulation are derived. The index of representations is analysed and sufficient conditions for the representations to be computationally efficient are given. This theory is applied to some models arising in automotive applications.

2 Bond graph models

A generic bond graph is shown in Figure 1. The bond graph formalism used here is based on the classification of physical variables rooted in thermodynamics. This leads to the generalised bond graph formalism that admits only one type of storage elements [2], [3].

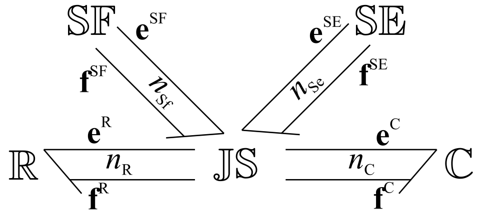

Figure 1: Generic bond graph

Here, $\mathbb{C}$ stands for the collection of C-type multiports (energy storage multiport), $\mathbb{R}$ for the collection of R-type multiports (energy dissipative element), $\mathbb{SE}$ for the collection of SE-type ports and $\mathbb{SF}$ for the collection of SF-type ports. These elements are called *power discontinuous elements* [2], [3]. $\mathbb{JS}$ stands for the junction structure. $\mathbb{JS}$ is connected to power discontinuous elements by multi-bonds. With every multibond we can associate two power variables: effort $\mathbf{e}^\alpha$ and flow $\mathbf{f}^\alpha$, $\alpha \in \{\mathrm{SE}, \mathrm{SF}, \mathrm{C}, \mathrm{R}\}$. They belong to dual spaces and their duality product $\langle \mathbf{e}^\alpha | \mathbf{f}^\alpha \rangle$ represents the power that goes from or to the junction structure, depending the positive orentation indicated by half-arrow, to or from the α-type multiport. The integer n^α inside a multibond denotes the dimension of the vectors $\mathbf{e}^\alpha$ and $\mathbf{f}^\alpha$. If $n^\alpha = 1$ then a multibond is simply called bond.

The constitutive relations for the $\mathbb{C}$ element are given by

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}^{\mathrm{C}}, \\ \mathbf{e}^{\mathrm{C}} &= \tfrac{\partial H}{\partial \mathbf{x}} (\mathbf{x}), \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathcal{X}$ is the vector of energy variables (such as displacement, momentum, charge, flux, volume, etc), $H \in C^\infty(\mathcal{X})$ is energy of the system and $\mathcal{X}$ is an n^{C}-dimensional smooth manifold.

The constitutive relation for the $\mathbb{R}$ element is given by

$$\Phi\left(\mathbf{e}^{\mathrm{R}}, \mathbf{f}^{\mathrm{R}}, \mathbf{x}\right) = 0, \tag{2}$$

where (2) defines n^{R}-dimensional set $\forall \mathbf{x} \in \mathcal{X}$. The set has the properties that every pair belonging to it satisfies the inequality $\left(\mathbf{e}^{\mathrm{R}}\right)^{\mathrm{T}} \mathbf{f}^{\mathrm{R}} \geq 0$.

Furthermore, the constitutive relations of $\mathbb{SE}$, $\mathbb{SF}$ elements are described by

$$\begin{aligned} \mathbf{e}^{\mathrm{SE}} &= \mathbf{u}^{\mathrm{SE}} \in \mathcal{R}^{n^{\mathrm{SE}}}, \\ \mathbf{e}^{\mathrm{SF}} &= \mathbf{u}^{\mathrm{SF}} \in \mathcal{R}^{n^{\mathrm{SF}}}. \end{aligned} \tag{3}$$

As said before, the junction structure is a power continuous part of a bond graph. It means that it can not accumulate, dissipate or generate power. It only distributes the power between the power discontinuous elements in a power continuous way. A generic junction structure is shown in Figure 2.

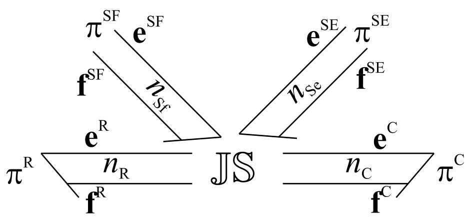

Figure 2: Junction structure

Here, π^{α} stands for α-mulitport (port if $n^{\alpha} = 1$). The ports represents the connection of JS to the power discontinuous elements. The main feature of the junction structure is power continuity, i.e. a zero power balance at its ports, that is

$$\left\langle \mathbf{e}^{\mathrm{SE}} \middle| -\mathbf{f}^{\mathrm{SE}} \right\rangle + \left\langle -\mathbf{e}^{\mathrm{SF}} \middle| \mathbf{f}^{\mathrm{SF}} \right\rangle + \left\langle \mathbf{e}^{\mathrm{C}} \middle| \mathbf{f}^{\mathrm{C}} \right\rangle + \left\langle \mathbf{e}^{\mathrm{R}} \middle| \mathbf{f}^{\mathrm{R}} \right\rangle = 0. \tag{4}$$

Therefore JS relates the flows $\mathbf{f} = \left(-\mathbf{f}^{\mathrm{SE}}, \mathbf{f}^{\mathrm{SF}}, \mathbf{f}^{\mathrm{C}}, \mathbf{f}^{\mathrm{R}}\right)$ and efforts $\mathbf{e} = \left(\mathbf{e}^{\mathrm{SE}}, -\mathbf{e}^{\mathrm{SF}}, \mathbf{e}^{\mathrm{C}}, \mathbf{e}^{\mathrm{R}}\right)$ of the ports to each other. The nature of these relations is examined in section 4.

3 Dirac structures

In this section we recall the definition of a Dirac structure as a general representation of a power conserving interconnection structure of a physical system [18].

We start with the space of power variables $(T_{\mathbf{x}}\mathcal{X} \times \mathcal{V}) \times (T^*_{\mathbf{x}}\mathcal{X} \times \mathcal{V}^*)$, for some finite dimensional linear space $\mathcal{V}$ and the smooth finite dimensional manifold $\mathcal{X}$, with the power defined by

$$P = \langle \mathbf{e} | \mathbf{f} \rangle, \ (\mathbf{e}, \mathbf{f}) \in (T_{\mathbf{x}}\mathcal{X} \times \mathcal{V}) \times (T^*_{\mathbf{x}}\mathcal{X} \times \mathcal{V}^*)$$

where $\langle \mathbf{e} | \mathbf{f} \rangle$ denotes the duality product. We call $T_{\mathbf{x}}\mathcal{X} \times \mathcal{V}$ the space of flows $\mathbf{f}$, and the dual space $T^*_{\mathbf{x}}X \times V^*$ the space of efforts. Closely related to the definition of power there exists a canonically defined bilinear form $\langle\langle , \rangle\rangle$ on the space of power variables $(T_{\mathbf{x}}\mathcal{X} \times \mathcal{V}) \times (T^*_{\mathbf{x}}\mathcal{X} \times \mathcal{V}^*)$, defined as

$$\left\langle\left\langle \left(\mathbf{f}^a, \mathbf{e}^a\right), \left(\mathbf{f}^b, \mathbf{e}^b\right) \right\rangle\right\rangle := \left\langle \mathbf{e}^a \middle| \mathbf{f}^b \right\rangle + \left\langle \mathbf{e}^b \middle| \mathbf{f}^a \right\rangle.$$

Definition 1 (Definition of Dirac structure [17], [20]) *A Dirac structure on a differentiable manifold* $\mathcal{X} \times \mathcal{V}$ *is given by a smooth vector subundle* $\mathcal{D} \subset (\mathcal{T}\mathcal{X} \times \mathcal{V}) \times (\mathcal{T}^*\mathcal{X} \times \mathcal{V}^*)$ *such that the linear space* $\mathcal{D}(\mathbf{x}) \subset (\mathcal{T}_{\mathbf{X}}\mathcal{X} \times \mathcal{V}) \times (\mathcal{T}^*_{\mathbf{X}}\mathcal{X} \times \mathcal{V}^*)$ *satisfies the relation*

$$\mathcal{D}^{\perp}(\mathbf{x}) = \mathcal{D}(\mathbf{x}),$$

where $\perp$ *denotes orthogonal compliment with respect to the bilinear form* $\langle\langle , \rangle\rangle$.

Locally about every point $(\mathbf{x}, \mathbf{v}) \in \mathcal{X} \times \mathcal{V}$, we may find $n \times n$ (n is dimension of the manifold $\mathcal{X} \times \mathcal{V}$) matrices $\mathbf{E}(\mathbf{x})$ and $\mathbf{F}(\mathbf{x})$ depending smoothly on $\mathbf{x}$, such that locally [19], [18]

$$\mathcal{D}(\mathbf{x}) = \{(\mathbf{f}, \mathbf{e}) \in (T_{\mathbf{x}}\mathcal{X} \times \mathcal{V}) \times (T^*_{\mathbf{x}}\mathcal{X} \times \mathcal{V}^*)\}$$

Here, the matrices $\mathbf{F}(\mathbf{x})$, $\mathbf{E}(\mathbf{x})$ satisfy the following two conditions

rank condition:

$$\operatorname{rank}[\mathbf{F}(\mathbf{x})\ \mathbf{E}(\mathbf{x})] = n,$$

power conservation:

$$\mathbf{E}(\mathbf{x})\mathbf{F}^{\mathrm{T}}(\mathbf{x}) + \mathbf{F}(\mathbf{x})\mathbf{E}^{\mathrm{T}}(\mathbf{x}) = 0.$$

This representation is called kernel representation. Closely related to the kernel representation there exists an image representation defined by [20]

$$\mathcal{D}(\mathbf{x}) = \operatorname{Im}\left(\begin{bmatrix} \mathbf{E}^{\mathrm{T}}(\mathbf{x}) \\ \mathbf{F}^{\mathrm{T}}(\mathbf{x}) \end{bmatrix}\right) = \left\{(\mathbf{f}, \mathbf{e}) : \mathbf{f} = \mathbf{E}^{\mathrm{T}}(\mathbf{x})\lambda, \mathbf{e} = \mathbf{F}^{\mathrm{T}}(\mathbf{x})\lambda\right\}$$

4 Geometric formulation of a bond Graphs

As said in the section 2, junction structure relates efforts and flows to each other. Suppose that this relation at the point $\mathbf{x}$ is given by

$$\mathbf{J}(\mathbf{e}, \mathbf{f}, \mathbf{x}) = 0$$

Define the space $\mathcal{D}(\mathbf{x})$ as

$$\mathcal{D}(\mathbf{x}) = \{(\mathbf{e}, \mathbf{f}) : \mathbf{J}(\mathbf{e}, \mathbf{f}, \mathbf{x}) = 0\}$$

Proposition 1 $\mathcal{D}(\mathbf{x})$ *is a Dirac structure on* $\mathcal{X} \times \mathcal{V}$ *where* $\mathcal{X}$ *is an* n^{C}*-dimensional smooth manifold and* $\mathcal{V} = \mathcal{R}^{n^{\mathrm{SE}}+n^{\mathrm{SF}}+n^{\mathrm{R}}}$.

Proof: The junction structure is composed of junctions (1-junctions, 0-junctions) and transducers(transformers and gyrators).

The constitutive relations of 1-junction (see Figure 3) are given by

$$-\sum_{i=1}^{j} e_i + \sum_{i=j+1}^{k} e_i = 0,$$

$$f_1 = \ldots = f_j = \ldots = f_k.$$

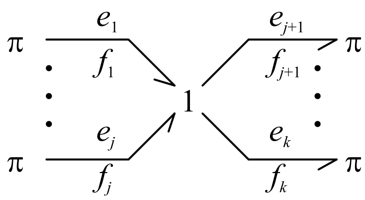

Figure 3: 1-junction

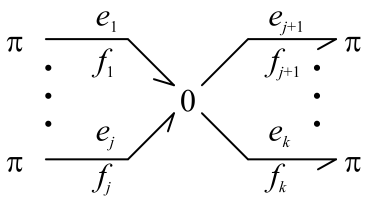

Figure 4: 0-junction

The constitutive relation of 0-junction (see Figure 4) are given by

$$-\sum_{i=1}^{j} f_i + \sum_{i=j+1}^{k} f_i = 0,$$
$$e_1 = ... = e_j = ... = e_k.$$

A transformer having n_1 incoming ports and n_2 outgoing ports is shown in Figure 5.

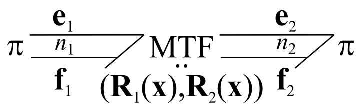

Figure 5: Transformer

The constitutive relations of the transformer are given by

$$[\mathbf{R}_1(\mathbf{x}) \;\; \mathbf{R}_2(\mathbf{x})] \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} = 0$$

$$(\ker([\mathbf{R}_1(\mathbf{x}) \;\; \mathbf{R}_2(\mathbf{x})]))^{\mathrm{T}} \begin{bmatrix} -\mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = 0$$

So the constitutive relations of transformer whose ratio is $(-\mathbf{R}(\mathbf{x}), \mathbf{I}_{n_2})$ are given by

$$\mathbf{e}_1 = \mathbf{R}^{\mathrm{T}}(\mathbf{x})\,\mathbf{e}_2,$$
$$\mathbf{f}_2 = \mathbf{R}(\mathbf{x})\,\mathbf{f}_1,$$

which represents the regular form of transformer. Finally, a gyrator having n_1 incoming ports and n_2 outgoing ports is shown in Figure 6. The constitutive relation of the gyrator are given

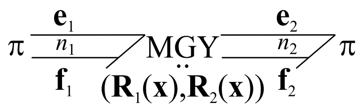

Figure 6: Gyrator

by

$$[\mathbf{R}_1(\mathbf{x}) \ \ \mathbf{R}_2(\mathbf{x})] \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{e}_2 \end{bmatrix} = 0,$$
$$(\ker([\mathbf{R}_1(\mathbf{x}) \ \ \mathbf{R}_2(\mathbf{x})]))^{\mathrm{T}} \begin{bmatrix} -\mathbf{e}_1 \\ \mathbf{f}_2 \end{bmatrix} = 0.$$

The elements of the matrices $\mathbf{R}_1(\mathbf{x}), \mathbf{R}_2(\mathbf{x})$ are the smooth function on the manifold $\mathcal{X}$ and rank $[\mathbf{R}_1(\mathbf{x}) \ \ \mathbf{R}_2(\mathbf{x})]$ is constant $\forall \mathbf{x} \in \mathcal{X}$. If the matrices $\mathbf{R}_1(\mathbf{x}), \mathbf{R}_2(\mathbf{x})$ do not depend on $\mathbf{x}$ then the symbols $\mathbf{MTF}, \mathbf{MGY}$ are replaced by $\mathbf{TF}, \mathbf{GY}$.

It may be proved that any of these elements can be related to a Dirac structure [5]. By using the fact that the composition of two or more Dirac structures is a Dirac structure [20], one concludes that $\mathcal{D}(\mathbf{x})$ is a Dirac structure on $\mathcal{X} \times \mathcal{V}$.

This means that relations describing junction structure are given by:

$$\underbrace{\begin{bmatrix} \left(\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{F}^{\mathrm{SF}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{F}^{\mathrm{R}}(\mathbf{x})\right)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}}_{\mathbf{F}(\mathbf{x})} \begin{bmatrix} -\mathbf{f}^{\mathrm{SE}} \\ \mathbf{f}^{\mathrm{SF}} \\ \mathbf{f}^{\mathrm{C}} \\ \mathbf{f}^{\mathrm{R}} \end{bmatrix} + \underbrace{\begin{bmatrix} \left(\mathbf{E}^{\mathrm{SE}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{E}^{\mathrm{SF}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{E}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{E}^{\mathrm{R}}(\mathbf{x})\right)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}}_{\mathbf{E}(\mathbf{x})} \begin{bmatrix} \mathbf{e}^{\mathrm{SE}} \\ -\mathbf{e}^{\mathrm{SF}} \\ \mathbf{e}^{\mathrm{C}} \\ \mathbf{e}^{\mathrm{R}} \end{bmatrix} = 0 \tag{5}$$

The graphical representation of a generic bond graph together with all constitutive relations is shown in Figure 7.

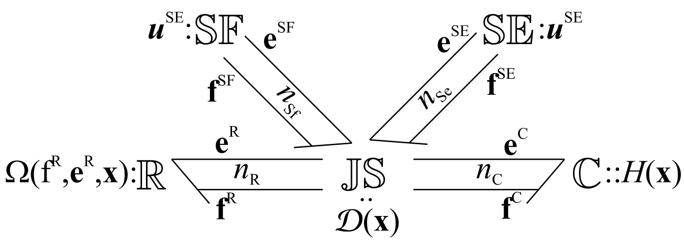

Figure 7: 1-junction

The equation (5) can be rewritten in the following way

$$\left(-\mathbf{f}^{\mathrm{SE}}, \mathbf{f}^{\mathrm{SF}}, \mathbf{f}^{\mathrm{C}}, \mathbf{f}^{\mathrm{R}}, \mathbf{e}^{\mathrm{SE}}, -\mathbf{e}^{\mathrm{SF}}, \mathbf{e}^{\mathrm{C}}, \mathbf{e}^{\mathrm{R}}\right) \in \mathcal{D}(\mathbf{x}). \tag{6}$$

By inserting (1), (2), (3) into (6), the equations describing the dynamics of a system, whose bond graph model is shown in Figure 7, are obtained:

$$\begin{gathered} \left(-\mathbf{f}^{\mathrm{SE}}, \mathbf{u}^{\mathrm{SF}}, \dot{\mathbf{x}}, \mathbf{f}^{\mathrm{R}}, \mathbf{u}^{\mathrm{SE}}, -\mathbf{e}^{\mathrm{SF}}, \frac{\partial H(\mathbf{x})}{\partial \mathbf{x}}, \mathbf{e}^{\mathrm{R}}\right) \in \mathcal{D}(\mathbf{x}), \\ \Omega\left(\mathbf{f}^{\mathrm{R}}, \mathbf{e}^{\mathrm{R}}, \mathbf{x}\right) = 0. \end{gathered} \tag{7}$$

Therefore, the equations describing the system whose bond graph model is shown in Figure 7 are in the form of implicit **p**ort-**c**ontrolled **H**amiltonian (PCH) system with the dissipation [19].

5 Well posedness and equation suitable for numerical simulation

The equation given by (7) are not in form suitable for numerical simulation. In this section we show how (7) may be transformed into more suitable form. Before that, we introduce some assumptions about the system described by (7). First we introduce the definitions of well-posedness of implicit PCH system with dissipation.

Definition 2 (Well-posedness of implicit PCH system with dissipation) *Implicit PCH system with dissipation is well-posed if*

$$\dim\left(\mathcal{D}^{\mathrm{U}}(\mathbf{x})\right) = n^{\mathrm{SE}} + n^{\mathrm{SF}},\ \forall \mathbf{x} \in \mathbf{X},$$

where

$$\mathcal{D}^{\mathrm{U}}(\mathbf{x}) = \left\{\left(\mathbf{e}^{\mathrm{SE}}, \mathbf{f}^{\mathrm{SF}}\right) : \exists \left(\mathbf{f}^{\mathrm{SE}}, \mathbf{f}^{\mathrm{C}}, \mathbf{f}^{\mathrm{R}}, \mathbf{e}^{\mathrm{SE}}, \mathbf{e}^{\mathrm{R}}\right) \text{ s.t. } (\mathbf{f},\mathbf{e}) \in \mathcal{D}(\mathbf{x}) \text{ and } \mathbf{e}^{\mathrm{C}} = 0\right\}.$$

Proposition 2 (Well-posed implicit PCH with dissipation) *The system described by (7) is well-posed if and only if*

$$\operatorname{rank}\left(\begin{bmatrix} \left(\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{E}^{\mathrm{SF}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}} \end{bmatrix}\right) = n^{\mathrm{SE}} + n^{\mathrm{SF}} + \operatorname{rank}\left(\left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}\right),\ \forall \mathbf{x} \in \mathcal{X}.$$

Proof: An admissible effort $\mathbf{e}^{\mathrm{C}}$ is represented by $\mathbf{e}^{\mathrm{C}} = \left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}\lambda,\ \lambda \in \mathcal{R}^{n}$. Since $\mathbf{e}^{\mathrm{C}} = 0$, then $\lambda \in \ker\left(\left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}\right)$. It means that the space $\mathcal{D}^{\mathrm{U}}(\mathbf{x})$ may be rewritten as

$$\mathcal{D}^{\mathrm{U}}(\mathbf{x}) = \begin{bmatrix} \left(\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{E}^{\mathrm{SF}}(\mathbf{x})\right)^{\mathrm{T}} \end{bmatrix} \ker\left(\left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}\right).$$

It is clear that

$$\operatorname{rank}\left(\begin{bmatrix} \left(\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{E}^{\mathrm{SF}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \left(\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{E}^{\mathrm{SF}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}} \end{bmatrix} \ker\left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}\right) + \operatorname{rank}\left(\left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}\right) =$$
$$\operatorname{rank}\begin{bmatrix} \left(\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\right)^{\mathrm{T}} \ker\left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}} \\ \left(\mathbf{E}^{\mathrm{SF}}(\mathbf{x})\right)^{\mathrm{T}} \ker\left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}} \\ 0 \end{bmatrix} + \operatorname{rank}\left(\left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}\right) = \dim\left(\mathcal{D}^{\mathrm{U}}(\mathbf{x})\right) + \operatorname{rank}\left(\left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}\right).$$

Now the claiming of the proposition is straightforward.

Remark: The system is not well-posed if either $\operatorname{rank}\left(\left[\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\,\mathbf{E}^{\mathrm{SF}}(\mathbf{x})\right]\right) < n^{\mathrm{SE}} + n^{\mathrm{SF}}$ or $\operatorname{rank}\left(\left[\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\,\mathbf{E}^{\mathrm{SF}}(\mathbf{x})\right]\right) = n^{\mathrm{SE}} + n^{\mathrm{SF}}$ but some columns of $\mathbf{F}^{\mathrm{C}}(\mathbf{x})$ linearly depend on columns of the matrices $\mathbf{F}^{\mathrm{SE}}(\mathbf{x}), \mathbf{E}^{\mathrm{SF}}(\mathbf{x})$. In the first case, the matrix $\left[\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\, \mathbf{E}^{\mathrm{SF}}(\mathbf{x})\right]^{\mathrm{T}}$ is not a full rank matrix and one can find a nonzero matrix $[\mathbf{A}(\mathbf{x})\, \mathbf{B}(\mathbf{x})]$ such that

$$[\mathbf{A}(\mathbf{x})\, \mathbf{B}(\mathbf{x})]\left[\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\, \mathbf{E}^{\mathrm{SF}}(\mathbf{x})\right]^{\mathrm{T}} = 0,\ \forall \mathbf{x} \in \mathcal{X}.$$

By postmultiplying the last equation with $\lambda \in \mathcal{R}^n$, one obtains that

$$\mathbf{A}(\mathbf{x}) \underbrace{\left(\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\right)^{\mathrm{T}} \lambda}_{\mathbf{e}^{\mathrm{SE}}} + \mathbf{B}(\mathbf{x}) \underbrace{\left(\mathbf{E}^{\mathrm{SF}}(\mathbf{x})\right)^{\mathrm{T}} \lambda}_{\mathbf{f}^{\mathrm{SF}}} = \mathbf{A}(\mathbf{x})\, \mathbf{e}^{\mathrm{SE}} + \mathbf{B}(\mathbf{x})\, \mathbf{f}^{\mathrm{SF}} = 0$$

It means that junction structure implies a dependency between the efforts of SE-ports and the flows of SF-ports. In other words, the input signals can not be chosen arbitrarily. In the second case, one can find a full rank matrix $[\mathbf{A}(\mathbf{x})\, \mathbf{B}(\mathbf{x})\, \mathbf{C}(\mathbf{x})]$ such that both $[\mathbf{A}(\mathbf{x})\, \mathbf{B}(\mathbf{x})]$ and $\mathbf{C}(\mathbf{x})$ are non-zero matrices and such that the following relation is satisfied

$$[\mathbf{A}(\mathbf{x})\, \mathbf{B}(\mathbf{x})\, \mathbf{C}(\mathbf{x})]\left[\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\, \mathbf{E}^{\mathrm{SF}}(\mathbf{x})\, \mathbf{F}^{\mathrm{C}}(\mathbf{x})\right]^{\mathrm{T}} = 0,\ \forall \mathbf{x} \in \mathcal{X}.$$

Similarly, the last relation implies $\mathbf{A}(\mathbf{x})\, \mathbf{e}^{\mathrm{SE}} + \mathbf{B}(\mathbf{x})\, \mathbf{f}^{\mathrm{SF}} + \mathbf{C}(\mathbf{x})\, \mathbf{f}^{\mathrm{C}} = 0$. If the power variables $\mathbf{e}^{\mathrm{SF}}, \mathbf{f}^{\mathrm{SF}}$ are discontinuous function in time, then also the energy variables (the states of the system) are discontinuous function. Therefore, the sources have to be capable to generate an infinite amount of power, which has no physical justification.

Suppose that the system described by (7) is well-posed. The set of admissible efforts of C-ports is represented by

$$\mathcal{D}^{\mathrm{C}}(\mathbf{x}) = \left\{\mathbf{e}^{\mathrm{C}} : \exists \left(\mathbf{e}^{\mathrm{SE}}, -\mathbf{e}^{\mathrm{SF}}, \mathbf{e}^{\mathrm{R}}, -\mathbf{f}^{\mathrm{SE}}, \mathbf{f}^{\mathrm{SF}}, \mathbf{f}^{\mathrm{C}}, \mathbf{f}^{\mathrm{R}} \text{ s.t. } (\mathbf{f}, \mathbf{e}) \in \mathcal{D}(\mathbf{x})\right)\right\}.$$

Assumption 1 (Constant dimensionality of $\mathcal{D}^{\mathrm{C}}(\mathbf{x})$) *It is assumed that*

$$\dim\left(\mathcal{D}^{\mathrm{C}}(\mathbf{x})\right) = n_1^{\mathrm{C}},\ \forall \mathbf{x} \in \mathcal{X}.$$

Now, we show how equations suitable for the numerical simulation can be derived.

Proposition 3 *Consider implicit PCH system with dissipation described by (7). Supposed that the system is well-posed and that the assumption 1 holds. Then (7) is equivalent with the following representation:*

$$\dot{\mathbf{x}} = \mathbf{J}^{\mathrm{C}}(\mathbf{x}) \frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\, \lambda + \mathbf{G}^{\mathrm{C,R}}(\mathbf{x})\, \bar{\mathbf{f}}^{\mathrm{R}} + \mathbf{G}^{\mathrm{C,U}}(\mathbf{x})\, \mathbf{u}, \tag{8}$$

$$0 = \mathbf{G}^{\mathrm{T}}(\mathbf{x}) \frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}), \tag{9}$$

$$\bar{\mathbf{e}}^{\mathrm{R}} = -\left(\mathbf{G}^{\mathrm{C,R}}(\mathbf{x})\right)^{\mathrm{T}} \frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}) + \mathbf{G}^{\mathrm{R,U}}(\mathbf{x})\, \mathbf{u}, \tag{10}$$

$$\bar{\Omega}\left(\bar{\mathbf{f}}^{\mathrm{R}}, \bar{\mathbf{e}}^{\mathrm{R}}, \mathbf{x}\right) = 0, \tag{11}$$

$$y = -\left(\mathbf{G}^{\mathrm{C,U}}(\mathbf{x})\right)^{\mathrm{T}} \frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}) - \left(\mathbf{G}^{\mathrm{R,U}}(\mathbf{x})\right)^{\mathrm{T}} \bar{\mathbf{f}}^{\mathrm{R}} + \mathbf{J}^{\mathrm{U}}(\mathbf{x})\, \mathbf{u}. \tag{12}$$

where

$$\mathbf{u} = \begin{bmatrix} \mathbf{e}^{\mathrm{SE}} \\ \mathbf{f}^{\mathrm{SF}} \end{bmatrix},\ \mathbf{y} = \begin{bmatrix} -\mathbf{f}^{\mathrm{SE}} \\ -\mathbf{e}^{\mathrm{SF}} \end{bmatrix},\ \begin{bmatrix} \bar{\mathbf{e}}^{\mathrm{R}} \\ \bar{\mathbf{f}}^{\mathrm{R}} \end{bmatrix} = \mathbf{T}(\mathbf{x}) \begin{bmatrix} \mathbf{f}^{\mathrm{R}} \\ \mathbf{e}^{\mathrm{R}} \end{bmatrix},$$

$$\bar{\Omega}\left(\bar{\mathbf{f}}^{\mathrm{R}}, \bar{\mathbf{e}}^{\mathrm{R}}, \mathbf{x}\right) = \bar{\Omega}\left(\mathbf{T}(\mathbf{x})\left(\mathbf{f}^{\mathrm{R}}, \mathbf{e}^{\mathrm{R}}\right), \mathbf{x}\right),$$

$\mathbf{T}(\mathbf{x})$ *is the regular and power-conserving transformation of power variables of R-ports and* $\mathbf{J}^{\mathrm{C}}(\mathbf{x}), \mathbf{J}^{\mathrm{U}}(\mathbf{x})$ *are the skew-symmetric matrices.*

Proof: The relation (5) can be rewritten as

$$\mathbf{E}^{\mathrm{w}}(\mathbf{x})\,\mathbf{e}^{\mathrm{w}}+\mathbf{F}^{\mathrm{w}}(\mathbf{x})\,\mathbf{f}^{\mathrm{w}}+\mathbf{E}^{\mathrm{R}}(\mathbf{x})\,\mathbf{e}^{\mathrm{R}}+\mathbf{F}^{\mathrm{R}}(\mathbf{x})\,\mathbf{f}^{\mathrm{R}}=0 \tag{13}$$

where

$$\mathbf{E}^{\mathrm{w}}(\mathbf{x})=\begin{bmatrix}\left(\mathbf{E}^{\mathrm{SE}}(\mathbf{x})\right)^{\mathrm{T}}\\ \left(\mathbf{F}^{\mathrm{SF}}(\mathbf{x})\right)^{\mathrm{T}}\\ \left(\mathbf{E}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}},\quad \mathbf{F}^{\mathrm{w}}(\mathbf{x})=\begin{bmatrix}\left(\mathbf{F}^{\mathrm{SE}}(\mathbf{x})\right)^{\mathrm{T}}\\ \left(\mathbf{E}^{\mathrm{SF}}(\mathbf{x})\right)^{\mathrm{T}}\\ \left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}},$$

$$\mathbf{e}^{\mathrm{w}}=\begin{bmatrix}\mathbf{e}^{\mathrm{SE}}\\ \mathbf{f}^{\mathrm{SF}}\\ \mathbf{e}^{\mathrm{C}}\end{bmatrix},\quad \mathbf{f}^{\mathrm{w}}=\begin{bmatrix}-\mathbf{f}^{\mathrm{SE}}\\ -\mathbf{e}^{\mathrm{SF}}\\ \mathbf{f}^{\mathrm{C}}\end{bmatrix}.$$

The well-posedness of the system and Assumption 1 guarantee that the matrix $\mathbf{F}^{\mathrm{w}}(\mathbf{x})$ is constant rank matrix. So by performing row like operation to (13), the following can be obtained

$$\begin{bmatrix}\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})\\ 0_{\left(n_2^{\mathrm{C}}+n^{\mathrm{R}}\right)\times\left(n-n^{\mathrm{R}}\right)}\end{bmatrix}\mathbf{f}^{\mathrm{w}}+\begin{bmatrix}\mathbf{E}_1^{\mathrm{w}}(\mathbf{x})\\ \tilde{\mathbf{E}}_2^{\mathrm{w}}(\mathbf{x})\end{bmatrix}\mathbf{e}^{\mathrm{w}}+\begin{bmatrix}\mathbf{E}_1^{\mathrm{R}}(\mathbf{x})\\ \tilde{\mathbf{E}}_2^{\mathrm{R}}(\mathbf{x})\end{bmatrix}\mathbf{e}^{\mathrm{R}}+\begin{bmatrix}\mathbf{F}_1^{\mathrm{R}}(\mathbf{x})\\ \tilde{\mathbf{F}}_2^{\mathrm{R}}(\mathbf{x})\end{bmatrix}\mathbf{f}^{\mathrm{R}}=0. \tag{14}$$

The matrix $\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})$ has the following form

$$\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})=\mathrm{diag}\left(\mathbf{I}_{n^{\mathrm{SE}}},\mathbf{I}_{n^{\mathrm{SF}}},\mathbf{F}_1^{\mathrm{C}}(\mathbf{x})\right)$$

and $\mathbf{F}_1^{\mathrm{C}}(\mathbf{x})$ is a full rank matrix $\forall\mathbf{x}\in\mathcal{X}$. Furthermore, $\mathbf{e}^{\mathrm{C}}$ can be expressed as $\mathbf{e}^{\mathrm{C}}=\left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}\lambda=\left[0_{n^{\mathrm{C}}\times n^{\mathrm{SE}}}\ 0_{n^{\mathrm{C}}\times n^{\mathrm{SF}}}\ \left(\mathbf{F}_1^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}\ 0_{n^{\mathrm{C}}\times n^{\mathrm{R}}}\right]\lambda$. It means that

$$\left(\ker\left(\mathbf{F}^{\mathrm{C}}(\mathbf{x})\right)\right)^{\mathrm{T}}\mathbf{e}^{\mathrm{C}}=0$$

Therefore, by performing row like operations on the last $n_2^{\mathrm{C}}+n^{\mathrm{R}}$ rows of (14), the following is obtained

$$\begin{bmatrix}\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})\\ 0_{n_2^{\mathrm{C}}\times\left(n-n^{\mathrm{R}}\right)}\\ 0_{n^{\mathrm{R}}\times\left(n-n^{\mathrm{R}}\right)}\end{bmatrix}\mathbf{f}^{\mathrm{w}}+\begin{bmatrix}\mathbf{E}_1^{\mathrm{w}}(\mathbf{x})\\ \mathbf{E}_2^{\mathrm{w}}(\mathbf{x})\\ \mathbf{E}_3^{\mathrm{w}}(\mathbf{x})\end{bmatrix}\mathbf{e}^{\mathrm{w}}+\begin{bmatrix}\mathbf{F}_1^{\mathrm{R}}(\mathbf{x})\\ 0_{n_2^{\mathrm{C}}\times n^{\mathrm{R}}}\\ \mathbf{F}_3^{\mathrm{R}}(\mathbf{x})\end{bmatrix}\mathbf{f}^{\mathrm{R}}+\begin{bmatrix}\mathbf{E}_1^{\mathrm{R}}(\mathbf{x})\\ 0_{n_2^{\mathrm{C}}\times n^{\mathrm{R}}}\\ \mathbf{E}_3^{\mathrm{R}}(\mathbf{x})\end{bmatrix}\mathbf{e}^{\mathrm{R}}=0 \tag{15}$$

and the matrix $\mathbf{E}_2^{\mathrm{w}}(\mathbf{x})$ has the following form

$$\mathbf{E}_2^{\mathrm{w}}(\mathbf{x})=\begin{bmatrix}0_{n_2^{\mathrm{C}}\times n^{\mathrm{SE}}} & 0_{n_2^{\mathrm{C}}\times n^{\mathrm{SF}}} & \mathbf{E}_2^{\mathrm{C}}(\mathbf{x})\end{bmatrix}.$$

The matrix $\mathbf{E}_2^{\mathrm{C}}(\mathbf{x})$ is a full rank matrix and $\mathbf{E}_2^{\mathrm{C}}(\mathbf{x})\left(\mathbf{F}_1^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}=0,\ \forall\mathbf{x}\in\mathcal{X}$. Since n^{R} power variables of $\mathbb{R}$-element have to be expressed as a function of other n^{R} ones then $\mathrm{rank}\left[\mathbf{F}_3^{\mathrm{R}}(\mathbf{x})\,\mathbf{E}_3^{\mathrm{R}}(\mathbf{x})\right]=n^{\mathrm{R}},\ \forall\mathbf{x}\in\mathcal{X}$. Suppose that

$$\mathbf{F}_3^{\mathrm{R}}(\mathbf{x})\left(\mathbf{F}_3^{\mathrm{R}}(\mathbf{x})\right)^{\mathrm{T}}+\mathbf{E}_3^{\mathrm{R}}(\mathbf{x})\left(\mathbf{E}_3^{\mathrm{R}}(\mathbf{x})\right)^{\mathrm{T}}=\mathbf{I}_{n^{\mathrm{R}}},\ \forall\mathbf{x}\in\mathcal{X}.$$

If it is not a case then, it can be achieved by multiplying the third row in (15) from the left side by

$$\left(\mathbf{F}_3^{\mathrm{R}}(\mathbf{x})\left(\mathbf{F}_3^{\mathrm{R}}(\mathbf{x})\right)^{\mathrm{T}}+\mathbf{E}_3^{\mathrm{R}}(\mathbf{x})\left(\mathbf{E}_3^{\mathrm{R}}(\mathbf{x})\right)^{\mathrm{T}}\right)^{-\frac{1}{2}}.$$

Power conservation yields (block matrices at the position (1,3) and (3,3) of the expression $\mathbf{E}(\mathbf{x})\mathbf{F}^{\mathrm{T}}(\mathbf{x})+\mathbf{F}(\mathbf{x})\mathbf{E}^{\mathrm{T}}(\mathbf{x})=0$)

$$\begin{aligned}&\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})\left(\mathbf{E}_3^{\mathrm{w}}(\mathbf{x})\right)^{\mathrm{T}}+\left[\mathbf{E}_1^{\mathrm{R}}(\mathbf{x})\ \ \mathbf{F}_1^{\mathrm{R}}(\mathbf{x})\right]\left[\mathbf{F}_3^{\mathrm{R}}(\mathbf{x})\ \ \mathbf{E}_3^{\mathrm{R}}(\mathbf{x})\right]^{\mathrm{T}}=0,\\&\mathbf{E}_3^{\mathrm{R}}(\mathbf{x})\left(\mathbf{F}_3^{\mathrm{R}}(\mathbf{x})\right)^{\mathrm{T}}+\mathbf{F}_3^{\mathrm{R}}(\mathbf{x})\left(\mathbf{E}_3^{\mathrm{R}}(\mathbf{x})\right)^{\mathrm{T}}=0.\end{aligned}\tag{16}$$

Consider the following transformation of power variables of R-ports

$$\begin{bmatrix}\mathbf{f}^{\mathrm{R}}\\\mathbf{e}^{\mathrm{R}}\end{bmatrix}=\underbrace{\begin{bmatrix}\mathbf{F}_3^{\mathrm{R}}(\mathbf{x}) & \mathbf{E}_3^{\mathrm{R}}(\mathbf{x})\\\mathbf{E}_3^{\mathrm{R}}(\mathbf{x}) & \mathbf{F}_3^{\mathrm{R}}(\mathbf{x})\end{bmatrix}^{\mathrm{T}}}_{\mathbf{T}^{\mathrm{T}}(\mathbf{x})}\begin{bmatrix}\bar{\mathbf{e}}^{\mathrm{R}}\\\bar{\mathbf{f}}^{\mathrm{R}}\end{bmatrix}\tag{17}$$

Straightforward computation shows that

$$\mathbf{T}(\mathbf{x})\mathbf{T}^{\mathrm{T}}(\mathbf{x})=\mathbf{I}_{n^{\mathrm{R}}},$$

$$\mathbf{T}(\mathbf{x})\begin{bmatrix}0 & \mathbf{I}_{n^{\mathrm{R}}}\\\mathbf{I}_{n^{\mathrm{R}}} & 0\end{bmatrix}\mathbf{T}^{\mathrm{T}}(\mathbf{x})=\begin{bmatrix}0 & \mathbf{I}_{n^{\mathrm{R}}}\\\mathbf{I}_{n^{\mathrm{R}}} & 0\end{bmatrix}.$$

The second condition guarantees that the transformation is power-conserving, i.e. $\langle\mathbf{e}^{\mathrm{R}}|\mathbf{f}^{\mathrm{R}}\rangle=\langle\bar{\mathbf{e}}^{\mathrm{R}}|\bar{\mathbf{f}}^{\mathrm{R}}\rangle$. By inserting (17) into (15) and taking into account (15) the following is obtained

$$\begin{aligned}&\begin{bmatrix}\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})\\0_{n_2^{\mathrm{C}}\times(n-n^{\mathrm{R}})}\\0_{n^{\mathrm{R}}\times(n-n^{\mathrm{R}})}\end{bmatrix}\mathbf{f}^{\mathrm{w}}+\begin{bmatrix}\bar{\mathbf{E}}_1^{\mathrm{w}}(\mathbf{x})\\\mathbf{E}_2^{\mathrm{w}}(\mathbf{x})\\\mathbf{E}_3^{\mathrm{w}}(\mathbf{x})\end{bmatrix}\mathbf{e}^{\mathrm{w}}+\begin{bmatrix}-\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})\left(\mathbf{E}_3^{\mathrm{w}}(\mathbf{x})^{\mathrm{T}}\right)\\0_{n_2^{\mathrm{C}}\times n^{\mathrm{R}}}\\0_{n^{\mathrm{R}}\times n^{\mathrm{R}}}\end{bmatrix}\bar{\mathbf{f}}^{\mathrm{R}}+\\&\begin{bmatrix}0_{\left(n^{\mathrm{SE}}+n^{\mathrm{SF}}+n_1^{\mathrm{C}}\right)\times n^{\mathrm{R}}}\\0_{n_2^{\mathrm{C}}\times n^{\mathrm{R}}}\\\mathbf{I}_{n^{\mathrm{R}}}\end{bmatrix}\bar{\mathbf{e}}^{\mathrm{R}}=0\end{aligned}\tag{18}$$

where

$$\bar{\mathbf{E}}_1^{\mathrm{w}}(\mathbf{x})=\mathbf{E}_1^{\mathrm{w}}(\mathbf{x})-\left[\mathbf{F}_1^{\mathrm{R}}(\mathbf{x})\ \ \mathbf{E}_1^{\mathrm{R}}(\mathbf{x})\right]\left[\mathbf{F}_3^{\mathrm{R}}(\mathbf{x})\ \ \mathbf{E}_3^{\mathrm{R}}(\mathbf{x})\right]^{\mathrm{T}}\mathbf{E}_3^{\mathrm{w}}(\mathbf{x}).$$

The power-conservation yields

$$\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})\left(\bar{\mathbf{E}}_1^{\mathrm{w}}(\mathbf{x})\right)^{\mathrm{T}}+\bar{\mathbf{E}}_1^{\mathrm{w}}(\mathbf{x})\left(\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})\right)^{\mathrm{T}}=0,\tag{19}$$

$$\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})\left(\mathbf{E}_2^{\mathrm{w}}(\mathbf{x})\right)^{\mathrm{T}}=0.\tag{20}$$

Since the matrix $\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})$ is full rank matrix $\forall x\in\mathcal{X}$, then the matrix $\bar{\mathbf{E}}_1^{\mathrm{w}}(\mathbf{x})$ may be represented as follows

$$\bar{\mathbf{E}}_1^{\mathrm{w}}(\mathbf{x})=-\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})\mathbf{J}(\mathbf{x}).\tag{21}$$

By inserting (21) into (18), one obtains

$$\mathbf{f}^{\mathrm{w}}=\mathbf{J}(\mathbf{x})\mathbf{e}^{\mathrm{w}}+\left(\mathbf{E}_3^{\mathrm{w}}(\mathbf{x})\right)^{\mathrm{T}}\bar{\mathbf{f}}^{\mathrm{R}}+\ker\left(\mathbf{F}_1^{\mathrm{w}}(\mathbf{x})\right)^{\mathrm{T}}\lambda$$

$$0=\mathbf{E}_2^{\mathrm{w}}(\mathbf{x})\mathbf{e}^{\mathrm{w}}$$

$$\bar{\mathbf{e}}^{\mathrm{R}}=-\mathbf{E}_3^{\mathrm{w}}(\mathbf{x})\mathbf{e}^{\mathrm{w}}$$

Inserting (21) into (20) gives

$$\mathbf{J}(\mathbf{x})+\mathbf{J}^{\mathrm{T}}(\mathbf{x})=0.$$

Splitting $\mathbf{J}(\mathbf{x})$ as

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \mathbf{J}^{\mathrm{U}}(\mathbf{x}) & -\left(\mathbf{G}^{\mathrm{C,U}}(\mathbf{x})\right)^{\mathrm{T}} \\ \mathbf{G}^{\mathrm{C,U}}(\mathbf{x}) & \mathbf{J}^{\mathrm{C}}(\mathbf{x}) \end{bmatrix}$$

$\mathbf{E}_3^{\mathrm{w}}(\mathbf{x})$ as

$$\mathbf{E}_3^{\mathrm{w}}(\mathbf{x}) = \begin{bmatrix} \mathbf{G}^{\mathrm{R,U}}(\mathbf{x}) \\ -\left(\mathbf{G}^{\mathrm{C,R}}(\mathbf{x})\right)^{\mathrm{T}} \end{bmatrix},$$

and replacing $\mathbf{E}_2^{\mathrm{C}}(\mathbf{x})$ by $\mathbf{G}^{\mathrm{T}}(\mathbf{x})$, $\ker\left(\mathbf{F}_1^{\mathrm{C}}(\mathbf{x})\right)^{\mathrm{T}}$ by $\mathbf{G}(\mathbf{x})$ the equations (8)-(12) are obtained.

6 Index of system

In this section, the computational issue of the system described by (8)-(12) is investigated. A measure for expected difficulties in the numerical simulation of the system of equations is the index. The definition of the (differential) index is given.

Definition 3 (Local differential index [21]) *Consider a system described by the following implicit equation*

$$\mathbf{Q}(\dot{\mathbf{z}}, \mathbf{z}, \mathbf{u}) = 0. \tag{22}$$

Equation (22) has a local differential index m at the point $(\mathbf{z}_{\mathrm{o}}, \mathbf{u}_{\mathrm{o}})$ if m is the minimal number such that there exists a neighbourhood of the point $(\mathbf{z}_{\mathrm{o}}, \mathbf{u}_{\mathrm{o}})$ in which the system of equations

$$\mathbf{Q}(\dot{\mathbf{z}}, \mathbf{z}, \mathbf{u}) = 0,$$

$$\frac{\mathrm{d}\mathbf{Q}(\dot{\mathbf{z}}, \mathbf{z}, \mathbf{u})}{\mathrm{d}t} = 0,$$

$$\vdots$$

$$\frac{\mathrm{d}^m\mathbf{Q}(\dot{\mathbf{z}}, \mathbf{z}, \mathbf{u})}{\mathrm{d}t^m} = 0,$$

can be uniquely solved for $\dot{\mathbf{z}}$ as a function of $\mathbf{z}, \mathbf{u}$ only.

The equation (12) is an output equation and it can be omitted from the index analysis. By inserting (10) into (11) the following set of equations are obtained

$$\dot{\mathbf{x}} = \mathbf{J}^{\mathrm{C}}(\mathbf{x})\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\lambda + \mathbf{G}^{\mathrm{C,R}}(\mathbf{x})\bar{\mathbf{f}}^{\mathrm{R}} + \mathbf{G}^{\mathrm{C,U}}(\mathbf{x})\mathbf{u}, \tag{23}$$

$$0 = \mathbf{G}^{\mathrm{T}}(\mathbf{x})\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}), \tag{24}$$

$$0 = \tilde{\Omega}\left(\bar{\mathbf{f}}^{\mathrm{R}}, \mathbf{x}, \mathbf{u}\right). \tag{25}$$

Proposition 4 (Index two system) *Consider the system described by (23)-(25). Assume that for $\mathbf{x} = \mathbf{x}_{\mathrm{o}}$ and $\mathbf{u} = \mathbf{u}_{\mathrm{o}}$ the equation (24) is satisfied and that there exists $\bar{\mathbf{f}}_{\mathrm{o}}^{\mathrm{R}}$ such that (25) is satisfied. If*

$$\det\left(\frac{\partial}{\partial \mathbf{x}^{\mathrm{T}}}\left(\mathbf{G}^{\mathrm{T}}(\mathbf{x})\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x})\right)\mathbf{G}(\mathbf{x})\right)_{\mathbf{x}=\mathbf{x}_{\mathrm{o}}} \neq 0 \tag{26}$$

$$\det\left(\frac{\partial\tilde{\Omega}\left(\bar{\mathbf{f}}^{\mathrm{R}},\mathbf{x},\mathbf{u}\right)}{\partial\left(\bar{\mathbf{f}}^{\mathrm{R}}\right)^{\mathrm{T}}}\right)_{\substack{\mathbf{x}=\mathbf{x}_{\mathrm{o}}\\ \bar{\mathbf{f}}^{\mathrm{R}}=\bar{\mathbf{f}}^{\mathrm{R}}_{\mathrm{o}}\\ \mathbf{u}=\mathbf{u}_{\mathrm{o}}}}\neq 0 \tag{27}$$

then the system (23)-(25) has a differential index two at the point $\left(\mathbf{x}_{\mathrm{o}},\bar{\mathbf{f}}^{\mathrm{R}}_{\mathrm{o}},\lambda_{\mathrm{o}},\mathbf{u}_{\mathrm{o}}\right)$.

Proof: In this case $\mathbf{z}=\left(\mathbf{x},\lambda,\mathbf{f}^{\mathrm{R}}\right)$. Differentiation of (25) gives

$$\frac{\partial\tilde{\Omega}\left(\bar{\mathbf{f}}^{\mathrm{R}},\mathbf{x},\mathbf{u}\right)}{\partial\left(\bar{\mathbf{f}}^{\mathrm{R}}\right)^{\mathrm{T}}}\dot{\bar{\mathbf{f}}}^{\mathrm{R}}+\mathbf{L}_1\left(\mathbf{x},\mathbf{u},\lambda,\bar{\mathbf{f}}^{\mathrm{R}}\right)=0.$$

Now it is clear that if the condition (27) is satisfied then the last equation can be solved for $\dot{\bar{\mathbf{f}}}^{\mathrm{R}}$. Differentiation of (24) gives

$$\frac{\partial}{\partial\mathbf{x}^{\mathrm{T}}}\left(\mathbf{G}^{\mathrm{T}}\left(\mathbf{x}\right)\frac{\partial H}{\partial\mathbf{x}}\left(\mathbf{x}\right)\right)\mathbf{G}\left(\mathbf{x}\right)\lambda+\mathbf{L}_2\left(\mathbf{x},\mathbf{u},\lambda,\bar{\mathbf{f}}^{\mathrm{R}}\right)=0$$

If the condition (26) is satisfied then last equation can be uniquely solved for λ. Therefore for the given value of $\mathbf{x}_{\mathrm{o}},\bar{\mathbf{f}}^{\mathrm{R}}_{\mathrm{o}},\mathbf{u}_{\mathrm{o}}$ λ_{o} can be uniquely computed. Differentiation of the last equation gives

$$\frac{\partial}{\partial\mathbf{x}^{\mathrm{T}}}\left(\mathbf{G}^{\mathrm{T}}\left(\mathbf{x}\right)\frac{\partial H}{\partial\mathbf{x}}\left(\mathbf{x}\right)\right)\mathbf{G}\left(\mathbf{x}\right)\dot{\lambda}+\mathbf{L}_3\left(\mathbf{x},\lambda,\bar{\mathbf{f}}^{\mathrm{R}},\mathbf{u},\dot{\mathbf{u}}\right)=0.$$

It is clear that if the condition (26) is satisfied then the last equation can be uniquely solved for $\dot{\lambda}$. So the index of (23)-(25) is two.

Remark 1 (Index two system) *Since (24) does not depend on* λ *then the index two system (23)-(25) is computationally efficient [21].*

Suppose that the matrix $\mathbf{G}^{\perp}\left(\mathbf{x}\right)$ is defined by

$$\mathbf{G}^{\perp}\left(\mathbf{x}\right)=\left(\ker\left(\mathbf{G}\left(\mathbf{x}\right)\right)^{\mathrm{T}}\right)^{\mathrm{T}}.$$

By premultiplying (23) with $\mathbf{G}^{\perp}\left(\mathbf{x}\right)$, the following system of equations is obtained

$$\mathbf{G}^{\perp}\left(\mathbf{x}\right)\dot{\mathbf{x}}=\mathbf{G}^{\perp}\left(\mathbf{x}\right)\mathbf{J}^{\mathrm{C}}\left(\mathbf{x}\right)\frac{\partial H}{\partial\mathbf{x}}\left(\mathbf{x}\right)+\mathbf{G}^{\perp}\left(\mathbf{x}\right)\mathbf{G}^{\mathrm{C,R}}\left(\mathbf{x}\right)\bar{\mathbf{f}}^{\mathrm{R}}+\mathbf{G}^{\perp}\left(\mathbf{x}\right)\mathbf{G}^{\mathrm{C,U}}\left(\mathbf{x}\right)\mathbf{u}, \tag{28}$$

$$0=\mathbf{G}^{\mathrm{T}}\left(\mathbf{x}\right)\frac{\partial H}{\partial\mathbf{x}}\left(\mathbf{x}\right), \tag{29}$$

$$0=\tilde{\Omega}\left(\bar{\mathbf{f}}^{\mathrm{R}},\mathbf{x},\mathbf{u}\right). \tag{30}$$

Proposition 5 (Index one system) *Consider the system described by (28)-(30). Assume that for* $\mathbf{x}=\mathbf{x}_{\mathrm{o}}$ *and* $\mathbf{u}=\mathbf{u}_{\mathrm{o}}$ *the equation (29) is satisfied and that there exists* $\bar{\mathbf{f}}^{\mathrm{R}}_{\mathrm{o}}$ *such that (30) is satisfied. If the conditions (26) and (27) are satisfied then the system (28)-(30) has a differential index one at the point* $\left(\mathbf{x}_{\mathrm{o}},\bar{\mathbf{f}}^{\mathrm{R}}_{\mathrm{o}},\mathbf{u}_{\mathrm{o}}\right)$.

Proof: In this case $\mathbf{z}=\left(\mathbf{x},\mathbf{f}^{\mathrm{R}}\right)$. The part of the proof regarding the calculation of $\dot{\bar{\mathbf{f}}}^{\mathrm{R}}$ is the same as in the proof of Proposition 4. Now, we concentrate on calculation of $\dot{\mathbf{x}}$. Differentiating (29) and regrouping the derivatives on the left side gives the following

$$\begin{bmatrix}\mathbf{G}^{\perp}\left(\mathbf{x}\right)\\ \frac{\partial}{\partial\mathbf{x}^{\mathrm{T}}}\left(\mathbf{G}^{\mathrm{T}}\left(\mathbf{x}\right)\frac{\partial H}{\partial\mathbf{x}}\left(\mathbf{x}\right)\right)\end{bmatrix}\dot{\mathbf{x}}=\mathbf{L}_4\left(\mathbf{x},\bar{\mathbf{f}}^{\mathrm{R}},\mathbf{u}\right).$$

If the condition (26) is fullfiled then the rank of the matrix on the left side is full $\forall \mathbf{x} \in \mathcal{X}$. For proving this, the following identity is used

$$\operatorname{rank}\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} = \operatorname{rank}(\mathbf{A}) + \operatorname{rank}\left(\mathbf{B}\mathbf{A}^{\perp}\right), \ \mathbf{A}^{\perp} = \ker(\mathbf{A}).$$

7 Example

A classical simplified half-car model [4], known as *bicycle model*, is shown in Figure 8. There is no suspension, nor any bushing to consider. It is assumed that the width of the system is neglectable. The front wheel can be steered over the angle δ.

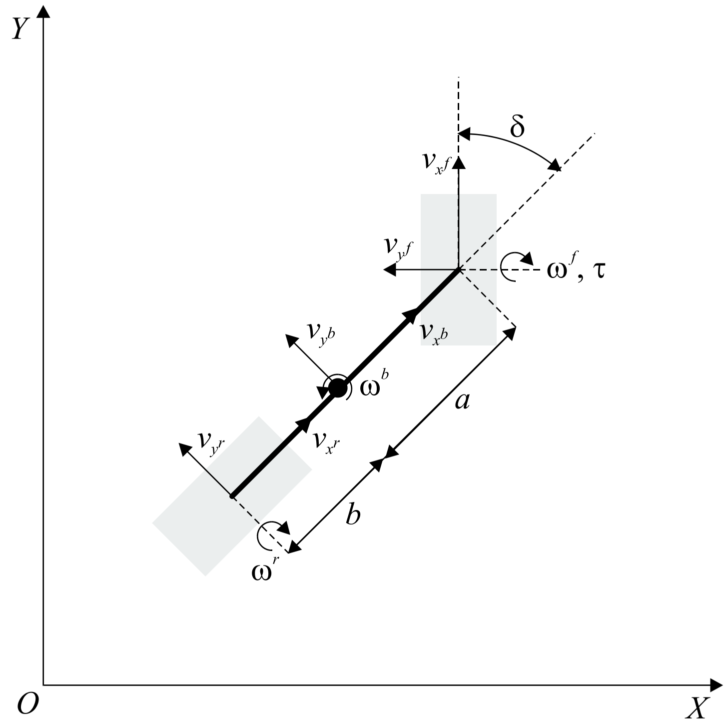

Figure 8: Bicycle model: Ideal physical model.

Inertial frame is XOY. The velocities v_{x^f}, v_{y^f} (v_{x^r}, v_{y^r}) are the velocities of the front wheel (rear wheel) and v_{x^b}, v_{y^b} are the velocities of the center of mass of the body. The angular velocities of the wheels are denoted by ω^f, ω^r and the angular velocity of the body is denote by ω^b and it is assumed that δ is constant. The model of wheel-tire system is considered. Schematic of tire is shown in Figure 9. Here, F_{x^α} is longitudinal force $(\alpha \in \{f, r\})$, F_{y^α} cornering force and $\lambda_{x^\alpha}, \lambda_{y^\alpha}$ are the constraint forces between the body and wheels. For $\alpha = f$, $\tau^\alpha = \tau$ and for $\alpha = r$, $\tau^\alpha = 0$. The forces $F_{x^\alpha}, F_{y^\alpha}$ are given by

$$\begin{aligned} F_{x^\alpha} &= d_{x^\alpha}\left(e^{\mathrm{R}}_{x^\alpha}, p_{x^\alpha}, p_{\omega^\alpha}\right), \\ F_{y^\alpha} &= d_{y^\alpha}\left(e^{\mathrm{R}}_{y^\alpha}, p_{x^\alpha}, p_{y^\alpha}\right), \end{aligned}$$

where $p_{x^\alpha}, p_{y^\alpha}$ are the momenta of the wheel with respect to body frame, p_{ω^α} is angular momentum of the wheel with respect to polar axis, $e^{\mathrm{R}}_{x^\alpha} = R\omega^\alpha - v_{x^\alpha}$ and $e^{\mathrm{R}}_{y^\alpha} = v_{y^\alpha}$.

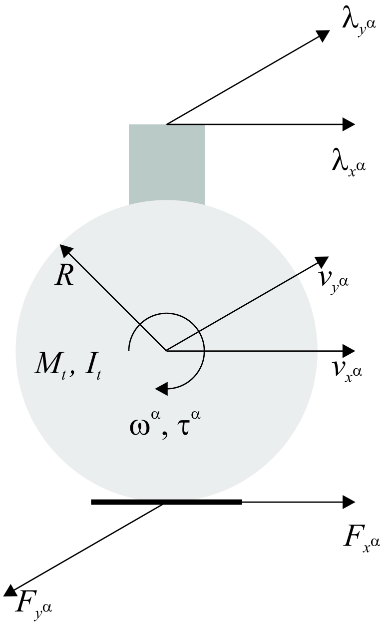

Figure 9: Wheel-tire system: Ideal physical model.

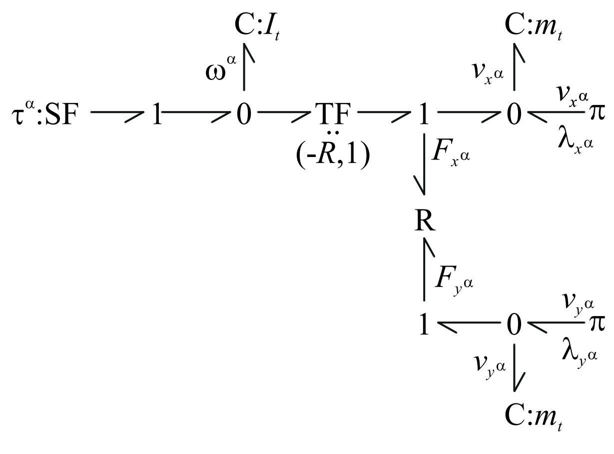

Figure 10: Bond graph model of wheel-tire system.

The mass of the tire is denoted by m_t, polar inertia by I_t and radius by R. The bond graph model of wheel-tire system is given in Figure 10. Schematic of the vehicle's body is shown in Figure 11. The mass of the body is m and its inertia is I. The bond graph model is shown in Figure 12.

The transformation matrices from the body frame to the wheel frames, $\mathbf{C}^f(\delta)$ and $\mathbf{C}^r$,

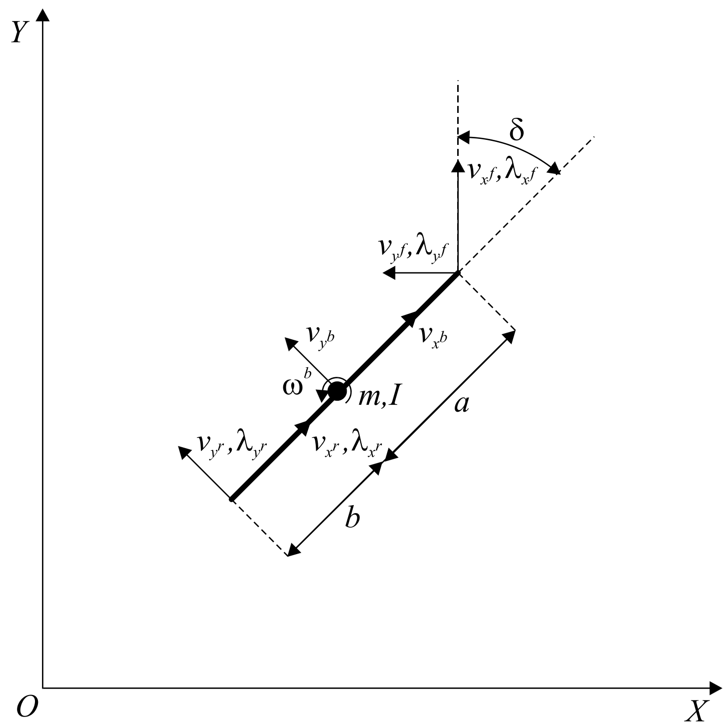

Figure 11: Body of the vehicle: The ideal physical model

are given by

$$\mathbf{C}^f(\delta) = \begin{bmatrix} \cos(\delta) & \sin(\delta) & a\sin(\delta) \\ -\sin(\delta) & \cos(\delta) & a\cos(\delta) \end{bmatrix},$$

$$\mathbf{C}^r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \end{bmatrix}.$$

The bond graph model of the whole system is given in Figure 13. The equations describing the system (8)-12 now become

$$\dot{\mathbf{x}} = \mathbf{G}(\delta)\lambda + \mathbf{G}^{\mathrm{C,R}}\bar{\mathbf{f}}^{\mathrm{R}} + \mathbf{G}^{\mathrm{C,U}}\mathbf{u}, \tag{31}$$

$$0 = \mathbf{G}^{\mathrm{T}}(\delta)\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}), \tag{32}$$

$$\bar{\mathbf{e}}^{\mathrm{R}} = -\left(\mathbf{G}^{\mathrm{C,R}}\right)^{\mathrm{T}}\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}), \tag{33}$$

$$y = -\left(\mathbf{G}^{\mathrm{C,U}}\right)^{\mathrm{T}}\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}), \tag{34}$$

$$0 = \bar{\Omega}\left(\bar{\mathbf{f}}^{\mathrm{R}}, \bar{\mathbf{e}}^{\mathrm{R}}, \mathbf{x}\right), \tag{35}$$

where

$$\mathbf{x}^{\mathrm{T}} = \begin{bmatrix} \mathbf{x}_f^{\mathrm{T}} & \mathbf{x}_r^{\mathrm{T}} & \mathbf{x}_b^{\mathrm{T}} \end{bmatrix},$$

$$\mathbf{x}_\alpha = \begin{bmatrix} p_{x^\alpha} & p_{y^\alpha} & p_{\omega^\alpha} \end{bmatrix}^{\mathrm{T}},$$

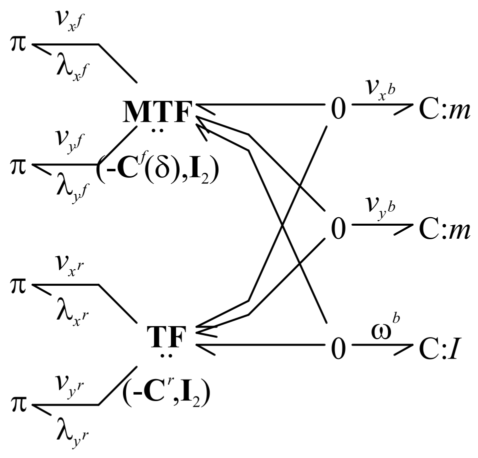

Figure 12: Bond graph model of vehicle's body

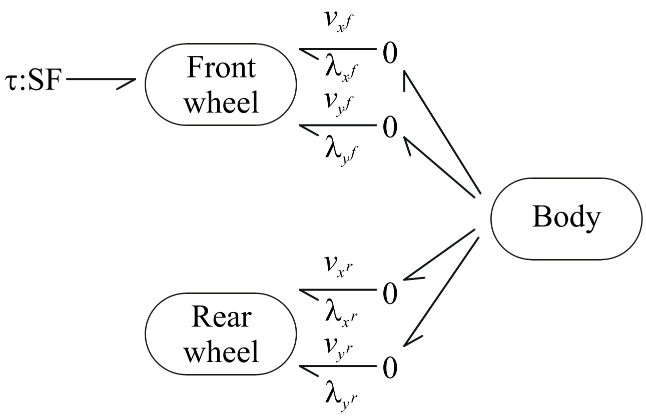

Figure 13: Bond graph model of vehicle's body

$$\mathbf{G}^{\mathrm{T}}(\delta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -\cos(\delta) & -\sin(\delta) & -a\sin(\delta) \\ 0 & 1 & 0 & 0 & 0 & 0 & \sin(\delta) & -\cos(\delta) & -a\cos(\delta) \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & b \end{bmatrix},$$

$$\left(\mathbf{G}^{\mathrm{C,R}}\right)^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & -R & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left(\mathbf{G}^{\mathrm{C,U}}\right)^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{x},$$

$$\mathbf{M} = \mathrm{diag}\left(m_t, m_t, I_t, m_t, m_t, I_t, m, m, I\right),$$

$$\bar{\mathbf{f}}^{\mathrm{R}} = \begin{bmatrix} F_{xf} & F_{yf} & F_{x^r} & F_{y^r} \end{bmatrix}^{\mathrm{T}},$$

$$\bar{\Omega}\left(\bar{\mathbf{f}}^{\mathrm{R}},\bar{\mathbf{e}}^{\mathrm{R}},\mathbf{x}\right)=\begin{bmatrix} -F_{xf}+d_{xf}\left(e^{\mathrm{R}}_{xf},p_{xf},p_{\omega f}\right)\\ -F_{yf}+d_{yf}\left(e^{\mathrm{R}}_{yf},p_{xf},p_{yf}\right)\\ -F_{x^r}+d_{x^r}\left(e^{\mathrm{R}}_{x^r},p_{x^r},p_{\omega^r}\right)\\ -F_{y^r}+d_{y^r}\left(e^{\mathrm{R}}_{y^r},p_{x^r},p_{y^r}\right)\end{bmatrix}.$$

Now the index of the system is analysed. The condition (26) is checked.

$$\frac{\partial}{\partial\mathbf{x}^{\mathrm{T}}}\left(\mathbf{G}^{\mathrm{T}}(\delta)\frac{\partial H}{\partial\mathbf{x}}(\mathbf{x})\right)\mathbf{G}(\delta)=\mathbf{G}^{\mathrm{T}}(\delta)\frac{\partial H}{\partial\mathbf{x}\partial\mathbf{x}^{\mathrm{T}}}(\mathbf{x})\,\mathbf{G}(\delta)=\mathbf{G}^{\mathrm{T}}(\delta)\,\mathbf{M}^{-1}\mathbf{G}(\delta)\,.$$

The rank of the matrix $\mathbf{G}(\delta)$ is full and $\mathbf{M}$ is a positive definite matrix. Therefore, the conditions (26) is satisfied. Now, the condition (27) is analysed. Since

$$\frac{\partial\bar{\Omega}\left(\bar{\mathbf{f}}^{\mathrm{R}},\bar{\mathbf{e}}^{\mathrm{R}},\mathbf{x}\right)}{\partial\bar{\mathbf{f}}^{\mathrm{R}}}=-\mathbf{I}_4$$

then also the condition (27) is satisfied. Therefore the index of the system is two. The index one system may be obtained by multiplying the equation (31) by $\mathbf{G}^{\perp}(\delta)$ where

$$\mathbf{G}^{\perp}(\delta)=\begin{bmatrix} a\sin(\delta) & a\cos(\delta) & 0 & 0 & -b & 0 & 0 & 0 & 1\\ \sin(\delta) & \cos(\delta) & 0 & 0 & 1 & 0 & 0 & 1 & 0\\ \cos(\delta) & -\sin(\delta) & 0 & 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

8 Conclusion

In this paper, a mathematical formulation of bond graphs has been developed. It has been proven that every junction structure can be associated with a Dirac structure. Also, it has been proven that equations describing a bond graph model correspond to an implicit port-controlled Hamiltonian system with dissipation. The condition for well-posedness of a modelled system has been given and representations suitable for the numerical simulation have been derived. The index of the representations have been analysed and sufficient conditions for the representations to be computationally efficient have been given. The theory has been applied to some models arising in automotive applications.

References

[1] H.M. Paynter. *Analysis and Design of Engineering Systems.* The M.I.T. Press, Cambridge, Massachusetts, 1961.

[2] P.C. Breedveld. Multibond graph elements in physical systems theory. *Journal of the Franklin Institute*, 319:1–36, 1985.

[3] P.C. Breedveld. *Physical Systems Theory in Terms of Bond Graphs.* PhD thesis, Univeristy of Twente, 1984.

[4] R.C. Rosenberg D.C. Karnopp, D.L. Margolis. *System Dynamics: A Unified Approach*. John Wiley and Sons, Inc., 1990.

[5] G. Golo, P.C. Breedveld, B.M. Maschke, and A.J. van der Schaft. Geometric formulation of generalized bond graphs models part i: Generalized junction structure. Technical report, Faculty of Mathematical Sciences, 2000.

[6] J.D. Lamb, D.R. Woodall, and G. M. Asher. Bond graphs i: Acausal equivalence. *Discrete Applied Mathematics*, 72:261–293, 1997.

[7] J.D. Lamb, D.R. Woodall, and G. M. Asher. Bond graphs ii: Causality and singularity. *Discrete Applied Mathematics*, 73:143–173, 1997.

[8] J.D. Lamb, D.R. Woodall, and G. M. Asher. Bond graphs iii: Bond graph and electrical networks. *Discrete Applied Mathematics*, 73:211–250, 1997.

[9] H. Birkett and P.H. Roe. The mathematical foundations of bond graphs-i. algebraic theory. *Journal of the Franklin Institute*, 326:329–350, 1989.

[10] H. Birkett and P.H. Roe. The mathematical foundations of bond graphs-ii. duality. *Journal of the Franklin Institute*, 326:691–708, 1989.

[11] H. Birkett and P.H. Roe. The mathematical foundations of bond graphs-iii. matroid theory. *Journal of the Franklin Institute*, 327:87–108, 1990.

[12] H. Birkett and P.H. Roe. The mathematical foundations of bond graphs-iv. matrix representations and causality. *Journal of the Franklin Institute*, 327:109–128, 1990.

[13] A.S. Perelson and G.F. Oster. Chemical reaction dynamics part ii: Reaction networks. *Archive of Rational Mech. Anal.*, 55:31–98, 1975.

[14] G.F. Oster and A.S. Perelson. Chemical reaction dynamics part i: Geometrical structure. *Archive of Rational Mech. Anal.*, 57:230–274, 1975.

[15] B.M. Maschke, A.J. van der Schaft, and P.C. Breedveld. An intrinsic hamiltonian formulation of networks dynamics: Non-standard poisson structures and gyrators. *Journal of the Franklin Institute*, 329:923–966, 1992.

[16] B.M. Maschke. *Modelling and Control of Mechanisms and Robots*, chapter Elements on the Modelling of Mechanical Systems, pages 1–38. Bertinaro, World Scientific Publishing Ltd, 1996.

[17] M. Dalsmo and A.J. van der Schaft. On representation and integrability of mathematical structures in energy-conserving physical systems. *SIAM Journal on Control and Optimization*, 37:54–91, 1999.

[18] A.J. van der Schaft and B.M. Maschke. The hamiltonian formulation of energy conserving physical systems with external ports. *Archiv fur Elektronik und Ubertragungstechnik*, 49:362–371, 1995.

[19] A.J. van der Schaft. *L_2-Gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag, 2000.

[20] A.J. van der Schaft. *The Mathematics of Systems and Control: From Intelegent Control to Behavioral Systems*, chapter Interconncetion and Geometry, pages 203–217. Foundations Systems and Control Groningen, 1999.

[21] E. Hairer and G. Wanner. *Numerical Solution of Nonlinear Diferential Equations II: Stiff and Differential-Algebraic Problems*. Springer-Verlag, 1991.