Equivalence of hybrid dynamical systems*†

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1 Introduction

A common theme in theoretical computer science (in particular, the theory of distributed processes and computer-aided verification) and in systems and control theory is to characterize systems which are 'externally equivalent'. The intuitive idea is that we only want to distinguish between two systems if the distinction can be detected by an external system interacting with these systems. This is a fundamental notion in design, enabling us to take a 'divide and rule' strategy, and in analysis, allowing us to switch between externally equivalent representations of the same system and to reduce sub-systems to externally equivalent but simpler sub-systems.

More specifically, a main issue in the theory of concurrent processes and computer-aided verification is to develop a mathematical framework that can handle the state explosion in complex systems. A crucial notion in this endeavor is the concept of *bisimulation* which expresses when a sub-process can be considered to be externally equivalent to another (hopefully simpler) process. On the other hand, classical notions in systems and control theory are *state space equivalence* of dynamical systems, and *reduction* of a dynamical system to an equivalent system with minimal state space dimension. These notions have been instrumental in e.g. linking input-output models to state space models, and in studying the properties of interconnected systems.

Developments in both areas have been rather independent, one of the reasons being that the mathematical formalisms for describing both types of systems (discrete processes on the one hand, and continuous dynamical systems on the other hand) are rather different. However, with the rise of interest in *hybrid systems*, which are systems with interacting discrete and continuous dynamics, there is a clear need to bring these theories together. This has spurred various work on the extension of the notion of bisimulation as originating in the theory concurrent processes to the hybrid case, see e.g. [3, 1, 5, 6, 15, 8, 2]. Furthermore, for continuous dynamical systems the notion of bisimulation has been closely linked to system-theoretic notions of equivalence, reduction and controlled invariance, see [5, 6, 1, 7, 8, 9, 14, 12, 11].

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The aim of this paper is to make another step in this reapproachment between the theory of concurrent processes and mathematical systems theory by further analyzing and extending the notion of *structural bisimulation* for hybrid systems as recently proposed in [11], see also [10].

2 Structural bisimulation of hybrid systems

In this paper we consider hybrid systems with discrete *and* continuous external variables. The discrete external variables are the actions corresponding to the discrete transitions, while the continuous external variables (which often may be split into input and output variables) are continuously evolving in time. The bisimulation relation should thus respect the *total* external behavior of the hybrid system, that is, with respect to the actions, *as well as* with respect to the continuous external variables. The inclusion of continuous external variables makes the setting different from previous notions of bisimulation of hybrid systems, which only involve the *discrete* external behavior, see e.g. [3, 1, 5, 6, 15].

Apart from checking the external equivalence of two different hybrid systems, the notion of bisimulation is also key in *reducing* (if possible) a hybrid system to an equivalent hybrid system with smaller hybrid (that is discrete *and* continuous) state space. This is done by constructing bisimulation equivalence relations between the hybrid system and itself, leading to an externally equivalent hybrid system with hybrid state space given by the set of equivalence classes.

We start from the definition of a hybrid system with continuous external variables as given in [13].

Definition 2.1 (Hybrid system)**.** A hybrid system is described by a six-tuple $\Sigma^{hyb} := (\mathcal{L}, \mathcal{X}, \mathcal{A}, \mathcal{W}, E, F)$, where the symbols have the following meanings.

- $\mathcal{L}$ is a discrete set, called the set of *discrete states* or *locations.*
- $\mathcal{X}$ is a finite-dimensional manifold called the *continuous state space.*
- $\mathcal{A}$ is a discrete set of symbols called the set of *discrete communication variables*, or *actions.*
- $\mathcal{W}$ is a finite-dimensional linear space called the space of *continuous communication variables.* Often the vector $w \in \mathcal{W}$ can be partitioned into an input u and an output vector y.
- E is a subset of $\mathcal{L} \times \mathcal{X} \times \mathcal{A} \times \mathcal{L} \times \mathcal{X}$; a typical element of this set is denoted by (l^-, x^-, a, l^+, x^+), with $^-$ denoting the value just *before* and $^+$ denoting the value just *after* the event.
- F is a subset $\mathcal{L} \times T\mathcal{X} \times \mathcal{W}$, where $T\mathcal{X}$ denotes the tangent bundle of $\mathcal{X}$; a typical element of this set is denoted by $(l, x, \dot{x}, w)$.

A hybrid trajectory or *run* of the hybrid system Σ^{hyb} on the time-interval $[0, T]$ consists of the following ingredients. First such a trajectory involves a discrete set $\mathcal{E} \subset [0, T]$ denoting the *event times* $t \in [0, T]$ associated with the trajectory. Secondly, there is a function $l : [0, T] \to \mathcal{L}$ which is constant on every subinterval between subsequent event times $t_a, t_b \in \mathcal{E}$, and which specifies the location of the hybrid system for $t \in (t_a, t_b)$. Thirdly, the trajectory involves admissible time-functions $x : [0,T] \to \mathcal{X}, \quad w : [0,T] \to \mathcal{W}$, satisfying for all $t \notin \mathcal{E}$ the dynamics

$$(l, x(t), \dot{x}(t), w(t)) \in F \tag{1}$$

with l the location between subsequent event times $t_a, t_b \in \mathcal{E}$. Finally, the trajectory includes a discrete function $a : \mathcal{E} \to \mathcal{A}$ such that for all $t \in \mathcal{E}$

$$(l(t^-), x(t^-), a(t), l(t^+), x(t^+)) \in E \tag{2}$$

Here $x(t^-)$ and $x(t^+)$ denote the limit values of the variables x when approaching t from the left, respectively from the right. (Hence we throughout assume that the class of admissible functions x is chosen in such a way that these left and right limits are defined.) Furthermore, $l(t^-)$ and $l(t^+)$ denote the values of l *before* and *after* the event time t. Thus a hybrid run is specified by a five-tuple

$$r = (\mathcal{E}, l, x, a, w) \tag{3}$$

Note that the subset F (the *flow* conditions) specifies the continuous dynamics of the hybrid system depending on the location the system is in, and this continuous dynamics remains the same between subsequent event times. On the other hand, E (the *event* conditions) stands for the event behavior at the event times, entailing the discrete state variables $l \in \mathcal{L}$ and the discrete communication variables $a \in \mathcal{A}$, together with a reset of the continuous state variables x. Furthermore, the flow conditions F incorporate the notion of *location invariant*, while the event conditions E include the notion of *guard*.

Remark 2.2. Much more can be said about the possible semantics of the hybrid system defined above. In particular, additional requirements can be imposed on the set $\mathcal{E} \subset [0,T]$ of event times, while on the other hand the notion of a trajectory can be further generalized by allowing for *multiple events* at the same event time. For a discussion of these issues we refer to [13].

In terms of the hybrid runs a natural definition of hybrid bisimulation can be given as follows, cf. [11]:

Definition 2.3 (Hybrid bisimulation relation)**.** Consider two hybrid systems $\Sigma_i^{hyb} = (\mathcal{L}_i, \mathcal{X}_i, \mathcal{A}_i, \mathcal{W}_i, E_i, F_i), i = 1, 2$, as above. A hybrid bisimulation relation between Σ_1^{hyb} and Σ_2^{hyb} is a subset

$$\mathcal{R} \subset (\mathcal{L}_1 \times \mathcal{X}_1) \times (\mathcal{L}_2 \times \mathcal{X}_2)$$

with the following property. Take any $(l_{10}, x_{10}, l_{20}, x_{20}) \in \mathcal{R}$. Then for every hybrid run $r_1 = (\mathcal{E}_1, l_1, x_1, a_1, w_1)$ of Σ_1^{hyb} with $(l_1(0), x_1(0)) = (l_{10}, x_{10})$ there should exist a hybrid run $r_2 = (\mathcal{E}_2, l_2, x_2, a_2, w_2)$ of Σ_1^{hyb} with $(l_2(0), x_2(0)) = (l_{20}, x_{20})$ such that for all times t for which the hybrid run r_1 is defined

- $\mathcal{E}_1 = \mathcal{E}_2 =: \mathcal{E}$

- $w_1(t) = w_2(t)$ for all $t \geq 0$ with $t \notin \mathcal{E}$

- $a_1(t) = a_2(t)$ for all $t \geq 0$ with $t \in \mathcal{E}$

- $(l_1(t), x_1(t), l_2(t), x_2(t)) \in \mathcal{R}$ for all $t \geq 0$ with $t \notin \mathcal{E}$,

and conversely for every hybrid run r_2 of Σ_2^{hyb} there should exist a hybrid run r_1 of Σ_1^{hyb} with the same properties.

A more checkable version of hybrid bisimulation is obtained by merging the algebraic characterization of bisimulation relations for dynamical systems as obtained in [12] with the common notion of bisimulation for concurrent processes. Hereto we throughout assume that the continuous state space parts of the bisimulation relation $\mathcal{R}$, namely all sets

$$\mathcal{R}_{l_1 l_2} := \{(x_1, x_2) \mid (l_1, x_1, l_2, x_2) \in \mathcal{R}\} \subset \mathcal{X}_1 \times \mathcal{X}_2 \tag{4}$$

are submanifolds.

Definition 2.4 (Structural hybrid bisimulation relation [11])**.** Consider two hybrid systems $\Sigma_i^{hyb} = (\mathcal{L}_i, \mathcal{X}_i, \mathcal{A}_i, \mathcal{W}_i, E_i, F_i), i = 1, 2$, as above. A structural hybrid bisimulation relation between Σ_1^{hyb} and Σ_2^{hyb} is a subset

$$\mathcal{R} \subset (\mathcal{L}_1 \times \mathcal{X}_1) \times (\mathcal{L}_2 \times \mathcal{X}_2)$$

with the following property. Take any $(l_1^-, x_1^-, l_2^-, x_2^-) \in \mathcal{R}$. Then for every l_1^+, x_1^+, a for which

$$(l_1^-, x_1^-, a, l_1^+, x_1^+) \in E_1,$$

there should exist l_2^+, x_2^+ such that

$$(l_2^-, x_2^-, a, l_2^+, x_2^+) \in E_2$$

while $(l_1^+, x_1^+, l_2^+, x_2^+) \in \mathcal{R}$, and conversely.
Furthermore, take any $(l_1, x_1, l_2, x_2) \in \mathcal{R}$. Then for every $\dot{x}_1, w$ for which

$$(l_1, x_1, \dot{x}_1, w) \in F_1$$

there should exist $\dot{x}_2$ such that

$$(l_2, x_2, \dot{x}_2, w) \in F_2$$

while $(\dot{x}_1, \dot{x}_2) \in T_{(x_1,x_2)}\mathcal{R}_{l_1 l_2}$, and conversely.

It is easily seen that any structural hybrid bisimulation relation is a hybrid bisimulation relation in the sense of Definition 2.3. The basic observation is that the infinitesimal invariance condition $(\dot{x}_1(t), \dot{x}_2(t)) \in T_{(x_1(t),x_2(t))}\mathcal{R}_{l_1 l_2}$ implies that the trajectory $(l_1, l_2, x_1(t), x_2(t))$ remains in $\mathcal{R}$. For the converse statement (a hybrid bisimulation relation is a structural hybrid bisimulation relation) in general additional conditions are necessary.

3 Structural bisimulation of hybrid systems described by linear equations and inequalities

Definition 2.4 provides checkable conditions for bisimulation once algebraic conditions can be found for $\mathcal{R}$ being a structural hybrid bisimulation relation. For the special case of *switching linear systems*, where the discrete dynamics is independent of the continuous dynamics (no invariants nor guards, reset map is the identity map) and all discrete transitions have the same action label, this has been worked out in [10]. In this paper it is also shown how to compute in this case the *maximal bisimulation relation* between two switching linear systems.

In the present paper conditions for structural bisimulation will be developed for hybrid systems as in Definition 2.1 where $\mathcal{X}$ is a *linear space*, and where both the event conditions E and the flow conditions F are specified by *linear equations* and *linear inequalities*.

Let us first consider bisimulation for the flow conditions F. We assume that the flow conditions F are such that the dynamics in every location is given by differential-algebraic equations (DAEs) (sometimes called the *pencil-form*, cf. [4])

$$\begin{aligned} E\dot{x} &= Ax \\ w &= Hx \end{aligned} \tag{5}$$

where the matrices E, A, H depend on the location l. Next to the linear constraint equations which may be present in (5) we allow the possibility of linear constraint inequalities (again depending on l)

$$Kx \leq 0 \tag{6}$$

Systems of the form (5), (6) naturally arise in switching physical systems modeling [13].

Consequently, the first step in order to characterize structural hybrid bisimulation is to generalize the theory of bisimulation as developed in [12] for continuous dynamical systems described in the form

$$\begin{aligned} \dot{x} &= Ax + Bu + Gd, \quad x \in \mathcal{X}, u \in \mathcal{U}, d \in \mathcal{D} \\ y &= Cx, \qquad\qquad\qquad\;\; y \in \mathcal{Y} \end{aligned} \tag{7}$$

with $w = (u, y) \in \mathcal{W} := \mathcal{U} \times \mathcal{Y}$ and d a disturbance generator, to systems (5). Note that indeed (5) is a generalization of (7). This can be seen by multiplying both sides of (7) by an annihilating matrix $G^{\perp}$ of maximal rank leading to the system without disturbances

$$\begin{aligned} G^{\perp}\dot{x} &= G^{\perp}Ax + G^{\perp}Bu, \quad x \in \mathcal{X}, u \in \mathcal{U} \\ y &= Cx, \qquad\qquad\qquad\;\; y \in \mathcal{Y} \end{aligned} \tag{8}$$

and then adding the input vector u to the state x so as to obtain the extended state vector (x, u), and rewriting (8) as

$$\begin{aligned} \begin{bmatrix} G^{\perp} & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} &= \begin{bmatrix} G^{\perp}A & G^{\perp}B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\ \begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \end{aligned} \tag{9}$$

with $w = (y, u)$.

Consider two dynamical systems of the form (5) (yet without inequalities (6)):

$$\Sigma_i : \begin{array}{rcl} E_i \dot{x}_i & = & A_i x_i, \quad x_i \in \mathcal{X}_i \\ w_i & = & H_i x_i, \quad w_i \in \mathcal{W} \quad i = 1, 2 \end{array} \tag{10}$$

The geometric characterization of a bisimulation relation as derived in [11] (Proposition 2.9) for systems of the form (7) is generalized to systems (5) as follows.

First we recall [4] that the *consistent subspace* $\mathcal{V}^*$ for a system Σ given by (5) is given as the maximal subspace $\mathcal{V} \subset \mathcal{X}$ satisfying

$$A\mathcal{V} \subset E\mathcal{V} \tag{11}$$

The subspace $\mathcal{V}^*$ is the set of all initial conditions x_0 for which there exists a continuous solution trajectory of Σ starting from $x(0) = x_0$. Any subspace $\mathcal{V}$ satisfying (11) is called a *controlled invariant* or *viability* subspace. The maximal subspace $\mathcal{V}^*$ is computed as follows, see e.g. [4]. Define recursively

$$\mathcal{V}^0 = \mathcal{X},\, \mathcal{V}^j = \{x \in \mathcal{X} \mid Ax = Ev \text{ for some } v \in \mathcal{V}^{j-1}\}, \quad j = 1, 2, \cdots . \tag{12}$$

It is readily seen that the sequence of subspaces $\mathcal{V}^j$ is non-increasing, and thus converges in a finite number of steps to the maximal subspace $\mathcal{V}^*$ satisfying (11).

Proposition 3.1. *Consider two systems Σ_i as in (10), with consistent subspaces $\mathcal{V}_i^*, i = 1, 2$. Denote by $\pi_i : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_i$ the canonical projections. A subspace $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ with $\pi_i(\mathcal{R}) \subset \mathcal{V}_i^*, i = 1, 2,$ is a bisimulation relation between Σ_1 and Σ_2 if and only if for all $(x_1, x_2) \in \mathcal{R}$ the following properties hold:*

(i) For all $\dot{x}_1 \in \mathcal{V}_1^*$ such that $E_1\dot{x}_1 = A_1 x_1$ there should exist $\dot{x}_2 \in \mathcal{V}_2^*$ such that $E_2\dot{x}_2 = A_2 x_2$ while

$$(\dot{x}_1, \dot{x}_2) \in \mathcal{R}, \tag{13}$$

and conversely for every $\dot{x}_2 \in \mathcal{V}_2^*$ such that $E_2\dot{x}_2 = A_2 x_2$ there should exist $\dot{x}_1 \in \mathcal{V}_1^*$ such that $E_1\dot{x}_1 = A_1 x_1$ while (13) holds.

(ii)

$$H_1 x_1 = H_2 x_2 \tag{14}$$

Based on this geometric characterization we obtain the following generalization of Theorem 2.10 in [12].

Theorem 3.2. *A subspace $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a bisimulation relation between Σ_1 and Σ_2 satisfying $\pi_i(\mathcal{R}) \subset \mathcal{V}_i^*, i = 1, 2,$ if and only if*

$$\begin{array}{ll} (a) & \mathcal{R} + \begin{bmatrix} \ker E_1 \cap \mathcal{V}_1^* \\ 0 \end{bmatrix} = \mathcal{R} + \begin{bmatrix} 0 \\ \ker E_2 \cap \mathcal{V}_2^* \end{bmatrix} \\[2ex] (b) & \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \mathcal{R} \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R} \\[2ex] (c) & \mathcal{R} \subset \ker \begin{bmatrix} H_1 \vdots - H_2 \end{bmatrix} \end{array} \tag{15}$$

Proof. (If) Condition (ii) follows trivially from (15c). From (15b) it follows that $\mathcal{R}$ is a controlled invariant subspace for the product dynamics of Σ_1 and Σ_2 given by

$$\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{16}$$

that is, for every $(x_1, x_2) \in \mathcal{R}$ there exists $(\dot{x}_1, \dot{x}_2) \in \mathcal{R}$ satisfying (16). This implies $\pi_i(\mathcal{R}) \subset \mathcal{V}_i^*, i = 1, 2,$.

Now let $(x_1, x_2) \in \mathcal{R}$. Again by (15b) there exists $(w_1, w_2) \in \mathcal{R}$ such that

$$\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{17}$$

Consider now any $\dot{x}_1 \in \mathcal{V}_1^*$ such that $E_1 \dot{x}_1 = A_1 x_1$. Then there exists $v_1 \in \ker E_1 \cap \mathcal{V}_1^*$ such that

$$\dot{x}_1 = w_1 + v_1 \tag{18}$$

Hence by (15a) there exists $v_2 \in \ker E_2 \cap \mathcal{V}_2^*$ and $u_1, \dot{x}_2 \in \mathcal{V}_2^*$ such that

$$\begin{pmatrix} \dot{x}_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ \dot{x}_2 \end{pmatrix} - \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \tag{19}$$

with $(u_1, \dot{x}_2) \in \mathcal{R}$. Hence $E_2 \dot{x}_2 = E_2(w_2 + v_2) = E_2 w_2 = A_2 x_2$, while

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} w_1 + v_1 \\ w_2 - v_2 \end{pmatrix} \in \mathcal{R} \tag{20}$$

Thus we have shown that for every $\dot{x}_1 \in \mathcal{V}_1^*$ with $E_1 \dot{x}_1 = A_1 x_1$ there exists $\dot{x}_2 \in \mathcal{V}_2^*$ with $E_2 \dot{x}_2 = A_2 x_2$ while $(\dot{x}_1, \dot{x}_2) \in \mathcal{R}$. Similarly it is shown that for every $\dot{x}_2 \in \mathcal{V}_2^*$ with $E_2 \dot{x}_2 = A_2 x_2$ there exists $\dot{x}_1 \in \mathcal{V}_2^*$ with $E_2 \dot{x}_2 = A_2 x_2$ and $(\dot{x}_1, \dot{x}_2) \in \mathcal{R}$. Hence Condition (i) has been proven.

(Only if) Since $\pi_i(\mathcal{R}) \subset \mathcal{V}_i^*, i = 1, 2,$, condition (i) implies (15b), while Condition (ii) is trivially equivalent with (15c). In order to prove (15a), consider $(x_1, x_2) = (0, 0) \in \mathcal{R}$. Then by Condition (i) there exists for all $\dot{x}_1 \in \mathcal{V}_1^*$ such that $E_1 \dot{x}_1 = 0$ an $\dot{x}_2 \in \mathcal{V}_1^*$ such that $E_2 \dot{x}_2 = 0$ while $(\dot{x}_1, \dot{x}_2) \in \mathcal{R}$. Clearly this implies that

$$\begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \dot{x}_2 \end{bmatrix} \in \mathcal{R} + \begin{bmatrix} 0 \\ \ker E_2 \end{bmatrix} \tag{21}$$

and thus $\begin{bmatrix} \ker E_1 \cap \mathcal{V}_1^* \\ 0 \end{bmatrix} \subset \mathcal{R} + \begin{bmatrix} 0 \\ \ker E_2 \cap \mathcal{V}_2^* \end{bmatrix}$. Similarly one gets $\begin{bmatrix} 0 \\ \ker E_2 \cap \mathcal{V}_2^* \end{bmatrix} \subset \mathcal{R} + \begin{bmatrix} \ker E_1 \cap \mathcal{V}_1^* \\ 0 \end{bmatrix}$, proving (15a). ■

The *maximal* subspace $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ satisfying (15) can be computed by the following

Algorithm 3.3. *Define the following sequence of subsets* $\mathcal{R}^j \subset \mathcal{X}_1 \times \mathcal{X}_2$

$$
\begin{aligned}
\mathcal{R}^0 &= \mathcal{V}_1^* \times \mathcal{V}_2^* \\
\mathcal{R}^1 &= \left\{ z \in \mathcal{R}^0 \mid z \in \ker \left[H_1 \vdots - H_2 \right], \mathcal{R}^1 + \begin{bmatrix} \ker E_1 \cap \mathcal{V}_1^* \\ 0 \end{bmatrix} = \mathcal{R}^1 + \begin{bmatrix} 0 \\ \ker E_2 \cap \mathcal{V}_2^* \end{bmatrix} \right\} \\
\mathcal{R}^2 &= \left\{ z \in \mathcal{R}^1 \mid \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} z \in \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R}^1,\, \mathcal{R}^2 + \begin{bmatrix} \ker E_1 \cap \mathcal{V}_1^* \\ 0 \end{bmatrix} = \mathcal{R}^2 + \begin{bmatrix} 0 \\ \ker E_2 \cap \mathcal{V}_2^* \end{bmatrix} \right\} \\
&\vdots \\
\mathcal{R}^j &= \left\{ z \in \mathcal{R}^{j-1} \mid \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} z \in \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R}^{j-1},\, \mathcal{R}^j + \begin{bmatrix} \ker E_1 \cap \mathcal{V}_1^* \\ 0 \end{bmatrix} = \mathcal{R}^j + \begin{bmatrix} 0 \\ \ker E_2 \cap \mathcal{V}_2^* \end{bmatrix} \right\}
\end{aligned} \tag{22}
$$

Proposition 3.4. *(i)* $\mathcal{R}^0 \supset \mathcal{R}^1 \supset \mathcal{R}^2 \supset \cdots$*, while* $\mathcal{R}^j$ *is a subspace if non-empty.*

(ii) $\exists j$ *such that* $\mathcal{R}^j = \mathcal{R}^{j-1} =: \mathcal{R}^*$*. If non-empty,* $\mathcal{R}^*$ *is the maximal subspace* $\mathcal{R}$ *satisfying*

$$
\mathcal{R} + \begin{bmatrix} \ker E_1 \cap \mathcal{V}_1^* \\ 0 \end{bmatrix} = \mathcal{R} + \begin{bmatrix} 0 \\ \ker E_2 \cap \mathcal{V}_2^* \end{bmatrix} \tag{23}
$$

$$
\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \mathcal{R} \subset \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix} \mathcal{R} \tag{24}
$$

$$
\mathcal{R} \subset \ker \left[H_1 \vdots - H_2 \right] \tag{25}
$$

Proof. Analogously to the proof of Theorem 3.4 in [12]. ■

It directly follows that $\mathcal{R}^*$, if non-empty, is the maximal bisimulation relation.

Proposition 3.5. *There exists a bisimulation relation between* Σ_1 *and* Σ_2 *if and only if* $\mathcal{R}^*$ *is non-empty. In this case,* $\mathcal{R}^*$ *is the maximal bisimulation relation between* Σ_1 *and* Σ_2*.*

Furthermore, we will call Σ_1 and Σ_2 to be *bisimilar* if and only if

$$
\pi_i \mathcal{R}^* = \mathcal{V}_i^*, i = 1, 2 \tag{26}
$$

where $\mathcal{V}_i^*$ is the consistent subspace of $\Sigma_i, i = 1, 2$. (Note that $\pi_i \mathcal{R}^*$, if non-empty, is a controlled invariant subspace of $\Sigma_i, i = 1, 2$.)

Now let us consider flow conditions F entailing next to the DAEs (5) linear inequalities (6). Thus consider two dynamical systems Σ_i given by (10) together with inequalities

$$
K_i x_i \leq 0 \tag{27}
$$

Then a subspace $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a bisimulation relation between Σ_1 and Σ_2 given by (10,27) if and only if $\mathcal{R}$ satisfies next to (15) the condition

$$
\text{For all } (x_1, x_2) \in \mathcal{R} \text{ it holds that } K_1 x_1 \leq 0 \Leftrightarrow K_2 x_2 \leq 0 \tag{28}
$$

Next we turn to the event conditions. E is a subset of $\mathcal{L} \times \mathcal{X} \times \mathcal{A} \times \mathcal{L} \times \mathcal{X}$; where a typical element of this set is denoted by (l^-, x^-, a, l^+, x^+). Associated with E we can define for every l^-, a, l^+ the subsets

$$E^{l^-,a,l^+} := \{(x^-, x^+) \mid (l^-, x^-, a, l^+, x^+) \in E\} \tag{29}$$

If E^{l^-,a,l^+} is empty, then there is no possible transition from location l^- to location l^+ with label a. On the other hand, if E^{l^-,a,l^+} is non-empty then it defines the *reset relation* of this transition, with

$$\{x^- \mid \exists x^+ \text{ such that } (l^-, x^-, a, l^+, x^+) \in E\} \tag{30}$$

being the *guard* of this transition. In order to remain within the linear framework it is natural to assume that the sets E^{l^-,a,l^+} are, if non-empty, described by linear equations and linear inequalities

$$E^{l^-,a,l^+} = \{(x^-, x^+) \mid M^-x^- + M^+x^+ = 0, N^-x^- + N^+x^+ \leq 0\} \tag{31}$$

where the matrices M^-, M^+, N^-, N^+ may all depend on l^-, a, l^+. Then a subspace $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ is a bisimulation relation between the event conditions E_i of $\Sigma_i, i = 1, 2$, if the following holds. Take any $(l_1^-, x_1^-, l_2^-, x_2^-) \in \mathcal{R}$. Then for every l_1^+, x_1^+, a for which

$$(l_1^-, x_1^-, a, l_1^+, x_1^+) \in E_1,$$

that is,

$$M_1^-x_1^- + M_1^+x_1^+ = 0, N_1^-x_1^- + N_1^+x_1^+ \leq 0 \tag{32}$$

there should exist l_2^+, x_2^+ such that

$$(l_2^-, x_2^-, a, l_2^+, x_2^+) \in E_2$$

that is

$$M_2^-x_2^- + M_2^+x_2^+ = 0, N_2^-x_2^- + N_2^+x_2^+ \leq 0 \tag{33}$$

(where the matrices $M_i^-, M_i^+, N_i^-, N_i^+$ may all depend on l_i^-, a, l_i^+) while $(l_1^+, x_1^+, l_2^+, x_2^+) \in \mathcal{R}$, and conversely.

Finally, by combining the bisimulation conditions (15, 28) for the flow conditions F_i with the bismulation conditions (32, 33) for the event conditions E_i we obtain (linear-algebraic) conditions for a subspace $\mathcal{R} \subset \mathcal{X}_1 \times \mathcal{X}_2$ being a bisimulation relation between Σ_1 and Σ_2.

4 Conclusions

We have laid out a framework for studying bisimulation of hybrid systems described by linear differential and algebraic equations and linear inequalities. First we extended the notion of bisimulation of continuous dynamical systems as developed in [12] to systems in pencil-form (5). Then as in [11] a notion of structural hybrid bisimulation is developed by merging the notion of bisimulation for continuous dynamical systems in pencil-form with the standard notion of bisimulation for concurrent processes.

An important challenge is to give an algorithm for computing the *maximal* structural hybrid bisimulation relation, extending the results obtained in [10] for switching linear systems without invariants and guards. Secondly, it is important to relate the proposed notion of structural hybrid bisimulation with previously proposed notions of bisimulation for hybrid systems without (or with 'abstracted') continuous external behavior, see e.g. [3, 1, 5, 6, 15, 8, 2].

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