Switched Networks and Complementarity

M.K. Çamlıbel, W.P.M.H. Heemels, A.J. van der Schaft, and J.M. Schumacher

***Abstract*— A modeling framework is proposed for circuits that are subject both to externally induced switches (time events) and to state events. The framework applies to switched networks with linear and piecewise linear elements, including diodes. We show that the linear complementarity formulation, which already has proved effective for piecewise linear networks, can be extended in a natural way to cover also switching circuits. To achieve this, we use a generalization of the linear complementarity problem known as the cone complementarity problem. We show that the proposed framework is sound in the sense that existence and uniqueness of solutions is guaranteed under a passivity assumption. We establish an equivalence between passivity and representability in port-Hamiltonian form. We prove that only first-order impulses occur and characterize all situations that give rise to a state jump; moreover, we provide rules that determine the jump. Finally, we show that within our framework energy cannot increase as a result of a jump, and we derive a stability result from this.**

***Index Terms*— Complementarity systems, piecewise linear systems, hybrid systems, ideal switches.**

I. Introduction

The standard literature on dynamical systems is mostly concerned with systems that evolve in time according to a set of rules depending smoothly on the current state of the system. However, in electrical engineering as well as in other fields, one is often confronted with systems that are most easily modeled as going through a succession of periods of smooth evolution separated by instantaneous events that mark transitions of one set of laws of evolution to another. Events may be externally induced (as in the case of switches) or internally induced (as in the case of diodes). To come up with a precise mathematical formulation of systems with events is a nontrivial matter, in particular because one has in general to allow for the possibility that a state jump is associated with events and so it would be too restrictive to require solutions to be continuous, let alone differentiable.

It is the main purpose of the present paper to propose a modeling framework for systems with events, designed in particular for switched piecewise linear networks. Our approach is based on the complementarity modeling that was used in [14] for dynamic networks with diodes. Here we extend the framework of [14] to include also external switches. It turns out that the extension can be carried out in a very natural way. Instead of working with the cone of elementwise nonnegative vectors as in [14], we use here cones of a more general type. This corresponds to the generalization of the linear complementarity problem of mathematical programming to a "cone complementarity problem" (cf. for instance [9, p. 31]). This generalization brings a more geometric flavor to the setting of [14] and may be useful as well in the modeling of mode-switching elements other than diodes. Essentially, we describe switched piecewise linear networks as cone complementarity systems that are switched in time from between several different cones from a given family.

In addition to the notion of cone complementarity, the concept of passivity is central to the development of this paper; in fact our main results all assume passivity. In this paper we establish a connection between passivity on the one hand and the class of port-Hamiltonian systems (see [26]) on the other. Specifically, we show that all linear port-Hamiltonian systems are passive and conversely, all linear passive systems that allow a strictly positive storage function can be written in port-Hamiltonian form. In this way our results can also be applied for instance to 1-D mechanical systems.

As already noted, one of the main problems in setting up a rigorous framework for switched systems is to take into account the possibility of state jumps. We need a sufficiently rich solution space that allows discontinuities in state trajectories and consequently even impulses in input trajectories. In this paper we choose a distributional framework. Although this choice effectively limits us to considering only (piecewise) linear networks, an advantage of using distributions is that we do not need to impose *a priori* a restriction on the nature of the jumps; rather we can *prove* that only first-order impulses arise, even though our setting in principle allows distributional solutions of arbitrarily high order.

II. Notation

The following notational conventions will be in force.

For any set S, 2^S denotes the power set, i.e. the set of all subsets of S. The n-tuples of elements of S will be denoted S^n as usual. The set of real numbers is denoted by $\mathbb{R}$. $\mathbb{R}_+$ stands for the set of nonnegative real numbers, i.e. $\{\rho \in \mathbb{R} \mid \rho \geqslant 0\}$. $\mathbb{C}$ denotes the set of complex numbers. For a complex number z, $\bar{z}$ and $\operatorname{Re} z$ stand for the complex conjugate and the real part of z, respectively. The notations v^T and v^* denote the transpose and conjugate transpose of a vector v. When two vectors v and w are orthogonal, i.e. $v^T w = 0$, we write $v \perp w$. Ineqaulities for real vectors must be understood componentwise.

Manuscript received January 20, 2002; revised November 18, 2002. This work was supported by the IEEE.

M.K. Çamlıbel is with the Dept. of Electronics Eng., Dogus University, 34722 Istanbul, Turkey (e-mail:kcamlibel@dogus.edu.tr), with the Dept. of Econometrics and OR, Tilburg University, 5000 LE Tilburg, The Netherlands and with the Dept. of Electrical Eng., Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands.

W.P.M.H. Heemels is with the Dept. of Electrical Eng., Eindhoven University of Technology, 5600 MB Eindhoven, The Netherlands (e-mail:w.m.p.h.heemels@tue.nl).

A.J. van der Schaft is with the Fac. of Mathematical Sciences, University of Twente, 7500 AE Enschede, The Netherlands (e-mail:twarjan@math.utwente.nl).

J.M. Schumacher with the Dept. of Econometrics and OR, Tilburg University, 5000 LE Tilburg, The Netherlands (e-mail:jms@uvt.nl).

The notation $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ matrices with real elements. The transpose of A is denoted by A^T. Let $A^{n\times m}$ be a matrix. We write A_{ij} for the (i,j)th element of A. For $J \subseteq \{1,2,\dots,n\}$, and $K \subseteq \{1,2,\dots,m\}$, A_{JK} denotes the submatrix $\{A_{jk}\}_{j\in J, k\in K}$. If $J = \{1,2,\dots,n\}$ $(K = \{1,2,\dots,m\})$, we also write $A_{\bullet K}$ $(A_{J\bullet})$. If $n = m$ and $J = K$, the submatrix A_{JJ} is called a *principal matrix* of A and the determinant of A_{JK} is called a *principal minor* of A. In order to avoid bulky notation, we use A^T_{JK} and A^{-1}_{JJ} instead of $(A^T)_{JK}$ and $(A_{JJ})^{-1}$, respectively.

Let M be square matrix. As usual, we say that M is *symmetric* if $M = M^T$ and *skew-symmetric* if $M = -M^T$. The matrix M (not necessarily symmetric) is said to be *nonnegative definite* if $v^T M v \geqslant 0$ for all vectors v. We say that M is *positive definite* if M is nonnegative definite and $v^T M v = 0$ implies $v = 0$. Sometimes, we write $M \geqslant 0$ $(M > 0)$ by meaning that M is nonnegative (positive) definite. In the obvious way, we define *nonpositive definite* and *negative definite* matrices. For two matrices M and N with the same number of columns, $\mathrm{col}(M,N)$ will denote the matrix obtained by stacking M over N. The identity matrix will be denoted by I, the zero matrix by 0.

A triple of matrices $(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ is said to be minimal if $\operatorname{rank}(B, AB, \dots, A^{n-1}B) = n$ and $\operatorname{rank}(\mathrm{col}(C, CA, \dots, CA^{n-1})) = n$.

A rational matrix $G(s)$ is said to be *proper* if $\lim_{s\uparrow\infty} G(s)$ is well-defined and finite. It is said to be *strictly proper* if it is proper and the above-mentioned limit is zero.

A subset of $\mathbb{R}^n$ is said to be *polyhedral* if it is of the form $\{v \in \mathbb{R}^n \mid Av \leqslant b\}$ for some matrix A and a vector b.

Let f be a function. We write $f|_\Omega$ for the restriction of f to the set Ω. The notation $f(\tau+)$ $(f(\tau-))$ will denote the limit $\lim_{t\uparrow\tau} f(t)$ $(\lim_{t\downarrow\tau} f(t))$ whenever it is well-defined.

The set of all Lebesgue measurable, square integrable functions $f : \Omega \leftarrow \mathbb{R}$ will be denoted $\mathcal{L}_2(\Omega)$. In case, $\Omega = \mathbb{R}_+$, we write only $\mathcal{L}_2$. The notation $\mathcal{L}_2^{loc}$ denotes locally $\mathcal{L}_2$-functions, i.e., the set $\{f \mid f|_{t,T} \in \mathcal{L}_2 \text{ for all } 0 \leqslant t < T\}$.

Dirac distribution supported at θ will be denoted by δ_θ and its k-th derivative by $\delta_\theta^{(k)}$. When it is supported at zero, we usually write δ and $\delta^{(k)}$.

We say that a proposition $\mathcal{P}(\rho)$ holds for all sufficiently small (large) ρ if there exists ρ' such that $\mathcal{P}(\rho)$ holds for all $0 < \rho \leqslant \rho'$ $(\rho' \leqslant \rho)$.

A. Cones and dual cones

Definition II.1 A set $\mathcal{C} \subseteq \mathbb{R}^\ell$ is said to be a *cone* if $v \in \mathcal{C}$ implies that $\alpha v \in \mathcal{C}$ for all $\alpha \geqslant 0$.

Definition II.2 For any nonempty set $\mathcal{Q} \subseteq \mathbb{R}^\ell$, we define the *dual cone* as the set $\{w \in \mathbb{R}^\ell \mid w^T v \geqslant 0 \text{ for all } v \in \mathcal{Q}\}$. It will be denoted by $\mathcal{Q}^*$.

Note that the dual cone of a set can be defined even if the set is not a cone. It is immediate that a dual cone is always closed and convex.

B. Complementarity problems

Linear complementarity problem (LCP) plays quite an important role in the sequel. In what follows, we will quote some well-known facts from complementarity theory.

Definition II.3 LCP(q,M): Given a vector $q \in \mathbb{R}^m$ and a matrix $M \in \mathbb{R}^{m\times m}$ find a vector $z \in \mathbb{R}^m$ such that

$$z \geqslant 0 \qquad q + Mz \geqslant 0 \tag{1a}$$

$$z^T(q + Mz) = 0. \tag{1b}$$

We say that the LCP(q,M) is *solvable* if such a z exists. In this case, we also say that z *solves (is a solution)* of LCP(q,M). The set of all solutions of LCP(q,M) is denoted by SOL(q,M). A weaker notion than solvability is feasibility. The LCP(q,M) is said to be *feasible* if there exists z such that (1a) is satisfied.

Theorem II.4 *The following statements hold.*

1) [9, Thm. 3.3.7] LCP(q,M) has a unique solution for all $q \in \mathbb{R}^m$ if and only if all principal minors of M are positive.
2) [9, Cor. 3.8.10 and Thm. 3.8.13] Suppose that M is nonnegative definite. Then, the following statements are equivalent.
   a) $q \in (\mathrm{SOL}(0,M))^*$.
   b) LCP(q,M) is feasible.
   c) LCP(q,M) is solvable.

Remark II.5 The matrices whose all principal minors are positive are known as P-matrices in complementarity theory (see e.g. [9, Def. 3.3.1]). In particular, positive definite matrices are in this class.

One interesting generalization of the LCP can be obtained by modifying the conditions (1a) as follows.

Definition II.6 LCP$_\mathcal{C}(q,M)$: Given a cone $\mathcal{C}$, a vector $q \in \mathbb{R}^m$, and a matrix $M \in \mathbb{R}^{m\times m}$ find a vector $z \in \mathbb{R}^m$ such that

$$z \in \mathcal{C} \qquad q + Mz \in \mathcal{C}^* \tag{2a}$$

$$z^T(q + Mz) = 0. \tag{2b}$$

We define solvability and feasibility as in Definition II.3. If $\mathcal{C} = \mathbb{R}^m_+$ then LCP$_\mathcal{C}(q,M)$ becomes the ordinary LCP defined in Definition II.3. The following theorem can be proven by following the footsteps of the proof of Theorem II.4.

Theorem II.7 *Let $\mathcal{C} = \mathbb{R}^m_+ \times \mathbb{R}^\ell$. Suppose that $M \in \mathbb{R}^{(m+\ell)\times(m+\ell)}$ is nonnegative definite. Then, the following statements are equivalent.*

1) $q \in (\mathrm{SOL}_\mathcal{C}(0,M))^*$.
2) LCP$_\mathcal{C}(q,M)$ *is feasible.*
3) LCP$_\mathcal{C}(q,M)$ *is solvable.*

Moreover, SOL$_\mathcal{C}(q,M)$ is polyhedral and equal to $\{z \in \mathbb{R}^{m+\ell} \mid z \in \mathcal{C},\ q + Mz \in \mathcal{C}^*,\ q^T(z - \bar{z}) = 0,\ (M + M^T)(z - \bar{z}) = 0\}$ where $\bar{z}$ is an arbitrary solution of LCP$_\mathcal{C}(q,M)$.

III. Linear network models

Consider a linear k-port electrical network consisting of only resistors (R), inductors (L), capacitors (C), gyrators (G), and transformers (T). Under suitable conditions (the network does not contain all-capacitor/voltage sources loops or nodes with the only elements incident being inductors/current sources, see [1, Ch. 4] for more details), this RLCGT circuit can be described by the state space model

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3a}$$
$$y(t) = Cx(t) + Du(t) \tag{3b}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times k}$ denote real matrices of appropriate dimensions, and x denotes the state variable of the network (typically consisting of linear combinations of the fluxes through the inductors and charges at the capacitors). The pair (u_i, y_i) denotes the voltage-current variables at the ports of the circuit.

Since (3) is a model for an RLCGT-multiport network, the matrix quadruple (A, B, C, D) is not arbitrary, but has a certain structure. Indeed, the state space description has the so-called port-Hamiltonian form

$$\dot{x}(t) = (J - R)Qx(t) + (G - P)u(t) \tag{4a}$$
$$y(t) = (G + P)^TQx(t) + (M + S)u(t) \tag{4b}$$

where J is a skew-symmetric $n \times n$ matrix, R is a symmetric $n \times n$ matrix with $R = R^T$, and Q is an $n \times n$ matrix with $Q = Q^T \geqslant 0$. The Hamiltonian $H(x)$ (the energy of the system) is given by $H(x) = \frac{1}{2}x^TQx$. Furthermore, G and P are $n \times k$ matrices, M is a skew-symmetric $k \times k$ matrix and S is a symmetric $k \times k$ matrix.

An important assumption concerning port-Hamiltonian linear systems is the following.

Assumption III.1 The system matrices of the port-Hamiltonian linear system (4) satisfy $\begin{pmatrix} R & P \\ P^T & S \end{pmatrix} \geqslant 0$.

As we will see more clearly later on, this assumption corresponds to a nonnegative internal energy dissipation. In particular, if $P = 0$, then Assumption III.1 reduces to $R \geqslant 0$ and $S \geqslant 0$.

In general linear systems (3) are called port-Hamiltonian linear systems (without algebraic constraints) if the system matrices A, B, C, and D have the above mentioned additional structure. We refer to e.g. [26] for a treatment of *general* (not necessarily linear) port-Hamiltonian systems, and how these systems naturally arise from a port-based network modeling of general (nonlinear) lumped-parameter physical systems. If there are no algebraic constraints in the system such a port-Hamiltonian linear system is given by (4).[1]

Important examples of port-Hamiltonian linear systems are 1D mechanical systems and electrical networks (see [26] for further references).

[1]This definition generalizes the definition of a port-Hamiltonian linear system given in [26] for $P = 0$ and $M = S = 0$; it does fit however within the general definition given in [26] of a port-Hamiltonian system with respect to a Dirac structure. This general definition also allows the presence of algebraic constraints.

A. Passivity

The port-Hamiltonian framework is explicitly visible in the system matrices of (4). However, the system matrices satisfy also a more implicit property called passivity, which is well-known in circuit theory.

Definition III.2 [32] A system (A, B, C, D) given by (3) is called *passive*, or *dissipative* with respect to the supply rate u^Ty, if there exists a nonnegative-valued function $V : \mathbb{R}^n \to \mathbb{R}_+$, (a *storage function*), such that for all $t_0 \leqslant t_1$ and all time functions $(u, x, y) \in \mathcal{L}_2^{k+n+k}(t_0, t_1)$ satisfying (3) the following inequality holds:

$$V(x(t_0)) + \int_{t_0}^{t_1} u^T(t)y(t)dt \geqslant V(x(t_1)). \tag{5}$$

The above inequality is called the *dissipation inequality*. The storage function represents a notion of "stored energy" in the network.

Proposition III.3 [32] *Consider a system (A, B, C, D) in which (A, B, C) is a minimal representation. The following statements are equivalent.*

- *(A, B, C, D) is passive.*
- *The transfer matrix $G(s) := C(sI - A)^{-1}B + D$ is* positive real, *i.e., $x^*[G(\lambda)+G^*(\lambda)]x \geqslant 0$ for all complex vectors x and all $\lambda \in \mathbb{C}$ such that Re $\lambda > 0$ and λ is not an eigenvalue of A.*
- *The matrix inequalities*

$$\begin{pmatrix} -A^TK - KA & -KB + C^T \\ -B^TK + C & D + D^T \end{pmatrix} \geqslant 0 \tag{6}$$

and $K = K^T \geqslant 0$ have a solution K.

Moreover, in case (A, B, C, D) is passive, all solutions to the linear matrix inequalities (6) are positive definite and a symmetric K is a solution to (6) if and only if $V(x) = \frac{1}{2}x^TKx$ defines a storage function of the system (A, B, C, D).

An assumption that we will often use is the following.

Assumption III.4 The matrix col$(B, D + D^T)$ has full column rank and the triple (A, B, C) is a minimal representation.

These assumptions imply that (specific kinds of) redundancy have been removed from the circuit (see [14] for a discussion).

We note the following consequence of passivity.

Lemma III.5 [14, Lemma III.4] *Consider a system (A, B, C, D) in which (A, B, C) is a minimal representation and (A, B, C, D) is passive. If $v \in \mathbb{R}^k$ satisfies $(D+D^T)v = 0$ (or equivalently, $v^TDv = 0$), then $C^Tv = KBv$ for any K satisfying (6).*

B. Equivalence of passive and port-Hamiltonian systems

One might wonder what the relationship between passive and port-Hamiltonian linear systems is and therefore, it forms the topic of this section. We will establish an equivalence result between those concepts, which is important because any statement for port-Hamiltonian linear systems on e.g. well-posedness and stability is now also valid for passive linear systems and vice versa.

Theorem III.6 *The following statements hold.*

1) *If the system (3) is passive with quadratic storage function $\frac{1}{2}x^T Qx$ satisfying $Q \geqslant 0$, and $Qx = 0$ implies $Ax = 0$ and $Cx = 0$, then (3) can be rewritten into the port-Hamiltonian form (4).*
2) *If $Q \geqslant 0$ then the port-Hamiltonian linear system (4) is passive.*

Remark III.7 Note that the condition ($Qx = 0 \Rightarrow Ax = 0, Cx = 0$) is automatically satisfied if $Q > 0$.

Proof: *1)* Because of the condition ($Qx = 0 \Rightarrow Ax = 0, Cx = 0$) it follows from linear algebra that there exists a matrix Σ such that

$$\begin{pmatrix} A & B \\ -C & -D \end{pmatrix} = \Sigma \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \tag{7}$$

In fact, if $Q > 0$ then such a Σ is uniquely defined. Now, passivity of the system (3) with quadratic storage function $\frac{1}{2}x^T Qx$ amounts to the differential dissipation inequality

$$x^T Q\dot{x} - u^T y \leqslant 0 \tag{8}$$

for all x, u. Substituting $\dot{x} = Ax + Bu$ and $y = Cx + Du$, and making use of (7), this can be rewritten as

$$\begin{pmatrix} x^T & u^T \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \Sigma \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \leqslant 0 \tag{9}$$

for all x, u, or equivalently

$$\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} (\Sigma + \Sigma^T) \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \leqslant 0 \tag{10}$$

It follows from basic linear algebra that we can choose Σ satisfying (7) in such a way that

$$\Sigma + \Sigma^T \leqslant 0 \tag{11}$$

Hence, if we write

$$\bar{R} = -\tfrac{1}{2}(\Sigma + \Sigma^T) \quad \bar{J} = \tfrac{1}{2}(\Sigma - \Sigma^T) \tag{12}$$

then $\bar{R} \geqslant 0$. Now, denote

$$\bar{J} = \begin{pmatrix} J & G \\ -G^T & -M \end{pmatrix}, \bar{R} = \begin{pmatrix} R & P \\ P^T & S \end{pmatrix} \tag{13a}$$

$$J = -J^T, \quad M = -M^T, \quad R = R^T, \quad S = S^T. \tag{13b}$$

Then (3) can be written as

$$\begin{pmatrix} \dot{x} \\ -y \end{pmatrix} = (\begin{pmatrix} J & G \\ -G^T & -M \end{pmatrix} - \begin{pmatrix} R & P \\ P^T & S \end{pmatrix}) \begin{pmatrix} Qx \\ u \end{pmatrix}, \tag{14}$$

or equivalently

$$\dot{x} = (J - R)Qx + (G - P)u \tag{15}$$

$$y = (G + P)^T Qx + (M + S)u \tag{16}$$

which is a system with Hamiltonian dynamics (4) satisfying Assumption III.1 due to (11) and (12).
2) We show that port-Hamiltonian linear systems (4) are passive with the Hamiltonian $H(x) = \frac{1}{2}x^T Qx$ being a storage function. Along trajectories of the port-Hamiltonian linear system we have (time arguments left out for brevity):

$$\begin{aligned} \frac{d}{dt}H(x) &= x^T Q\dot{x} = x^T Q(J - R)Qx + x^T Q(G - P)u \\ &= x^T QJQx - x^T QRQx + x^T Q(G - P)u \\ &= -x^T QRQx + x^T Q(G - P)u \quad (J = -J^T) \\ &= -x^T QRQx + y^T u - u^T (M + S)^T u - 2x^T QPu \\ &\quad (S = S^T \text{ and } M = M^T) \\ &= y^T u - ((Qx)^T \; u^T) \begin{pmatrix} R & P \\ P^T & S \end{pmatrix} \begin{pmatrix} Qx \\ u \end{pmatrix} \\ &\leqslant y^T u \quad (\text{Assumption III.1 }). \end{aligned} \tag{17}$$

Integration leads to the dissipation inequality (5). ■

IV. Switched network models

In the previous section we concentrated on the linear networks of the form (3). Adding the switches, diodes and sources will lead to the class of circuits that form the object of study of the paper.

A. Adding diodes, switches and sources

The equations that are added to (3) if the terminals are terminated by diodes, switches and sources are given as follows:

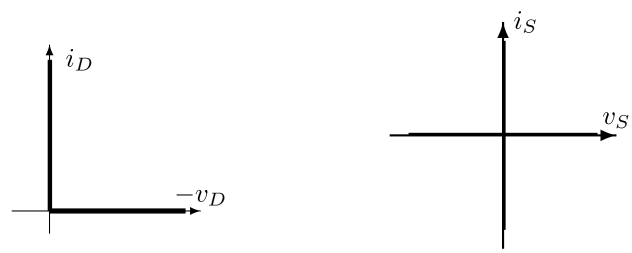

Fig. 1. Voltage-current characteristic of an ideal diode and an ideal switch

- If the i-th port is connected to a *diode*:

  $$(u_i = -V_i \;\wedge\; y_i = I_i) \;\vee\; (u_i = I_i \;\wedge\; y_i = -V_i),$$

  where V_i and I_i are the voltage across and current through the i-th diode, respectively, and $\vee$ denotes the Boolean "or" and $\wedge$ the Boolean "and"-operator. The ideal diode characteristics are described by the relations

  $$V_i \leqslant 0 \;\wedge\; I_i \geqslant 0 \;\wedge\; (V_i = 0 \;\vee\; I_i = 0) \tag{18}$$

  as shown in Figure 1. Putting the above equations together leads to $0 \leqslant u_i \perp y_i \geqslant 0$ where $u_i \perp y_i$ means that the product $u_i y_i$ is zero or stated otherwise, that either $u_i = 0$ or $y_i = 0$.
- If the i-th port is connected to a *switch*:

  $$u_i = 0 \wedge y_i = 0$$

  or stated differently, $u_i \perp y_i$ as shown in Figure 1.
- If the i-th port is connected to a *source*: u_i is actually being described by a suitable function of time, which reflects the applied voltage or current related to the port.

For the moment, we exclude voltage/current sources, but we will return to this at the end of the paper, where the required modifications are indicated.

Based on the previous discussion, we obtain network models of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) + Du(t) \tag{19a}$$
$$0 \leqslant y_i(t) \perp u_i(t) \geqslant 0,\ i = 1, 2, \ldots, m \tag{19b}$$
$$y_i(t) \perp u_i(t),\ i = m+1, m+2, \ldots, k \tag{19c}$$

where we assumed that the first m ports are terminated with diodes and the last $k-m$ ports by pure switches. The variable $t \in \mathbb{R}_+$ denotes time, $x(t)$ the state, and $u(t)$ and $y(t)$ the switch variables at time t.

System (19) without the switch conditions (19c) is called a *linear complementarity system* (LCS). System descriptions of this form were introduced in [27] and were further studied in [6], [14–16], [28]. Systems without complementarity conditions (19b) have been studied in [12] under a Hamiltonian structure and were called *switched Hamiltonian systems* (SHS). This paper provides a unified framework that has LCS and SHS as special cases and therefore encompasses a large class of switching circuits. We will use the terminology *switched complementarity systems* for systems of the form (19) together with the notation SCS(A, B, C, D).

B. Cone complementarity systems

A certain similarity between diodes and switches can be made apparent by using a formulation in terms of *cones*. The constitutive equations for a k-tuple of diodes may be written in the form

$$u \in \mathcal{C}, \quad y \in \mathcal{C}^*, \quad y \perp u \tag{20}$$

where $\mathcal{C}$ denotes the nonnegative cone $\mathbb{R}^k_+$ in $\mathbb{R}^k$, i.e. the set of k-vectors with nonnegative entries. The conditions (20) however become the specification of a set of switches in a particular configuration if we let $\mathcal{C}$ denote a set of the form $\Pi_{i=1}^k \mathcal{C}_i$ where each $\mathcal{C}_i$ is either $\mathbb{R}$ or $\{0\}$. This set is a subspace and so in particular it is a cone. The cones corresponding to diodes and to switches may be taken together in a product cone. Consequently, linear RCLTG networks with diodes and switches can always be written in the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{21a}$$
$$y(t) = Cx(t) + Du(t) \tag{21b}$$
$$\mathcal{C}^*_{\pi(t)} \ni y(t) \perp u(t) \in \mathcal{C}_{\pi(t)} \tag{21c}$$

where $\pi(\cdot)$ is a switching sequence taking values in a finite set $\{\pi_1, \ldots, \pi_N\}$, and for each i the set $\mathcal{C}_{\pi_i}$ is a closed convex cone in $\mathbb{R}^k$.

Remark IV.1 The perpendicularity relation in (20) indicates absence of dissipation. Therefore, resistors cannot be modeled as in (20).

V. Dynamics in a given mode

At this point, we could start studying the properties of solution trajectories to (19) either put an explicit structure on the system matrices (A, B, C, D) by using the Hamiltonian formalism or an implicit structure using the concept of passivity. We will opt for the latter and mainly work with a passivity assumption, but due to Theorem III.6 all results can be transformed into Hamiltonian SCSs as well. Possible other relaxations or reformulations will be discussed later on.

Note that (19b)-(19c) implies that for all $i \in \{1, 2, \ldots, k\}$ either $y_i(t) = 0$ or $u_i(t) = 0$ must be satisfied. In other words, each diode is either conducting or blocking, and each switch is either open or closed. Accordingly, diodes and switches can be replaced by a short or an open circuit.

This results in a multimodal system with 2^k modes, where each mode is characterized by a subset $\mathcal{I}$ of $\{1, 2, \ldots, k\}$, indicating that $y_i(t) = 0$ if $i \in \mathcal{I}$ and $u_i(t) = 0$ if $i \in \mathcal{I}^c$ with $\mathcal{I}^c := \{1, 2, \ldots, k\} \setminus \mathcal{I}$. We split $\mathcal{I}$ as $\mathcal{I} = \mathcal{D} \cup \mathcal{S}$ with $\mathcal{D} \subseteq \{1, 2, \ldots, m\}$ and $\mathcal{S} \subseteq \{m+1, m+2, \ldots, k\}$, where $\mathcal{D}$ denotes the status of the diodes and $\mathcal{S}$ of the switches[2].

For each such mode (also called "topology," "configuration," or "discrete state") the laws of motion are given by differential and algebraic equations (DAEs). Specifically, in mode $\mathcal{I}$ they are given by

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) + Du(t) \tag{22a}$$
$$u_i(t) = 0,\ i \in \mathcal{I}^c \qquad y_i(t) = 0,\ i \in \mathcal{I}. \tag{22b}$$

During the time evolution of the system, the mode will vary whenever some of the diodes and/or switches change their state (i.e. diodes go from conducting to blocking or vice versa and/or switches from open to close or vice versa). The switch can be considered as time events since an external device triggers the mode change, while the mode transition of the diodes are due to state events: the current mode remains active as long as the inequality conditions in (19b) are satisfied. If they tend to be violated (e.g. the current through the diode tends to become negative) a mode transition occurs.

VI. Solution concept

The time evolution of SCS is a sequence of smooth continuations followed by mode transitions.

During the smooth continuations, system trajectories satisfy the DAEs (22) for some mode $\mathcal{I}$ in the classical sense. Hence, it suffices to consider the so-called Bohl functions (see [13]). More precisely, a function f is called a *Bohl function* (or *Bohl type*) if $f(t) = He^{Ft}G$ for some matrices F, G, and H of appropriate sizes. We denote the set of all Bohl function by $\mathcal{B}$.

At the event of a mode transition, the system may in principle display jumps in the state variable x. Jumping phenomena are well-known in the theory of unilaterally constrained mechanical systems [4], where at impacts the change of velocity of the colliding bodies is often modelled as being instantaneous. These discontinuous and impulsive motions are also observed in electrical networks (see e.g. [10], [21], [23–25], [30], [31]).

To obtain a mathematically precise solution concept, we will use a distributional framework. In particular, the Dirac distribution and its derivatives will play a key role.

[2] In the sequel of the paper when we write $\mathcal{D}$ or $\mathcal{S}$, we always mean a subset of $\{1, 2, \ldots, m\}$ or $\{m+1, m+2, \ldots, k\}$, respectively. By $\mathcal{D}^c$ and $\mathcal{S}^c$, we will denote the sets $\{1, 2, \ldots, m\} \setminus \mathcal{D}$ and $\{m+1, m+2, \ldots, k\} \setminus \mathcal{S}$, respectively.

Definition VI.1 A *Bohl distribution* is a distribution $\mathtt{u}$ of the form $\mathtt{u} = \mathtt{u}_{imp} + \mathtt{u}_{reg}$, where

- $\mathtt{u}_{imp}$ is a linear combination of δ and its derivatives, i.e., $\mathtt{u}_{imp} = \sum_{i=0}^{l} u^{-i}\delta^{(i)}$ for real numbers u^{-i}, $i = 0, \ldots, l$ and
- $\mathtt{u}_{reg}$ is a Bohl function on $[0, \infty)$.

The class of Bohl distributions is denoted by $\mathcal{B}_{imp}$. For a distribution $\mathtt{u} \in \mathcal{B}_{imp}$, $\mathtt{u}_{imp}$ is called the impulsive part and $\mathtt{u}_{reg}$ is called the regular or smooth part. In case $\mathtt{u}_{imp} = 0$ we call $\mathtt{u}$ a *regular* or *smooth* distribution.

Note that the Laplace transform of a Bohl distribution is a rational function. It can be easily verified that a Bohl distribution is regular if and only if its Laplace transform is strictly proper. In what follows, Bohl distributions having a proper Laplace transform will play an important role. We call them *first order* Bohl distributions. Note that a Bohl distribution is of first order if and only if its impulsive part does not contain the derivatives of Dirac distribution.

With this machinery we can now introduce the concept of an initial solution given an initial state $x(0) = x_0$ and a switch configuration $\mathcal{S}$ for the pure switches. This actually implies that $u(t)$ is contained in the cone

$$\mathcal{C}_{\mathcal{S}} = \{v \mid v_i \geqslant 0,\ i = 1, \ldots, m, \text{ and } v_i = 0,\ i \in \mathcal{S}^c\} \quad (23)$$

and $y(t)$ should be in the dual cone $\mathcal{C}_{\mathcal{S}}^*$. Note that

$$\mathcal{C}_{\mathcal{S}}^* = \{v \mid v_i \geqslant 0,\ i = 1, \ldots, m, \text{ and } v_i = 0,\ i \in \mathcal{S}\} \quad (24)$$

Hence, that means that given $\mathcal{S}$ the governing equations (19) are reduced to

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) + Du(t) \quad (25a)$$
$$\mathcal{C}_{\mathcal{S}} \ni u(t) \perp y(t) \in \mathcal{C}_{\mathcal{S}}^* \quad (25b)$$

Note that this system can be considered as an extension of the standard linear complementarity system (LCS) in [16] as it used general positive cones $\mathcal{C}_{\mathcal{S}}$. The equations (25) become an ordinary LCS when $\mathcal{C}_{\mathcal{S}} = \mathbb{R}_+^k$.

Note that the "modes" $\mathcal{D}$ of the diodes are not specified by the formulation (25), i.e. $\mathcal{I} = \mathcal{D} \cup \mathcal{S}$ in (22) is not completely known. Hence, a solution in a mode $\mathcal{I}$ being governed by (22) is valid as long as $\mathcal{D}$ does not change. This means that mode $\mathcal{I}$ will only be valid for a limited amount of time in general, since a change of mode (diode going from conducting to blocking or vice versa) may be triggered by the inequality constraints. Therefore, we would like to express some kind of "local satisfaction of the constraints."

We call a (smooth) Bohl function v *initially in the cone* $\mathcal{C}$ if there exists an $\varepsilon > 0$ such that $v(t) \in \mathcal{C}$ for all $t \in [0, \varepsilon)$. We know from the initial value theorem (see e.g. [11]) that there is a connection between small time values of time functions and large values of the indeterminate s in the Laplace transform. In fact, one can show that v is initially in the cone $\mathcal{C}$ if and only if there exists a $\sigma_0 \in \mathbb{R}$ such that its Laplace transform $\hat{v}(\sigma) \in \mathcal{C}$ for all $\sigma \geqslant \sigma_0$.

The definition of being initially in the cone $\mathcal{C}$ for Bohl distributions will be based on this observation (see also [15]).

Definition VI.2 We call a Bohl distribution $\mathtt{v}$ *initially in the cone* $\mathcal{C}$ if its Laplace transform $\hat{\mathtt{v}}(s)$ satisfies $\hat{\mathtt{v}}(\sigma) \in \mathcal{C}$ for all sufficiently large real σ.

Remark VI.3 To relate the definition to the time domain, note that a scalar-valued[3] first order Bohl distribution $\mathtt{v}$ (i.e., $\mathtt{v}_{imp} = v^0\delta$ for some $v^0 \in \mathbb{R}$) is initially in the cone $\mathcal{C}$ if and only if

1) $v^0 \in \mathcal{C}$, or
2) $v^0 = 0$ and there exists an $\varepsilon > 0$ such that $v_{reg}(t) \in \mathcal{C}$ for all $t \in [0, \varepsilon)$.

Now, we are in a position to define a local solution concept.

Definition VI.4 We call a Bohl distribution $(\mathtt{u}, \mathtt{x}, \mathtt{y}) \in \mathcal{B}_{imp}^{k+n+k}$ an *initial solution* to (19) with initial state x_0 and pure switch configuration $\mathcal{S}$ if

1) there is a diode configuration $\mathcal{D}$ such that $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ satisfies (22) for mode $\mathcal{I} = \mathcal{D} \cup \mathcal{S}$ and initial state x_0 in the distributional sense, i.e. satisfies

$$\dot{\mathtt{x}} = A\mathtt{x} + B\mathtt{u} + x_0\delta \qquad \mathtt{y} = C\mathtt{x} + D\mathtt{u} \quad (26a)$$
$$\mathtt{u}_i = 0,\ i \notin \mathcal{I} \qquad \mathtt{y}_i = 0,\ i \in \mathcal{I} \quad (26b)$$

as equalities of distributions, and
2) $(\mathtt{u}, \mathtt{y})$ are initially in the cone $(\mathcal{C}_{\mathcal{S}} \times \mathcal{C}_{\mathcal{S}}^*)$.

Note that the condition 2, together with real-analiticity of Bohl functions, already implies that (26b) hold for $i \in \mathcal{S}$ and $i \in \mathcal{S}^*$, respectively.

For examples of initial solutions in networks without pure switches one can consider Example V.4 and V.5 in [14].

Theorem VI.5 *Consider an SCS given by* (19) *such that Assumption III.4 is satisfied and* (A, B, C, D) *represents a passive system. Let a pure switch configuration* $\mathcal{S}$ *be given and let* $\mathcal{Q}_{\mathcal{S}}$ *be the solution set of* LCP$_{\mathcal{C}_{\mathcal{S}}}(0, D)$*, i.e.,* $\mathcal{Q}_{\mathcal{S}} = \{v \in \mathbb{R}^k \mid v \in \mathcal{Q}^*, Dv \in \mathcal{Q}_{\mathcal{S}}^*$ *and* $v \perp Dv\}$*. Then, the following statements hold.*

1) *For each initial state* x_0*, there exists exactly one initial solution to SCS.*
2) *This solution is of first order. Stated differently, its impulsive part is of the form* $(u^0\delta, 0, Du^0\delta)$ *for some* $u^0 \in \mathcal{Q}_{\mathcal{S}}$*.*
3) *This impulsive part results in a re-initialization (jump) -if applicable- of the state from* x_0 *to* $x_0 + Bu^0$*.*
4) *For all* $x_0 \in \mathbb{R}^n$*,* $C(x_0 + Bu^0) \in \mathcal{Q}_{\mathcal{S}}^*$*.*
5) *The initial solution is smooth (i.e.,* $u^0 = 0$*) if and only if* $Cx_0 \in \mathcal{Q}_{\mathcal{S}}^*$*.*

Proof: 1: If $\mathcal{S} = \varnothing$, the proof follows from [5, Thm. 6.1]. In the general case, we will employ the ideas and techniques that are used in this reference. Define $\mathcal{M} := \{1, 2, \ldots, m\}$, $q(s) := C(sI - A)^{-1}x_0$, and $G(s) := D + C(sI - A)^{-1}B$. Further, define $q'(s) = q_{\mathcal{M}}(s) - G_{\mathcal{MS}}(s)G_{\mathcal{SS}}^{-1}(s)q_{\mathcal{S}}(s)$ and

[3] In this case the cone $\mathcal{C}$ can only be equal to $\mathbb{R}$, $\mathbb{R}_+$, $-\mathbb{R}_+$ or $\{0\}$.

$G'(s) = G_{\mathcal{M}\mathcal{M}}(s) - G_{\mathcal{M}\mathcal{S}}(s)G_{\mathcal{S}\mathcal{S}}^{-1}(s)G_{\mathcal{S}\mathcal{M}}(s)$. Now, consider the following complementarity problem:

$$\hat{\mathrm{y}}_{\mathcal{M}}(s) = q'(s) + G'(s)\hat{\mathrm{u}}_{\mathcal{M}}(s)$$
$$\hat{\mathrm{u}}_{\mathcal{M}}(\sigma) \geqslant 0 \text{ and } \hat{\mathrm{y}}_{\mathcal{M}}(\sigma) \geqslant 0 \text{ for all sufficiently large real } \sigma$$
$$\hat{\mathrm{u}}_{\mathcal{M}}(s) \perp \hat{\mathrm{y}}_{\mathcal{M}}(s) \text{ for all } s \in \mathbb{C}.$$

Problems of this type are called *rational complementarity problems* (RCPs). The RCP has been introduced in [27] and further studied in [15]. It is already well-known that there is a one-to-one correspondence between the initial solutions of LCSs and the solutions of RCPs (see [15]). We first suppose that the RCP has a solution $(\hat{\mathrm{u}}_{\mathcal{M}}(s), \hat{\mathrm{y}}_{\mathcal{M}}(s))$. Define $\hat{\mathrm{u}}_{\mathcal{S}}(s) := -G_{\mathcal{S}\mathcal{S}}^{-1}(s)[q_{\mathcal{S}}(s) + G_{\mathcal{S}\mathcal{M}}(s)\hat{\mathrm{u}}_{\mathcal{M}}(s)]$, $\hat{\mathrm{u}}_{\mathcal{S}^c}(s) := 0$, $\hat{\mathrm{x}}(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{\mathrm{u}}(s)$, and finally $\hat{\mathrm{y}}_{\mathcal{S}^c}(s) = C_{\mathcal{S}^c\bullet}\hat{\mathrm{x}}(s) + D_{\mathcal{S}^c\bullet}\hat{\mathrm{u}}(s)$. Now, we claim that the inverse Laplace transform of the triple $(\hat{\mathrm{u}}(s), \hat{\mathrm{x}}(s), \hat{\mathrm{y}}(s))$, say $(\mathrm{u}, \mathrm{x}, \mathrm{y})$, is an initial solution to SCS (19) with the initial state x_0 and pure switch configuration $\mathcal{S}$. Indeed, one can verify that $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ satisfies all the requirements of Definition VI.4 for the diode configuration $\mathcal{D} := \{i \in \mathcal{M} \mid \mathrm{y}_i(s) \equiv 0\}$. So far, we proved existence of an initial solution provided that the RCP has a solution. Note that $G'(s)$ is the Schur complement of $G(s)$ with respect to $G_{\mathcal{S}\mathcal{S}}(s)$. It follows from [5, Lem. 3.2, (v)] that $G(\sigma)$ and hence $G'(\sigma)$ are positive definite for all sufficiently large real σ. This implies, together with Theorem II.4 item 1, Remark II.5 and [15, Thm. 4.1 and Thm. 4.9], that the RCP has a unique solution. At this point, we already showed the existence. Suppose now there are two different initial solutions. Their Laplace transforms should satisfy the relations of the RCP. However, we know that it has a unique solution. Note that $(\mathrm{u}_{\mathcal{M}}, \mathrm{y}_{\mathcal{M}})$ determines u uniquely since $G_{\mathcal{S}\mathcal{S}}(s)$ is invertible due to [5, Lem. 3.2, (v)]. This concludes the uniqueness proof.

2: Let $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ be the unique initial solution with the Laplace transform $(\hat{\mathrm{u}}(s), \hat{\mathrm{x}}(s), \hat{\mathrm{y}}(s))$. Define $\mathcal{D} := \{i \in \mathcal{M} \mid \mathrm{y}_i(s) \equiv 0\}$, and $\mathcal{I} := \mathcal{D} \cup \mathcal{S}$. The equations (26) yield $0 = C_{\mathcal{I}\bullet}(sI - A)^{-1}x_0 + G_{\mathcal{I}\mathcal{I}}(s)\hat{\mathrm{u}}_{\mathcal{I}}(s)$ in the Laplace domain. Note that the first summand of the right hand side is strictly proper, $G_{\mathcal{I}\mathcal{I}}(s)$ is invertible as a rational matrix (due to [5, Lem. 3.2, (v)]), and $s^{-1}G_{\mathcal{I}\mathcal{I}}^{-1}(s)$ is proper (due to [5, Lem. 3.2, (vi)]). Consequently, $\hat{\mathrm{u}}_{\mathcal{I}}(s) = -G_{\mathcal{I}\mathcal{I}}^{-1}(s)C_{\mathcal{I}\bullet}(sI - A)^{-1}x_0$ is proper. We can conclude from (26a) that both $\hat{\mathrm{x}}(s)$ and $\hat{\mathrm{y}}(s)$ are also proper. Therefore, $(\mathrm{u}, \mathrm{x}, \mathrm{y})$ is of first order. Let the impulsive part of u be of the form $u^0\delta$ for some $u^0 \in \mathbb{R}^k$. It is clear from (26a) that x has no impulsive part and $Du^0\delta$ is the impulsive part of y. Note that $u^0 \in \mathcal{C}_{\mathcal{S}}$ and $Du^0 \in \mathcal{C}_{\mathcal{S}}^*$ due to Definition VI.4 item 2, and u^0 is orthogonal to Du^0 due to (26b). Therefore, u^0 solves LCP$_{\mathcal{C}_{\mathcal{S}}}(0, D)$. Theorem II.7 implies that $u^0 \in \mathcal{Q}_{\mathcal{S}}$.

3: Immediately follows from (26a).

4: Note that (26b) imply that $\hat{\mathrm{u}}^T(s)\hat{\mathrm{y}}(s) = 0$ for all values of $s \in \mathbb{C}$. Take any $v \in \mathcal{Q}_{\mathcal{S}}$. Then, for all sufficiently large σ we have $(\hat{\mathrm{u}}(\sigma) - v)^T(\hat{\mathrm{y}}(\sigma) - Dv) = -v^T\hat{\mathrm{y}}(\sigma) - \hat{\mathrm{u}}^T(\sigma)Dv \leqslant 0$ since $(\hat{\mathrm{u}}(\sigma), \hat{\mathrm{y}}(\sigma))$, $(v, Dv) \in \mathcal{C}_{\mathcal{S}} \times \mathcal{C}_{\mathcal{S}}^*$, and they are pairwise orthogonal. Substituting $\hat{\mathrm{y}}(s) = C(sI - A)^{-1}x_0 + [D + C(sI - A)^{-1}B]\hat{\mathrm{u}}(s)$, we get $(\hat{\mathrm{u}}(\sigma) - v)^T[C(\sigma I - A)^{-1}x_0 + C(\sigma I - A)^{-1}B\hat{\mathrm{u}}(\sigma) + D(\hat{\mathrm{u}}(\sigma) - v)] \leqslant 0$ for all sufficiently large σ. Since D is nonnegative definite due to the hypotheses (see [5, Lem. 3.2 (i)]), we have even $\hat{\mathrm{u}}(\sigma) - v)^T[C(\sigma I - A)^{-1}x_0 + C(\sigma I - A)^{-1}B\hat{\mathrm{u}}(\sigma)] \leqslant 0$. Multiplying this relation by σ and letting σ tend to infinity yields

$$(u^0 - v)^T(Cx_0 + CBu^0) \leqslant 0. \tag{27}$$

Now, let the series expansion of $\hat{\mathrm{u}}(s)$ around infinity be $\hat{\mathrm{u}}(s) = u^0 + s^{-1}u^{-1} + \cdots$. Hence, we get

$$\begin{aligned} &\hat{\mathrm{u}}^T(s)\hat{\mathrm{y}}(s) = (u^0)^T Du^0 \\ &+ s^{-1}[(u^0)^T(Cx_0 + CBu^0 + (D + D^T)u^{-1})] + \cdots. \end{aligned} \tag{28}$$

Note that $(u^0)^T Du^0 = 0$ as proven in 2. Since D is nonnegative definite due to the hypotheses , we have even $(D + D^T)u^0 = 0$. This means that (28) implies

$$(u^0)^T(Cx_0 + CBu^0) = 0. \tag{29}$$

Together with (27), this results in $v^T(Cx_0 + CBu^0) \geqslant 0$. Since $v \in \mathcal{Q}_{\mathcal{S}}$ is arbitrary, we get $Cx_0 + CBu^0 \in \mathcal{Q}_{\mathcal{S}}^*$.

5: The 'only if' part follows from 4. If $Cx_0 \in \mathcal{Q}_{\mathcal{S}}^*$ then we get $(u^0)^T Cx_0 \geqslant 0$ since $u^0 \in \mathcal{Q}_{\mathcal{S}}$ as shown in 2. From the proof of the previous item, we already know $(u^0)^T Du^0 = 0$ and $(u^0)^T(Cx_0 + CBu^0) = 0$. This implies from [7, Lem. 20] that $(u^0)^T Cx_0 = -(u^0)^T CBu^0 \leqslant 0$. Hence, we get $(u^0)^T Cx_0 = 0$ and hence $(u^0)^T CBu^0 = 0$. Finally, [7, Lem. 20] gives $u^0 = 0$. ■

The fact that solutions of linear passive networks with ideal diodes and pure switches do not contain derivatives of Dirac impulses is widely believed true on "intuitive" grounds, but the authors are not aware of any previous rigorous proof. The framework proposed here makes it possible to prove the intuition. Only for the diode-case it was proven in [14].

A direct implication of the statements 3, 4 and 5 in Theorem VI.5 is that if smooth continuation is not possible for x_0, it is possible after one re-initialization. Indeed, by 3 the state after the re-initialization is equal to $x_0 + Bu^0$ where u^0 as in 2. Since $C(x_0 + Bu^0) \in \mathcal{Q}_{\mathcal{S}}^*$ due to 4, it follows from statement 5 that from $x_0 + Bu^0$ there exists a smooth initial solution. This immediately implies local existence (on a time interval $[0, \epsilon)$) of a solution.

In [15], [16] a (global) solution concept for LCS has been introduced that is based on concatenation of initial solutions. In principle, this allows impulses at any mode transition time (necessary for e.g. unilaterally constrained mechanical systems). However, it has been shown in [14] that such a general notion of solution will not be needed in the context of linear passive electrical networks with diodes.

At this point, we need to introduce some nomenclature. The function space $\mathcal{L}_\delta(0, T)$ consists of the distributions of the form $\mathrm{u} = \mathrm{u}_{imp} + \mathrm{u}_{reg}$, where $\mathrm{u}_{imp} = u^0\delta$ with $u^0 \in \mathbb{R}$ and $\mathrm{u}_{reg} \in \mathcal{L}_2(0, T)$.

The following theorem shows the existence and uniqueness of solutions to SCS for a fixed switch configuration.

Theorem VI.6 *Consider an SCS given by* (19) *such that Assumption III.4 is satisfied and* (A, B, C, D) *represents a passive system. Let a pure switch configuration* $\mathcal{S}$ *be given. For all initial states* x_0 *and all* $T > 0$, *there exists a unique triple* $(\mathrm{u}, \mathrm{x}, \mathrm{y}) \in \mathcal{L}_\delta^{k+n+k}(0, T)$ *such that*

1) *there exists an initial solution* $(\bar{\mathtt{u}}, \bar{\mathtt{x}}, \bar{\mathtt{y}})$ *such that*

$$(\mathtt{u}_{imp}, \mathtt{x}_{imp}, \mathtt{y}_{imp}) = (\bar{\mathtt{u}}_{imp}, \bar{\mathtt{x}}_{imp}, \bar{\mathtt{y}}_{imp}),$$

2) $\mathtt{x}_{reg}(0+) = x_0 + Bu^0$ *with* $u^0 \in \mathbb{R}^k$ *given by* $\bar{\mathtt{u}}_{imp} = u^0\delta$*, and*
3) *for almost all* $t \in (0, T)$

$$\mathtt{x}_{reg}(t) = \mathtt{x}_{reg}(0+) + \int_0^t [A\mathtt{x}_{reg}(\tau) + B\mathtt{u}_{reg}(\tau)]d\tau$$
$$\mathtt{y}_{reg}(t) = C\mathtt{x}_{reg}(t) + D\mathtt{u}_{reg}(t)$$
$$\mathcal{C}_{\mathcal{S}} \ni \mathtt{u}_{reg}(t) \perp \mathtt{y}_{reg}(t) \in \mathcal{C}^*_{\mathcal{S}}.$$

Proof: Since the set of initial states that lead to a smooth initial solution (i.e. $\{x_0 \mid Cx_0 \in \mathcal{Q}^*_{\mathcal{S}}\}$) is a closed set, one can follow the same line of argumentation of the proof of [14, Thm. VII.2] step by step. ∎

So far, we were interested in the behavior of SCSs for a fixed switch configuration $\mathcal{S}$. Our next step is to allow changes in switch configuration. To do so, we first describe the allowed switching sequences.

Definition VI.7 A function $\pi : \mathbb{R}_+ \to 2^{\{m+1,m+2,\dots,k\}}$ is said to be an *admissible switching function* if it is piecewise constant and it changes value at most finitely many times on every finite length interval. The set of point at which π changes value will be denoted by Γ_π.

Note that Γ_π is set of isolated points due to the fact that there are finitely many points at which π changes value on every interval of finite length. By considering only admissible switching sequences, we exclude the so-called Zeno-behaviour[4].

As we showed earlier jumps may occur only at switching instants. In what follows, we will adopt a global solution concept which allows jumps at isolated points in time. First, the definition of the trajectory set that we consider is in order.

Definition VI.8 The distribution space $\mathcal{L}_{2,\delta}$ is defined as the set of all $\mathtt{u} = \mathtt{u}_{imp} + \mathtt{u}_{reg}$, where $\mathtt{u}_{imp} = \sum_{\theta\in\Gamma} u^\theta\delta_\theta$ for $u^\theta \in \mathbb{R}$ with $\Gamma \subset \mathbb{R}_+$ a set of isolated points, and $\mathtt{u}_{reg} \in \mathcal{L}_2^{loc}$.

The isolatedness of the points of the set Γ is required to prevent the occurrence of an accumulation of Dirac impulses in the solution trajectories. One could very well relax this requirement by making some extra assumptions. However, we prefer to keep the definition simpler and avoid technical details which might blur the main picture.

Definition VI.9 Let the impulsive part of the distribution $(\mathtt{u}, \mathtt{x}, \mathtt{y}) \in \mathcal{L}_{2,\delta}^{m+n+m}$ be supported on a set of isolated points Γ, i.e., $(\mathtt{u}_{imp}, \mathtt{x}_{imp}, \mathtt{y}_{imp}) = \sum_{\theta\in\Gamma}(u^\theta, x^\theta, y^\theta)\delta_\theta$ for $(u^\theta, x^\theta, y^\theta) \in \mathbb{R}^{k+n+k}$. Then we call $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ a *(global) solution* to SCS (19) for the initial state x_0 and the admissible switching function π if the following properties hold.

[4]Zeno-behaviour denotes the phenomenon of an infinite number of events (mode transitions) in a finite length time interval.

1) For any interval (a, b) such that $(a, b) \cap \Gamma = \varnothing$ the restriction $\mathtt{x}_{reg}|_{(a,b)}$ is absolutely continuous and satisfies for almost all $t \in (a, b)$

$$\dot{\mathtt{x}}_{reg}(t) = A\mathtt{x}_{reg}(t) + B\mathtt{u}_{reg}(t)$$
$$\mathtt{y}_{reg}(t) = C\mathtt{x}_{reg}(t) + D\mathtt{u}_{reg}(t)$$
$$\mathcal{C}_{\pi(t)} \ni \mathtt{u}_{reg}(t) \perp \mathtt{y}_{reg}(t) \in \mathcal{C}^*_{\pi(t)}.$$

2) For each $\theta \in \Gamma$ the corresponding impulse $(u^\theta\delta_\theta, x^\theta\delta_\theta, y^\theta\delta_\theta)$ is equal to the impulsive part of the unique initial solution to (19) with initial state $\mathtt{x}_{reg}(\theta-) := \lim_{t\uparrow\theta}\mathtt{x}_{reg}(t)$ (taken equal to x_0 for $\theta = 0$).
3) For times $\theta \in \Gamma$ it holds that $\mathtt{x}_{reg}(\theta+) = \mathtt{x}_{reg}(\theta-) + Bu^\theta$.

Note that the solution in the above sense satisfies the equations $\dot{\mathtt{x}} = A\mathtt{x} + B\mathtt{u}$ and $\mathtt{y} = C\mathtt{x} + D\mathtt{u}$ in the distributional sense.

The following theorem establishes existence and uniqueness of solutions to SCS.

Theorem VI.10 *Consider an SCS given by* (19) *such that Assumption III.4 is satisfied and* (A, B, C, D) *represents a passive system. The SCS* (19) *has a unique (global) solution* $(\mathtt{u}, \mathtt{x}, \mathtt{y}) \in \mathcal{L}_{2,\delta}^{m+n+m}$ *for any initial state* x_0 *and admissible switching function* π*. Moreover,* $\mathtt{x}_{imp} = 0$ *and impulses in* $(\mathtt{u}, \mathtt{y})$ *only show up at the initial time and times for which* π *changes value (i.e.* Γ *in Definition VI.9 should be a subset of* $\{0\} \cup \Gamma_\pi$*).*

Proof: A global solution for the switching function π can be easily constructed by using Theorem VI.6 repeatedly. For the uniqueness proof, let $(\mathtt{u}', \mathtt{x}', \mathtt{y}')$ and $(\mathtt{u}'', \mathtt{x}'', \mathtt{y}'')$ be two different global solutions of SCS (19) for the initial state x_0 and the switching function π. Let $T \in \Gamma_\pi$ be such that

$$(\mathtt{u}', \mathtt{x}', \mathtt{y}')|_{[0,T)} = (\mathtt{u}'', \mathtt{x}'', \mathtt{y}'')|_{[0,T)} \tag{30}$$
$$(\mathtt{u}', \mathtt{x}', \mathtt{y}')|_{[T,\Delta)} \neq (\mathtt{u}'', \mathtt{x}'', \mathtt{y}'')|_{[T,\Delta)} \tag{31}$$

for some Δ with $[T, \Delta) \cap \Gamma_\pi = \varnothing$. It follows from [14, Thm. VII.2] that both $\lim_{t\uparrow T}\mathtt{x}'(t)$ and $\lim_{t\uparrow T}\mathtt{x}''(t)$ are well-defined. Moreover, (30) implies that they are equal. Uniqueness of initial solutions for a given initial state (Theorem VI.5 item 1), together with Definition VI.9 item 2, implies that the impulsive parts of both solutions are the same at $t = T$. Hence, (31) results in

$$(\mathtt{u}'_{reg}, \mathtt{x}'_{reg}, \mathtt{y}'_{reg})|_{[T,\Delta)} \neq (\mathtt{u}''_{reg}, \mathtt{x}''_{reg}, \mathtt{y}''_{reg})|_{[T,\Delta)}. \tag{32}$$

Note that $\mathtt{x}'_{reg}(T) = \mathtt{x}''_{reg}(T)$. This means that

$$(\mathtt{u}'_{reg}(t) - \mathtt{u}''_{reg}(t), \mathtt{x}'_{reg}(t) - \mathtt{x}''_{reg}(t), \mathtt{y}'_{reg}(t) - \mathtt{y}''_{reg}(t))|_{[T,\Delta)}$$

is a trajectory of the linear system (3) with zero initial state. By using the dissipation inequality, we get $\int_T^t[\mathtt{u}'_{reg}(s) - \mathtt{u}''_{reg}(s)]^T[\mathtt{y}'_{reg}(s) - \mathtt{y}''_{reg}(s)]\,ds \geqslant [\mathtt{x}'_{reg}(t) - \mathtt{x}''_{reg}(t)]^T K[\mathtt{x}'_{reg}(t) - \mathtt{x}''_{reg}(t)]$. Definition VI.9 item 1 implies that the left hand side is nonpositive. However, the right hand side is nonnegative due to the fact that K is positive definite. Therefore, $\mathtt{x}'_{reg}(t) = \mathtt{x}''_{reg}(t)$ for all $t \in [T, \Delta)$. This immediately results in

$$B[\mathtt{u}'_{reg}(t) - \mathtt{u}''_{reg}(t)] = 0 \tag{33}$$

$$\mathtt{y}'_{reg}(t) - \mathtt{y}''_{reg}(t) = D[\mathtt{u}'_{reg}(t) - \mathtt{u}''_{reg}(t)] \tag{34}$$

due to Definition VI.9. Premultiplying (34) by $[\mathtt{u}'_{reg}(t) - \mathtt{u}''_{reg}(t)]^T$, one can show that

$$[\mathtt{u}'_{reg}(t) - \mathtt{u}''_{reg}(t)]^T D[\mathtt{u}'_{reg}(t) - \mathtt{u}''_{reg}(t)] = 0.$$

Since D is nonnegative definite, this implies

$$(D + D^T)[\mathtt{u}'_{reg}(t) - \mathtt{u}''_{reg}(t)] = 0. \tag{35}$$

We can conclude from Assumption III.4, (33), and (35) that $\mathtt{u}'_{reg}(t) - \mathtt{u}''_{reg}(t) = 0$ for all $t \in [T, \Delta)$. Finally, (34) gives $\mathtt{y}'_{reg}(t) - \mathtt{y}''_{reg}(t) = 0$ for all $t \in [T, \Delta)$. As a consequence, we reached a contradiction with (31). ■

VII. Regular states

Another consequence of Theorem VI.5 is the characterization of so-called *regular states* (sometimes also called consistent states) as introduced in the following definition.

Definition VII.1 A state x_0 is called *regular* for SCS(A, B, C, D) with respect to a pure switch configuration if the corresponding initial solution for the same pure switch configuration is smooth. The collection of regular states for a given quadruple (A, B, C, D) with respect to the pure switch configuration $\mathcal{S}$ is denoted by $\mathcal{R}_{\mathcal{S}}$.

We have the following equivalent characterizations of regular states.

Theorem VII.2 *Consider an SCS given by* (19) *such that Assumption III.4 is satisfied and* (A, B, C, D) *represents a passive system. Let a pure switch configuration* $\mathcal{S}$ *be given and let* $\mathcal{Q}_{\mathcal{S}}$ *be the solution set of LCP*$_{\mathcal{C}_{\mathcal{S}}}(0, D)$*, i.e.,* $\mathcal{Q}_{\mathcal{S}} = \{v \in \mathbb{R}^k \mid v \in \mathcal{C}_{\mathcal{S}}, Dv \in \mathcal{C}^*_{\mathcal{S}} \text{ and } v \perp Dv\}$*. The following statements are equivalent.*

1) x_0 *is a regular state for (19) with respect to the pure switch configuration* $\mathcal{S}$.
2) $Cx_0 \in \mathcal{Q}^*_{\mathcal{S}}$.
3) *LCP*$_{\mathcal{C}_{\mathcal{S}}}(Cx_0, D)$ *has a solution.*
4) *There exist two vectors* $v_1 \in \mathcal{C}^*_{\mathcal{S}}$ *and* $v_2 \in \mathcal{C}_{\mathcal{S}}$ *such that* $Cx_0 = v_1 - Dv_2$.[5]

Proof: 1 $\Leftrightarrow$ 2: This is clear from Theorem VI.5 item 5.

2 $\Leftrightarrow$ 3: It follows from Theorem II.7.

3 $\Rightarrow$ 4: Note that if v is a solution of LCP$_{\mathcal{C}_{\mathcal{S}}}(Cx_0, D)$ then we can choose $v_1 := Cx_0 + Dv \in \mathcal{C}^*_{\mathcal{S}}$ and $v_2 := v \in \mathcal{C}_{\mathcal{S}}$.

4 $\Rightarrow$ 2: Let $v_1 \in \mathcal{C}^*_{\mathcal{S}}$ and $v_2 \in \mathcal{C}_{\mathcal{S}}$ be such that $Cx_0 = v_1 - Dv_2$. Take any $w \in \mathcal{Q}_{\mathcal{S}}$. Then, we have

$$\begin{aligned} w^T Cx_0 &= w^T v_1 - w^T Dv_2 = w^T v_1 + (Dw)^T v_2 \\ &\qquad (\text{since } w^T Dw = 0 \text{ implies } D^T w = -Dw) \\ &\geqslant 0 \ (\text{since } w \in \mathcal{C}_{\mathcal{S}},\ v_1 \in \mathcal{C}^*_{\mathcal{S}},\ Dw \in \mathcal{C}^*_{\mathcal{S}}, \text{ and } v_2 \in \mathcal{C}_{\mathcal{S}}). \end{aligned}$$

[5]When $\mathcal{Q}_{\mathcal{S}}$ is the usual positive cone (i.e. equals to $\mathbb{R}^k_+$), this comes down to saying that Cx_0 is a positive linear combination of the columns of $(I \quad -D)$.

As a consequence, $Cx_0 \in \mathcal{Q}^*_{\mathcal{S}}$. ■

Hence, several tests are available for deciding the regularity of an initial state x_0. In [2] it is stated that a well-designed circuit does not exhibit impulsive behavior. As a consequence, the characterization of regular states forms a verification of the synthesis of the network.

In the next section, it will be shown that the characterization of the regular states plays a key role in the proof of global existence of solutions as the set of such initial states will be proven to be invariant under the dynamics.

VIII. Jump rules

If a state jump occurs at time $t = 0$, the new state is given by $x(0+) = x_0 + Bu^0$, see Theorem VI.5 item 3. We now give a characterization of this jump multiplier u^0 for SCS.

Theorem VIII.1 (Characterization of u^0) *Let a switch configuration* $\mathcal{S}$ *and an initial state* x_0 *be given. The following characterizations can be obtained for* u^0.

1) *The jump multiplier* u^0 *is the unique solution to*

$$\mathcal{Q}_{\mathcal{S}} \ni v \perp C(x_0 + Bv) \in \mathcal{Q}^*_{\mathcal{S}} \tag{36}$$

2) *The cone* $\mathcal{Q}^*$ *is equal to pos* $N := \{N\lambda \mid \lambda \geqslant 0\}$ *and* $\mathcal{Q}^*_{\mathcal{S}} = \{v \mid N^T v \geqslant 0\}$ *for some real matrix* N*. The re-initialized state* $\mathtt{x}_{reg}(0+)$ *is equal to* $x_0 + BN\lambda^0$ *and* $u^0 = N\lambda^0$ *where* λ^0 *is a solution of the following ordinary LCP.*

$$0 \leqslant \lambda \perp (N^T Cx_0 + N^T CBN\lambda) \geqslant 0. \tag{37}$$

3) *The re-initialized state* $\mathtt{x}_{reg}(0+)$ *is the unique minimum of*

$$\text{minimize } \tfrac{1}{2}(x - x_0)^T K(x - x_0) \tag{38a}$$

$$\text{subject to } Cx \in \mathcal{Q}^*_{\mathcal{S}} \tag{38b}$$

and the multiplier u^0 *is uniquely determined by* $\mathtt{x}_{reg}(0+) = x_0 + Bu^0$.

4) *The jump multiplier* u^0 *is the unique minimizer of*

$$\text{minimize } \tfrac{1}{2}(x_0 + Bv)^T K(x_0 + Bv) \tag{39}$$

$$\text{subject to } v \in \mathcal{Q}_{\mathcal{S}} \tag{40}$$

Proof:

1: It is already known from Theorem VI.5 items 2 and 4 that

$$u^0 \in \mathcal{Q}^* \tag{41}$$

$$Cx_0 + CBu^0 \in \mathcal{Q}^*_{\mathcal{S}}. \tag{42}$$

Furthermore, (29) readily shows

$$u^0 \perp C(x_0 + Bu^0).$$

It remains to prove that u^0 is uniquely determined by (36). Suppose that z^i is a solution of the generalized linear complementarity problem

$$\begin{gathered} z \in \mathcal{Q}^* \\ Cx_0 + CBz \in \mathcal{Q}^*_{\mathcal{S}} \\ z^T (Cx_0 + CBz) = 0 \end{gathered}$$

for $i = 1, 2$. Note that $(z^1 - z^2)^T CB(z^1 - z^2) = (z^1 - z^2)^T[(Cx_0 + CBz^1) - (Cx_0 + CBz^2)] = -(z^1)^T(Cx_0 + CBz^2) - (z^2)^T(Cx_0 + CBz^1)$ and hence

$$(z^1 - z^2)^T CB(z^1 - z^2) \leqslant 0. \tag{43}$$

Since $\mathcal{Q}^* \subseteq \ker(D + D^T)$, we have $z^1 - z^2 \in \ker(D + D^T)$. Hence, $(z^1 - z^2)^T CB(z^1 - z^2) = (z^1 - z^2)^T B^T KB(z^1 - z^2) \geqslant 0$ due to [5, Lem. 3.2 (iii)]. Together with the above inequality, this gives $(z^1 - z^2)^T CB(z^1 - z^2) = (z^1 - z^2)^T B^T KB(z^1 - z^2) = 0$. Since $\mathrm{col}(B, D + D^T)$ is of full column rank and K is positive definite, we get $z^1 = z^2$. Consequently, the jump multiplier u^0 is uniquely determined by (36).

2: Since (A, B, C, D) is passive, D is necessarily nonnegative definite. It follows from Theorem II.7 that $\mathrm{SOL}(0, D)$ is a polyhedral cone, i.e., the solution set of a homogeneous system of inequalities of the form $Hx \geqslant 0$ for some matrix H. Minkowski's theorem [29, Theorem 2.8.6] states that every polyhedral cone has a finite set of generators. Therefore, one can find a matrix N such that $\mathcal{Q}_\mathcal{S} = \mathrm{pos}\, N = \{N\lambda \mid \lambda \geqslant 0\}$. It can be checked that the dual cone can be given in the form $\mathcal{Q}_\mathcal{S}^* = \{v \mid N^T v \geqslant 0\}$. Since $u^0 \in \mathcal{Q}_\mathcal{S}$, there exists $\lambda^0 \geqslant 0$ such that $u^0 = N\lambda^0$. Note that $Cx_0 + CBN\lambda^0 \in \mathcal{Q}_\mathcal{S}^*$. Hence, $N^T(Cx_0 + CBN\lambda^0) \geqslant 0$. Note that we have

$$(\lambda^0)^T N^T (Cx_0 + Fw(0) + CBN\lambda^0) = 0$$

due to previous item. This means that λ^0 is a solution of the LCP (37).

3: The minimization problem (38) admits a unique solution since $\{x \mid Cx \in \mathcal{Q}_\mathcal{S}^*\}$ is a polyhedron and K is positive definite. Let $\bar{x}$ be the solution of (38). Dorn's duality theorem [20, Thm. 8.2.4] implies that there exists a $\bar{\lambda}$ such that the pair $(\bar{x}, \bar{\lambda})$ solves

$$\text{minimize } x^T K x \tag{44a}$$

$$\text{subject to } \lambda \geqslant 0 \text{ and } x = x_0 + BN\lambda. \tag{44b}$$

Since $N\lambda \in \mathcal{Q}_\mathcal{S} \subseteq \ker(D + D^T)$ for all $\lambda \geqslant 0$, it follows that $KBN\lambda = C^T N\lambda$ for all $\lambda \geqslant 0$ due to Lemma III.5. Thus,

$$\begin{aligned} x^T K x &= (x_0 + BN\lambda)^T K (x_0 + BN\lambda) \\ &= \lambda^T N^T CBN\lambda + 2x_0^T C^T N\lambda + x_0^T K x_0 \end{aligned} \tag{45}$$

whenever $\lambda \geqslant 0$. So the vector $\bar{\lambda}$ solves the minimization problem

$$\text{minimize } \tfrac{1}{2}\lambda^T N^T CBN\lambda + (Cx_0)^T N\lambda \tag{46a}$$

$$\text{subject to } \lambda \geqslant 0. \tag{46b}$$

Since $N^T CBN$ is nonnegative definite, the Karush-Kuhn-Tucker conditions

$$\bar{\lambda} \geqslant 0 \tag{47a}$$

$$N^T(Cx_0 + CBN\bar{\lambda}) \geqslant 0 \tag{47b}$$

$$\bar{\lambda}^T N^T (Cx_0 + CBN\bar{\lambda}) = 0 \tag{47c}$$

are necessary and sufficient for the vector $\bar{\lambda}$ to be a globally optimal solution of (46). For a detailed discussion on this equivalence, the reader is referred to [8] or [9, Sec. 1.2]. Note that the LCP given by (47) is the same as the one in (ii). It follows from (ii) that $u^0 = N\bar{\lambda}$ and $\bar{p} = x_0 + Bu^0$. Since $u^0 \in \ker(D + D^T)$ and $\mathrm{col}(B, D + D^T)$ is of full column rank (due to Assumption III.4), the equation $\mathtt{x}_{reg}(0+) = x_0 + Bu^0$ determines the multiplier u^0 uniquely. ■

IX. Stability

In this section we discuss the stability of Switched Complementarity Systems (SCS) under a passivity (or equivalently, a Hamiltonian structure) assumption. The Lyapunov stability of hybrid and switched systems in general has already received considerable attention [3], [17–19], [22], [33]. We have narrowed down the definitions and theorems on the stability of general hybrid systems from [18] and [33] to apply to SCS. From now on, we denote the unique global trajectory for a given switch function π and initial state x_0 of an SCS by $(\mathtt{u}^{\pi,x_0}, \mathtt{x}^{\pi,x_0}, \mathtt{y}^{\pi,x_0})$. For the study of stability we consider the source-free case.

Definition IX.1 (Equilibrium point) A state $\bar{x}$ is an equilibrium point of the SCS (19), if for all admissible switching functions π $\mathtt{x}_{reg}^{\pi,\bar{x}}(t) = \bar{x}$ for almost all $t \geqslant 0$ and all π, i.e. for all solutions starting in $\bar{x}$ the state stays in $\bar{x}$.

Note that in an equilibrium point $\dot{x} = 0$, which leads in a simple way to the following characterization of equilibria of an SCS.

Lemma IX.2 *A state $\bar{x}$ is an equilibrium point of the SCS* (19), *if and only if for all $\mathcal{S} \subset \{m+1, \ldots, k\}$ there exist $u^\mathcal{S} \in \mathbb{R}^k$ and $y^\mathcal{S} \in \mathbb{R}^k$ satisfying*

$$0 = A\bar{x} + Bu^\mathcal{S} \tag{48a}$$

$$y^\mathcal{S} = C\bar{x} + Du^\mathcal{S} \tag{48b}$$

$$\mathcal{C}_\mathcal{S} \ni u^\mathcal{S} \perp y^\mathcal{S} \in \mathcal{C}_\mathcal{S}^*. \tag{48c}$$

Moreover, this means that $\bar{x} \in \mathcal{R}_\mathcal{S}$ for all $\mathcal{S}$, i.e. $\bar{x}$ is a regular state for all switch configurations.

From this lemma it follows that $\bar{x} = 0$ is an equilibrium. Note that if A is invertible we get $\bar{x} = -A^{-1}Bu^\mathcal{S}$ and

$$\mathcal{C}_\mathcal{S} \ni u^\mathcal{S} \perp [-CA^{-1}B + D]u^\mathcal{S} \in \mathcal{C}_\mathcal{S}^*,$$

which is a homogeneous LCP over a cone.

Definition IX.3 Let $\bar{x}$ be an equilibrium point of the SCS (19) and d denote a metric on $\mathbb{R}^n$.

1) $\bar{x}$ is called *stable*, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(\mathtt{x}_{reg}^{\pi,x_0}(t), \bar{x}) < \varepsilon$ for almost all $t \geqslant 0$ whenever $d(x_0, \bar{x}) < \delta$ and π being an admissible switching function.
2) $\bar{x}$ is called asymptotically stable if $\bar{x}$ is stable and there exists an $\eta > 0$ such that $\lim_{t\to\infty} d(\mathtt{x}_{reg}^{\pi,x_0}(t), \bar{x}) = 0$ whenever $d(x_0, \bar{x}) < \delta$ and π being an admissible switching function. By $\lim_{t\to\infty} d(\mathtt{x}_{reg}^{\pi,x_0}(t), \bar{x}) = 0$ we mean that for every $\varepsilon > 0$ there exists a t_ε such that $d(\mathtt{x}_{reg}^{\pi,x_0}(t), \sigma), \bar{x}) < \varepsilon$ whenever $t \geqslant t_\varepsilon$.

In the proof of the main theorem on stability we will need the following lemma.

Lemma IX.4 *For given $\mathcal{S}$ and vectors $w_1 \in \mathcal{Q}_{\mathcal{S}} = SOL_{\mathcal{C}_{\mathcal{S}}}(0, D)$ and $w_2 \in \mathcal{Q}_{\mathcal{S}}$, it holds that $w_1^T D w_2 = 0$.*

Proof: Since $Dw_2 \in \mathcal{C}_{\mathcal{S}}^*$ and $w_1 \in \mathcal{C}_{\mathcal{S}}$, it holds that $w_1^T D w_2 \geqslant 0$. Note that $w_2 \in \mathcal{Q}_{\mathcal{S}}$ implies that $w_2^T D w_2 = 0$ and thus $Dw_2 = -D^T w_2$. Hence, $w_1^T D w_2 = -w_1^T D^T w_2 = -\underbrace{(Dw_1)}_{\in \mathcal{C}_{\mathcal{S}}^*}{}^T \underbrace{w_2}_{\in \mathcal{C}_{\mathcal{S}}} \leqslant 0$, the result follows. ∎

Theorem IX.5 *Consider an SCS given by* (19) *such that Assumption III.4 is satisfied and (A, B, C, D) represents a passive system. This SCS has only stable equilibrium points $\bar{x}$. Moreover, if $A^T K + KA < 0$ is invertible*[6] *$\bar{x} = 0$ is the only equilibrium point, which is asymptotically stable.*

Proof: Let $\bar{x}$ be an equilibrium. The proof will be based on taking $V(x) = \frac{1}{2}(x - \bar{x})^T K (x - \bar{x})$ as a Lyapunov function with K a positive definite solution to (6). Take an initial state x_0 and an admissible switching function π and denote the corresponding solution by $(\mathtt{u}^{\pi,x_0}, \mathtt{x}^{\pi,x_0}, \mathtt{y}^{\pi,x_0})$. If we apply the same switching function to SCS(A, B, C, D) with initial state $\bar{x}$, then the solution is equal to $t \mapsto (u^{\pi(t)}, \bar{x}, y^{\pi(t)})$, where $u^{\pi(t)}$ and $y^{\pi(t)}$ are the vectors that satisfy the conditions in Lemma IX.2. Note that the difference trajectory $(\mathtt{u}^{\pi,x_0} - u^{\pi(\cdot)}, \mathtt{x}^{\pi,x_0} - \bar{x}, \mathtt{y}^{\pi,x_0} - y^{\pi(\cdot)})$ is a (distributional) solution to the linear system (3). From Definition VI.9 it follows that that jumps of this trajectory only take place at the initial time 0 and the discontinuity points of π being Γ_π. In intervals between these "jump times", the difference trajectory is smooth and satisfies the dissipation inequality meaning that for $t_0 \leqslant t_1$ (we drop the "reg"-subscript as we consider times intervals $[t_0, t_1]$ in which no impulses are active) $\frac{1}{2}(\mathtt{x}^{\pi,x_0}(t_0) - \bar{x})^T K (\mathtt{x}^{\pi,x_0}(t_0) - \bar{x}) + \int_{t_0}^{t_1} [\mathtt{u}^{\pi,x_0}(t) - u^{\pi(t)}]^T [\mathtt{y}^{\pi,x_0}(t) - y^{\pi(t)}] dt \geqslant \frac{1}{2}(\mathtt{x}^{\pi,x_0}(t_1) - \bar{x})^T K (\mathtt{x}^{\pi,x_0}(t_1) - \bar{x})$. Since $\mathtt{u}^{\pi,x_0}(t) \perp \mathtt{y}^{\pi,x_0}(t)$, $u^{\pi(t)} \perp y^{\pi(t)}$, $-(u^{\pi(t)})^T \mathtt{y}^{\pi,x_0}(t) \leqslant 0$ and $-(\mathtt{u}^{\pi,x_0}(t))^T y^{\pi(t)} \leqslant 0$, it follows that $\frac{1}{2}(\mathtt{x}^{\pi,x_0}(t_0) - \bar{x})^T K (\mathtt{x}^{\pi,x_0}(t_0) - \bar{x}) \geqslant \frac{1}{2}(\mathtt{x}^{\pi,x_0}(t_1) - \bar{x})^T K (\mathtt{x}^{\pi,x_0}(t_1) - \bar{x})$ for all intervals $[t_0, t_1]$ not containing jumps and impulses. Hence, the Lyapunov function cannot increase on these intervals.

The only issue left to prove, to obtain stability according to the standard theorems from [18] and [33], is the fact that the $V(x)$ decreases during jumps of the state trajectory satisfying the equations of SCS(A, B, C, D). If a jump occurs it obeys the rules as indicated in item 4 in Theorem VIII.1. Let a jump take place from x_0 (or any other state) and v the corresponding multiplier. As $0 \in \mathcal{Q}_{\mathcal{S}}$ it follows from item 4, that $(x_0 + Bv)^T K (x_0 + Bv) \leqslant x_0^T K x_0$, or stated differently,

$$v^T B^T K x_0 + x_0^T K B v + v^T B^T K B v \leqslant 0. \tag{49}$$

Consider the difference between the value of the Lyapunov function after and before the jump: $2(V(x_0 + Bv) - V(x_0)) = -v^T B^T K (x_0 - \bar{x}) - (x_0 - \bar{x})^T K B v + v^T B^T K B v$. Then, we get

$$\begin{aligned} 2(V(x_0 + Bv) - V(x_0)) &\leqslant -v^T B^T K \bar{x} - \bar{x} K B v \text{ [from (49)]} \\ &= -v^T C \bar{x} - \bar{x}^T C^T v \text{ [from Lemma III.5 as } v^T D v = 0] \\ &= -2v^T [y^{\mathcal{S}} - D u^{\mathcal{S}}] \text{ [due to Lemma IX.2]} \\ &\leqslant 0 \text{ [Lemma IX.4 and } v^T y^{\mathcal{S}} \geqslant 0.]. \end{aligned}$$

This means that during jumps and smooth continuation the Lyapunov function never increases.

Consider the Lyapunov function $V(x)$ for $\bar{x} = 0$. It can actually be shown that $\frac{dV}{dt} \leqslant \frac{1}{2} x^T(t) [A^T K + KA] x(t)$ along a solution trajectory, which implies that only the origin is an equilibrium and it is asymptotically stable. ∎

X. Including sources

So far, our development excluded the presence of external inputs. Now, we discuss how one can derive well-posedness results for SCS with external inputs. Consider an SCS with (external) inputs w of the form:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t) \tag{50a}$$
$$y(t) = Cx(t) + Du(t) + Fw(t) \tag{50b}$$
$$0 \leqslant y_i(t) \perp u_i(t) \geqslant 0, \; i = 1, 2, \ldots, m \tag{50c}$$
$$y_i(t) \perp u_i(t), \; i = m+1, m+2, \ldots, k. \tag{50d}$$

This system will be denoted by SCSwI. Similar to Definition VI.4, we can define an initial solution concept for SCSwI as follows.

Definition X.1 We call a Bohl distribution $(\mathtt{u}, \mathtt{x}, \mathtt{y}) \in \mathcal{B}_{imp}^{k+n+k}$ an *initial solution* to (50) with initial state x_0, input $w \in \mathcal{B}^p$, and pure switch configuration $\mathcal{S}$ if

1) there is a diode configuration $\mathcal{D}$ such that $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ satisfies

$$\dot{\mathtt{x}} = A\mathtt{x} + B\mathtt{u} + Ew + x_0 \delta \tag{51a}$$
$$\mathtt{y} = C\mathtt{x} + D\mathtt{u} + Fw \tag{51b}$$
$$\mathtt{y}_i = 0, \; i \in \mathcal{I} \tag{51c}$$
$$\mathtt{u}_i = 0, \; i \notin \mathcal{I} \tag{51d}$$

as equalities of distributions, and
2) $(\mathtt{u}, \mathtt{y})$ are initially in the cone $(\mathcal{C}_{\mathcal{S}} \times \mathcal{C}_{\mathcal{S}}^*)$.

Next, we state the analogue of Theorem VI.5 without a proof. The proof can be derived by modifying the proof of Theorem VI.5 in accordance with the proof of [5, Thm. 6.1].

Theorem X.2 *Consider an SCSwI given by* (50) *such that Assumption III.4 is satisfied and (A, B, C, D) represents a passive system. Let a pure switch configuration $\mathcal{S}$ be given and let $\mathcal{Q}_{\mathcal{S}}$ be the solution set of LCP$_{\mathcal{C}_{\mathcal{S}}}(0, D)$, i.e., $\mathcal{Q}_{\mathcal{S}} = \{v \in \mathbb{R}^k \mid v \in \mathcal{C}_{\mathcal{S}}, Dv \in \mathcal{C}_{\mathcal{S}}^* \text{ and } v \perp Dv\}$. Then, the following statements hold.*

1) *For each initial state $x_0 \in \mathbb{R}^n$ and input $w \in \mathcal{B}^p$, there exists exactly one initial solution to SCSwI.*
2) *This solution is of first order. Stated differently, its impulsive part is of the form $(u^0 \delta, 0, Du^0 \delta)$ for some $u^0 \in \mathcal{Q}_{\mathcal{S}}$.*

[6] This implies that the LMI (6) is strict in the x-variable and thus that A is stable. In the case of a Hamiltonian framework this means in the current setting that $R > 0$.

3) *This impulsive part results in a re-initialization (jump) -if applicable- of the state from* x_0 *to* $x_0 + Bu^0$.
4) *For all* $x_0 \in \mathbb{R}^n$, $Cx_0 + Fw(0) + CBu^0 \in \mathcal{Q}^*_{\mathcal{S}}$.
5) *The initial solution is smooth (i.e.,* $u^0 = 0$*) if and only if* $Cx_0 + Fw(0) \in \mathcal{Q}^*_{\mathcal{S}}$.

Analogues of Theorems VI.6, and VI.10 can be given in a similar fashion by introducing inputs in the corresponding solution concepts. Note that being a Bohl function is quite restrictive for an input which is supposedly arbitrary. By following the steps of the proof of [5, Thm. 7.5], one can state an analogue of Theorem VI.10 for a larger set of inputs as defined below.

Definition X.3 A function $w : \mathbb{R}_+ \mapsto \mathbb{R}$ is called *piecewise Bohl*,[7] if w is right-continuous[8] and there exists a collection $\Gamma_w = \{\tau_i\} \subset \mathbb{R}_+$ such that

- Γ_w is a set of isolated points, and
- for every i there exists a $v \in \mathcal{B}$ such that $w(t) = v(t)$ for all $t \in (\tau_i, \tau_{i+1})$.

The set of piecewise Bohl functions is denoted by $\mathcal{PB}$.

In addition to initial jump and jumps that are caused by the pure switches, one might expect jumps at the discontinuity points of Fw as [5, Thm. 7.5] suggests.

Still, one might argue that being piecewise Bohl is restrictive for inputs. Ongoing research reveals that one can take piecewise locally Lipschitz continuous functions as the input space and can still prove well-posedness.

After establishing Theorem X.2, the analogues of Theorems VII.2 and VIII.1 can be established straightforwardly.

XI. Conclusions

Our aim in this paper has been to demonstrate that a suitable framework for switched piecewise linear networks is provided by the notion of cone complementarity systems. The dynamics described by cone complementarity systems can be very complicated but nevertheless is given by two simple components, to wit a linear system and a closed convex cone. Switching may be described within this context in a conceptually straightforward way as switching between cones, while the underlying linear system remains the same.

Making use of impulsive-smooth distributions to define a sufficiently flexible notion of solution, we have shown that the framework of cone complementarity systems is sound in the sense that, under the passivity assumption, it produces unique solutions for any given initial state. Moreover, the framework allows formal proofs for intuitive properties concerning jumps and stability. We have obtained a characterization of the situations in which jumps occur as well as of the extent of the jump in these cases; this information should be useful both for theoretical and for simulation purposes.

The cones that we have considered are in fact of a special type in which each component is either unconstrained, constrained to be zero, or constrained to be nonnegative. The formulation of cone complementarity systems however invites a less coordinate-based and more geometric perspective, which helps to achieve a focus on basic issues. Some of the results that we have obtained in this paper still make use of the special properties of cones obtained from diodes and switches; it is a natural question to ask whether these results can be obtained at a more general level, and we intend to return to this in future work.

Another possible direction of generalization is concerned with nonlinear networks. The notion of passivity of course does not depend on linearity and so it seems reasonable to expect that many of the results in this paper can be generalized to the nonlinear case. However, the distributional framework seems less suited in connection with nonlinear dynamics and so a different setting will have to be chosen.

The notion of passivity has been crucial in this paper. In fact it is remarkable that this energy-related concept turns out to play an important role even in establishing existence and uniqueness of solutions in a context that involves switching.

[7] Strictly speaking, we define a subspace of the class of piecewise Bohl functions. For reasons of brevity we will refer to this subspace as the space of piecewise Bohl functions.

[8] This means that $\lim_{t \downarrow \tau} w(t) = w(\tau)$ for all $\tau \in \mathbb{R}_+$.

References

[1] B. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis. A Modern Systems Theory Approach*. Englewood Cliffs, New Jersey: Pentice-Hall, 1973.

[2] D. Bedrosian and J. Vlach, "Time-domain analysis of networks with internally controlled switches," *IEEE Trans. Circuits and Systems-I*, vol. 39, no. 3, pp. 199–212, 1992.

[3] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, 1998.

[4] B. Brogliato, *Nonsmooth Impact Mechanics. Models, Dynamics and Control*, ser. Lecture Notes in Control and Information Sciences. London: Springer, 1996, vol. 220.

[5] M. Çamlıbel, W. Heemels, and J. Schumacher, "On linear passive complementarity systems," Special issue dedicated to V.M. Popov on 'Dissipativity of Dynamical Systems: Applications in Control' of *European Journal of Control*, vol. 8, no. 3, 2002.

[6] M. Çamlıbel, M. Cevik, W. Heemels, and J. Schumacher, "From Lipschitzian to non-Lipschitzian characteristics: continuity of behaviors," in *39-th IEEE Conf. on Decision and Control,* Sydney (Australia), 2000, pp. 4373–4378.

[7] M. Çamlıbel, W. Heemels, and J. Schumacher, "Consistency of a time-stepping method for a class of piecewise-linear networks," *IEEE Transactions on Circuits and Systems-I*, vol. 49, no. 3, pp. 349–357, 2002.

[8] R. Cottle and G. Dantzig, "Complementarity pivot theory of mathematical programming," in *Studies in Optimization*, ser. MAA Studies in Mathematics, G. Dantzig and B. Eaves, Eds. The Mathematical Association of America, 1974, vol. 10, pp. 27–51.

[9] R. Cottle, J.-S. Pang, and R. Stone, *The Linear Complementarity Problem*. Boston: Academic Press, Inc., 1992.

[10] A. Dervisoglu, "State equations and initial values in active RLC networks," *IEEE Trans. Circuit Theory*, vol. 18, pp. 544–547, 1971.

[11] J. DiStefano, A. Stubberud, and I. Williams, *Theory and problems of feedback and control systems*, ser. Schaum's outline series. McGraw-Hill, 1967.

[12] K. Gerritsen, A. van der Schaft, and W. Heemels, "On switched Hamiltonian systems," in *Proceedings 15th International Symposium on Mathematical Theory of Networks and Systems (MTNS2002)*, South Bend, USA, 2002.

[13] M. Hautus and L. Silverman, "System structure and singular control," *Linear Algebra and its Applications*, vol. 50, pp. 369–402, 1983.

[14] W. Heemels, M. Çamlıbel, and J. Schumacher, "On the dynamic analysis of piecewise-linear networks," *IEEE Transactions on Circuits and Systems – I*, vol. 49, no. 3, pp. 315–327, 2002.

[15] W. Heemels, J. Schumacher, and S. Weiland, "The rational complementarity problem," *Linear Algebra and its Applications*, vol. 294, pp. 93–135, 1999.

[16] ——, "Linear complementarity systems," *SIAM Journal on Applied Mathematics*, vol. 60, no. 4, pp. 1234–1269, 2000.

[17] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 555–559, 1998.

[18] Z. Li, C. B. Soh, and X. Xu, "Lyapunov stability of a class of hybrid dynamic systems," *Automatica*, vol. 36, no. 2, pp. 297–302, 2000.

[19] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Systems Magazine*, vol. 19, no. 5, pp. 59–70, 1999.

[20] O. Mangasarian, *Nonlinear Programming*. New York: McGraw-Hill, 1969.

[21] A. Massarini, U. Reggiani, and K. Kazimierczuk, "Analysis of networks with ideal switches by state equations," *IEEE Trans. Circuits and Systems-I*, vol. 44, no. 8, pp. 692–697, 1997.

[22] A. N. Michel, "Recent trends in the stability analysis of hybrid dynamical systems," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 46, no. 1, pp. 120–134, 1999.

[23] Y. Murakami, "A method for the formulation and solution of circuits composed of switches and linear RLC networks," *IEEE Trans. Circuits and Systems*, vol. 34, pp. 496–509, 1987.

[24] A. Opal and J. Vlach, "Consistent initial conditions of nonlinear networks with switches," *IEEE Trans. Circuits and Systems*, vol. 38, no. 7, pp. 698–710, 1991.

[25] S. Sastry and C. Desoer, "Jump behaviour of circuits and systems," *IEEE Trans. Circuits and Systems*, vol. 28, no. 12, pp. 1109–1124, 1981.

[26] A. J. v. d. Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, 2nd ed., ser. Communications and Control Engineering. Springer, London, 2000.

[27] A. v. d. Schaft and J. Schumacher, "The complementary-slackness class of hybrid systems," *Mathematics of Control, Signals and Systems*, vol. 9, pp. 266–301, 1996.

[28] ——, "Complementarity modeling of hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 483–490, 1998.

[29] J. Stoer and C. Witzgall, *Convexity and Optimization in Finite Dimensions I*. Berlin: Springer, 1970.

[30] J. Tolsa and M. Salichs, "Analysis of linear networks with inconsistent initial conditions," *IEEE Trans. Circuits and Systems-I*, vol. 40, no. 12, pp. 885–894, 1993.

[31] J. Vlach, J. Wojciechowski, and A. Opal, "Analysis of nonlinear networks with inconsistent initial conditions," *IEEE Transactions on Circuits and Systems-I*, vol. 42, no. 4, pp. 195–200, 1995.

[32] J. Willems, "Dissipative dynamical systems," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 321–393, 1972.

[33] H. Ye, A. N. Michel, and L. Hou, "Stability theory for hybrid dynamical systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 461–474, 1998.

PLACE
PHOTO
HERE

Kanat Çamlıbel Kanat Çamlıbel was born in Istanbul, Turkey, in 1970. He received the B.Sc. and M.Sc. degrees in Control and Computer Engineering from Istanbul Technical University in 1991 and 1994, respectively. In May 2001, he obtained Ph.D. degree from Tilburg University, the Netherlands. Between May 2001 and November 2002, he was with the Department of Mathematics, University of Groningen as a postdoc. Currently, he holds a part-time assistant professor position at Dogus University, Istanbul and a part-time postdoc position at Tilburg University. His main research interest is the analysis and control of nonsmooth dynamical systems, in particular complementarity systems.

PLACE
PHOTO
HERE

Maurice Heemels Maurice Heemels was born in St. Odilienberg, The Netherlands, in 1972. He received the M.Sc. degree (with honours) from the dept. of Mathematics and the Ph.D. degree (cum laude) from the dept. of Electrical Engineering of the Eindhoven University of Technology (The Netherlands) in 1995 and 1999, respectively. He was awarded the ASML price for the best Ph.D. thesis of the Eindhoven University of Technology in 1999/2000 in the area of fundamental research. Currently, he is working as an assistant professor in the Control Systems group of the dept. of Electrical Engineering of the Eindhoven University of Technology. His research interests include modeling, analysis and control of hybrid systems and dynamics under inequality constraints (especially complementarity problems and systems).

PLACE
PHOTO
HERE

Arjan van der Schaft A.J.(Arjan) van der Schaft (1955) received the undergraduate and Ph.D. degrees in Mathematics from the University of Groningen, The Netherlands, in 1979 and 1983, respectively. In 1982 he joined the Department of Applied Mathematics, University of Twente, Enschede, The Netherlands, where he is presently a full professor in Mathematical Systems and Control Theory. His research interests include the mathematical modeling of physical and engineering systems and the control of nonlinear and hybrid systems. He has served as Associate Editor for Systems and Control Letters, Journal of Nonlinear Science, IEEE Transactions on Automatic Control, and SIAM Journal on Control and Optimization. Currently he is Associate Editor for Systems and Control Letters and Editor-at-Large for the European Journal of Control. He is the (co-)author of the following books: System Theoretic Descriptions of Physical Systems (Amsterdam, The Netherlands: CWI,1984), (with P.E. Crouch) Variational and Hamiltonian Control Systems (Berlin, Germany: Springer-Verlag, 1987), (with H. Nijmeijer) Nonlinear Dynamical Control Systems (Berlin, Germany: Springer-Verlag, 1990), L_2-Gain and Passivity Techniques in Nonlinear Control (second edition, Springer Communications and Control Engineering Series, 2000) and (with J.M. Schumacher) An Introduction to Hybrid Dynamical Systems (London, UK: Springer-Verlag, LNCIS 251, 2000). Arjan van der Schaft is Fellow of the IEEE.

PLACE
PHOTO
HERE

Hans Schumacher J.M. (Hans) Schumacher was born in Heemstede, the Netherlands, in 1951. He obtained the M.Sc. and Ph.D. degrees in Mathematics, both from the Vrije Universiteit in Amsterdam, in 1976 and 1981 respectively. After spending a year as a visiting scientist at the Laboratory for Information and Decisions Sciences of MIT, he was a research associate at the Department of Econometrics of Erasmus University in Rotterdam and held a one-year ESA fellowship at ESTEC, the European Space Agency's research center in Noordwijk, the Netherlands. He moved to the Centre for Mathematics and Computer Science (CWI) in Amsterdam in 1984 and combined his position there with a part-time professorship in Mathematics at the Department of Econometrics and Operations Research of Tilburg University in 1987. In 1999 he accepted a full-time position at the same department as Professor of Mathematics. Dr. Schumacher was corresponding editor of SIAM Journal on Control and Optimization and is an associate editor of Systems and Control Letters. His current research interests are in the areas of mathematical finance and hybrid dynamical systems.