Port Hamiltonian Formulation of Infinite Dimensional Systems I. Modeling

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***Abstract*— In this paper, some new results concerning the modeling of distributed parameter systems in port Hamiltonian form are presented. The classical finite dimensional port Hamiltonian formulation of a dynamical system is generalized in order to cope with the distributed parameter and multi-variable case. The resulting class of infinite dimensional systems is quite general, thus allowing the description of several physical phenomena, such as heat conduction, piezoelectricity and elasticity. Furthermore, classical PDEs can be rewritten within this framework. The key point is the generalization of the notion of finite dimensional Dirac structure in order to deal with an infinite dimensional space of power variables. In this way, also in the distributed parameter case, the variation of total energy within the spatial domain of the system can be related to the power flow through the boundary. Since this relation deeply relies on the Stokes theorem, these structures are called Stokes–Dirac structures**

I. INTRODUCTION

Following the same ideas behind the bond graph formalism [11], a finite dimensional physical system can be modeled as the result of the interconnection of a small set of atomic elements, each of them characterized by a particular energetic behavior (e.g. energy storing, dissipation or conversion). Each element can interact with the environment by means of a *port*, that is a couple of input and output signals whose combination gives the *power flow*. The network structure allows a power exchange between these components and describes the power flows within the system and between the system and the environment. This network can be mathematically described by means of a Dirac structure [1], [2], [7], [14], generalization of the well-known Kirchoff laws of circuit theory, [8].

Once the Dirac structure is defined, the dynamics of the system is specified when the space of energy (state) variables and the energy (Hamiltonian) function are given. The port Hamiltonian formalism [7], [14] is based on these ideas and allows the description of a wide class of finite dimensional non-linear systems, such as mechanical, electro-mechanical, hydraulic and chemical ones.

The port Hamiltonian representation of a finite dimensional system has been recently extended in order to cope with the infinite dimensional case, [15], thus generalizing the *classical* Hamiltonian formulation of a distributed parameter system which is a well-established mathematical result, [10], [13]. From the network modeling perspective, the dynamics of an infinite dimensional system with spatial domain $\mathcal{Z}$ and boundary $\partial\mathcal{Z}$ is the result of the interaction among (at least) two energy domains within $\mathcal{Z}$ and/or between the system and its environment through $\partial\mathcal{Z}$. This interaction is mathematically described by a generalization of the Dirac structure to the distributed parameter case. Since this new class of power conserving interconnection deeply relies on the Stokes theorem, we speak about Stokes–Dirac structure.

In [15], a simple Stokes–Dirac structure has been introduced and it has been shown that it is can be the starting point for the description in port Hamiltonian form of the telegrapher equation, of Maxwell's equations and of the vibrating string equation. Moreover, in [9], this Stokes-Dirac structure has been modified in order to model fluid dynamical systems and in [3], [6] to model the Timoshenko beam equation. In any case, it is not completely clear how a general formulation of a multi-variable distributed parameter system within the port Hamiltonian formalism could be obtained.

In this paper, some new results in this direction are presented. In particular, a novel class of Dirac structures over an infinite dimensional space of power variables are introduced. The interconnection, damping and input/output matrices are replaced by matrix differential operators which are assumed to be constant, that is no explicit dependence on the state (energy) *variables* is considered. As in finite dimensions, given the Stokes–Dirac structure, the model of the system easily follows once the Hamiltonian function is specified. The resulting class of infinite dimensional systems in port Hamiltonian form is quite general, thus allowing the interpretation of classical PDEs within this framework and the description of several physical phenomena, as the heat conduction, piezo electricity and elasticity.

This work is organized as follows: after a short background about finite dimensional Dirac structures and port Hamiltonian system in Sect. II, the infinite dimensional Stokes–Dirac structures are introduced in Sect. III and the corresponding port Hamiltonian formulation of multi-variable infinite dimensional system (mdpH systems) is discussed in Sect. IV. In Sect. V, some simple examples are presented, the Harry–Dym equation, a *classical* non-linear PDE, and the heat equation. Finally, conclusions are discussed in Sect. VI.

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II. DIRAC STRUCTURES AND FINITE DIMENSIONAL PORT HAMILTONIAN SYSTEMS

A. Background on Dirac structures

The interconnection of physical system basically is power exchange. In order to mathematically model these phenomena, it is necessary to give a definition of power and to introduce a proper set of tools that will be useful to treat and describe the network structure behind a physical system.

Consider an n-dimensional linear space $\mathcal{F}$ and denote by $\mathcal{E} \equiv \mathcal{F}^*$ its dual, that is the space of linear operator $e : \mathcal{F} \to \mathbb{R}$. The elements belonging to $\mathcal{F}$ are called *flows* (e.g. velocities and currents), while the elements in $\mathcal{E}$ are called *efforts* (i.e. forces and voltages). Flows and efforts are the *port variables*, that is the input and output signals, whose combination gives the power flowing inside the physical system. The space $\mathcal{F}\times\mathcal{E}$ is called space of power variables.

Given an effort $e \in \mathcal{E}$ and a flow $f \in \mathcal{F}$, define the associated power P as

$$P := \langle e, f \rangle = e(f) \quad (\in \mathbb{R})$$

where $\langle \cdot, \cdot \rangle$ is the *dual product* between f and e. Based on the dual product, the following linear operator is well-defined.

Definition 2.1 (+pairing operator): Consider the space of power variables $\mathcal{F}\times\mathcal{E}$. The following symmetric bilinear form is well-defined:

$$\ll (f_1, e_1), (f_2, e_2) \gg \; := \langle e_1, f_2 \rangle + \langle e_2, f_1 \rangle \qquad (1)$$

with $(f_i, e_i) \in \mathcal{F} \times \mathcal{E}$, $i = 1, 2$; $\ll \cdot, \cdot \gg$ is called +pairing operator.

Consider a linear subspace $\mathbb{S} \subset \mathcal{F} \times \mathcal{E}$ of dimension m and denote by $\mathbb{S}^{\perp}$ its orthogonal complement with respect to the +pairing operator (1), which is again a linear subspace of $\mathcal{F}\times\mathcal{E}$ with dimension $2n-m$ since (1) is a non-degenerate form. Based on the +pairing operator (1), it is possible to give the fundamental definition of Dirac structure, that is the basic mathematical tool that is used to describe the interconnection structure between physical systems.

Definition 2.2 (Dirac structure): Consider the space of power variables $\mathcal{F} \times \mathcal{E}$ and the symmetric bilinear form (1). A (constant) Dirac structure on $\mathcal{F}$ is a linear subspace $\mathbb{D} \subset \mathcal{F} \times \mathcal{E}$ such that

$$\mathbb{D} = \mathbb{D}^{\perp}$$

Note 2.1: It is possible to prove that the dimension of a Dirac structure $\mathbb{D}$ on an n-dimensional space $\mathcal{F}$ is equal to n. This result is related to an interesting property of physical systems. Consider, for example, the interconnection of electrical networks: it is well known that it is not possible to impose both currents and voltages. By generalization, a physical interconnection cannot determine both the flow either the effort.

Moreover, suppose that $(f, e) \in \mathbb{D}$; from (1), we have that

$$0 = \ll (f, e), (f, e) \gg = 2 \langle e, f \rangle$$

Then, it can be deduced that, for every $(f, e) \in \mathbb{D}$,

$$\langle e, f \rangle = 0$$

or, equivalently, that every Dirac structure $\mathbb{D}$ on $\mathcal{F}$ defines a power-conserving relation between power variables $(f, e) \in \mathcal{F} \times \mathcal{E}$.

With the following proposition, a quite general class of Dirac structures is introduced, [14].

Proposition 2.1: Consider the space of power variables $\mathcal{F}\times\mathcal{E}$ and denote by $\mathcal{X}$ an n-dimensional space, the space of energy variables. Suppose that $\mathcal{F} := (\mathcal{F}_s, \mathcal{F}_r, \mathcal{F}_e)$ and that $\mathcal{E} := (\mathcal{E}_s, \mathcal{E}_r, \mathcal{E}_e)$, with $\dim \mathcal{F}_s = \dim \mathcal{E}_s = n$, $\dim \mathcal{F}_r = \dim \mathcal{E}_r = n_r$ and $\dim \mathcal{F}_e = \dim \mathcal{E}_e = m$. Moreover, denote by $J(x)$ a skew-symmetric matrix of dimension n and by $G_r(x)$ and $G(x)$ two matrices of dimension $n_r \times n$ and $m \times n$ respectively. Then, the set

$$\begin{aligned} \mathbb{D} := \{ \; (f_s, f_r, f_e, e_s, e_r, e_e) \in \mathcal{F} \times \mathcal{E} \;\; | \\ f_s = -J(x)e_s - G_r(x)f_r - G(x)f_e \\ e_r = G_r^{\mathrm{T}}(x)e_s \\ e_e = G^{\mathrm{T}}(x)e_s \; \} \end{aligned} \qquad (2)$$

is a Dirac structure on $\mathcal{F}$

Note 2.2: In Def. 2.2, the pairs (f_s, e_s) and (f_r, e_r) represent the port variables of the *storage* and *dissipative* elements respectively, while (f_e, e_e) are the port variables through which the *environment* can exchange power with the system. Given the interconnection structure (2), the dynamics of the system can be specified once the port behavior of the energy storage elements is specified and when the dissipative ports are *terminated*.

B. Finite dimensional port Hamiltonian systems

The Dirac structure introduced in Def. 2.2 is quite general. Based on that, a general formulation of non-linear system in port Hamiltonian form can be easily given. As discussed in Note 2.2, a dynamical system can be interpreted as the result of the combination of the Dirac structure (2) with the port behavior of the energy storing and of the dissipative elements.

Under the same hypothesis of Prop. 2.1, denote by $H : \mathcal{X} \to \mathbb{R}$ a real valued function bounded from below defined over the space of energy variables $\mathcal{X}$. Then, define the port behavior of the energy storing elements as follows:

$$f_s = -\dot{x} \qquad\qquad e_s = \frac{\partial H}{\partial x} \qquad (3)$$

where the minus sign is necessary in order to have a consistency in the power flow. If restricted to the linear case, dissipative effects can be taken into account by imposing the following relation on the variables (f_r, e_r) of the Dirac structure (2):

$$f_r = -Y_r e_r \qquad (4)$$

where $Y_r = Y_r^{\mathrm{T}} \geq 0$. By substitution of (3) and (4) in (2), the representation of a port Hamiltonian system with dissipation can be deduced [7], [14] and the following definition makes sense.

Definition 2.3 (finite dim. port Ham. systems): Denote by $\mathcal{X}$ an n-dimensional space of state (energy) variables and by $H : \mathcal{X} \to \mathbb{R}$ a scalar energy function (Hamiltonian) bounded from below. Denote by $\mathcal{U} \equiv \mathcal{F}_e$ an m-dimensional (linear) space of input variables and by its dual $\mathcal{Y} \equiv \mathcal{E}_e$ the space of output variables. Then,

$$\begin{cases} \dot{x} &= [J(x) - R(x)]\dfrac{\partial H}{\partial x} + G(x)u \\ y &= G^{\mathrm{T}}(x)\dfrac{\partial H}{\partial x} \end{cases} \tag{5}$$

with $J(x) = J^{\mathrm{T}}(x)$, $R(x) = R^{\mathrm{T}}(x) \geq 0$ and $G(x)$ matrices of proper dimensions, is a port Hamiltonian system with dissipation. The $n \times n$ matrices J and R are called *interconnection* and *damping* matrix respectively.

Note 2.3: Given a dynamical system in port Hamiltonian from (5), the variation of internal energy equals the dissipated power plus the power provided to the system by the environment, that is:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\frac{\partial^{\mathrm{T}} H}{\partial x} R(x) \frac{\partial H}{\partial x} + y^{\mathrm{T}} u \leq y^{\mathrm{T}} u$$

This relation expresses a fundamental property of port Hamiltonian systems, their *passivity*. Roughly speaking, the internal energy of the unforced system ($u = 0$) is non-increasing along system trajectories or, if the port variable are closed on a dissipative element, that is a relation similar to (4) is imposed between u and y, then the energy function is always a decreasing function. If the definition of Lyapunov stability is recalled, together with the sufficient condition for the stability of an equilibrium point, then it can be deduced that the Hamiltonian is a good candidate for being a Lyapunov function.

III. Power conserving interconnections in infinite dimensions

A. Constant matrix differential operators

In the finite dimensional formulation (5) of a port Hamiltonian system, an important role is played by the interconnection, damping and input matrices. These operators are strictly related to the properties of the Dirac structure defining the power flows within the dynamical system and between the system and its environment. In infinite dimensions, these *objects* are generalized and they are mathematically described by matrix differential operators. In this paper, only the constant case is taken into account. In the finite dimensional framework, this means that the dependence on the x variable of the elements of the Dirac structure (2) is neglected.

Denote by $\mathcal{Z}$ a compact subset of $\mathbb{R}^d$ representing the spatial domain of the distributed parameter system. Then, denote by $\mathcal{U}$ and $\mathcal{V}$ two sets of *smooth* functions from $\mathcal{Z}$ to $\mathbb{R}^{q_u}$ and $\mathbb{R}^{q_v}$ respectively.

Definition 3.1 (constant matrix differential operator): A constant matrix differential operator of order N is a map L from $\mathcal{U}$ to $\mathcal{V}$ such that, given $u = (u^1, \ldots, u^{q_u}) \in \mathcal{U}$ and $v = (v^1, \ldots, v^{q_v}) \in \mathcal{V}$

$$v = Lu \quad \iff \quad v^b := \sum_{\#\alpha=0}^{N} P_{a,b}^{\alpha} D^{\alpha} u^a \tag{6}$$

where $\alpha := \{\alpha_1, \ldots, \alpha_d\}$ is a multi-index of order $\#\alpha := \sum_{i=1}^{d} \alpha_i$, P^α are a set of constant $q_u \times q_v$ matrices and $D^\alpha := \partial_{z_1}^{\alpha_1} \cdots \partial_{z_d}^{\alpha_d}$ is an operator resulting from a combination of spatial derivatives. Note that, in (6), the sum is intended over all the possible multi-indexes α with order 0 to N and, implicitly, on a from 1 to q.

*Definition 3.2 (*formal *adjoint):* Consider the constant matrix differential operator (6). Its formal adjoint is the map L^* from $\mathcal{V}$ to $\mathcal{U}$ such that

$$u = L^* v \quad \iff \quad u^b := \sum_{\#\alpha=0}^{N} (-1)^{\#\alpha} P_{b,a}^{\alpha} D^{\alpha} v^a \tag{7}$$

Definition 3.3 (skew-adjoint matrix diff. op.): Denote by J a constant matrix differential operator. Then, J is *skew-adjoint* if and only if

$$J = -J^*$$

Note 3.1: It is easy to prove that, L is a skew-adjoint matrix differential operator if and only if

$$P_{a,b}^{\alpha} = (-1)^{\#\alpha} P_{b,a}^{\alpha}$$

for every multi-index α from order 0 to N.

An important relation between a differential operator and its adjoint is expressed by the following lemma, which generalizes an analogous result presented in [12] to the multi variable case. As it will be discussed in Sect. III-B, this result is fundamental in the definition of Stokes–Dirac structure and, basically, it generalizes the well-known integration by parts formula.

Lemma 3.1: Consider a matrix differential operator L and denote by L^* its formal adjoint. Then, for every functions $u \in \mathcal{U}$ and $v \in \mathcal{V}$, we have that

$$\int_{\mathcal{Z}} \left[v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v \right] \mathrm{dV} = \int_{\partial \mathcal{Z}} B_L(u, v) \cdot \mathrm{dA} \tag{8}$$

where B_L is a differential operator induced on $\partial\mathcal{Z}$ by L.

Note 3.2: Given $u \in \mathcal{U}$ and $v \in \mathcal{V}$, from the Stokes' theorem, it is well known that relation (8) can be equivalently written as

$$v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v = \operatorname{div} B_L(u, v)$$

that is $v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v$ can be expressed in divergence form. From (6) and (7), we have that

$$\begin{aligned} &v^{\mathrm{T}} L u - u^{\mathrm{T}} L^* v = \\ &\quad = \sum_{\#\alpha=0}^{N} P_{a,b}^{\alpha} \left[(D^\alpha u^a) v^b - (-1)^{\#\alpha} \left(D^\alpha v^b \right) u^a \right] \end{aligned} \tag{9}$$

whose divergence form is

$$\sum_{\#\beta=1} D^\beta \sum_{\alpha \geq \beta}^{N} \sum_{\gamma \leq \alpha - \beta} (-1)^{\#\gamma} P_{a,b}^{\alpha} \left(D^\gamma v^b \right) \left(D^{\alpha-\beta-\gamma} u^a \right) \tag{10}$$

in which the first sum is extended to all the multi index β of order 1.

Note 3.3: It is important to note that B_L is a constant differential operator. The quantity $B_L(u,v)$ is a constant linear combination of the functions u and v together with their spatial derivatives up to a certain order and depending on L. Consequently, denote by $B_{\mathcal{Z}}$ an operator providing a vector with all the spatial derivatives in (10) and by B_L^i, $i=1,\dots,d$, a set of constant square matrices of a certain order given by a proper combinations of all the P^α matrices. Then,

$$\int_{\partial\mathcal{Z}} B_{\mathcal{Z}}^{\mathrm{T}}(u)\left[B_L^1 B_{\mathcal{Z}}(v)\ \cdots\ B_L^d B_{\mathcal{Z}}(v)\right]\cdot \mathrm{dA}$$

gives the integral over $\partial\mathcal{Z}$ in (8).

Corollary 3.2: Consider a skew-adjoint matrix differential operator J. Then, for every functions $u\in\mathcal{U}$ and $v\in\mathcal{V}$ with $q_u=q_v$, we have that

$$\int_{\mathcal{Z}}\left[v^{\mathrm{T}}Ju+u^{\mathrm{T}}Jv\right]\,\mathrm{dV}=\int_{\partial\mathcal{Z}} B_J(u,v)\cdot\mathrm{dA} \qquad (11)$$

where B_J is a non-degenerate symmetric differential operator on $\partial\mathcal{Z}$ depending on the differential operator J.

Proof: It is immediate from Def. 3.3 and the previous lemma. ■

Note 3.4: From Note 3.3, the integral on $\partial\mathcal{Z}$ in (11) can be alternatively written as

$$\int_{\partial\mathcal{Z}} B_{\mathcal{Z}}^{\mathrm{T}}(u)\left[B_J^1 B_{\mathcal{Z}}(v)\ \cdots\ B_J^d B_{\mathcal{Z}}(v)\right]\cdot \mathrm{dA}$$

B. Constant Stokes–Dirac structures

As in finite dimensions, the definition of a power conserving interconnection structure is possible once the notion of power is properly introduced. Denote by $\mathcal{F}$ the space of flows and assume that $\mathcal{F}$ is the space of *smooth* functions from the compact set $\mathcal{Z}\subset\mathbb{R}^d$ to $\mathbb{R}^q$. As far as concerns the space of efforts $\mathcal{E}$, assume for simplicity that $\mathcal{E}\equiv\mathcal{F}$. Then, given $f=(f^1,\dots,f^q)\in\mathcal{F}$ and $e=(e^1,\dots,e^q)\in\mathcal{E}$, define the dual product as follows:

$$\langle e,f\rangle := \int_{\mathcal{Z}}\sum_{i=1}^{q} e^i f^i\,\mathrm{dV}=\int_{\mathcal{Z}} e^{\mathrm{T}} f\,\mathrm{dV}$$

From Def. 2.1, the +pairing operator on $\mathcal{F}\times\mathcal{E}$ is given by

$$\ll (f_1,e_1),(f_2,e_2)\gg\ := \int_{\mathcal{Z}}\left[e_1^{\mathrm{T}}f_2+e_2^{\mathrm{T}}f_1\right]\,\mathrm{dV}$$

where $(f_1,e_1),(f_2,e_2)\in\mathcal{F}\times\mathcal{E}$.

Denote by J a skew-adjoint differential operator and consider the following subset of the space of power variables:

$$\tilde{\mathbb{D}} := \{\ (f,e)\in\mathcal{F}\times\mathcal{E}\ \mid\ f=-Je\ \} \qquad (12)$$

Then, for every $(f_i,e_i)\in\tilde{\mathbb{D}}$, $i=1,2$, we have that

$$\begin{aligned}\ll (f_1,e_1),(f_2,e_2)\gg &= \int_{\mathcal{Z}}\left[e_1^{\mathrm{T}}f_2+e_2^{\mathrm{T}}f_1\right]\,\mathrm{dV}\\ &= -\int_{\mathcal{Z}}\left[e_1^{\mathrm{T}}Je_2+e_2^{\mathrm{T}}Je_1\right]\,\mathrm{dV}\\ &= -\int_{\partial\mathcal{Z}} B_J(e_1,e_2)\cdot\mathrm{dA}\end{aligned} \qquad (13)$$

If only the elements of $\tilde{\mathbb{D}}$ with compact support on $\mathcal{Z}$ are considered, then the resulting subset of $\mathcal{F}\times\mathcal{E}$ is a Stokes–Dirac structure on $\mathcal{F}$, as it can be directly deduced from Def. 2.2 since the integral over $\partial\mathcal{Z}$ is equal to 0. In general, when an exchange of power between system and environment through the boundary of the spatial domain is present, (12) is not a Stokes–Dirac structure because also the boundary terms have to be taken into account. These boundary terms are the restriction of the efforts and their spatial derivatives on $\partial\mathcal{Z}$.

Denote by $w:=B_{\mathcal{Z}}(e)$ the boundary terms, where $B_{\mathcal{Z}}$ the operator providing the restriction on $\partial\mathcal{Z}$ of the effort e and of its spatial derivatives of *proper* order as discussed in Note 3.3 and Note 3.4. In this way, it is possible to write (with some abuse in notation):

$$\int_{\partial\mathcal{Z}} B_J(e_1,e_2)\cdot\mathrm{dA}=\int_{\partial\mathcal{Z}} w_1^{\mathrm{T}}\left[B_J^1 w_2\ \cdots\ B_J^d w_2\right]\cdot\mathrm{dA}$$

with $w_i = B_{\mathcal{Z}}(e_i)$, $i=1,2$ and where, in the last integral, B_J^j, $j=1,\dots,d$, are the square constant matrices introduced in Note 3.4. Furthermore, based on $B_{\mathcal{Z}}$, the following set representing the space of boundary conditions can be introduced:

$$\mathcal{W} := \{w\ \mid\ w=B_{\mathcal{Z}}(e),\ \ \forall\, e\in\mathcal{E}\} \qquad (14)$$

Then, the following proposition can be proved.

Proposition 3.3: Consider the extended space of power variables $\mathcal{F}\times\mathcal{E}\times\mathcal{W}$ and denote by J a skew-adjoint differential operator. Then, the following subset

$$\begin{aligned}\mathbb{D}_J := \{\ (f,e,w)\in\mathcal{F}\times\mathcal{E}\times\mathcal{W}\ \mid\\ f=-Je,\ w=B_{\mathcal{Z}}(e)\ \}\end{aligned} \qquad (15)$$

is a Stokes–Dirac structure on $\mathcal{F}$ with respect to the pairing

$$\begin{aligned}&\ll (f_1,e_1,w_1),(f_2,e_2,w_2)\gg_J\ :=\\ &:= \int_{\mathcal{Z}}\left[e_1^{\mathrm{T}}f_2+e_2^{\mathrm{T}}f_1\right]\,\mathrm{dV}+\int_{\partial\mathcal{Z}} B_J(w_1 w_2)\cdot\mathrm{dA}\end{aligned} \qquad (16)$$

Proof: The proof is immediate from (13) and (14). ■

Note 3.5: From the properties of a Stokes–Dirac structure, summarized in Note 2.1 for the finite dimensional case, if $(f,e,w)\in\mathbb{D}$, then $\ll (f,e,w),(f,e,w)\gg_J=0$, that is

$$-\int_{\mathcal{Z}} e^{\mathrm{T}} f\,\mathrm{dV}=\frac{1}{2}\int_{\partial\mathcal{Z}} w^{\mathrm{T}}\left[B_J^1 w\ \cdots\ B_J^d w\right]\cdot\mathrm{dA}$$

This relation, beside expressing the power conservation property of the Stokes–Dirac structure, is able to relate the variation of internal energy (the integral on the spatial domain $\mathcal{Z}$) with the power flowing inside the domain through the boundary (the integral on $\partial\mathcal{Z}$).

The Stokes–Dirac structure introduced in Prop. 3.3 is developed around a skew-adjoint differential operator which *induces* a non-degenerate differential operator on the boundary. In finite dimensions, this situation can be obtained by assuming $G_r=G=0$ in the Dirac structure of Prop. 2.1, that is by assuming that the power conserving network interconnects only a set of energy storing elements. It is interesting to completely generalize the result of Prop. 2.1

to the distributed parameter case or, equivalently, to properly modify the Stokes–Dirac structure (15) of Prop. 3.3 in order to take into account dissipative effects and an interaction between system and environment along the spatial domain $\mathcal{Z}$ and not only through the boundary $\partial\mathcal{Z}$. The last situation can be encountered, for example, in the case of Maxwell's equations when a current density different from 0 is present, [5], [15].

Theorem 3.4 (constant Stokes–Dirac structure): Denote by $\mathcal{Z} \subset \mathbb{R}^d$ a compact set and by $\mathcal{F} = (\mathcal{F}_s, \mathcal{F}_r, \mathcal{F}_d)$ a space of vector values smooth functions on $\mathcal{Z}$, the space of flows. For simplicity, suppose that $\mathcal{E} = (\mathcal{E}_s, \mathcal{E}_r, \mathcal{E}_d) \equiv \mathcal{F}$ is the space of efforts. Moreover, assume that J, G_r and G_d are constant matrix differential operator such that $J : \mathcal{E}_s \to \mathcal{F}_s$ and $J = -J^*$, $G_r : \mathcal{F}_r \to \mathcal{F}_s$ and $G_d : \mathcal{F}_d \to \mathcal{F}_s$. Then,

$$\begin{aligned} \mathbb{D} := \{ \ (f, e, w) \in \mathcal{F} \times \mathcal{E} \times \mathcal{W} \ \mid & \\ f_s &= -J e_s - G_r f_r - G_d f_d \\ e_r &= G_r^* e_s \\ e_d &= G_d^* e_s \\ w &= B_{\mathcal{Z}}(e_s, f_r, f_d) \ \} \end{aligned} \tag{17}$$

is a Stokes–Dirac structure with respect to the pairing

$$\begin{aligned} \ll (f_1, e_1, w_1), (f_2, e_2, w_2) \gg_{\{J, G_r, G_d\}} &:= \\ := \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1 \right] \mathrm{dV} & \\ + \int_{\partial\mathcal{Z}} B_{\{J, G_r, G_d\}}(w_1, w_2) \cdot \mathrm{dA} & \end{aligned} \tag{18}$$

where $B_{\mathcal{Z}}$ is the analogous of the boundary operator of Prop. 3.3 and $B_{\{J,G_r,G_d\}}$ is the boundary differential operator induced by J, G_r and G_d on $\partial\mathcal{Z}$.

Proof: Consider $(f_i, e_i) \in \mathcal{F} \times \mathcal{E}$, $i = 1, 2$. Then,

$$\begin{aligned} \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1 \right] \mathrm{dV} = &- \int_{\mathcal{Z}} \left[e_{s,1}^{\mathrm{T}} J e_{s,2} + e_{s,2}^{\mathrm{T}} J e_{s,1} \right] \mathrm{dV} \\ &- \int_{\mathcal{Z}} \left(e_{s,1}^{\mathrm{T}} G_r f_{r,2} - f_{r,2}^{\mathrm{T}} G_r^* e_{s,1} \right) \mathrm{dV} \\ &- \int_{\mathcal{Z}} \left(e_{s,2}^{\mathrm{T}} G_r f_{r,1} - f_{r,1}^{\mathrm{T}} G_r^* e_{s,2} \right) \mathrm{dV} \\ &- \int_{\mathcal{Z}} \left(e_{s,1}^{\mathrm{T}} G_d f_{d,2} - f_{d,2}^{\mathrm{T}} G_d^* e_{s,1} \right) \mathrm{dV} \\ &- \int_{\mathcal{Z}} \left(e_{s,2}^{\mathrm{T}} G_d f_{d,1} - f_{d,1}^{\mathrm{T}} G_d^* e_{s,2} \right) \mathrm{dV} \end{aligned}$$

From Lemma 3.1 and its Corollary 3.2, all the quantities under integration can be expressed in divergence form, that is as the divergence of some differential form which is non-degenerate. In particular, denote by B_J, B_{G_r}, $B_{-G_r^*}$, B_{G_d} and $B_{-G_d^*}$ the differential operators induced on $\partial\mathcal{Z}$ by J, G_r and G_d and their adjoint. Then,

$$\begin{aligned} \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1 \right] \mathrm{dV} = &- \int_{\partial\mathcal{Z}} B_J(e_{s,1}, e_{s,2}) \cdot \mathrm{dA} \\ &- \int_{\partial\mathcal{Z}} \left\{ B_{G_r}(e_{s,1}, f_{r,2}) + B_{-G_r^*}(f_{r,1}, e_{s,2}) \right\} \cdot \mathrm{dA} \\ &- \int_{\partial\mathcal{Z}} \left\{ B_{G_d}(e_{s,1}, f_{d,2} + B_{-G_d^*}(f_{d,1}, e_{s,2}) \right\} \cdot \mathrm{dA} \end{aligned}$$

If $w_i = (e_{s,i}, f_{r,i}, f_{d,i})$, $i = 1, 2$, and

$$B^i_{\{J, G_r, G_d\}} = \begin{pmatrix} B^i_J & B^i_{G_r} & B^i_{G_d} \\ B^i_{-G_r^*} & 0 & 0 \\ B^i_{-G_d^*} & 0 & 0 \end{pmatrix}$$

with $i = 1, \dots, d$, then it is possible to write that

$$\int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1 \right] \mathrm{dV} + \int_{\partial\mathcal{Z}} B_{\{J, G_r, G_d\}}(w_1, w_2) \cdot \mathrm{dA} = 0$$

which, beside providing the expression (18) of the pairing $\ll \cdot, \cdot \gg_{\{J, G_r, G_d\}}$, proves that the set defined in (17) is Stokes–Dirac structure on $\mathcal{F}$ with respect to the bilinear form (18). ■

Note 3.6: The previous theorem is the generalization of the result presented in Prop. 2.1 to the constant infinite dimensional case. It is possible, eventually, to introduce the dependence on the energy *variables* and their spatial derivatives in the differential operators J, G_r and G_d. The result is the definition of nonlinear and *state* modulated Dirac structure in infinite dimensions. The way in which this result can be obtained relies on the generalization to the nonlinear case of matrix differential operator and, in particular, of the result expressed by Lemma 3.1. This result still holds in the nonlinear case, but there are no results concerning the properties of the *boundary* differential operator B_L, in particular about its non degeneracy property.

Note 3.7: Suppose that $(f, e, w) \in \mathbb{D}$. From (18), we have that

$$\begin{aligned} -\int_{\mathcal{Z}} e_s^{\mathrm{T}} f_s = \int_{\mathcal{Z}} e_r^{\mathrm{T}} f_r \,\mathrm{dV} + \int_{\mathcal{Z}} e_d^{\mathrm{T}} f_d \,\mathrm{dV} & \\ + \frac{1}{2} \int_{\partial\mathcal{Z}} B_{\{J, G_r, G_d\}}(w_1, w_2) \cdot \mathrm{dA} & \end{aligned} \tag{19}$$

This relation, which is a direct consequence of the definition of Dirac structure, expresses the property that the variation of internal energy is equal to the sum of the dissipated power with the power provided to the system through the domain $\mathcal{Z}$ and the boundary $\partial\mathcal{Z}$.

IV. Multi-variable infinite dimensional port Hamiltonian systems

A. General definition

As in finite dimensions, the dynamics of a distributed parameter system can be obtained from its Stokes–Dirac structure once the power ports are terminated on the corresponding elements, that is the input/output behavior of the *components* are specified.

Denote by $\mathcal{X}$ the space of smooth real valued functions on $[0, +\infty) \times \mathcal{Z}$ representing the space of energy *configuration*. The total energy is a functional $\mathcal{H} : \mathcal{X} \to \mathbb{R}$ such that

$$\mathcal{H}(x) = \int_{\mathcal{Z}} H(z, x) \,\mathrm{dV}$$

where H is the energy density. As proposed in [15], the port behavior of the energy storing element is given by

$$f_s = -\frac{\partial x}{\partial t} \qquad\qquad e_s = \delta_x \mathcal{H} \tag{20}$$

where $\delta_x\mathcal{H}$ is the variational derivative of the Hamiltonian with respect to the energy configuration. Linear dissipation can be introduced by imposing that

$$f_r = -Y_r e_r, \quad \text{with} \quad \int_{\mathcal{Z}} e_r^{\mathrm{T}} Y_r e_r \,\mathrm{dV} \geq 0 \tag{21}$$

where $Y_r : \mathcal{E}_r \to \mathcal{F}_r$ is a linear operator. If $\tilde{B}_{\mathcal{Z}}$ is the boundary operator introduced in (17), from (20) we have that

$$\begin{aligned}\tilde{B}_{\mathcal{Z}}(e_s, f_r, f_d) &= \tilde{B}_{\mathcal{Z}}(e_s, -Y_r G_r^* e_s, f_d) \\ &=: B_{\mathcal{Z}}(e_s, f_d)\end{aligned} \tag{22}$$

and then the boundary terms can be computed as $w = B_{\mathcal{Z}}(e_s, f_d)$. Consequently, taking into account (17), (20), (21) and (22), the following definition makes sense.

Definition 4.1 (mdpH system): Denote by $\mathcal{X}$ the space of vector value smooth functions on $[0,+\infty) \times \mathcal{Z}$ (energy configurations), by $\mathcal{F}_d$ the space of vector value smooth functions on $\mathcal{Z}$ (distributed flows) and assume that $\mathcal{E}_d \equiv \mathcal{F}_d$ is its dual (distributed efforts) and by $\mathcal{W}$ the space of vector value smooth functions on $\partial\mathcal{Z}$ representing the boundary terms. Moreover, denote by J a skew-adjoint differential operator, by G_d a differential operator and by $B_{\mathcal{Z}}$ the boundary operator defined in (22). If $\mathcal{H} : \mathcal{X} \to \mathbb{R}$ is the Hamiltonian function, the general formulation of a multi-variable distributed port Hamiltonian system with constant Stokes–Dirac structure is

$$\left\{\begin{array}{rcl} \dfrac{\partial x}{\partial t} &=& (J - R)\, \delta_x\mathcal{H} + G_d f_d \\ e_d &=& G_d^* \,\delta_x\mathcal{H} \\ w &=& B_{\mathcal{Z}}(\delta_x\mathcal{H}, f_d) \end{array}\right. \tag{23}$$

where $R := G_r Y_r G_r^*$ is a differential operator taking into account energy dissipation and $(f_d, e_d) \in \mathcal{F}_d \times \mathcal{E}_d$.

Note 4.1: It is important to note that there is no *a priori* distinction between flows and efforts in the boundary terms w. These variables result from the restriction on $\partial\mathcal{Z}$ of the variational derivative of $\mathcal{H}$ and of its spatial derivatives and, consequently, they are not characterized by an explicit physical meaning. In other words, given a generic multi-variable distributed port Hamiltonian system, the classical structure of power port, i.e. a couple of signals (flow and effort) whose *combination* gives the power flow, has been lost on the boundary. Only if the boundary operator $B_{\{J, G_r, G_d\}}$ has a particular structure, it is possible to split the boundary variable w into two components, that is into a flow and an effort.

Proposition 4.1: Consider the mdpH system (23). Then, the following energy balance inequality holds:

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} &= -\int_{\mathcal{Z}} (\delta_x\mathcal{H})^{\mathrm{T}} R\, \delta_x\mathcal{H}\,\mathrm{dV} + \int_{\mathcal{Z}} e_d^{\mathrm{T}} f_d \,\mathrm{dV} \\ &\quad + \frac{1}{2}\int_{\partial\mathcal{Z}} B_{\{J, G_r, G_d\}}(w, w) \cdot \mathrm{dA} \\ &\leq \int_{\mathcal{Z}} e_d^{\mathrm{T}} f_d \,\mathrm{dV} + \frac{1}{2}\int_{\partial\mathcal{Z}} B_{\{J, G_r, G_d\}}(w, w) \cdot \mathrm{dA}\end{aligned} \tag{24}$$

Proof: From (20), we have that

$$-\int_{\mathcal{Z}} e_s^{\mathrm{T}} f_s \,\mathrm{dV} = \int_{\mathcal{Z}} (\delta_x\mathcal{H})^{\mathrm{T}} \frac{\partial x}{\partial t}\,\mathrm{dV} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}$$

Then, (24) is immediate from (19) and (21). ■

Note 4.2: Relation (24) expresses an obvious property of physical systems, that is the variation of internal energy is less or equal (if no dissipation is present) to the power provided to the system. In the case of distributed parameter system, the power can flow inside the system either through the boundary and/or the spatial domain.

V. SIMPLE EXAMPLES

A. Harry–Dym equation

The Harry–Dym equation is

$$\frac{\partial x}{\partial t} = \frac{\partial^3}{\partial z^3}\left(x^{-1/2}\right) \tag{25}$$

Denote by $\mathcal{Z} = [0,1]$ the spatial domain and by $\mathcal{X} = L^2([0,+\infty) \times \mathcal{Z})$ the space of energy configurations. The differential operator $J = \frac{\partial^3}{\partial z^3}$ is skew-adjoint and, then, it is possible to define a Stokes–Dirac structure based on J as discussed in Prop. 3.3. We give the following proposition.

Proposition 5.1: Denote by $\mathcal{Z} = [0,1]$ the spatial domain and by $\mathcal{F} = L^2(\mathcal{Z})$ the space of flows and assume that $\mathcal{E} \equiv \mathcal{F}$ is the space of efforts. Then

$$\begin{aligned}\mathbb{D}_{HD} := \{\ (f, e, w) \in \mathcal{F} \times \mathcal{E} \times \mathcal{W} \ \mid\ & f = \partial_z^3 e \\ & w = B_{\mathcal{Z}}(e) = (e\,|_{\partial\mathcal{Z}}, \partial_z e\,|_{\partial\mathcal{Z}}, \partial_z^2 e\,|_{\partial\mathcal{Z}})\ \}\end{aligned}$$

is a Stokes–Dirac structure with respect to the pairing

$$\begin{aligned}&\ll (f_1, e_1, w_1), (f_2, e_2, w_2) \gg_{HD} := \\ &:= \int_0^1 \left[e_1 f_2 + e_2 f_1\right] \mathrm{dz} + w_1^{\mathrm{T}} B_J w_2 \Big|_0^1\end{aligned}$$

with

$$B_J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $\mathcal{W} = \mathbb{R}^3$.

Proof: Since ∂_z^3 is a skew-adjoint differential operator, from Prop. Prop. 3.3 we deduce that it can define a Stokes–Dirac structure. Then, it is necessary only to compute $B_{\mathcal{Z}}$ and B_J. Given $(f_i, e_i) \in \mathcal{F} \times \mathcal{E}$, $i = 1, 2$, we have that

$$\begin{aligned}e_1 f_2 + e_2 f_1 &= -\left(e_1 \frac{\partial^3 e_2}{\partial z^3} + e_2 \frac{\partial^3 e_1}{\partial z^3}\right) \\ &= -\frac{\partial}{\partial z}\left(e_1 \frac{\partial^2 e_2}{\partial z^2} - \frac{\partial e_1}{\partial z}\frac{\partial e_2}{\partial z} + e_2 \frac{\partial^2 e_1}{\partial z^2}\right)\end{aligned}$$

which gives $B_{\mathcal{Z}}$ and B_J thus concluding the proof. ■

The mdpH formulation of the Harry–Dym equation is completed once the Hamiltonian function is specified. In this case, we have that

$$\mathcal{H}(x) := 2\int_0^1 x^{1/2}(z)\,\mathrm{dz}$$

then (25) can be obtained if, as in (20), we assume that $f = -\dot{x}$ and $e = \delta_x \mathcal{H} = x^{-1/2}$. Clearly, the following energy balance relation holds:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \left[\delta_x \mathcal{H} \frac{\partial^2 \delta_x \mathcal{H}}{\partial z^2} - \frac{1}{2} \left(\frac{\partial \delta_x \mathcal{H}}{\partial z} \right)^2 \right]_0^1$$

Note that, in this case, it is not possible to define a pair of flow and effort variables on the boundary of the spatial domain (see Note 4.1) and that the model is nonlinear.

B. Heat equation

The one-dimensional heat equation is

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} \tag{26}$$

This system is not Hamiltonian in the classical sense [4], but it can be written in mdpH form.

Denote by $\mathcal{Z} = [0, 1]$ the spatial domain and by $\mathcal{X} = L^2([0, +\infty) \times \mathcal{Z})$ the space of energy configurations. The differential operator $R = \frac{\partial^2}{\partial z^2}$ is *not* skew-adjoint and, then, it is not possible to refer to the result of Prop. 3.3 in order to define a Stokes–Dirac structure.

Define the energy $\mathcal{H}$ of the system as

$$\mathcal{H}(x) = \frac{1}{2} \int_0^1 x^2(z)\, \mathrm{d}z \tag{27}$$

and then

$$\begin{aligned} \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} &= \int_{\mathcal{Z}} x \dot{x}\, \mathrm{d}z = \int_{\mathcal{Z}} x \frac{\partial^2 x}{\partial z^2}\, \mathrm{d}z \\ &= \int_{\mathcal{Z}} \frac{\partial}{\partial z} \left(x \frac{\partial x}{\partial z} \right) \mathrm{d}z - \int_{\mathcal{Z}} \left(\frac{\partial x}{\partial z} \right)^2 \mathrm{d}z \\ &\leq x \frac{\partial x}{\partial z} \bigg|_0^L \end{aligned} \tag{28}$$

This relation can be interpreted as an energy balance equation: the variation of internal energy is less or equal to the power provided to the system through the boundary. In this way, the diffusion phenomenum modeled by (26) can be described as pure dissipation. Clearly, a mdpH formulation of (26) is possible only once a proper Stokes–Dirac structure is determined.

We give the following proposition:

Proposition 5.2: Denote by $\mathcal{Z} = [0, 1]$ the spatial domain and by $\mathcal{F} = (L^2(\mathcal{Z}))^2$ the space of flows and suppose that $\mathcal{E} \equiv \mathcal{F}$ is the space of efforts. Then, the set

$$\begin{aligned} \mathbb{D}_H := \{ \ (f_s, f_r, e_s, e_r, w) \in \mathcal{F} \times \mathcal{E} \times \mathcal{W} \ \ | \\ f_s &= -\, \partial_z f_r \\ e_r &= -\, \partial_z e_s \\ w &= (e_s \,|_{\partial \mathcal{Z}}, f_r \,|_{\partial \mathcal{Z}}) \ \} \end{aligned} \tag{29}$$

is a Stokes–Dirac structure on $\mathcal{F}$ with respect to the pairing

$$\begin{aligned} &\ll (f_1, e_1, w_1), (f_2, e_2, w_2) \gg_H = \\ &= \int_{\mathcal{Z}} \left[e_1^{\mathrm{T}} f_2 + e_2^{\mathrm{T}} f_1 \right] \mathrm{d}z + w_1^{\mathrm{T}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} w_2 \bigg|_0^1 \end{aligned} \tag{30}$$

where $\mathcal{W} = \mathbb{R}^2$.

Proof: The proof can be found in [15] since (29) is the same Stokes–Dirac structure of the telegrapher equation or, equivalently, it can be deduced from Theorem 3.4 if $J = 0$, $G_r = \partial_z$ and $G_d = 0$. ■

The heat equation (26) can be obtained from the Stokes–Dirac structure by imposing that $f_s = -\dot{x}$ and $e_s = \delta_x \mathcal{H} = x$, where the Hamiltonian function is given in (27). Moreover, it is necessary to properly terminate the resistive port (f_r, e_r) in (29) by supposing that

$$f_r = -e_r$$

Finally, the energy balance relation (28) can be obtained from (30) since given $(f_s, f_r, e_s, e_r; w) = (-\dot{x}, \partial_z \delta_x \mathcal{H}, \delta_x \mathcal{H}, -\partial_z \delta_x \mathcal{H};\, \delta_x \mathcal{H} \,|_{\partial \mathcal{Z}}, \partial_z \delta_x \mathcal{H}|_{\partial \mathcal{Z}}) \in \mathbb{D}_H$, then $\ll (f_s, f_r, e_s, e_r; w), (f_s, f_r, e_s, e_r; w) \gg_H = 0$.

VI. Conclusions

In this paper, the classical finite dimensional port Hamiltonian formulation of a dynamical system is generalized in order to cope with the distributed parameter and multivariable case and some new results concerning modeling and control of distributed parameter systems in port Hamiltonian form have been presented. In this way, the description of several physical phenomena, such as heat conduction, is now possible within this new port-based framework. The central result is the generalization of the notion of finite dimensional Dirac structure to the distributed parameter case in order to deal with an infinite dimensional space of power variables.

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