Achievable behavior of general systems

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Abstract

A basic question in systems and control theory concerns the characterization of the set of all achievable closed-loop systems for a given plant system and a controller system to be designed. This problem is addressed in a general behavioral context. Sufficient, and often necessary, conditions for a behavior to be achievable are given, and for any achievable behavior a canonical controller is defined. The results generalize previously obtained results obtained for finite-dimensional linear systems. The application to nonlinear differential systems is outlined.

Keywords

Behaviors, interconnection, specifications, linear systems, nonlinear systems.

1 Introduction

In this paper we generalize and extend the following result obtained in [12]; see also [4, 9]. Consider a linear time-invariant differential plant system represented by linear higher-order differential equations

$$R\left(\frac{d}{dt}\right) w(t) = M\left(\frac{d}{dt}\right) z(t) \quad w \in \mathbb{R}^q, z \in \mathbb{R}^k \tag{1}$$

where $R(s)$ and $M(s)$ are $\ell \times q$, respectively $\ell \times k$, polynomial matrices (ℓ is the number of equations). Here z denotes the variables which are accessible to controller action, and w denotes the variables that represent the interaction of the system with (the rest of) its environment, and whose behavior we intend to shape. This is done by interconnecting the plant system (1) to a controller system

$$H\left(\frac{d}{dt}\right) z(t) = 0 \tag{2}$$

with $H(s)$ a $h \times k$ polynomial matrix, leading to the interconnected ('closed-loop') system

$$\begin{aligned} R\left(\tfrac{d}{dt}\right) w(t) &= M\left(\tfrac{d}{dt}\right) z(t) \\ H\left(\tfrac{d}{dt}\right) z(t) &= 0 \end{aligned} \tag{3}$$

We look at (3) as defining a *dynamical behavior* in the *w-variables* (with z being auxiliary variables). The basic question which is addressed in [12] is to characterize all achievable behaviors (3) of w by considering all possible controller systems (2).

Formally, let us define the *plant behavior* $\mathcal{P}$, respectively, *controller behavior* $\mathcal{C}$ as

$$\begin{aligned} \mathcal{P} &:= \{(w,z) : \mathbb{R} \to \mathbb{R}^q \times \mathbb{R}^k, C^\infty \mid (1) \text{ is satisfied for all } t\} \\ \mathcal{C} &:= \{z : \mathbb{R} \to \mathbb{R}^k, C^\infty \mid (2) \text{ is satisfied for all } t\} \end{aligned} \tag{4}$$

and the interconnected behavior $\mathcal{P} \parallel_z \mathcal{C}$ (with shared variables z)

$$\mathcal{P} \parallel_z \mathcal{C} = \left\{ w : \mathbb{R} \to \mathbb{R}^q, C^\infty \mid \exists z : \mathbb{R} \to \mathbb{R}^k, C^\infty, \text{such that (3) is satisfied for all } t \right\} \tag{5}$$

It can be readily shown (the so-called Elimination Theorem, [5]) that $\mathcal{P} \parallel_z \mathcal{C}$ can be also represented as a set of differential equations solely in w

$$G\left(\frac{d}{dt}\right) w(t) = 0 \tag{6}$$

for a certain $g \times q$ polynomial matrix $G(s)$. That is, there exists $G(s)$ such that $\mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$ where

$$\mathcal{S} := \{w : \mathbb{R} \to \mathbb{R}^q, C^\infty \mid (6) \text{ is satisfied for all } t\} \tag{7}$$

In [12] the following result is obtained concerning the achievable interconnected behaviors $\mathcal{P} \parallel_z \mathcal{C}$, with $\mathcal{C}$ ranging over all controller systems. Define the following behaviors derived from the plant behavior $\mathcal{P}$:

$$\begin{aligned} \pi_w(\mathcal{P}) &:= \left\{ w : \mathbb{R} \to \mathbb{R}^q, C^\infty \mid \exists z : \mathbb{R} \to \mathbb{R}^k, C^\infty, \text{s.t. } (w,z) : \mathbb{R} \to \mathbb{R}^q \times \mathbb{R}^k \text{ is in } \mathcal{P} \right\} \\ \mathcal{P}_0 &:= \left\{ w : \mathbb{R} \to \mathbb{R}^q, C^\infty \mid (w,0) \text{ is in } \mathcal{P}, \text{ with } 0 : \mathbb{R} \to \mathbb{R}^k \text{ the zero-function} \right\} \end{aligned} \tag{8}$$

Theorem 1.1. *Let $\mathcal{P}$ be a given plant behavior as in (4). Let $\mathcal{S}$ be a desired behavior (7). Then there exists a controller system $\mathcal{C}$ as in (4) such that $\mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$ if and only if*

$$\mathcal{P}_0 \subset \mathcal{S} \subset \pi_w(\mathcal{P}). \tag{9}$$

Remark 1.2. The analogous statement can be derived for the case where the plant, controller, and desired system behavior are defined as $\mathcal{L}_1^{loc}$ behaviors, by considering solutions of (1), (2), respectively (6), in a distributional sense. We will come back to this situation in Section 3 (Remark 3.2).

Theorem 1.1 elegantly formalizes the set of achievable interconnected behaviors, and provides a starting point for assessing the 'limits of performance' of the plant system $\mathcal{P}$ when controlled by any controller system $\mathcal{C}$. (Follow-up questions concern the *construction* of the controller $\mathcal{C}$ in order to achieve a desired behavior $\mathcal{S}$ satisfying (9), and the 'realizability' of such a controller. Note in this regard that we did not make any distinction between inputs and outputs in the vector z, nor did we impose any properness condition on $\mathcal{C}$ with respect to such a division; see [11, 1] for developments in this direction.)

The aim of the present paper is to generalize Theorem 1.1 to a *general system setting*, applicable to various types of systems (nonlinear, infinite-dimensional, $n-D$, discrete event, hybrid, ...).

Clearly, the first inclusion in (9), namely $\mathcal{P}_0 \subset \mathcal{S}$, is a typical 'linear condition'. Therefore we may expect that this condition needs to be generalized in order to be applicable to a general system context. Indeed, we shall derive a generalization of the first inclusion, which together with the second inclusion is sufficient (and almost necessary) for the existence of a controller in the general case. Furthermore, it will turn out that in the course of doing so we obtain some general results concerning the *construction* of a controller $\mathcal{C}$ achieving $\mathcal{S}$. This will be discussed in Section 2.

The results obtained in Section 2 also provide some new insights in the finite-dimensional linear case. This will be addressed in Section 3. In Section 4 we shall sketch how the general results obtained in Section 2 can be specialized to continuous-time systems described by *nonlinear* differential equations. Finally, in Section 5 we indicate some other relevant system cases where this approach may be applied and further developed.

2 General results

Consider a system $\mathcal{P}$ (the 'plant') with two types of external variables, namely the variables z which can be interconnected to another system $\mathcal{C}$ (the 'controller') sharing the same variables z, and remaining variables w which represent the interaction (or communication) of the system with (the rest of) its environment; see Figure 1.

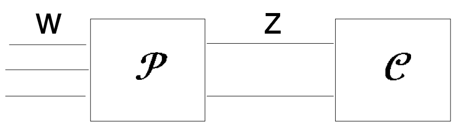

Figure 1: Plant controller configuration

We consider $\mathcal{P}$ and $\mathcal{C}$ to be systems in a general *behavioral* sense, that is, as a collection of allowable system trajectories. Also, we do not impose any conditions of linearity or finite-dimensionality. Later on in Sections 3 and 4 we specialize the results to more structured situations.

Formally, let W be a general set where the variables w take value, and let Z be the set where the variables z take value. Furthermore, let T be a general set denoting the time-axis. (Note that although we primarily think of T as $\mathbb{R}$ or $\mathbb{Z}$ we do not impose any conditions on the set T.) The plant system $\mathcal{P}$ is given as a collection of time-functions (w, z) with

$$\begin{aligned} w &: T \to W, \\ z &: T \to Z, \end{aligned} \tag{10}$$

that is, $\mathcal{P} \subset (W \times Z)^T$.

Similarly, the controller system $\mathcal{C}$ is given as a collection of time-functions

$$z : T \to Z \tag{11}$$

that is, $\mathcal{C} \subset Z^T$. The *composition* of $\mathcal{P}$ and $\mathcal{C}$ via the shared variables z, denoted $\mathcal{P} \parallel_z \mathcal{C}$, is given by

$$\mathcal{P} \parallel_z \mathcal{C} = \{w : T \to W |\ \exists z : T \to Z \text{ such that } (w, z) \in \mathcal{P}, z \in \mathcal{C}\} \tag{12}$$

(Note that the shared variables z become *hidden* variables in the composition.) The central question studied in this paper is to characterize the set of composed behaviors $\mathcal{P} \parallel_z \mathcal{C}$ that are achievable by selecting the controller system $\mathcal{C}$ in an appropriate way. As already discussed in the Introduction, this can be regarded as a fundamental issue in characterizing the 'limits of performance' of a given plant system $\mathcal{P}$ by considering all possible controller systems $\mathcal{C}$.

The main theorem reads as follows. Denote, similarly to (8), by $\pi_w(\mathcal{P}) \subset W^T$ the plant behavior projected on W^T, that is

$$\pi_w(\mathcal{P}) = \{w : T \to W \mid\ \exists z : T \to Z \text{ such that } (w, z) \in \mathcal{P}\} \tag{13}$$

Theorem 2.1. *Let $\mathcal{P} \subset (W \times Z)^T$ be a given plant system, and let $\mathcal{C} \subset Z^T$ be a controller system to be designed. Let $\mathcal{S} \subset W^T$ be a desired behavior. Then there exists $\mathcal{C}$ such that $\mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$ if*

(i) $\mathcal{S} \subset \pi_w(\mathcal{P})$

(ii) The following implication holds: for any $(w, z), (\tilde{w}, z) \in \mathcal{P}$ whenever $\tilde{w} \in \mathcal{S}$ then also $w \in \mathcal{S}$.

Proof Define the controller system $\mathcal{C}_{can}$ in the following implicit way; see Figure 2

$$\mathcal{C}_{can} := \{z : T \to Z \mid\ \exists \tilde{w} : T \to W \text{ such that } (\tilde{w}, z) \in \mathcal{P} \text{ and } \tilde{w} \in \mathcal{S}\} \tag{14}$$

We prove that $\mathcal{P} \parallel_z \mathcal{C}_{can} = \mathcal{S}$; see Figure 3.
$\supset$: Let $w \in \mathcal{S}$. Because of $(i) \exists z : T \to Z$ such that $(w, z) \in \mathcal{P}$. Hence also $z \in \mathcal{C}_{can}$ (take $\tilde{w} = w$), and thus $w \in P \parallel_z \mathcal{C}$.
$\subset$: Let $w \in \mathcal{P} \parallel_z \mathcal{C}_{can}$. Thus $\exists z : T \to Z, \tilde{w} : T \to W$ such that $(w, z) \in \mathcal{P}, (\tilde{w}, z) \in \mathcal{P}$ and $\tilde{w} \in \mathcal{S}$. By (ii) this implies that $w \in \mathcal{S}$. ☐

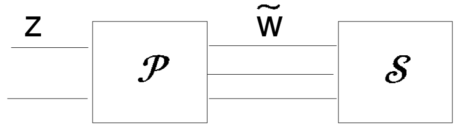

Figure 2: Canonical controller $\mathcal{C}_{can}$

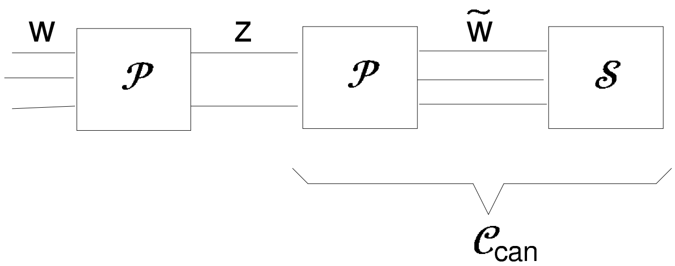

Figure 3: Composed behavior

Remark 2.2. In Section 3 we shall make explicit how condition (ii) generalizes the first inclusion in (9) of Theorem 1.1.

Remark 2.3. We call $\mathcal{C}_{can} := \mathcal{P} \parallel_w \mathcal{S}$ the *canonical controller* (depending on the desired system $\mathcal{S}$). The definition of $\mathcal{C}_{can}$ was inspired by a similar construction in network interconnection structures appearing in [2].

Remark 2.4. It immediately follows from the proof of Theorem 2.1 that if $\mathcal{S}$ only satisfies condition (i) then still $\mathcal{S} \subset \mathcal{P} \parallel_z \mathcal{C}_{can}$, while if $\mathcal{S}$ only satisfies condition (ii) then $\mathcal{P} \parallel_z \mathcal{C}_{can} \subset \mathcal{S}$. The first case guarantees a kind of liveness property (the composed system contains a desired behavior $\mathcal{S}$), while in the second case the composed system $\mathcal{P} \parallel_z \mathcal{C}_{can}$ satisfies at least the 'specifications' given by $\mathcal{S}$ (see also [10]).

Remark 2.5. Note that the canonical controller $\mathcal{C}_{can}$ contains an 'internal model' of the plant $\mathcal{P}$, as well as of the desired behavior $\mathcal{S}$. (Although elimination of the variables $\tilde{w}$ from $\mathcal{C}_{can}$ will result in a controller representation of quite a different form.) This raises interesting questions about the *robustness*[1] of the controller configuration with respect to e.g. uncertainty in the plant model $\mathcal{P}$. One easy observation to be made is that if the plant model $\mathcal{P}_{nom}$ in the canonical controller $\mathcal{C}_{can}$ differs from the actual plant $\mathcal{P}$, then by Remark 2.4 one will still have $\mathcal{S} \subset \mathcal{P} \parallel_z \mathcal{C}_{can}$ if $\mathcal{P}_{nom} \subset \mathcal{P}$, while $\mathcal{P} \parallel_z \mathcal{C}_{can} \subset \mathcal{S}$ if $\mathcal{P} \subset \mathcal{P}_{nom}$.

Remark 2.6. What can be done if $\mathcal{S}$ does *not* satisfy conditions (i) or (ii) of Theorem 2.1, but still we want to design a controller $\mathcal{C}$ such that $\mathcal{P} \parallel_z \mathcal{C}$ approximates $\mathcal{S}$ 'as good as possible'? The obvious construction seems to *shrink* $\mathcal{S}$ by leaving out all those w for which there does not exist z such that $(w, z) \in \mathcal{P}$ (so as to ensure that condition (i) is satisfied), and then to *enlarge* $\mathcal{S}$ by adding all those w for which

[1] I thank the anonymous reviewer for pointing out this issue.

there exist $\tilde{w} \in \mathcal{S}$ and z such that $(w,z),(\tilde{w},z) \in \mathcal{P}$ (resulting in the satisfaction of condition (ii)). (Note that these two operations do not conflict, and may be performed in arbitrary order.) Then the newly defined behavior $\mathcal{S}'$ does satisfy the conditions of Theorem 2.1, and so the corresponding canonical controller $\mathcal{C}_{can}$ will result in $\mathcal{P} \parallel_z \mathcal{C}_{can} = \mathcal{S}'$.

The conditions of Theorem 2.1 are close to be *necessary* as well. Indeed, let $\mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$ for some controller $\mathcal{C}$. Then it immediately follows that for every $w \in \mathcal{S} = \mathcal{P} \parallel_z \mathcal{C}$ there exists $z \in \mathcal{C}$ such that $(w,z) \in \mathcal{P}$, and hence $w \in \pi_w(\mathcal{P})$. Thus condition (i) is a *necessary condition* as well.

Necessity of condition (ii) is more subtle. Let $\mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$. Then for every $\tilde{w} \in \mathcal{S} = \mathcal{P} \parallel_z \mathcal{C}$ there exists $z' \in \mathcal{C}$ such that $(\tilde{w},z') \in \mathcal{P}$. Let now $(w,z') \in \mathcal{P}$. Then also $w \in \mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$. Hence condition (ii) is necessary for a *non-empty subset* of $z' \in Z^T$ such that $(w,z'),(\tilde{w},z') \in \mathcal{P}$.

Complete necessity of condition (ii) is ensured if the plant $\mathcal{P}$ satisfies the following additional 'homogeneity' property:

$\mathcal{P}$ satisfies **Property** H^W if: Let $(\tilde{w},z),(w,z) \in \mathcal{P}$. Then if $(\tilde{w},z') \in \mathcal{P}$ also $(w,z') \in \mathcal{P}$.

Indeed, let $(w,z),(\tilde{w},z) \in \mathcal{P}$ and $\tilde{w} \in \mathcal{S}$, and let $\mathcal{P}$ satisfy H^W. We have seen above that because $\tilde{w} \in \mathcal{S} = \mathcal{P} \parallel_z \mathcal{C}$ there exists $z' \in \mathcal{C}$ such that $(\tilde{w},z') \in \mathcal{P}$. Then by Property H^W also $(w,z') \in \mathcal{P}$. Hence as above $w \in \mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$.

We summarize the above discussion on the necessity of conditions (i) and (ii) of Theorem 2.1 in

Proposition 2.7. Let $\mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$ for some $\mathcal{C}$. Then $\mathcal{P}$ satisfies condition (i). Furthermore, condition (ii) holds for a non-empty subset of $z' \in Z^T$ such that $(w,z'),(\tilde{w},z') \in \mathcal{P}$. If additionally $\mathcal{P}$ satisfies H^W, then condition (ii) holds everywhere.

For later use (Sections 3 and 4) we include the following Proposition ensuring homogeneity (we leave the obvious proof to the reader).

Proposition 2.8. Let the plant P be given as

$$\mathcal{P} = \{(w,z) : T \to W \times Z \mid R(w) = M(z)\} \tag{15}$$

for certain mappings $R : W^T \to K$, $M : Z^T \to K$. Then $\mathcal{P}$ satisfies property H^W. Furthermore, in this case condition (ii) is equivalent to the following implication:
$(ii)'$ Let $R(\tilde{w}) = R(w)$ and $\tilde{w} \in \mathcal{S}$, then also $w \in \mathcal{S}$.

Let us now investigate the features of the canonical controller $\mathcal{C}_{can}$ defined in (14) for some desired system $\mathcal{S}$. In general, if there exists a controller system $\mathcal{C}$ such that $\mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$ then there will be many *different* controller systems $\mathcal{C}'$ also yielding $\mathcal{P} \parallel_z \mathcal{C}' = \mathcal{S}$. Among all these controllers the canonical controller $\mathcal{C}_{can}$ has the property of being the *least restrictive* controller, in the following sense:

Proposition 2.9. Consider the controller system $\mathcal{C}_{can}$ such that $\mathcal{P} \parallel \mathcal{C}_{can} = \mathcal{S}$. Let $\mathcal{C}$ be another controller such that $\mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$. Then for every $z \in \mathcal{C}$ with $(w,z) \in \mathcal{P}$, also $z \in \mathcal{C}_{can}$.

Proof Let $(w, z) \in \mathcal{P},\ z \in \mathcal{C}$. Then $w \in \mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$, and hence $z \in \mathcal{C}_{can}$ (take $\tilde{w} = w$). $\square$

Another distinguishing feature of the canonical controller $\mathcal{C}_{can}$ is that it can be shown to satisfy conditions $(i), (ii)$ of Theorem 2.1 *dualized* with respect to the variables w and z. Define analogously to $\pi_w(\mathcal{P})$ (see (13)), $\pi_z(\mathcal{P})$ as the projection of the plant behavior on Z^T, that is, $\pi_z(\mathcal{P}) = \{z : T \to Z \mid \exists w : T \to W \text{ such that } (w, z) \in \mathcal{P}\}$.

Theorem 2.10. *Consider for any system $\mathcal{S}$ the canonical controller $\mathcal{C}_{can}$. Then*

(i) $\mathcal{C}_{can} \subset \pi_z(\mathcal{P})$

(ii) For any $\tilde{z} \in \mathcal{C}_{can}$ there exists $w \in \mathcal{S}$ such that $(w, \tilde{z}) \in \mathcal{P}$. Then if $(w, z) \in \mathcal{P}$ also $z \in \mathcal{C}_{can}$. In general, if $(w, \tilde{z}) \in \mathcal{P},\ (w, z) \in \mathcal{P}, \tilde{z} \in \mathcal{C}_{can}$, for some $w \in \mathcal{S}$, then also $z \in \mathcal{C}_{can}$.

Proof By construction of $\mathcal{C}_{can}$. $\square$

Statement (ii) in Theorem 2.10 can be again strenghtened by imposing the following "dualized" homogeneity property on the plant behavior $\mathcal{P}$:

$\mathcal{P}$ satisfies **Property** H^Z if: Let $(w, \tilde{z}), (w, z) \in \mathcal{P}$. Then if $(w', \tilde{z}) \in \mathcal{P}$ also $(w', z) \in \mathcal{P}$.

Proposition 2.11. Let $\mathcal{P}$ satisfy property H^Z. Define $\mathcal{C}_{can}$ for some $\mathcal{S}$ as in (14). Then, whenever $(w, \tilde{z}) \in \mathcal{P}, (w, z) \in \mathcal{P}, \tilde{z} \in \mathcal{C}_{can}$, also $z \in \mathcal{C}_{can}$.

Proof $\tilde{z} \in \mathcal{C}_{can} \Rightarrow \exists w' \in \mathcal{S}$ such that $(w', \tilde{z}) \in \mathcal{P}$. Then by Property H^Z $(w', z) \in \mathcal{P}$. Hence by (ii) $z \in \mathcal{C}_{can}$. $\square$

Furthermore, the sufficient condition for Property H^W as formulated in Proposition 2.8 is easily seen to be sufficient for Property H^Z as well:

Proposition 2.12. Let $\mathcal{P}$ be given as in (15). Then $\mathcal{P}$ satisfies property H^Z. Furthermore, condition (ii) of Theorem 2.10 can be strenghtened to:
$(ii)'$ Let $M(\tilde{z}) = M(z)$ and $\tilde{z} \in \mathcal{C}_{can}$, then also $z \in \mathcal{C}_{can}$.

Remark 2.13. The canonical controllers $\mathcal{C}_{can} := \mathcal{P} \parallel_w \mathcal{S}$, with $\mathcal{S}$ any system, are 'universal' in the following sense. Let $\mathcal{C}$ be any controller, and denote $\mathcal{S} := \mathcal{P} \parallel_z \mathcal{C}$. Then define $\mathcal{C}_{can} := \mathcal{P} \parallel_w \mathcal{S}$. If $\mathcal{P}$ satisfies Property H^Z it follows that

$$\mathcal{P} \parallel_z \mathcal{C}_{can} = \mathcal{S}$$

Indeed, let $w \in \mathcal{S}$. Then $\exists z \in \mathcal{C}_{can}$ with $(w, z) \in \mathcal{P}$. Therefore $w \in \mathcal{P} \parallel_z \mathcal{C}_{can}$ (see Figure 4).

Conversely, let $w \in \mathcal{P} \parallel_z \mathcal{C}_{can}$. Then there exist $z, \tilde{w}$ and $\tilde{z}$ such that $(w, z) \in \mathcal{P},\ (\tilde{w}, z) \in \mathcal{P},\ (\tilde{w}, \tilde{z}) \in \mathcal{P},\ \tilde{z} \in \mathcal{C}_{can}$; see Figure 5.

Since $\mathcal{P}$ satisfies Property H^Z, it follows that also $z \in \mathcal{C}$, and hence $w \in \mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$.

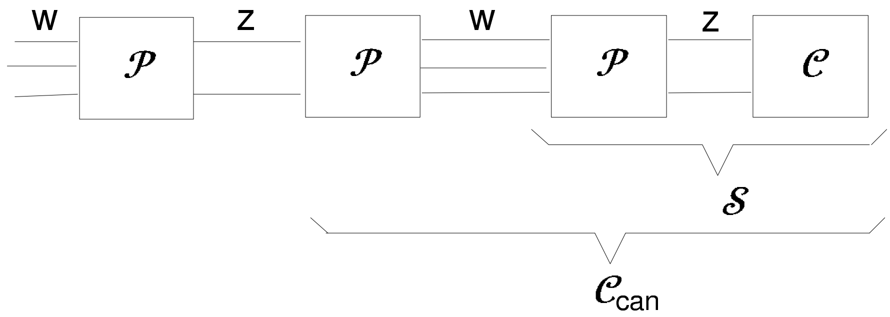

Figure 4: $\mathcal{P} \parallel_z \mathcal{C}_{can} \supset \mathcal{S}$

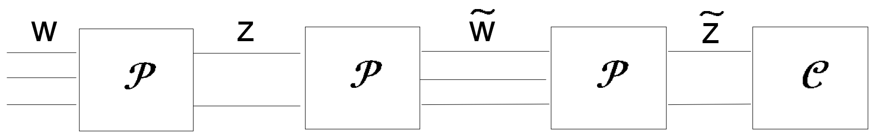

Figure 5: $\mathcal{P} \parallel_z \mathcal{C}_{can} \subset \mathcal{S}$

From an implementation point of view a basic problem in the construction of the canonical controllers concerns the presence of the auxiliary variables $\tilde{w}$. Indeed, we would like to have an algorithmic procedure for eliminating these latent variables, and so to obtain an equivalent *explicit* controller. For the linear time-invariant case this can be easily done (see Section 3), using operations on polynomial matrices. In general, we expect that elimination algorithms will strongly depend on the particular class of systems at hand. Furthermore, important questions remain concerning the nature and 'realizability' of the obtained controllers (introduction of algebraic constraints, reduction to input-output format,..; see e.g. [11])

3 The linear time-invariant finite-dimensional case

Consider the linear time-invariant finite-dimensional plant system $\mathcal{P}$ represented by (1). Clearly, $\mathcal{P}$ satisfies condition (15). Hence in view of Proposition 2.8 and Theorem 2.1 there exists a controller system $\mathcal{C}$ such that $\mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$ if and only if

$$\begin{aligned} &(i) \quad \mathcal{S} \subset \pi_w(\mathcal{P}) \\ &(ii) \quad R\left(\tfrac{d}{dt}\right)\tilde{w}(t) = R\left(\tfrac{d}{dt}\right)w(t), \quad \tilde{w} \in \mathcal{S} \Rightarrow w \in \mathcal{S} \end{aligned} \tag{16}$$

Now, let $\mathcal{S}$ be a linear desired behavior represented by (6), (7). Then by linearity it immediately follows that (ii) in (16) is equivalent to the implication

$$(ii)' \quad R\left(\frac{d}{dt}\right)\bar{w}(t) = 0 \Rightarrow G\left(\frac{d}{dt}\right)\bar{w}(t) = 0 \tag{17}$$

and hence to the inclusion

$$(ii)'' \quad \mathcal{P}_0 \subset \mathcal{S} \tag{18}$$

Thus we have recovered Theorem 1.1. Furthermore, the canonical controller $\mathcal{C}_{can}$ is the linear system (with auxiliary variables $\tilde{w}$)

$$\mathcal{C}_{can}: \begin{array}{rcl} G\left(\frac{d}{dt}\right)\tilde{w}(t) & = & 0 \\ R\left(\frac{d}{dt}\right)\tilde{w}(t) & = & M\left(\frac{d}{dt}\right)z(t) \end{array} \tag{19}$$

In this case it is easy to see how an equivalent *explicit* controller representation (without auxiliary variables $\tilde{w}$) can be obtained. Indeed, by (17) there exists a polynomial matrix $L(s)$ of appropriate dimensions such that

$$G(s) = L(s)R(s) \tag{20}$$

Hence by subtracting from the first set of equations of (19) the second set 'premultiplied' by the differential operator $L\left(\frac{d}{dt}\right)$ we obtain the equivalent representation of $\mathcal{C}_{can}$

$$\mathcal{C}_{can}: \begin{array}{l} L\left(\frac{d}{dt}\right)M\left(\frac{d}{dt}\right)z(t) = 0 \\ R\left(\frac{d}{dt}\right)\tilde{w}(t) = M\left(\frac{d}{dt}\right)z(t) \end{array} \tag{21}$$

Hence $\mathcal{P} \parallel_z \mathcal{C}_{can}$ is also represented as

$$\begin{array}{l} R\left(\frac{d}{dt}\right)w(t) = M\left(\frac{d}{dt}\right)z(t) \\ L\left(\frac{d}{dt}\right)M\left(\frac{d}{dt}\right)z(t) = 0 \\ R\left(\frac{d}{dt}\right)\tilde{w}(t) = M\left(\frac{d}{dt}\right)z(t) \end{array} \tag{22}$$

Since clearly the third set of equations in (22) does not impose any constraints on the time-functions $w(\cdot), z(\cdot)$ which is not already contained in the first and second set of equations (take $\tilde{w} = w$), it follows that $\mathcal{P} \parallel_z \mathcal{C}_{can}$ is already represented as

$$\begin{array}{l} R\left(\frac{d}{dt}\right)w(t) = M\left(\frac{d}{dt}\right)z(t) \\ L\left(\frac{d}{dt}\right)M\left(\frac{d}{dt}\right)z(t) = 0 \end{array} \tag{23}$$

and thus $\mathcal{C}_{can}$ can be represented simply by

$$L\left(\frac{d}{dt}\right)M\left(\frac{d}{dt}\right)z(t) = 0 \tag{24}$$

This final representation of $\mathcal{C}_{can}$ clearly illustrates Theorem 2.1 and Proposition 2.12.

Remark 3.1. The above reasoning leads to what is probably the simplest proof of the sufficiency part of Theorem 1.1 for obtaining an explicit representation (2) of a linear controller $\mathcal{C}$ such that $\mathcal{P} \parallel_z \mathcal{C} = \mathcal{S}$, with $\mathcal{S}$ given by (6). The proof consists of the following two direct steps.

Step 1

Since $\mathcal{S} \subset \pi_w(\mathcal{P})$ it follows that

$$\mathcal{S} = \{w : \mathbb{R} \to \mathbb{R}^q, C^\infty \mid G\left(\frac{d}{dt}\right) w(t) = 0\} = \{w : \mathbb{R} \to \mathbb{R}^q, C^\infty \mid$$

$$\exists z : \mathbb{R} \to \mathbb{R}^k, C^\infty, \text{ s.t. } R\left(\frac{d}{dt}\right) w(t) = M\left(\frac{d}{dt}\right) z(t), G\left(\frac{d}{dt}\right) w(t) = 0\} \qquad (25)$$

Step 2

Since $\mathcal{P}_0 \subset \mathcal{S}$ it follows that there exists $L(s)$ such that $G(s) = L(s)R(s)$ (cf. (17), (20)). Therefore, $w(t), z(t)$ satisfy

$$R\left(\frac{d}{dt}\right) w(t) = M\left(\frac{d}{dt}\right) z(t), \quad G\left(\frac{d}{dt}\right) w(t) = 0$$

if and only if they satisfy

$$R\left(\frac{d}{dt}\right) w(t) = M\left(\frac{d}{dt}\right) z(t), \quad L\left(\frac{d}{dt}\right) M\left(\frac{d}{dt}\right) z(t) = 0 \qquad (26)$$

Hence, by using this equivalence in (25) it follows that $\mathcal{S} = \mathcal{P} \parallel_z \mathcal{C}$ for $\mathcal{C}$ defined by (24).

Remark 3.2. Instead of considering the smooth (C^∞) solutions of the linear higher-order differential equations we may also *enlarge* the solution set to $\mathcal{L}_1^{loc}$ time-functions which are solutions in a *distributional sense.* In this case we consider the following enlarged plant and controller behaviors (compare with (4))

$$\begin{aligned} \tilde{\mathcal{P}} &:= \{(w, z) : \mathbb{R} \to \mathbb{R}^q \times \mathbb{R}^k, \mathcal{L}_1^{loc} \mid (1) \text{ is satisfied in distributional sense}\} \\ \tilde{\mathcal{C}} &:= \{z : \mathbb{R} \to \mathbb{R}^k, \mathcal{L}_1^{loc} \mid (2) \text{ is satisfied in distributional sense}\} \end{aligned} \qquad (27)$$

and the interconnected behavior

$$\tilde{\mathcal{P}} \parallel_z \tilde{\mathcal{C}} := \left\{w : \mathbb{R} \to \mathbb{R}^q, \mathcal{L}_1^{loc} \mid \exists z : \mathbb{R} \to \mathbb{R}^k, \mathcal{L}_1^{loc}, \text{ s.t.(3) is satisfied in distr. sense}\right\} \qquad (28)$$

A technical complication which arises in this setting (see [3]) is that $\tilde{\mathcal{P}} \parallel_z \tilde{\mathcal{C}}$ cannot be always represented as $\tilde{\mathcal{S}}$ with

$$\tilde{\mathcal{S}} := \left\{w : \mathbb{R} \to \mathbb{R}^q, \mathcal{L}_1^{loc} \mid (6) \text{ is satisfied in distributional sense}\right\} \qquad (29)$$

for some polynomial matrix $G(s)$. On the other hand, $(\tilde{\mathcal{P}} \parallel_z \tilde{\mathcal{C}})^{closure}$, with closure taken in the $\mathcal{L}_1^{loc}$- topology, *can* be always represented this way.

Nevertheless, similarly as above, we derive in this setting the following analogous statement: *Let $\tilde{\mathcal{P}}$ be a given plant system as in (27). Let $\mathcal{S}$ be any desired linear system consisting of $\mathcal{L}_1^{loc}$ time-functions. Then there exists a controller system $\tilde{\mathcal{C}}$ as in (27) such that $\tilde{\mathcal{P}} \parallel_z \tilde{\mathcal{C}} = \mathcal{S}$ if and only if*

$$\tilde{\mathcal{P}}_0 \subset \mathcal{S} \subset \pi_w(\tilde{\mathcal{P}}). \qquad (30)$$

(Note that although the equality $R\left(\frac{d}{dt}\right)w(t) = M\left(\frac{d}{dt}\right)z(t)$ only holds in the distributional sense, still condition (ii) in Theorem 2.1 can be replaced by the analogue of (16, part (ii)) in the distributional sense, that is

$$(ii)' R\left(\frac{d}{dt}\right)\tilde{w}(t) = R\left(\frac{d}{dt}\right)w(t) \text{ (in the distributional sense)}, \ \tilde{w} \in \mathcal{S} \Rightarrow w \in \mathcal{S} \quad (31)$$

and hence by

$$(ii)'' R\left(\tfrac{d}{dt}\right)\bar{w}(t) = 0 \quad \text{(distributional sense)} \Rightarrow \bar{w} \in \mathcal{S},$$

whence the first inclusion in (30)).

Furthermore, the canonical controller $\mathcal{C}_{can}$ is given as

$$\mathcal{C}_{can} = \{z : \mathbb{R} \to \mathbb{R}^k, \mathcal{L}_1^{loc} \mid \exists \tilde{w} \in \mathcal{S} \text{ s.t. (1) is satisfied in distributional sense}\} \quad (32)$$

In particular, if $\mathcal{S} = \tilde{\mathcal{S}}$ with $\tilde{\mathcal{S}}$ as in (29), then $\mathcal{C}_{can}$ is given as

$$\mathcal{C}_{can} = \{z : \mathbb{R} \to \mathbb{R}^k, \mathcal{L}_1^{loc} \mid \exists \tilde{w} : \mathbb{R} \to \mathbb{R}^q, \mathcal{L}_1^{loc}, \text{s.t. (19) is satisfied in distr. sense}\}, \quad (33)$$

which has the equivalent representation (see (24))

$$\mathcal{C}_{can} = \{z : \mathbb{R} \to \mathbb{R}^k, \mathcal{L}_1^{loc} \mid L\left(\frac{d}{dt}\right) M\left(\frac{d}{dt}\right) z(t) = 0 \text{ in distributional sense}\} \quad (34)$$

Compare condition (30) with the slightly different version in [1], where everywhere the closure is being taken with respect to the $\mathcal{L}_1^{loc}$ topology.

4 On the nonlinear case

In this section we briefly point out a few specializations of the general framework established in Section 2 to finite-dimensional nonlinear systems. In particular, we consider nonlinear plant systems $\mathcal{P}$ represented by higher-order nonlinear differential equations in the finite dimensional variables w and z

$$\mathcal{P}: \qquad F(w, \dot{w}, \ddot{w}, \cdots, z, \dot{z}, \ddot{z}, \cdots) = 0, \quad (35)$$

with $w \in W, z \in Z$, where W and Z are smooth manifolds of dimension q, respectively k. Analogously, the desired behavior $\mathcal{S}$ is represented as

$$\mathcal{S}: \qquad G(w, \dot{w}, \ddot{w}, \cdots) = 0. \quad (36)$$

At this point there are two options to proceed. The first option is to define the *solution set* of the systems (35) and (36) in some suitable function space. In this option we may directly use the behavioral results of Section 2 (in particular Theorem 2.1), in order to characterize the set of achievable behaviors $\mathcal{S}$ and the corresponding canonical controllers $\mathcal{C}_{can}$. Drawback of this approach, however, is that checking the conditions of Theorem 2.1 may not be easy (because we have to explicitly solve the differential equations (35) and (36)), and that the results may depend on the choice of the solution function space.

Second option is to proceed in an *algebraic* way, by concentrating on the *equations* (35) and (36) instead of their solution sets. Thus we interpret 'solutions' of (36) in the algebraic sense of an element $(w, \dot{w}, \ddot{w}, \cdots, w^{(k)})$ in the higher-order tangent bundle T^kW of W. That is, the element $(w, \dot{w}, \ddot{w}, ...)$ of T^kW (for some k large enough) is an *algebraic* solution of the equations $G(w, \dot{w}, \ddot{w}, \cdots) = 0$ in the indeterminates $w, \dot{w}, \ddot{w}, \cdots$. Similarly for the algebraic solutions of (35).

By doing so we deviate from the strictly behavioral approach of Section 2. On the other hand, although the results of Section 2 are formulated in terms of the behaviors $\mathcal{P} \subset (W \times Z)^T, \mathcal{C} \subset Z^T, \mathcal{S} \subset W^T$, we did not really use the time-function structure of these sets. Indeed, all statements remain equally valid if we replace $(W \times Z)^T = W^T \times Z^T, Z^T, W^T$ by general sets $\mathcal{W} \times \mathcal{Z}, \mathcal{Z}, \mathcal{W}$, and consider $\mathcal{P} \subset \mathcal{W} \times \mathcal{Z}, \mathcal{C} \subset \mathcal{Z}, \mathcal{S} \subset \mathcal{W}$. Hence, we can alternatively consider the systems in the following algebraic sense

$$\begin{array}{llcll} \mathcal{P} & \subset & \mathcal{W} \times \mathcal{Z} & := & T^kW \times T^kZ \\ \mathcal{C} & \subset & \mathcal{Z} & := & T^kZ \\ \mathcal{S} & \subset & \mathcal{W} & := & T^kW \end{array} \tag{37}$$

where the integer k is taken sufficiently large in order to accomodate all the higher-order derivatives of w and z appearing in the higher-order differential equations.

In this setting Theorem 2.1 and Propositions 2.7, 2.8 yield

Proposition 4.1. Let $\mathcal{P}$ and $\mathcal{S}$ represented by (35), respectively (36), be such that

(i) For all $(w, \dot{w}, \ddot{w}, \cdots)$ such that $G(w, \dot{w}, \ddot{w}, \cdots) = 0$ there exists $(z, \dot{z}, \ddot{z}, \cdots)$ such that $F(w, \dot{w}, \ddot{w}, \cdots, z, \dot{z}, \ddot{z}, \cdots) = 0$.

(ii) For any $(w, \dot{w}, \ddot{w}, \cdots, z, \dot{z}, \ddot{z}, \cdots)$, $(\tilde{w}, \dot{\tilde{w}}, \ddot{\tilde{w}}, \cdots, z, \dot{z}, \ddot{z}, \cdots)$ such that $F(w, \dot{w}, \ddot{w}, \cdots, z, \dot{z}, \ddot{z}, \cdots) = 0$, $F(\tilde{w}, \dot{\tilde{w}}, \ddot{\tilde{w}}, \cdots, z, \dot{z}, \ddot{z}, \cdots) = 0$, whenever $G(\tilde{w}, \dot{\tilde{w}}, \ddot{\tilde{w}}, \cdots) = 0$ then also $G(w, \dot{w}, \ddot{w}, \cdots) = 0$

Then the implicitly defined nonlinear controller

$$\mathcal{C}_{can} : \begin{cases} F(\tilde{w}, \dot{\tilde{w}}, \ddot{\tilde{w}}, \cdots, z, \dot{z}, \ddot{z}, \cdots) = 0 \\ G(\tilde{w}, \dot{\tilde{w}}, \ddot{\tilde{w}}, \cdots) = 0 \end{cases} \tag{38}$$

is such that the composed system $\mathcal{P} \parallel_z \mathcal{C}_{can}$ represented by

$$\begin{cases} F(w, \dot{w}, \ddot{w}, \cdots, z, \dot{z}, \ddot{z}, \cdots) = 0 \\ F(\tilde{w}, \dot{\tilde{w}}, \ddot{\tilde{w}}, \cdots, z, \dot{z}, \ddot{z}, \cdots) = 0 \\ G(\tilde{w}, \dot{\tilde{w}}, \ddot{\tilde{w}}, \cdots) = 0 \end{cases} \tag{39}$$

is equivalent to $\mathcal{S}$, in the sense that all algebraic solutions $(w, \dot{w}, \ddot{w}, \cdots)$ to (36) are solutions of (39) for some $(z, \dot{z}, \ddot{z}, \cdots)$, $(\tilde{w}, \dot{\tilde{w}}, \ddot{\tilde{w}}, \cdots)$, and vice versa all solutions $(w, \dot{w}, \ddot{w}, \cdots)$ to (39) are solutions to (36).

Conversely, if F satisfies the homogeneity property

$$\left.\begin{array}{r} F(\tilde{w},\dot{\tilde{w}},\ddot{\tilde{w}},\cdots,z,\dot{z},\ddot{z},\cdots)=0 \\ F(w,\dot{w},\ddot{w},\cdots,z,\dot{z},\ddot{z},\cdots)=0 \\ F(\tilde{w},\dot{\tilde{w}},\ddot{\tilde{w}},\cdots,z',\dot{z}',\ddot{z}'\cdots)=0 \end{array}\right\} \Rightarrow F(w,\dot{w},\ddot{w},\cdots,z',\dot{z}',\ddot{z}'\cdots)=0 \tag{40}$$

then if there exists a controller $\mathcal{C}$ represented as

$$H(z,\dot{z},\ddot{z},\cdots)=0 \tag{41}$$

such that

$$\begin{array}{r} F(\tilde{w},\dot{\tilde{w}},\ddot{\tilde{w}},\cdots,z,\dot{z},\ddot{z},\cdots)=0 \\ H(z,\dot{z},\ddot{z},\cdots)=0 \end{array} \tag{42}$$

has the same set of algebraic solutions as $\mathcal{S}$, then (i) and (ii) hold.

Furthermore, if $\mathcal{P}$ can be represented as

$$R(w,\dot{w},\ddot{w},\cdots)=M(z,\dot{z},\ddot{z},\cdots) \tag{43}$$

for certain mappings R, M, then F satisfies the homogeneity property (40), and condition (ii) can be expressed as

$(ii)'$ $R(w,\dot{w},\ddot{w},\cdots)=R(\tilde{w},\dot{\tilde{w}},\ddot{\tilde{w}},\cdots)$, $G(\tilde{w},\dot{\tilde{w}},\ddot{\tilde{w}},\cdots)=0 \Rightarrow G(w,\dot{w},\ddot{w},\cdots)=0.$

We leave the specialization of the remaining results of Section 2 to the nonlinear case to the reader.

An important question in this context is how the canonical nonlinear controller $\mathcal{C}_{can}$ defined in (38) can be converted into an explicit controller *without* auxilary variables $\tilde{w}$. In principle, this problem can be attacked using the approach outlined in [6, 7]. This approach extends the concept of equivalence transformation by unimodular transformations of polynomial matrices (which is underlying the results of Section 3) to the nonlinear domain.

Finally, as alluded to above, additional (and non-trivial) work needs to be done regarding the interpretation of the algebraic results obtained above in terms of solutions to the differential equations.

5 Conclusions and open problems

While the results of Section 2 have been derived in a general behavioral context it is important to specialize and, whenever possible, strengthen the results within more structured situations, ranging from discrete-event systems, hybrid systems, infinite-dimensional systems, to $n-D$ systems. In Section 3 we have performed this task for linear time-invariant finite-dimensional systems, thereby recovering the results of [12]. In Section 4 we have indicated how the general results of Section 2 can be applied to nonlinear systems, taking an algebraic point of view.

A real challenge is the application of this methodology to discrete-event and hybrid systems. In particular, it seems that a purely behavioral point of view will not be satisfactory in this case, and that instead of achievable behaviors we should look at achievable systems modulo equivalence under (weak) bisimulation. Nevertheless, we expect that the basic ideas exposed in this paper carry over to this problem setting. Preliminary work in this direction can be found in [8].

References

[1] M.N. Belur, H.L. Trentelman. Stabilization, pole placement and regular implementability. *IEEE Transactions on Automatic Control, 47*, pp. 735–744, 2002.

[2] H. Narayanan. Some applications of an Implicit Duality Theorem to connections of structures of special types including Dirac and reciprocal structures, *Systems and Control Letters, 45*, pp. 87–96, 2002.

[3] J.W. Polderman. Proper elimination of latent variables, *Systems & Control Letters, 32*, pp.261-269, 1997.

[4] J.W. Polderman. Sequential continuous time adaptive control: a behavioral approach, pp. 2484-2487 in Proc. 39th IEEE Conf. Decision and Control, Sydney, Australia, 2000.

[5] J.W. Polderman, J.C. Willems, *Introduction to Mathematical Systems Theory: A Behavioral Approach*, Springer-Verlag, 1997.

[6] A.J. van der Schaft. Transformations of nonlinear systems under external equivalence, pp. 33-43, in *New trends in nonlinear control theory*, (eds. J. Descusse, M. Fliess, A. Isidori, D. Leborgne), Lect. Notes in Control and Inform. Sci., vol. 122, Springer, Berlin, 1989.

[7] A.J. van der Schaft. Transformations and representations of nonlinear systems, pp. 297-314 in *Perspectives in Control Theory*, (eds. B. Jakubczyk, K. Malanowski, W. Respondek), Birkhaüser, Boston, 1990.

[8] A.J. van der Schaft, A.A. Julius. Achievable behavior by composition. Proc.41st IEEE Conf. Decision and Control, Las Vegas, USA, 2002, to appear.

[9] H.L. Trentelman. A truly behavioral approach to the H_∞- control problem, pp. 177-190 in *The Mathematics of Systems and Control: From Intelligent Control to Behavioral Systems*, (editors J.W. Polderman and H.L. Trentelman), Foundations Systems and Control Groningen, 1999.

[10] S. Weiland, A.A. Stoorvogel, B.D. de Jager. A behavioral approach to the H_∞ optimal control problem, *Systems and Control Letters, 32*, pp. 323–334, 1997.

[11] J.C. Willems. On interconnections, control, and feedback. *IEEE Transactions on Automatic Control, 42*, pp.326–339, 1997.

[12] J.C. Willems, H.L. Trentelman. Synthesis of dissipative systems using quadratic differential forms, Part I, *IEEE Transactions on Automatic Control, 47*, pp. 53–69, 2002.