A.J. STAM

Binominal identities
with old-fashioned proofs
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A.J. STAM

BINOMIAL IDENTITIES

WITH OLD-FASHIONED PROOFS

PART I^A
INTRODUCTION AND PLAN

Sums involving binomial coefficients arise in many parts of mathematics, in particular in combinatorics and probability theory. Though many of these sums result in simple expressions, their evaluation is not always easy. The researcher confronted by such a problem may consult several books, e.g. Netto (1927), Riordan (1968), Gould (1972c), Kaucký (1973), Egorychev (1984), Graham, Knuth and Patashnik (1989).

Over the past years the author collected, as a hobby, many binomial coefficient sums with proofs and / or references. Notwithstanding the existence of the literature mentioned above he feels that it is useful to present these data. The only lists of such identities arranged systematically known to the author are Gould (1972c) and part of Kaucký (1973), but Gould gives practically no proofs and few references and Kaucký is in Slovak and not readily available.

The book by Egorychev concentrates on a single technique ( contour integration) that has the advantage of providing a systematic attack on binomial sums, but our proofs in many cases are simpler, at least in PART I.

Having mentioned this literature from the years before 1990 we have to say something about, mainly more recent, literature that might render obsolete the greater part of the present work, viz. publications on manipulation of formulas by computer. A complete theory of the evaluation of binomial and hypergeometric sums by computer was developed by Wilf, Zeilberger, Gosper, Nemes and Petkovsek.

They give algorithms for deciding whether a sum satisfies a recurrence relation and then finding it and solving it in closed form when possible. See our list of references.

Still we think that our collection is useful. Many of our proofs are simpler than verifying by pencil and paper the recurrence and its solution, at least those in PART I. Also one may see the formulas in wider context, e.g. by showing how some of them fit into the theory of polynomials of convolution type (Chapter C). Generalizations, variations and companions may be
found. Sometimes we are able to give combinatorial or probabilistic proofs and interpretations.

Proving the identities we met a number of more general subjects: general summation formulas, finite differences, polynomials of convolution type, inverse relations and Fibonacci numbers and related polynomials. Because of their help in proving formulas and their inspiration in finding new ones these subjects are treated extensively in separate chapters.

All sums listed in PART I have, essentially, a single summation variable, say \( k \), and besides elementary functions of \( k \) they contain one or two binomial coefficients, e.g. \( \binom{x}{n} \), \( \binom{x+k}{k} \), \( \binom{2k-l}{n-k} \), etc. Here \( \binom{x}{n} \), \( x \in C, m \in N_0 \), sometimes called a generalized binomial coefficient, is defined as \( x(x-1)\ldots(x-m+1)/m! \), \( m > 0 \), \( \binom{0}{0} = 1 \). So

\[
(1) \quad \binom{x}{n} = x!/(m!(x-m)!), \quad m-x \not\in N_1
\]

Here \( x! = \Gamma(x+1) \). See Chapter D. We have some good reasons for taking \( x \in C \). First, these numbers occur in practice, e.g. as negative binomial probabilities \( \binom{x}{k} p^k (p-1)^{x-k} \), \( p > 0, 0 < p < 1, k \in N_0 \). Further, \( \binom{x}{n} \) is a polynomial in \( x \) of degree \( m \) and \( \binom{x}{n} \), \( n = 0,1,2,\ldots \) is a sequence of polynomials of convolution type. The elegant theory of such sequences provides us with a unifying principle and some beautiful proofs. See Chapter C. Finally we may avoid the values \( +\infty, -\infty \) and 0 for the factorials in the binomial coefficients, e.g. when we want to cancel or rearrange factorials such as in \( \binom{x}{k} \binom{x-k}{l} = \binom{x}{l} \binom{x-l}{k} \). We then assume \( x \not\in Z \) and at the end of our derivation we extend \( x \) to \( Z \), when that makes sense, by continuity.

Here a remark is in order. The above relation is but a small example of the many ways in which the appearance of a binomial identity may be changed. One then might decide to write the binomial coefficients as (1) above. But even then the appearance of such identities is not unique as is seen e.g. from D(24) and B(13):

\[
\binom{X^k}{k} = (-1)^k \binom{X-k}{k-1}/k!, \quad \binom{-l^k}{k} = (-l^k)/k!.
\]

One also might write binomial sums as finite or infinite hypergeometric series as advocated by Ranjan Roy (1987), see also Graham, Knuth and Patashnik (1989), Ch.5. The rich theory of hyper-
geometric functions then may be applied to evaluate or transform sums. We do not use this method in PART I. The sums do not have unique form even in this way, because of the many ways in which hypergeometric functions may be transformed. We also note that the form of the binomial coefficients in the sum may convey its combinatorial meaning.

A note on definition and notation. Some authors define \( \binom{x}{k} = 0 \) for \( k = -1, -2, \ldots \), and some do not include summation boundaries, letting the binomial coefficients take care of themselves, i.e. implying that the sum extends over all values of the summation variable that give nonzero terms. We do NOT do this: \( \binom{x}{k} \) with \( k = -1, -2, \ldots \) is NOT defined and the summation boundaries exclude such terms (unless stated explicitly in exceptional cases). On the other hand our summation boundaries allow zero terms such as those with \( k > n \) in \( \sum_{k=0}^{m} \binom{n}{k} \binom{x}{k} y^k \) when \( m > n \). The factor \( \binom{n}{k} \) then takes care of itself. However, in proofs, before canceling and rearranging factorials as above, the summation boundaries will be restricted so that terms with \( \infty / \infty \) are excluded. We also are careful with sums including terms with factors such as \( \binom{n-2k}{k} \) which is zero for \( n/3 < k \leq n/2 \) but nonzero for \( k > n/2 \).

The plan of this work is as follows.

Chapter N is a list of notations and conventions.

Chapter D gives definitions and some of the principal properties of the gamma and factorial functions, falling and rising factorials, Stirling numbers, binomial coefficients, etc.

Chapter G lists and proves a number of general formulas about summation and finite differences.

Chapter B proves a number of identities for binomial coefficients (not sums).

Chapter S collects some facts on special functions, partly with proofs.

Chapter C treats part of the theory of polynomials of convolution type, with attention for the special cases that are of interest for our subject.

Chapter IR is on inverse relations, by which new identities may be derived from old ones.

Chapters F and \( \Phi \) contain part of the theory of Fibonacci and Lucas numbers and related polyno-
mials. This theory gives rise to a number of binomial identities.

Chapter M discusses, with examples, a number of methods for evaluating binomial sums and proving identities between such sums.

The principal part of this work is Chapters T and P. Chapter T is a set of tables of binomial identities, arranged as systematically as the author could contrive, so that, when one wants the evaluation of a sum with binomial coefficients, it should be easily found when present. Identities between sums also are in the tables. The system of arranging the formulas is similar to the one in Gould (1972c). It may be found in the introduction to Chapter T.

When the proof of a formula may be found by a short reference to another formula or to the literature, this reference is given in the table. If not, one may find the proof in Chapter P under the same number as in the table. That chapter contains all proofs not given in the tables, arranged in the same order. It also has the remarks, alternative proofs, combinatorial or probabilistic interpretations and additional references to each table number that were found by the author.

For a small number of entries only probabilistic interpretations are given. These may be useful for asymptotic estimates by probability techniques.

Where did the formulas and proofs come from? More than half was found in or generalized from H.W. Gould, Combinatorial Identities, Morgantown, W. Va. 1972, that contains more than 500 of them. Since no proofs and few references are given there we seldom mention this work in our tables and proofs, but that does no right to the inspiration that we obtained from it. Other sources are The Fibonacci Quarterly, The American Mathematical Monthly, Netto (1927), Egorychev (1984), Riordan (1958, 1968, especially for Chapter IR), Vajda (1989, for Chapters F and Ψ), Mullin and Rota (1970), Rota e.a. (1973) and Di Bucchianico (1991) for Chapter C. These are not the only ones. Part of the identities and a greater part of the proofs was found by the author but not all of them might be new.

References to formulas are denoted e.g. as (17) within a chapter and e.g. IR (46) when referring to a formula in Chapter IR from another chapter. Identities in the tables are referred to as (3.445)
etc. for identity 445 in Table 3 or as (3.445), Chapter P, for proofs or remarks on (3.445). A deplorable ambiguity in this respect occurs in Chapter Φ:

Φ13 means: page 13 of Chapter Φ.

Φ(13) means: formula (13) in Chapter Φ,

Φn(13) means: Value of the function Φn(x) for x = 13.

Abbreviations of author and journal names are given at the beginning of the list of references.
\( \mathbb{R} = \) set of all real numbers.
\( \mathbb{R}_+ = \) set of all nonnegative real numbers.
\( \mathbb{C} = \) set of all complex numbers.

\( h, i, j, k, m, n, r, s, H, K, M, N \) nearly always, with exceptions stated explicitly, denote nonnegative integers. We use \( h, i, j, k \) most-

\[ \sum_{i+j=r} \text{ is a sum over all } i \text{ and } j \text{ with } i+j=r, \]

\[ \sum_{i} \text{ is the sum over } i \text{ from zero to } n \text{ when } n \rightarrow \infty. \]

\( x, y, z, \ldots \) are complex numbers.

A formula containing, say, \( x \) and \( n \) with no
restrictions added is meant to hold for all \( x \in \mathbb{C} \) and \( n \in \mathbb{N}_0 \).

\[
[x] = \max \{ t \in \mathbb{R} : t \leq x \}, \quad x \in \mathbb{R}.
\]

\[
\delta_{ij} = 0, \quad i \neq j, \quad \delta_{ii} = 1.
\]

\[f, g, h, \ldots \] denote functions \( \mathbb{D} \to \mathbb{C} \) with \( \mathbb{D} \subseteq \mathbb{C} \), maybe special ones, e.g. polynomials. We use the following operators (when applicable) on such functions:

- \( I = \text{identity operator} \),
- \( E^0 f(x) = f(x+a) \), \( E^0 = I \),
- \( \text{falling factorial, } D(q) \):
  \[
  (x)_0 = 1, \quad (x)_k = x(x+1)\ldots(x+k-1), \quad k \geq 1,
  \]
- \( \text{rising factorial, } D(2) \):
  \[
  (x)^0 = 1, \quad (x)^k = x(x+1)\ldots(x+k-1), \quad k \geq 1.
  \]
Note that $\Delta f(x)$ is the function $\Delta f$ with argument $x$. So $\Delta f(ax) = \Delta f(ax)$ and it is NOT $\Delta(ax)$ which is $\Delta(ax)$. Hence

$$\Delta \sin ax = \sin((ax + u) - \sin ax).$$


When necessary we indicate the variable on which $\Delta$ operates by writing $\Delta_x$, $\Delta_y$, e.g.

$$\Delta_y f(x, y) = f(x, y+1) - f(x, y).$$

$D_f(x) = \frac{d}{dx} f(x)$. 

$$H_n = \sum_{l=1}^{n-1} \frac{1}{l}, \ n \geq 1, \ H_0 = 0.$$ 

$s(n, k), S(n, k)$: Stirling numbers, see D (k8)-(k2).

$\mathcal{P}_n = \text{set of polynomials } \mathbb{C} \to \mathbb{C} \text{ of degree at most } n.$
\((a \ast b)_n, a^m_n\): convolutions. When \(a_k, b_k \in \mathbb{N}\),
and \(b_j, j \in \mathbb{N}_0\), are sequences, their con-

...efficient sequence with the corresponding generating function. See Chapter C, p. C/12.

\(X, Y, X_i, \ldots\), in Chapter P, denote random variables, mostly integer valued, \(E_g(X)\)
\(F\{g(X)\}\) denotes the expectation of \(g(X)\)
and \(\mathbb{P}(X = k)\) the probability that \(X\)
assumes the value \(k\).

\(F_n, L_n\): Fibonacci and Lucas numbers, Ch. F.

\(\Phi_n(x), \Lambda_n(x)\): Fibonacci- and Lucas-like polynomials, Ch. \(\Phi\).
D. Definitions and fundamental formulas.

The gamma function is defined by

\[ \Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} \, dt, \quad \text{Re} \, z > 0, \]

\[ z! = \Gamma(z+1), \quad -z \notin \mathbb{N}, \]

From (2)

\[ z! = (z-1)! \cdot z, \quad -z \notin \mathbb{N}, \]

\[ z! / (z-k)! = (z)_k, \quad k \in \mathbb{N}, z \notin \{k-1, k-2, \ldots \}. \]
where the falling factorial is defined by
\[(9) \quad (z)_0 = 1, \quad (z)_k = z(z-1) \cdots (z-k+1), \quad k \geq 1,\]
\[(10) \quad (n)_k = 0, \quad k > n, \quad n \in \mathbb{N}.\]
The binomial coefficients (for \(x \notin \mathbb{N}\) sometimes called generalized binomial coefficients) are defined by

\[(14) \quad \binom{x}{k} = \frac{x(x-1) \cdots (x-k+1)}{k!}, \quad k \in \mathbb{N}, \quad x \notin \{k-1, k-2, \ldots\},\]

\[(13) \quad \binom{n}{k} = \binom{n}{n-k}, \quad k = 0, \ldots, n, \quad n \in \mathbb{N},\]

\[(14a) \quad \binom{x}{0} = 1, \quad \binom{x}{1} = x, \quad \binom{x}{2} = \frac{1}{2} x(x-1), \quad x \in \mathbb{C},\]

\[(14b) \quad \binom{x+n}{n} = \frac{(x+1) \cdots (x+n)}{n!}, \quad x \in \mathbb{C}.\]

It is customary to define
\[(15) \quad \binom{x}{k} = 0, \quad k \in \{-1, -2, \ldots\},\]
in accordance with \( k! = \infty \) for \( -k \in \mathbb{N} \),
but we will avoid this convention, unless stated otherwise, by our choice of sum-

\[(16) \ \binom{y}{x} = \frac{\cdots}{y! (x-y)!}, \quad x, y, x-y \neq n-i, -\infty \ldots ,
\]

But we will not use this notation.
From (9) and (11) we have the important observations

\[(17) \ \binom{x}{k} \text{ and } \binom{x}{k} \text{ are polynomials in } x
\]
of degree \( k \).

From (14) and (7), first for \( x \notin \mathbb{Z} \) and then by continuity, we have the fundamental relation

that may be used to prove by induction on \( n \) the binomial formula

\[(19) \ \binom{u+v}{n} = \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k}, \quad n \in \mathbb{N}, \ u, v \in \mathbb{C}.
\]

Its extension is the binomial series

\[(20) \ \binom{1+z}{t} = \sum_{k=0}^{\infty} \binom{t}{k} z^k,\]
where \((1+z)^t = \exp \{ t \log(1+z) \}\) with \(\log(1+z)\) the principal value. Let \(t \notin \mathbb{N}\). Then the series in (10) converges absolutely for \(|z| < 1\). It also converges absolutely for \(|z| = 1\) when \(\text{Re} \ z > 0\) by B(q). It converges conditionally for \(|z| = 1\), \(z \neq -1\), when \(-1 < \text{Re} z \leq 0\) and it di-

\[(21) \quad z^{(0)} = 1, \quad z^{(k)} = z(z+1)\cdots(z+k-1), \quad k \in \mathbb{N},\]

as in Riordan (1958), Roman (1984). With (21)

From (9) and (21) and then with (11)

\[(23) \quad (-z)^k = (-1)^k z^{(k)} = (-1)^k (z+k-1),\]

\[(24) \quad (-x)^k = (-1)^k (x+k-1), \quad k \in \mathbb{N}, \quad x \in \mathbb{C},\]

also the remark following (10).
This relation is known as the Vandermonde convolution.

Let a random sample of size \( n \) without

\[ \text{whine bags and } n \text{-in mean sans,} \]

\[ (26) \quad \binom{a}{k} \binom{b}{n-k} / \binom{a+b}{n}, \quad k = 0, \ldots, n, \]

\!, called because of its generating function, (Kendall and Stuart I, Ch. 5, 18) and (26) states that its probabilities sum to 1. This provides a combinatorial proof of (26) for \( a \in \mathbb{N}, b \in \mathbb{N} \).

But then (26) follows for \( a \in \mathbb{C}, b \in \mathbb{C} \) because of (17). Cf. e.g. Feller (1957). For an equivalent form of this distribution see (3.44), (3.45).

From (13) we have the useful relation

\[ (27) \quad \binom{n}{k} = \binom{n}{n-k}, \quad k = 0, \ldots, n. \]
(28) \( (x)_n = \sum_{k=0}^{n} s(n,k) x^k, \quad x \in \mathbb{C}, \; n \in \mathbb{N}, \) 

See Kjordan (1958), Graham, Knuth and Patashnik (1988), Stanley (1986). One also defines

\[ a(n, m) \text{ form an inverse pair in the sense of Chapter 1R}. \]

From (28), (29), (31)

\[ (x)_n \mid_{x=1} = \sum_{k=0}^{n} s(n,k) x^k, \quad x \in \mathbb{C}, \; n \in \mathbb{N} \]
The Beta integral. See e.g. Rainville (1960), Ch. 2.
This integral is

$$ (34) \int_0^1 t^a (1-t)^b \, dt = a! b! / (a+b+1)! = \left( \frac{1}{a+b+1} \right)^{-1} \left( \frac{1}{a+b+1} \right)^{-1} = \frac{1}{a+b} $$
are useful in evaluating sums with binomial coefficients. Some of them are generalities on sums. Another class of these formulas consists of relations between polynomials leading to interesting results when specific choices are made for these polynomials. An important role is played by the theory of finite differences.

\[ \sum_{i=0}^{n-1} a_i b_{i+1} = a_n b_n - a_0 b_0 - \sum_{i=1}^{n-1} a_i (b_{i+1} - b_i), \]

when \( n \geq 1 \).

\[ \sum_{k=0}^{n} (a_{k+1} - a_k) b_k = \sum_{k=1}^{n+1} a_k b_{k-1} - \sum_{k=0}^{n} a_k b_{k+1} = a_{n+1} b_n - a_0 b_0 - \sum_{k=1}^{n} a_k (b_k - b_{k-1}), \]

when \( n \geq 1 \).
\[ b_{2m} = \sum_{k=0}^{2m} a_k a_{2m-k} = \sum_{m}^{2m} \phantom{m} \]

\[ \sum_{k=0}^{m} a_k a_{2m-k} = \frac{a^2}{m} \]

\[ a_{2m+1} = \sum_{k=0}^{2m+1} a_k a_{2m+1-k} = \sum_{k=0}^{m} a_k a_{2m+1-k} + \sum_{k=m+1}^{2m+1} a_k a_{2m+1-k} \]

\[ \sum_{k=0}^{m} a_k a_{2m+1-k} + \sum_{h=0}^{m} a_{2m+1-h} a_h = \]
\[ 2 \sum_{k=0}^{m} a_k a_{2m+1-k} \]

We have for any sequence \( a_0, \ldots, a_N \) noting \( D(19), (12), \)

\[ (6) \sum_{m=0}^{N} a_m z^m = \sum_{m=0}^{N} a_m \sum_{k=0}^{m} (\mu_k) (z-1)^k = \]

Applying the binomial theorem to \( (z-1) \)

and equating coefficients of \( z^m \) on both sides we find the "inclusion-exclusion"

formula. Let \( \text{TD}(1) \) or \( \text{TD}(17) \)

ate the sums of the terms with index \( \equiv p \bmod d \) in a series, mostly a

power series. The simplest and most

applied case is bisection \( (d=2) \). Let, for \( |z| < R \),

\[ (7) \sum_{k=0}^{\infty} c_k z^k = F(z) \]

Since \( F(-z) = \sum_{k=0}^{\infty} (-1)^k c_k z^k \) we have

\[ (8) F(z) + F(-z) = 2 \sum_{h=0}^{\infty} c_{2h} z^{2h} \]

\[ (9) F(z) - F(-z) = \sum_{h=0}^{\infty} c_{2h+1} z^{2h+1} \]
with \((7)\), for \(p = \ldots, \ldots, q-1\),

\[
\sum_{j=1}^{d} p^{-j} F(s^{j}z) = \sum_{i=1}^{d} p^{-i} \sum_{k=0}^{\infty} c_{k} s^{jk} z^{k} =
\]

\[
\sum_{j=1}^{d} s^{j} (k-p) \sum_{j=1}^{d} s^{j} (q-p) j =
\]

The last sum is zero when \(q \neq p\) and is equal to \(d\) when \(q = p\). So

\[
(12) \quad \sum_{j=1}^{d} p^{-j} F(s^{j}z) = d \sum_{h=0}^{\infty} c_{h} d^{p} z^{hd+p}
\]

This is the multisection. In particular
Finite differences. See e.g. Miller (1960), Fort (1948), Gelfond (1959).

Let \( f, g, h, \ldots \) denote real or complex functions of a real or complex variable. We consider the following operators transforming functions into functions:

(15) \( \mathcal{E} f = E^0 f = f \),

(16) \( E^t f(x) = (E^t f)(x) = f(x+t) \),

(17) \( \Delta = E - I \), \( \Delta^n f = \Delta \Delta^{n-1} f, \quad n \in \mathbb{N} \),

(18) \( \Delta^0 = I \).

The notations \( \Delta^m f(x) \) and \( E^m f(x) \) stand for the values at \( x \) of the functions \( \Delta^m f \) and \( E^m f \). Strictly the latter should be written as \( \Delta^m f(x) \) and \( E^m f(x) \). When no confusion can arise we use notations like \( \Delta^m x^r \) meaning \( \Delta^m f(x) \) with \( f(x) = x^r \). But we should be careful here. In our notation \( \Delta^m f(ax) \) means the value at \( ax \) of the function \( \Delta^m f \), so \( \Delta^m f(ax) = f(ax+m) - f(ax) \).

In the older literature, e.g. Jordan (1947) \( \Delta^m f(ax) \) may stand for the first difference of the function \( h(x) = f(ax) \), so that \( \Delta^m f(ax) \) would be equal to \( f(ax+m) - f(ax) \). We will avoid this notation. But we may write \( \Delta \sin ax \) or \( \Delta \left( \frac{ax}{m} \right) \) for \( \Delta f(x) \) with \( f(x) = \sin ax \) or \( f(x) = \left( \frac{ax}{m} \right) \). When writing \( \Delta^n f(x) \) we
We assume that with $x$, also $x+k$, $k=0, \ldots, n$, is in the domain of $f$. In many cases our functions are polynomials. We write

\[(19) \quad \mathcal{P}_n = \text{set of polynomials in a complex variable, with complex coefficients and} \]

\[
\sum_{h=0}^{n} (-1)^h \binom{n}{h} f(x+n-h),
\]

since $\Delta x = \sum_{j=0}^{\lfloor j \rfloor} x^j$, $m \leq x$, and $\Delta x^0 = 0$, we have by induction on $n$:
also when \( a_m = 0 \). Interesting notes on these formulas are in Gould (1978). In the same way,

\[
(E^b - I)^n g(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} g(x + kb) = 
\]
From (30) with \( \Delta \), for \( n \leq 1 \),

\[
\Delta^{n+e} f(x) = \sum_{i=0}^{n+e} \binom{n+e}{i} (-1)^i f(x + n + \varepsilon - i) = \\
= \sum_{j=0}^n \binom{n-j}{j} (-1)^j f(x + \varepsilon + j) + \sum_{j=1}^{n+1} \binom{n+j-1}{j-1} (-1)^j f(x + \varepsilon - j) = 
\]
\[
(-1)^n \binom{n + \kappa}{n+\kappa} \left\{ \sum_{j=0}^{\kappa} (-1)^j \binom{n}{j} \binom{n+j}{\kappa} f(x + \kappa + j) \right. \\
+ \left. \sum_{j=1}^{\kappa} (-1)^{j-1} \binom{n}{j-1} \binom{n+j-1}{\kappa-1} f(x + \kappa - j) \right\}.
\]

When \( f \in \mathcal{P}_m \) with \( m \leq n+\kappa-1 \), we have from (36) with \( x + \kappa = y \) and (24a),

\[
(31) \quad \sum_{j=0}^{\kappa} (-1)^j \binom{n}{j} \binom{n+j}{\kappa+1} f(y+j) + \sum_{j=0}^{\kappa} (-1)^j \binom{n}{j} \binom{n+j}{\kappa} f(y-j) = f(y),
\]

holding also for \( \kappa = 0 \) by (24a). Or, with D(13),

\[
(31a) \quad \sum_{j=0}^{\kappa} (-1)^j \binom{n}{j} \binom{n+j}{\kappa} f(y+j) + \sum_{j=0}^{\kappa} (-1)^j \binom{n}{j} \binom{n+j}{\kappa-j} f(y-j) = \binom{n+\kappa}{\kappa} f(y).
\]

For \( \kappa = n \geq 1 \) and \( f \in \mathcal{P}_m \) with \( m \leq 2n-1 \) this gives

\[
(32) \quad \sum_{j=0}^{\kappa} (-1)^j \binom{2n}{n-j}(f(y+j) + f(y-j)) = \binom{2n}{n} f(y).
\]

From (32) when \( f \) is even and \( f \in \mathcal{P}_m \) with \( m \leq 2n-1 \), \( n \geq 1 \),

\[
(33) \quad \sum_{j=0}^{\kappa} (-1)^j \binom{2n}{n-j} f(j) = \frac{1}{\kappa} \binom{2n}{n} f(0).
\]
We derive two generalizations of (33), cf. Verde-Star (1997).

First, let \( f \) be an even function. Then with (13) and with (22)

\[
(34) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{n+k}{k} \right)^{-1} f(k) = \\
(2n)^{-1} \sum_{k=0}^{n} (-1)^k \frac{(2n)!}{(n-k)! (n+k)!} f(k) = \\
\frac{1}{2} \left( \frac{2n}{n} \right)^{-1} \sum_{k=0}^{n} (-1)^k \binom{2n}{n-k} f(k) + \\
\frac{1}{2} \left( \frac{2n}{n} \right)^{-1} \sum_{k=0}^{n} (-1)^k \binom{2n}{n+k} f(k) = \\
\frac{1}{2} \left( \frac{2n}{n} \right)^{-1} \sum_{h=0}^{n} (-1)^{n-h} \binom{2n}{h} f(h-n) + \\
\frac{1}{2} \left( \frac{2n}{n} \right)^{-1} \sum_{j=n}^{2n} (-1)^{j-n} \binom{2n}{j} f(j-n) = \\
\frac{1}{2} f(0) + \frac{1}{2} \left( \frac{2n}{n} \right)^{-1} \sum_{j=0}^{2n} \binom{2n}{j} f(j-n) (-1)^{n-j} = \\
\frac{1}{2} f(0) + \frac{1}{2} \left( \frac{2n}{n} \right)^{-1} \Delta^{2n} f(-n). 
\]

Then let the function \( f \) satisfy \( f(x) = -f(-x+1) \). We have with (13) and with (22)

\[
(35) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{n+k+1}{k} \right)^{-1} f(k) = \\
\frac{1}{2n} \left( \frac{2n+1}{n} \right)^{-1} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} f(k) + \\
\frac{1}{2n} \left( \frac{2n+1}{n} \right)^{-1} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n+k} f(k). 
\]
\[ \frac{1}{2} \binom{2n+1}{n} \sum_{k=0}^{n} (-1)^k \binom{2n+1}{n+k+1} f(k) = \]

\[ \frac{1}{(2n+1)^{-1}} \sum_{j=0}^{n+1-j} \binom{2n+1}{n+j} f(n+j - 1) \]

see Verde-Star (1991), Prop. 5.2, by starting from (22), splitting the sum into \( k \leq m-1, k = m \) and \( k \geq m+1 \) and putting \( k = m-h, k = m+j \).

We have the following variant of (29):
Some specific finite differences. Here we compute the $n$th differences of some functions. By (22) we then obtain binomial sums. Some of the following results will be applied later.

From (22) we obtain (cf. E (21))

\[ F^{(j)}(x) = \Delta^{-j} x^n, \quad j = 0, 1, 2, \ldots \]
(42) \[ f(x+y) = \sum_{k=0}^{\infty} c_k(y) \frac{x}{x+ka} \binom{x+ka}{k}, \]

\( k=0, \ldots, n, \) form a basis of \( P_n. \) So the first relation in (42) holds, and we have:

\[ f(x) = E^{-x} \Delta f(x), \]

Putting \( x=0 \) gives \( c_0(y) = E^{-y} \Delta f(y). \)

The relations (38) - (42) are special cases of the theory of polynomials of convolution type, see Ch. C. The delta operators \( \Delta \) and \( E^{-x} \Delta \) have basic sequences \( \frac{x}{n} \) and \( \frac{x+na}{n}, \) see C, (21) and (84). The relations (41) and (42) are special cases of the expansion theorem, see C, (26), (88), (89).

In particular, from (40) and (42) since

by D (14) \[ \frac{x}{x+n} \binom{x+n}{n} = \binom{x+n-1}{n}, \]
(43) \[ E^{-\epsilon} \Delta^\epsilon \left( \frac{x+n-1}{n} \right) = \left( \frac{x+n-\epsilon-1}{n-\epsilon} \right), \] 
\[ \epsilon \leq n, \]
\[ = 0, \epsilon > n. \] Or directly, induction on \( \epsilon \). For \( f \in \mathcal{P}_n \) :

\[ \text{and then by induction on } \epsilon \]
\[ r(x) \Delta^\epsilon (a-x) = r(x) (a-x-\epsilon) \quad \epsilon \leq n \]

\[ \text{If some rational function } \quad \begin{equation} \Delta^n (x+a)^{-1} = \frac{(-1)^n n! (x+a-1)!}{(x+a+n)!} = \frac{(-1)^n n! (x+a)^{-1} (x+a+1)^{-1} \cdots (x+a+n)^{-1}}{n+1} = \frac{(-1)^n (x+a+n)^{-1}}{n+1} \frac{1}{x+a} \left( \frac{x+a+n}{n} \right)^{-1}, \quad n \in \mathbb{N}, \nonumber \end{equation} \]
\[ x+a \notin \{-n, -n+1, \ldots, -13, 0\}. \]

The first equality follows with induction on \( n \), applying \( \Delta^n (7) \). The other ones then follow with \( \Delta^n (14) \).

From (47), with application of \( \Delta (24) \),
\[(48) \quad \Delta^n (a-x)^{-1} = - \Delta^n (x-a)^{-1} =
\]
\[n! (a-x)^{-1} (a-x-1)^{-1} \cdots (a-x-n)^{-1} =
\]

Iterating this relation we find with \(D(9)\)

\[u+x \not\in \mathcal{N}_1, \quad v+x \not\in \mathcal{N}_1. \quad \text{Similarly}
\]
\[\Delta \frac{(u-x)!}{(v-x)!} = \frac{(u-x-1)!}{(v-x-1)!} \left(1 - \frac{u-x}{v-x}\right) = (v-u) \frac{(u-x-1)!}{(v-x)!}.
\]

Iteration now gives with \(D(21)\)

\[(50) \quad \Delta^n \frac{(u-x)!}{(v-x)!} = (v-u)^{(n)} \frac{(u-x-n)!}{(v-x)!},
\]
\[x-u+n \not\in \mathcal{N}_1, \quad x-v \not\in \mathcal{N}_1.
\]

From (49) and \(D(6), (22), (23)\) or directly by iteration of
\[-a \notin \mathcal{N}_0, \quad -x - a \notin \mathcal{N}_0.\]

\[
\begin{align*}
\sum_{\ell=0}^{m+n} (x+a+n) \cdot x \cdot (x+\ell) = \sum_{\ell=0}^{m+n} (m+n) \cdot x \cdot (x+\ell).
\end{align*}
\]

\[x+a \notin \{0, \ldots, m-1\}, \quad m \geq 1. \text{ Also for } m=0, \quad n \geq 1.\]

From (52) and D (24), or from (50) similarly to (52), for \(m \geq 1,
\]

\[
\begin{align*}
\Delta^n \left( \frac{a-x}{m} \right)^{-l} &= (-1)^m \Delta^n \left( \frac{x-a+m-1}{m} \right)^{-l} \\
&= (-1)^{m+n} \frac{m}{m+n} \left( \frac{x-a+m+n-1}{m+n} \right)^{-l} = \frac{m}{m+n} \left( \frac{a-x}{m+n} \right)^{-l},
\end{align*}
\]

\[a-x \notin \{0, \ldots, m+n-1\}. \text{ Also for } m=0, \quad n \geq 1.\]
Some general sums. For \( x \in \{0, \ldots, n\} \) we have

\[
(55) \quad g(x) = \sum_{k=0}^{n} g(k) \binom{x}{k} \binom{n-x}{n-k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{x}{k} \binom{x-k-1}{n-k} g(k).
\]

The second equality always holds by \( D \) (24). For the first one, note that for \( x \in \{0, \ldots, n\} \)

\[
\binom{x}{k} \binom{n-x}{n-k} = \delta_{xk}.
\]

When \( g \in \mathcal{P}_n \), the relation (55) holds for \( x \in \mathcal{P}_0 \), since by \( D \) (17) the second member also is in \( \mathcal{P}_n \) and these two polynomials coincide for \( x \in \{0, \ldots, n\} \).

For \( x = n+1 \) and \( x = -1 \) the second and first sum in (55) lead to

\[
(56) \quad g(n+1) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{k} g(k), \quad g \in \mathcal{P}_n,
\]

\[
(57) \quad g(-1) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+1}{n-k} g(k)
\]

\[
= \sum_{h=0}^{n} (-1)^{n-h} \binom{n+1}{h} g(n-h), \quad g \in \mathcal{P}_n.
\]

For \( f \in \mathcal{P}_n \) and \( -x \notin \{0, \ldots, n\} \) we have

\[
(58) \quad f(x) = x \binom{x+n}{n} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} f(-k)(x+k)^{-1}.
\]

Proof. From D(9), (11) by partial fraction expansion
\[ x^{-1} f(x) \left( \frac{x+n}{n} \right)^{-1} = n! f'(x) x^{-1} (x+1)^{-1} \cdots (x+n)^{-1} = \]
\[ \sum_{k=0}^{n} A_k (x+k)^{-1}, \]
also for \( k=0 \) and \( k=n \).

\[ f'(0) = f(0) H_n + O(x), \quad \text{as} \quad x \to 0, \]
with \( H_n = \sum_{j=1}^{n} j^{-1} \). So for \( x \to 0 \)
\begin{equation}
(59) \quad \sum_{k=1}^{n} (-1)^k (\binom{n}{k}) k^{-1} f(k) = f'(0) - f(0) H_n,
\quad f \in \mathcal{P}_n, \quad n \geq 1.
\end{equation}

Differentiating (58) with respect to \( x \), using B (39), and applying (58) one finds, for
\[ -x \notin \{0, \ldots, n\}, \quad n \geq 1, \quad \text{and trivially for} \quad n=0. \]
\[(60) \quad x^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x+k) (x+k)^{-x} = \]
\[- f'(x) + f(x) \sum_{j=0}^{n} (x+j)^{-1}, \quad f \in \mathcal{P}, \]

Let \(|z| < 1\) and \(-\infty < k < \infty\).

\[(62) \quad \sum_{n=0}^{\infty} z^n \Delta^n f(x) = \sum_{k=0}^{\infty} z^k f(x+k) (1+z)^{-k-1}, \]
both sides converging absolutely.

Proof. We have, formally, by (22) and (25),
\[
\sum_{n=0}^{\infty} z^n \Delta^n f(x) = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k) = \\
\sum_{k=0}^{\infty} f(x+k) \sum_{n=k}^{\infty} (-1)^{n-k} \binom{n}{k} z^n = \\
\sum_{k=0}^{\infty} f'(x+k) \sum_{m=0}^{\infty} (-1)^{m+k} \binom{m+k}{m} z^{m+k} = \\
\sum_{k=0}^{\infty} z^k f(x+k) (1+z)^{-k-1}.
\]

The interchange of summations and the absolute convergence in (62) may be justified by (61). In the same way, as above
\[
\sum_{n=0}^{\infty} |z|^n \sum_{k=0}^{n} \binom{n}{k} |f(x+k)| = \\
\sum_{k=0}^{\infty} |z|^k |f(x+k)| (1-|z|)^{-k-1} < \infty.
\]
We have
\[ (63) \quad e^{-z} \sum_{j=0}^{\infty} \frac{z^j}{j!} f(x+k) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \Delta^j f(x), \]

Proof. Let the left-hand side of (63) converge absolutely. Then with (23)
\[ \sum_{j=0}^{\infty} \frac{z^j}{j!} \Delta^j f(x) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} f(x+k) = \]
\[ \sum_{k=0}^{\infty} \frac{f(x+k)}{k!} \frac{z^k}{z^{j-k}} = e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{k!} f(x+k), \]
where the interchange of summations and the absolute convergence follow from the absolute convergence of the left-hand side in (63).

Let the right-hand side of (63) converge absolutely. Then by (23)
\[ \sum_{k=0}^{\infty} \frac{z^k}{k!} f(x+k) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{j=0}^{k} \binom{k}{j} \Delta^j f(x) = \]
\[ \sum_{j=0}^{\infty} \frac{1}{j!} \Delta^j f(x) \sum_{k=0}^{\infty} \frac{z^k}{z^{j-k}} = e^{-z} \sum_{j=0}^{\infty} \frac{z^j}{j!} \Delta^j f(x), \]
where the absolute convergence of the right-hand side of (63) justifies the interchange of the summations and implies the absolute convergence of the left-hand side.
of Gaussian interpolation formula. In
our notation and slightly generalized
they are

\[ g(y + a + b) = \sum_{i=0}^{m-1} \left( \frac{y + b + i}{2i+1} \right) \Delta^{2i+1} g(a - i - 1) \]

\[ g(y + a + b) = \sum_{i=0}^{m-1} \left( \frac{y + b + i}{2i+1} \right) \Delta^{2i+1} g(a - i) \]

\[ + \sum_{j=0}^{m} \left( y + b + j - 1 \right) \Delta^{2j} g(a - j), \quad m \geq 1, \]

for polynomials \( g \) of degree \( \leq 2m \). Our nota-

\[ \Delta \]

we give...
(69) \( f(x) = \sum_{i=0}^{n} c_i \left( \frac{x^{2i+1}}{2i+1} \right) + \sum_{j=0}^{m} d_j \left( \frac{x^{2j}}{2j} \right) \),

since the set of polynomials \( \left\{ \frac{x^{2i+1}}{2i+1}, i=0, \ldots, n \right\} \) is a basis for \( P_n \).

From (68) and (38) for \( m \leq m-1 \)

\[ a_{x} = \Delta f(-x-1). \]

Also, for \( m \leq m \),

\[ \Delta^m f(x) = \sum_{i=x}^{m-1} a_i \left( \frac{x+i}{2i-2x+1} \right) + \sum_{j=x}^{m} b_j \left( \frac{x+j}{2j-2x} \right), \]

with \( \sum_{i=x}^{m-1} = 0 \) when \( x = m \). All terms here are zero when \( x = -x \), except the term with \( j = x \). So

\[ \rho_x = \Delta^m f(-x). \]
So (68) leads to

\[ f(x) = \sum_{i=0}^{m-1} (\frac{x+i}{2i+1}) \Delta^{2i+1} f(-i) + \sum_{j=0}^{m} (\frac{x+j}{2j}) \Delta^{2j} f(-j). \]

Applying this relation to the function \( f(x) = g(x+a) \), and then putting \( x = y + b \), proves

\[ f(x) = \sum_{i=0}^{m} a_i (\frac{x+i}{2i+1}) + \sum_{j=0}^{m} d_j (\frac{x+j}{2j}). \]

When \( g \) has degree \( 2m-1 \) the relations (68) and (69) apply, with the sums over \( j \) restricted to \( 0, 1, \ldots, m-1 \). The relations for polynomials \( g \) of odd and even degree

\[ (71) \quad g(y+a+b) = \sum_{2i+1 \leq n} (\frac{y+b+i}{2i+1}) \Delta^{2i+1} g(a-i) \]
The next formulas connect certain binomial sums to absolute central moments of symmetric hypergeometric probability distributions (see D(26a)) and will be used to derive some identities with binomial coefficients.

\[(72) \sum_{k=0}^{m} f(k) \binom{m-k}{m+k} = \frac{1}{2} f'(0) \binom{m}{y}^2 \]

\[+ \frac{1}{2} \sum_{k=0}^{2m} f(1-k-m) \binom{y}{k} \binom{y}{2m-k} \]

**Proof.** The l.h.s. is equal to

\[\sum_{h=0}^{m} f(m-h) \binom{y}{h} \binom{2m-h}{y} \]

and also to

\[\sum_{j=m}^{2m} f(j-m) \binom{y}{2m-j} \binom{y}{j} \]

and therefore also to

\[\frac{1}{2} \sum_{h=0}^{m} f(m-h) \binom{y}{h} \binom{y}{2m-h} + \frac{1}{2} \sum_{j=m}^{2m} f(j-m) \binom{y}{j} \binom{y}{2m-j} \]

\[(73) \sum_{k=0}^{m} f\left(k + \frac{1}{2}\right) \binom{m-k}{m+k} = \frac{1}{2} \sum_{k=0}^{2m+1} f\left(k - m - \frac{1}{2}\right) \binom{y}{k} \binom{y}{2m+1-k} \]

**Proof.** The l.h.s. is equal to
\[ \sum_{h=0}^{m} f(m + \frac{1}{2} - h) \binom{\frac{1}{2}}{h} (2m+1-h), \]

and also to

and therefore also to

\[ \sum_{h=0}^{m} f(m+1 + \frac{1}{2} - h) \binom{\frac{1}{2}}{h} (2m+1-h), \]

We also have

\[ \sum_{h=0}^{m} f(m+1 + \frac{1}{2} - h) \binom{\frac{1}{2}}{h} (2m+1-h). \]

We also have

\[ \sum_{k=0}^{2m} f(k+\frac{1}{2}) \binom{m-k}{k} (m-k), \]

\[ \sum_{k=0}^{m} f(k+\frac{1}{2}) \binom{m-k}{k} (m-k) = -\frac{1}{2} \sum_{n=1}^{2m+1} f\left(\frac{1}{2}m-\frac{1}{2}\right) \binom{m-x}{0} \binom{m-x}{x}, \]

with \( m \) an integer.
A recurrence. We consider the sums

\[(76) \quad S_{nm} = \sum_{k=0}^{\infty} \left( \frac{n+a(k)}{m+b(k)} \right) f(k), \]

where

\[(77) \quad a(k) \in \mathbb{N}_0, \quad b(k) \in \mathbb{N}_0, \quad b(k) - a(k) \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty, \]

so that the sum in (76) contains only

\[\sum \quad \text{finite number of terms}.\]

Hence

\[\text{(77) } \quad \text{for } \quad n_{m} \quad \text{and } \quad m_{m}, \quad n_{m_{1}}, \quad m_{m_{1}} \quad \text{and } \quad \ldots \]

the same recurrence as is satisfied by the binomial coefficients \( \binom{n}{m} \), \( n \geq 0, \ m \geq 0 \).

When \( a(k) \leq b(k), \ k \geq 0, \) we have

\[(78) \quad S_{nm} = 0, \quad m > n, \]

and in particular

\[(80) \quad S_{0m} = 0, \quad m \geq 1. \]

The \( S_{nm} \) are determined uniquely by the recurrence (78) and the boundary values \( S_{no}, \ n \in \mathbb{N}, \) \( S_{00} \) and \( S_{0m}, \ m \in \mathbb{N}_{0} \). In solving

\[(81) \quad \delta_{nm} \quad \text{for } \quad m = 0, \quad n, \quad \ldots, \]

we might use one of the
Here we will not bother about convergence. Under (79) there is no problem for (81) and in our applications convergence may be verified or the solution found may be shown to satisfy (78) and the boundary conditions. From (78) we obtain

\[(82) \quad U_n(z) = (1+z) U_{n-1}(z) + S^t_{n_0} - S^t_{n-1}, \quad n \geq 1,\]

\[(83) \quad V_m(z) = (1-z) V_{m-1}(z) + S^d_{0m}, \quad m \geq 1,\]

Under (80) we have from (84) of G.K.P. ex. 71. p. 237.

\[(85) \quad V_m(z) = z^m (1-z)^{m-1} V_0(z),\]

showing that (80) implies (79), as also may be derived from (78).

A special case, used in the proof of (3.5.81), is obtained when \(S^t_{n_0} = 0, \quad n \geq 1, \quad S^d_{0m} = 0, \quad m \geq 1.\)

Then from (85) and D. (85)

\[V_m(z) = S^d_{00} z^m (1-z)^{m-1} = S^d_{00} \sum_{j=0}^{\infty} \binom{m+j-1}{j} z^{m+j},\]

so that then

\[(86) \quad S^d_{nm} = S^d_{00} \binom{n-1}{m-1}, \quad n \geq 1, \quad m \geq 1.\]

Under (80) we have from (85) with the
convolution property of generating functions (Theorem M1) and D(25) \(1\)

(87) \(S'_{nm} = \sum_{h=0}^{n-m} S'_{h0} \binom{n-h-1}{n-m-h}, \quad m \leq n.\)

Example: When

\[ S'_{nm} = \sum_{k=0}^{\infty} \binom{n+k}{m+2k} t^k \quad \text{or} \]

\[ S'_{nm} = \sum_{k=0}^{\infty} \binom{n+k+1}{m+2k+1} t^k, \]

we have \(S'_{0m} = \delta_{0m}\) and, respectively, by \(\Phi(127)\) and \(\Phi(122)\)

\[ S'_{n0} = t^n \Phi_{2n} (t^{-1}) \quad \text{or} \quad S'_{n0} = t^n \Phi_{2n+1} (t^{-1}). \]

So from (87)

(88) \[\sum_{k=0}^{n-m} \binom{n+k}{m+2k} t^k = \sum_{h=0}^{n-m} t^h \Phi_{2h+1} (t^{-1}) \binom{n-h-1}{n-m-h}, \quad m \leq n,\]

(89) \[\sum_{k=0}^{n-m} \binom{n+k+1}{m+2k+1} t^k = \sum_{h=0}^{n-m} t^h \Phi_{2h+1} (t^{-1}) \binom{n-h-1}{n-m-h}, \quad m \leq n.\]

From (88) and (89)

(90) \[\sum_{h=0}^{n-m} t^h \Phi_{2h+1} (t^{-1}) \binom{n-h-1}{n-m-h} = \]

\[ \sum_{h=0}^{n-m} t^h \Phi_{2h} (t^{-1}) \binom{n-h}{n-m-h}, \quad m \leq n.\]
In Problem 8, Nieuw Arch. (3), 13, 1965, 127-128 the following identity is proved for polynomials \( f \) of degree \( \leq n \):

\[
(91) \quad f(x) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} (-1)^{n+j+k} \binom{n+j}{n} \binom{n+1}{n-j-k} f(k),
\]

where the sum is over all \( j \in \mathbb{N}_0, k \in \mathbb{N}_0 \) with \( j+k \leq n \).

Proof. Since both sides are polynomials in \( x \) of degree \( \leq n \), it is sufficient to prove (91) for \( x \in \{0, 1, \ldots, n\} \). For \( x \leq n \) we have, with \( D(27), D(24), D(26) \)

\[
\sum_{j=0}^{n} \binom{n+j}{n} \sum_{k=0}^{n-j} (-1)^{n-j-k} \binom{n+1}{n-j-k} f(k) =
\]

\[
\sum_{i=0}^{n} \binom{n+i}{n} \sum_{k=0}^{i} (-1)^{i-k} \binom{n+1}{i-k} f(k) =
\]

\[
\sum_{i=0}^{n} \binom{n+i}{n} \sum_{k=0}^{n-i} (-1)^{n-i-k} \binom{n+1}{n-i-k} f(k) =
\]

\[
\sum_{k=0}^{n} f(k)(-1)^{n-k} \sum_{l=k}^{n} \frac{1}{l-k} \binom{n+1}{n-l} f(l) =
\]

\[
\sum_{k=0}^{n} f(k)(-1)^{n-k} \sum_{h=0}^{n-k} \binom{n+1}{h} \binom{-n-1}{n-k-h} =
\]

\[
\sum_{k=0}^{n} f(k)(-1)^{n-k} \frac{0}{n-k} = f(n).}
\]
Writing \((q_1)\) as

\[
(92) \quad f(x) = \sum_{k=0}^{n} f(k) \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{x+j}{n} \binom{n+1}{n-k-j}
\]

and comparing \((92)\) and \((55)\) — with \(q = f\) — we see that we must have

\[
\sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{x+j}{n} \binom{n+1}{n-k-j} = \binom{x}{k} \binom{n-x}{n-k}, \quad k \leq n,
\]

since

\[
\sum_{k=0}^{n} a_k f(k) = \sum_{k=0}^{n} b_k f(k)
\]

for all polynomials of degree \(< n\) implies

\[
a_k = b_k, \quad k = 0, \ldots, n.\]

Putting \(n-k = m\), we may write this identity as

\[
(93) \quad \sum_{j=0}^{m} (-1)^{m-j} \binom{x+j}{n} \binom{n+1}{m-j} = \binom{x}{n-m} \binom{n-x}{m}, \quad m \leq n.
\]

The author has no direct proof of \((93)\).
One of the causes of the enormous variety in binomial sums is that seemingly different

\[
(1) \quad \binom{x}{k}\binom{x+n-k}{n-k} = \binom{n}{k}\binom{x+n-k}{n}, \quad k = 0, \ldots, n,
\]
we obtain the alternative form of the hypergeometric probabilities, see \( \text{see } D \) (26), (26a):

\[
\frac{a! b! c! (a+b-c)!(a+c-b)!}{(a+b+c)!}
\]
The next identities are less trivial. We use them mainly to deal with \((k)\) where \(\frac{1}{2} \leq k \leq 2\). We start with Legendre's duplica-

\[
p^{\frac{1}{2}} \Gamma(z) = 2^{-z} \Gamma(z) \Gamma(z + \frac{1}{2}).
\]

With \(\Omega(6), (7)\) this gives, for arguments not in \(-\mathbb{N}\),

\[\pi^{\frac{1}{2}} (2z)! = 2^{2z} z! (z - \frac{1}{2})!,\]

\[\pi^{\frac{1}{2}} (2z+1)! = 2^{2z+1} z! (z + \frac{1}{2})!.\]

For \(z \in \mathbb{N}\), these relations follow by induction, starting with \(\Omega(4)\). From (9) and (10)

\[\frac{(v-\frac{1}{2})!}{(u-\frac{1}{2})!} = \frac{u-v}{(2u)!} \frac{(2v)! u!}{v!},\]

\[\frac{(v+\frac{1}{2})!}{(u+\frac{1}{2})!} = \frac{u-v}{(2u+1)!} \frac{(2v+1)! u!}{v!},\]

forbidden arguments excluded. From (11) with \(v = k\), \(u = 0\), starting with \(\Omega(2k)\), [or directly]

\[\left(\frac{-\frac{1}{2}}{k}\right) = (-1)^k \binom{k-\frac{1}{2}}{k} = (-1)^k \frac{(k-\frac{1}{2})!}{k! (-\frac{1}{2})!} =
\]

\[(-1)^k \frac{(2k)!}{k! (2k)!} = (-1)^k \frac{(-k)!}{k! k!}, \quad k \in \mathbb{N}_0.\]
By

In the same way as (13), with (12) or (10)

\[(14) \left( -\frac{3}{2} \right)^k \binom{k+1}{k} = (-1)^k \frac{(-k)}{k} (2k+1) \binom{2k}{k}, \quad k \in \mathbb{N}_0. \]

[or from (13) and D(14)].

From (11), (12) and D(14) we have

\[(16) \left( \frac{v - \frac{1}{2}}{k} \right) = y^k \frac{(2v)! (v-k)!}{k! (2v-2k)! v!} = \]

\[= y^k \left( \frac{2v}{k} \right) \binom{2v-k}{k} (v)^{-1} = y^k \left( \frac{2k}{k} \right) \binom{2v}{2k} \binom{v}{k}, \]

\[(17) \left( \frac{v + \frac{1}{2}}{k} \right) = y^k \frac{(2v+1)! (v-k)!}{k! (2v-2k+1)! v!} = \]

\[= y^k \left( \frac{2v+1}{k} \right) \binom{2v+1-k}{k} (v)^{-1} = y^k \left( \frac{2k}{k} \right) \binom{2v+1}{2k} \binom{v}{k}, \]

forbidden arguments excluded. We may write

(16) and (17) in a form holding for \( v \in \mathbb{C}^* :\)

\[(18) y^k \left( \frac{v}{k} \right) \binom{v - \frac{1}{2}}{k} = \left( \frac{2v}{k} \right) \binom{2v-k}{k} = \left( \frac{2k}{k} \right) \binom{2v}{2k}, \]

\[(19) y^k \left( \frac{v}{k} \right) \binom{v + \frac{1}{2}}{k} = \left( \frac{2v+1}{k} \right) \binom{2v+1-k}{k} = \left( \frac{2k}{k} \right) \binom{2v+1}{2k}. \]
We need the relation

$$\tag{20} (-1)^m (m - \frac{1}{2})! (-m - \frac{1}{2})! = \frac{1}{\pi} \frac{(-\frac{1}{2})!}{\pi} = \frac{1}{\pi}, \ m \in \mathbb{N}_0.$$  

The first equality follows by induction on m, the second one by D(4). The relation (20)

\[ (-1)^{\varepsilon} \frac{4^{-\varepsilon}}{\varepsilon} \binom{2m}{\varepsilon} \binom{2\varepsilon - 2m}{\varepsilon - m}, \ \varepsilon \geq m, \ \varepsilon, m \in \mathbb{N}_0. \]

The first equality follows with (20) by applying D(4) to the binomial coefficients. The second one follows by (13). By a proof similar to the one of (20) and (21) we find, using (14) and (15),

$$\tag{22} (-1)^m (m + \frac{1}{2})! (-m - \frac{3}{2})! = (\frac{1}{2})! \left( -\frac{3}{2} \right)! = -\pi, \ m \in \mathbb{N}_0.$$  

$$\tag{23} (-1)^m \left( \frac{\varepsilon}{m} \right) \left( \frac{\varepsilon + 1/2}{m + 1/2} \right) = \left( -\frac{3/2}{m} \right) \left( \frac{1/2}{\varepsilon - m} \right) =$$  

$$(-1)^{\varepsilon} \frac{4^{-\varepsilon}}{\varepsilon} \frac{2m+1}{2m-2\varepsilon+1} \binom{2m}{\varepsilon} \binom{2\varepsilon - 2m}{\varepsilon - m}, \ \varepsilon \geq m, \ \varepsilon, m \in \mathbb{N}_0.$$  

The relation (23) is found also by replacing m in (20) by m + 1.

Taking \( \varepsilon = 2m + 1 \) in (21) and (23) we find with D(4)
\[(24) \binom{m - \frac{1}{2}}{2m + 1} = (-1)^m \frac{2^{-2m - 1}}{m} \binom{2m}{m}, \quad m \in \mathbb{N}_0.
\]

\[(27) \binom{m - \frac{1}{2}}{2m} = (-1)^m \frac{1}{2^m} \binom{2m}{m}, \quad m \in \mathbb{N}_0.
\]

From (16) and (17) for \(k = v = n\) or from (11), (12) or from \(D(24)\) and (13), (14), respectively.

\[(30) \binom{n + \frac{1}{2}}{n} = (2n + 1) \frac{1}{2^n} \binom{2n}{n}.
\]

We have
\[ \binom{n+1}{2} - \frac{(n+3/2)(n+1/2)}{(2n+2)(2n+1)} \binom{n}{2} = (-1)^n 2^{n} \frac{(2n-1/2)!}{n! (n-1/2)!} = \]

\[ (-1)^{n+1} \frac{-4n-4}{(n+1)(n+1/2)} \frac{(2n+1/2)!}{n! (n-1/2)!} = \]

\[ (-1)^{n+1} \frac{-4(n+1)}{(n+1)! (n+1/2)!} = (-1)^{n+1} \frac{-4(n+1)}{2^{n+1}} \binom{2n+2-1/2}{n+1}. \]
(Cf. Rockett (1981)) and for \( x \notin \{-k, \ldots, -1\}, \)

\[
\binom{2k}{2k} + 1 = \binom{2k}{2k-1},
\]

\[
= \frac{2x}{x+k} \binom{x+k}{2k}, \quad k \in \mathbb{N}_0, \ x \neq -k.
\]
\[
\sum_{x=0}^{k-1} \frac{x(x-1) \cdots (x-k+1)}{(x-k)!} = \left(\frac{x}{x}\right) \sum_{x=0}^{k-1} \frac{1}{x}, \quad k \geq 1,
\]

Following Davis (1968, Ch. 5.2) we define the logarithmic numbers

\[
\lambda_n = \int_0^1 \left(\frac{t}{n}\right) dt.
\]

We do not write \( \lambda_n \), a notation already
justified. Using (45) below and (25), by

\[\sum_{n=0}^{\infty} |z|^n \int_0^1 (|t|) \, dt \leq \sum_{n=0}^{\infty} |z|^n \int_0^1 (t+n-1) \, dt \]

\[= \int_{-1}^{0} \sum_{n=-\infty}^{\infty} |z|^n (t+n-1) \, dt = \int_{-1}^{0} (1-|z|)^{-t} \, dt < \infty.\]

With (31), we find

\[\sum_{n=0}^{\infty} \frac{|z|^n}{n!} \left( \sum_{k=0}^{\infty} \frac{1}{k+1} \right).\]

See also Steffensen (1950), §12.
\( n \geq 1 \), but also for \( n = 0 \). When \( x \leq 0 \)

\[
\binom{-x}{n} = (-1)^n \frac{|x| (|x| + 1) \cdots (|x| + n - 1)}{n!} = \left( \frac{-1}{x} \right)^n \Gamma \left( x \right) \Gamma \left( -x \right),
\]

These relations may be useful for interchanging sum and integral in two sums, such as

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(-x)
\]

(\text{and zero for } n > x \text{ when } x \in \mathbb{N} ). \text{ See Cobson.}
For \( x \in \mathbb{Z} \) the l.h.s. is zero for \( k \) sufficiently large.

From D(24) and Stirling's formula

\[
(2^k) \left( \frac{x-k}{k} \right) = \frac{(2k-x-1)!}{k!(k-x-1)!} \sim (2\pi)^{1/2} k^{-1/2} e^{2k-x-1/2},
\]

as \( k \to \infty \).

Similar to (45) and (46), by D (9), (11),

\[
\left| \binom{x+n}{n} \right| = (x+1)(x+2)\ldots(x+n)/n! \leq \\
(1x)(1x+1)(1x+2)\ldots(1x+n)/n! = \left( \frac{1x+n}{n} \right),
\]

\[
\left| \binom{x+n}{n} \right| = 11\ldots\ldots\ldots\; x\; \ldots = \\
(1x)(1x+1)\ldots(1x+n-1)/n! = \left( \frac{1x+n-1}{n} \right).
\]

See Copson (1965), Ch. 6, §27.)
Jacobi polynomials, may be written as binomial sums, often in more than one way. A number of these sums is listed in our tables. We could refer there to the literature and often do, but for the proofs of the equivalence of the formulas given we also might draw on the formulas and techniques contained in this work. In this chapter we do this and prove a number of formulas for some special functions. These proofs are not new, of course.

**Legendre Polynomials**

See Courant–Hilbert (1931), Rainville (1960),
\[ 2^{-n} \sum_{k \leq n} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \times n^{-2k}. \]

\[ \frac{1}{x^n} \sum_{k=0}^{n} \binom{n}{k} (n-k)(x-1)^{-k} (n-k)(x+1)^{k} \]

\[ 2^{-n} \sum_{k=0}^{n} \binom{n}{k} (x+1)^{k} (x-1)^{-k}. \]

For \( x \) sufficiently small we have

(4) \[ \sum_{n=0}^{\infty} P(x) z^n = (1-2x z + z^2)^{-\frac{1}{2}}. \]

\[ (1-2x z + z^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{2k}{k} y^{-k} (2xz-z^2)^{k} = \]

\[ -\infty < k < \infty \quad L \left( \left. \binom{2k}{k} \right| L \right) \quad i < k \quad k+i. \]
\[(7) \quad P_n(x) = \sum_{2k \leq n} \binom{\frac{n}{2}}{k} x^{n-i} \sum_{i \leq k \leq n/2} \binom{n-i}{n-k} \binom{n-k}{k} x^{-k} = \]
\[ 2^{-n} \sum_{x_i \leq n} (-1)^i \binom{n}{i} x^{n-2i} \sum_{0 \leq k \leq n/2} \frac{(n-k)(n-i)}{k} 2^{n-2k}, \]

where we used D (13), and canceled and rearranged factorials. With (3.411) and (2) the last sum is equal to

\[ 2^{-n} \sum_{k=0}^{n}(\binom{2k}{k}/\binom{2n-2k}{n-k}) x^k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}(k-\frac{1}{2}) x^k. \]
(9) \[ \int f_m(x) f_n(x) \, dx = 0, \quad m \neq n \]

By (2) the coefficient of \( x^n \) in \( P_n(x) \) is \( 2^{-n} \binom{2n}{n} \).

So from (11) and (10)

(12) \[ \int_{-1}^{1} x^n P_n(x) \, dx = 2^n (2n)^{-1} \int_{-1}^{1} P_n^2(x) \, dx =
\frac{2^{n+1} n! n!}{(2n+1)!} \]
HERMITE POLYNOMIALS

These are not always defined in the same

\[ \begin{align*}
(15) & \quad H_n(x) = (-1)^n e^{-x} T_n(x) e^x. \\
& \quad \text{Since } f(w) = \exp(- (x+w)^2) \text{ is an entire} \\
& \quad \text{function, we have } f(0) = e^{-x}.
\end{align*} \]

\[ \begin{align*}
(17) & \quad -n=0 - \infty \sum_{n=0}^\infty (-z)^n f^{(n)}(0)/n! = e^{x^2} f(-z) = \\
& \quad e^{x^2} e^{-(x-z)^2} = e^{2xz-z^2}.
\end{align*} \]
Work was rendered largely obsolete by the development of computer algebra (manipulation of formulas by computers). The definition of a sum was then given by

By recurrence or by the use of auxiliary functions that change sums into telescoping ones. These proofs, however, are not always easily verified by pencil and paper, even for simple sums. Therefore the proofs given here, generally of a dif-

to evaluate a sum in closed, or more manageable form. Or a formula giving a sum in closed form (or an identity between sums) is found in the literature or suggested by a combinatorial argument, and one wants an analytical proof. The second type is generally easier than the first one, e.g. by the possibility of a proof by induction. Taken strictly, this difference is not visible in the present work, since
systematic methods to attack these problems and ad hoc ones. The first ones apply to large classes of sums. The most systematic are the above-mentioned algorithms. We do not discuss them. Other examples (see below) are the application of generating functions, Faà di Bruno's method, and reduction to a unique tour de force.

We try to keep our proofs as simple and elementary as possible, with the result that we do not apply, at least in PART I, the theory of hypergeometric functions and the method of Egorychev.

theory of fibonacci and lucas numbers and related polynomials (Chapters E and F).

Many proofs are done by simple transformations of sums and — an important point — application of identities already proved: the more formulas one already has, the more one may prove. This reduction, sometimes in several steps, might have the effect that the way from basics to final results is rather long.

* But in PART II hypergeometric functions are used
In what follows we discuss, with examples, the methods of finding and proving identities that we used or know of, with emphasis on PART I.

1. Reduction to known or simpler formulas.

What is meant here is reduction of a sum, by a simple transformation, to a sum already present in the list. This may also occur half-way in a proof. Some simple ways are:

Application of \( D(27) \), \( \binom{n}{k} = \binom{n}{n-k} \), \( k=0, \ldots, n \), often in combination with a new summation variable. Examples:

\[(3.1) : \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{n-k} = \binom{2n}{n},\]

with Vandermonde's convolution \( D(26) \).

\[(1.5) : \sum_{k=0}^{m} \binom{2m}{k} = 2^{2m-1} + \frac{1}{2} \binom{2m}{m},\]

From (1.1):

\[2^{2m} = \sum_{k=0}^{2m} \binom{2m}{k} = \sum_{k=0}^{2m} \binom{2m}{k} + \sum_{k=m}^{2m} \binom{2m}{2m-k} = \binom{2m}{m},\]

\[= \sum_{k=0}^{m} \binom{2m}{k} + \sum_{k=0}^{m} \binom{2m}{2k} - \binom{2m}{m} = 2 \sum_{k=0}^{m} \binom{2m}{k} - \binom{2m}{m}.\]
(3.156): \[ \sum_{h=0}^{n} (-1)^{h} \binom{n}{h} \binom{v-2h}{m} = \Delta^n \binom{v-2n+2x}{m} \bigg|_{x=0} \]

with \(G(n,2)\), and this is a special case of (3.154).

Another transformation is application of \(D(2y)\): \(\binom{x}{k} = (-1)^{k} \binom{x+k-1}{k}\). Examples:

(3.506): \[ \sum_{k=0}^{n} \binom{x+k}{k} \binom{y-k}{n-k} = \]

(3.486): \[ \sum_{k=\varepsilon}^{n} \binom{k}{k} \binom{y-k}{n-k} = \sum_{k=\varepsilon}^{n} \binom{k}{k} \binom{y-k}{n-k} = \]

\[ (-1)^{n-\varepsilon} \sum_{k=\varepsilon}^{n} \binom{-\varepsilon-1}{k-\varepsilon} \binom{n-\varepsilon-1}{n-k} = \]

\[ (-1)^{n-\varepsilon} \sum_{h=0}^{n-\varepsilon} \binom{-\varepsilon}{h} \binom{n-\varepsilon-\varepsilon}{n-\varepsilon-h} = (-1)^{n-\varepsilon} \binom{n-\varepsilon-\varepsilon-2}{n-\varepsilon} = \]

\[ (n+1), \text{ with } D(\varepsilon), D(2y), \text{ Vandermonde's convolution } D(26) \text{ and again } D(2y). \]
(3.96) With (3.89) as the first step

\[ \sum_{k=0}^{n} \binom{n}{k} (n-k)^{t} = \sum_{j=0}^{n} \binom{n}{j} \left( \frac{n-x+j}{n-j} \right)^t \]

\[ \sum_{j=0}^{n} (-1)^{-j} \binom{n}{j} \left( \frac{y-x}{n-j} \right)^t \]

which may be useful for small \( y-x \in N \).

A simple technique for more radical transformation of sums is application of \( D(13) \) or \( D(14) \) and canceling and/or reordering factorials. Examples:

Special case of (3.571):

\[ \sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \frac{t}{k+1} = \sum_{k=0}^{n} \frac{(-1)^k}{(n+k)!} \frac{t}{k! (k+1)!} \]

\[ \frac{1}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{n-1} = \frac{(-1)^n}{n} \Delta^n \left[ \binom{n+x}{n-1} \right]_{x=0}^{x=n} = 0, \quad n \geq 1, \]

with \( G(22) \) and \( G(38) \).

(3.142): \[ \sum_{j=0}^{n} \binom{n}{j} (j+1) = \frac{n+1}{n+1} \binom{n+1}{\kappa} \]

For \( 1 \leq \kappa \leq n+1 \) with \( D(12) \) the l.h.s. is equal to

\[ \sum_{j=\kappa-1}^{n} \frac{n!}{(n-j)!} \frac{1}{(j+1-\kappa)!} = \]

\[ \frac{1}{n+1} \binom{n+1}{\kappa} \sum_{j=\kappa-1}^{n} (j+1-\kappa)(j+1) = \]
\[
\frac{1}{n+1} \binom{n+1}{m} \sum_{i=0}^{n-m+1} \binom{n-m+1}{i} (i+1),
\]
and \(3.142\) follows with \((1.66)\) and \((1.1)\).

Note that we restricted the sum with \(D(12)\) to avoid infinite factorials. One also might use \(D(12)\):
\[
\binom{j+1}{k} = \binom{j}{k} + \binom{j}{k-1}
\]

and then \((3.130)\) or \((1.66)\).

For \(k > n+1\) both sides of \((3.142)\) vanish by \(D(12)\). For \(k = 0\) it is equal to \((1.1)\):
\[
(3.76) \sum_{k=-\infty}^{m+n} (-1)^k \binom{2m}{m-k} \binom{2n}{n-k} = \frac{(2m)! (2n)!}{m! n! (m+n)!}.
\]

For \(m < n\) the l.h.s. is equal to
\[
\sum_{h=0}^{2m} (-1)^{h-m} \binom{2m}{2m-h} \binom{2n}{n+m-h} =
\]
\[
\frac{(2m)! (2n)!}{(m+n)!} \sum_{h=0}^{2m} (-1)^{h-m} \binom{m+n}{h} \binom{2m-h}{h},
\]
and \((3.76)\) follows with \((3.67)\).

In Probl. 656, Nieuw Arch, 4th Ser. 1(2), 1983, 256–258 the sum
\[
\binom{yn}{2n} \sum_{i=0}^{n} \binom{2n}{2i} (yn)^{-i}
\]
looks difficult. But since
\[
\binom{yn}{2n} \binom{2n}{2i} (yn)^{-i} = \binom{2i}{2i} \frac{yn}{2i} \binom{2n-2i}{2i},
\]

\[
\binom{yn}{2n} \sum_{i=0}^{n} \binom{2n}{2i} (yn)^{-i} = \binom{2i}{2i} \frac{yn}{2i} \binom{2n-2i}{2i},
\]

\[
\binom{2i}{2i} \frac{yn}{2i} \binom{2n-2i}{2i}.
\]
our sum is a convolution, to be evaluated below, p. M36.

Theoretically the simplest way of reducing a sum to a known formula is recognizing it as a special case of the l.h.s. for the r.h.s. of a known identity. We will meet many occasions of it, e.g. in the following chapters:

The theory of Fibonacci and Lucas numbers and related polynomials (Chapters F and G).

These numbers and polynomials satisfy many identities involving binomial coefficients (which is the reason for including these chapters). Examples:

\[(1.374) = \Phi(68) \sum_{k \leq n} \binom{n-k}{k} x^k = \Phi_n(x),\]

\[(1.374) = \Phi(124) \sum_{h=0}^{m} \frac{2m+1}{m+1+h} \binom{m+1+h}{2h+1} x^{m-h} = \Lambda_{2m+1}(x),\]

where we may use recurrences and generating functions of the sequences of polynomials \(\Phi_n\) and \(\Lambda_n\). One finds \(\Phi(x)\) and \(\Lambda(x)\) for special values of \(x\) in \(\Phi(68), (96) - (99), (144) - (147)\).

The theory of polynomials of convolution type (Chapter C). A sequence \(q_n, n \geq 0\), of polynomials is basic for a so-called delta operator \(Q\) if \(Q q_n = q_{n+1}\), \(n \geq 1\), \(q_0(0) = 0\), \(q_0(x) = 1\). These \(\Phi_n\) sequences and their
related Sheffer sequences satisfy a number of interesting relations, e.g. the convolution identity

\[ \sum_{k=0}^{n} \binom{n}{k} q_n(x) q_{n-k}(y) = q_n(x+y), \quad x, y \in \mathbb{C}, \quad n \in \mathbb{N}. \]

Our interest is in the special cases \( q_n(x) = x^n n! \), \( q_n(x) = \binom{x}{n} \) and the Abel and Gould polynomials \( p_n(x) = x(x+n)^{n-1}/n! \) and \( q_n(x) = \frac{x}{x+n} \binom{x+n}{n} \),

with delta operators \( \Delta, \Delta^a, E^{-a} \Delta \) and \( E^{-a} \).

The general theory then immediately leads to a number of identities with binomial coefficients:

- \((1.71) = (1.192), (1.408) = (1.446) \) and \((3.591) = (3.645) \).

Example:

\[(3.614): \text{With } \Delta(24) \text{ and } C(91) \]

\[
\sum_{k=0}^{n-1} \frac{1}{h^k} \left( \begin{array}{c} n-k \\end{array} \right) \left( \frac{h}{h-k} \right)^{-2} = \\
(1) \sum_{k=0}^{n-1} \frac{-1}{h^k} \left( \begin{array}{c} n-k \\end{array} \right) \left( \frac{-h^k+h^k}{h-1-k} \right) = \\
(1) \frac{(-1-n h b)}{n-1} = \left( \frac{n h b+n-1}{n-1} \right), \quad n \geq 1.
\]

Two examples of specialization in a general summation formula are

\[(1.199): \sum_{k=0}^{\infty} \binom{x+k}{m} \frac{z^k}{k!} = e^z \sum_{j=0}^{m} \binom{x}{m-j} \frac{z^j}{j!}. \]
From (63) with \( f(x) = \binom{x}{m} \) and (38). 

From (58) with \( f(x) = \binom{ax+b}{m} \) one obtains

\[
\frac{1}{x} \binom{x+n}{n}^{-1} \binom{ax+b}{m}, \quad m \leq n, \quad -x \not\in \mathbb{N}_0.
\]

An important special case is recognizing a sum as a hypergeometric function. This in fact is a systematic method for finding sums with binomial coefficients, see e.g. Roy (1987).

2. The complex argument. As already remarked in the Introduction, p. 42, we have good reasons to define \( \binom{x}{m} \) for \( x \in \mathbb{C} \). Then \( \binom{ax+b}{m} \) is a polynomial in \( x \) of degree \( m \) and \( \binom{x}{n}, n \in \mathbb{N}_0, x(x+na)^{-1} \binom{x+na}{n}, n \in \mathbb{N}_0 \), are polynomials of convolution type and the theory of such polynomials in Chapter C immediately gives a number of identities, see p. 47 above.

There is another advantage. It is often useful to write the definition of \( \binom{x}{m} \) as \( x!/(m!(x-m)!) \). To avoid infinite factorials in our formulas we start proofs for \( x \not\in \mathbb{Z} \) and afterwards extend our identities to \( x \in \mathbb{Z} \) by continuity, as far as this makes sense. When
starting with \( x \in \mathbb{Z} \), we would have to be careful to exclude 'forbidden' values. The argument may, however, work the other way round as in the proof of

\[
\sum_{k=0}^{n} (-1)^k \binom{k}{n-k} \binom{2n-x}{n-k} = (-y)^n \binom{\frac{1}{2}x - \frac{1}{2}y}{n}.
\]

Both sides are polynomials in \( x \) of degree \( \leq n \). So it is sufficient to prove (3.70) for \( x=x_1, x_2, \ldots, x_n \). Then by D(12), D(13) and rearranging factorials

\[
\sum_{k=0}^{n} (-1)^k \binom{k}{n-k} \binom{2n-x_1}{n-k} = 
\sum_{k=0}^{n} (-1)^k \binom{k}{n-k} \binom{2n-x_2}{n-k} = 
\[x_1!(2n-x_1)!(n!)^{-2} \sum_{k=0}^{n} (-1)^k \binom{k}{n-k} \binom{n}{n-k},
\]

and the sum is reduced to (3.67). Then B(21).

3. Induction, mostly with respect to the number \( n \) of terms. This looks as a typical example of proving, not finding, binomial sums. However, when looking at the sum for small \( n \), one might guess the answer, especially when the sum contains a free parameter. Examples

\[
\sum_{k=0}^{n} \binom{k}{k} (-1)^k = (n+\beta) \binom{n-\beta}{n}.
\]

For the step \( n \to n+1 \), with \( \beta \notin \mathbb{Z} \), using D(24) and D(14).
\[
\binom{n+\beta}{n} \binom{n-\beta}{n} + \binom{\beta}{n+1} \binom{-\beta}{n+1} = \\
\binom{n+\beta}{n} \binom{n-\beta}{n} + \binom{n-\beta}{n+1} \binom{n+\beta}{n+1} = \\
\frac{(n+\beta)! (n-\beta)!}{n! (\beta-1)! (-\beta-1)!} \left\{ 1 - \frac{1}{\beta^2} + \frac{1}{(n+1)^2} \right\} = \\
\frac{(n+1+\beta)! (n+1-\beta)!}{(n+1)! (n+1)! \beta! (-\beta)!} = \binom{n+1+\beta}{n+1} \binom{n+1-\beta}{n+1}
\]

Induction w.r. to the number \(n\) of terms is easiest when the summand does not depend on \(n\). Otherwise one might try

\[\binom{\beta}{k} = \binom{\beta}{k-1} + \binom{\beta}{k-1}, \quad k \geq 1.\]

Example, with the exceptional notation \(\binom{\lambda}{\gamma} = \frac{x!}{\gamma! (\lambda - \gamma)!}\):

\[\binom{\lambda}{\gamma} = \binom{\lambda - 1}{\gamma} + \binom{\lambda}{\gamma - 1} \quad \text{where} \quad \frac{\lambda}{\gamma} \neq \binom{\lambda}{\gamma}.\]

(3.62): \[\sum_{k=0}^{n} \binom{\lambda}{k} \binom{\nu}{u-k} = \binom{\lambda + \nu}{u} .\]

For the step \(n-1 \rightarrow n \geq 1\) we have with \(D(12),\)

\[\sum_{k=0}^{n} \binom{\lambda}{k} \binom{\nu}{u-k} = \\
\binom{\nu}{u} + \sum_{k=1}^{n} \binom{\lambda - 1}{k} \binom{\nu}{u-k} + \sum_{k=1}^{n} \binom{\lambda - 1}{k-1} \binom{\nu}{u-k} =
\]
\[
\sum_{k=0}^{n-1} \binom{n-1}{k} (v_{n-k}) + \sum_{h=0}^{n-1} \binom{n-1}{h} (v_{n-1-h}) = (v_n + n) - (v_{n-1}) ,
\]

with a generalisation of D(18), that also helps in guessing the result from \( n = 0, 1, 2 \). When the summand depends on \( n \) more strongly, induction becomes more difficult. E.g. in

\[(1.180): \sum_{k=1}^{n} \binom{n}{k} k^1 k n^{-k-1} = 1 , \quad n \geq 1 ,
\]

induction on \( n \) does not seem feasible. If we could replace \( n \) in the summand, once or twice, by a free variable, the situation would be better. Here this is possible. The relation (1.180) is the special case \( x = n \) of

\[(1.183): \sum_{k=0}^{n} \binom{n}{k} k! k x^{-k-1} = \sum_{k=1}^{n} \binom{n}{k-1} k! x^{-k} = 1 - \frac{x}{(n+1)(n+1)!} x^{-n} = 1 - n! (\frac{x-1}{n}) x^{-n} , \quad n \geq 1 ,
\]

that is easily proved by induction on \( n \).

4. Recurrence: The difference with induction is small. One tries to find a recurrence relation w.r. to some integer valued parameter in the sum, and then one has to solve this recurrence. Examples:

\[(1.30): A_n(j) = \sum_{3k+j \leq n} \binom{2n}{n-3k-j} , \quad j = 0, 1, 2 ,
\]
\[(1.31): B_n(i) = \sum_{3k+j \leq n} \left( \frac{2n+1}{n-3k-j} \right), \quad j = 0, 1, 2.\]

With \(D(18)\), for \(n \geq 1\),

\[A_n(0) = \sum_{3k \leq n} \left( \frac{2n-1}{n-3k} \right) + \sum_{3k \leq n-1} \left( \frac{2n-1}{n-3k-1} \right) = \]

\[\sum_{0 \leq 3k \leq n} \left( \frac{2(n-1)+1}{n-3k-1} \right) + \sum_{3k \leq n-1} \left( \frac{2(n-1)+1}{n-3k} \right) = \]

\[\sum_{-3 \leq 3h \leq n-3} \left( \frac{2(n-1)+1}{n-3h-2} \right) + B_{n-1}(0) = \]

\[\left( \frac{2n-1}{n} \right) + B_{n-1}(2) + B_{n-1}(0).\]

In this way a set of six linear first-order joint recurrent relations for the \(A_n(i)\) and \(B_n(i)\), \(i = 0, 1, 2, j = 0, 1, 2\), is obtained. These are solved, fairly easily, in the proof of (1.30) and (1.31). By using the relation obtained directly with (1.5), from the definitions of the \(A_n(i)\):

\[\sum_{j=0}^{n} A_n(i) = \sum_{h=0}^{n} \left( \frac{2n}{n-h} \right) = 2^{n-1} + \frac{1}{2} \left( \frac{2n}{n} \right).\]

\[(2.1): \quad S_n = \sum_{k=0}^{n} \left( \frac{n}{k} \right)^{-1} = (n+1) \sum_{j=0}^{n} 2^{j-n}/(j+1).\]

From \(B(32)\):
\[ S_{n+1} = 1 + \frac{\sum_{k=1}^{n+1} (n+1)^{k-1}}{(k-1)} - \sum_{k=1}^{n+1} \frac{n+1-k}{n+2-k} \left( \frac{n+1}{k-1} \right)^{1} \]

\[ = 1 + \sum_{h=0}^{n} \left( \frac{1}{h} \right)^{1} - \sum_{h=0}^{n} \left( 1 - \frac{1}{n+1-h} \right) \left( \frac{n+1}{h} \right)^{1} \]

\[ = 1 + \left( \frac{S_{n+1}}{n+1} \right) - \sum_{h=0}^{n} \left( \frac{n+1}{h} \right)^{1} \]

\[ = 2 + \frac{n+2}{n+1} S_{n} - S_{n+1}, \quad \text{or} \]

\[ S_{n+1} = 1 + \frac{1}{2} \frac{n+2}{n+1} S_{n}, \]

which may be solved with \( S_{0} = 1 \) or applied to prove (2.1) by induction.

Put

\[ S_{nm} = \sum_{k=0}^{\infty} \left( \frac{n+a(k)}{m+b(k)} \right) \delta(k), \]

where \( a(k) \in \mathbb{N}, \ b(k) \in \mathbb{N} \) and \( b(k) - a(k) \to \infty \) as \( k \to \infty \), so that the sum contains only finitely many nonzero terms. From (2.8) we see that

\[ S_{nm} = S_{n-1, m} + S_{n-1, m-1}, \quad n \geq 1, \ m \geq 1, \]

the same recurrence as is satisfied by the binomial coefficients \( \binom{n}{m} \). On its application see pp. 27–29.
5. Finite differences. The \( n \)th difference of a function \( f: \mathbb{C} \to \mathbb{C} \) is given by

\[
G(\Delta^n) f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k) = \sum_{h=0}^{n} (-1)^{h} \binom{n}{h} f(x+n-h), \quad \text{See Chapter G.}
\]

The simplest derivation is by operators

1. \( E^a f(x) = f(x+a) \), \( E = E_1 \), \( I f = f \), \( \Delta = E - I \),

so that \( \Delta^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} E^k \).

In this way we find with \( G(38) \) - or iteration of \( D(18) \),

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{x+k}{m} = \binom{x}{m-n}, \quad n \leq m,
\]

\[= 0, \quad n > m.\]

By repeatedly applying \( \Delta \) one finds

\[
G(47) \Delta^n x^{-1} = (-1)^n n! \ x^{-1} (x+1)^{-1} \ldots (x+n)^{-1} = (-1)^n x^{-1} \binom{x+n}{n}^{-1}, \quad -x \not\in \mathbb{N}_0, \quad \text{so}
\]

\[
G(1.100) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^{-1} = x^{-1} \binom{x+n}{n}^{-1}, \quad -x \not\in \mathbb{N}_0.
\]

The operators in (1) offer quick derivations of some formulas with binomial coefficients:

Let \( g \) be a polynomial of degree \( m \). Then
\((E^b - I)^n g\) is a polynomial of degree \(m\) when \(m = 0\). Then

\[
G(25^n) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} g(x + kb) = (E^b - I)^n g(lx) = 0, \quad m < n,
\]

We have \(E^a - I = (E + I)^{\Delta a}\),

\[
(E^a - I)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} E^{2k},
\]

\[
(E + I)^n \Delta^n = \sum_{j=0}^{n} \binom{n}{j} E^j \Delta^n,
\]
giving the general identity

\[
G(29) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + 2k) = \sum_{j=0}^{n} \binom{n}{j} \Delta^n f(x + j).
\]

Many general identities involving the operators \((i)\), from which sums with binomial coefficients may be found, are collected in Chapter G. When a summand starts with \((-1)^k \binom{n}{k}\), one may suspect a \(n\)th finite difference.

A different application of the operators \((i)\) is in Chapters F and G on Fibonacci and Lucas numbers and related polynomials. Let \(E\) operate on sequences \(x_n\), \(n \in \mathbb{N}\) or \(n \in \mathbb{Z}\) as \(Ex_n = x_{n+1}\). Let the sequence \(A_n\) satisfy the Fibonacci recurrence \(A_{n+2} = A_{n+1} + A_n\), so that \((I + E)A_n = E^2 A_n\). Then
\[ \sum_{k=0}^{\infty} \binom{r}{k} A_{n+k} = (1+E)^{r} A_{n} = E^{r} A_{n} = A_{n+r} \]

For a lot of such identities see \( F(48) = \{ 6e^a \} \), \( F(127), (128), \Phi(103) - (115^a), \Phi(198), (199) \).

As an example, of \( G(22) \) combined with rearranging and canceling factorials by \( D(14) \) we have, using \( G(38) \)

\[
(3.53) \sum_{i=0}^{\infty} (-1)^{i} \binom{r}{i} (x-i)(x-i') = \\
\sum_{i=0}^{\infty} (-1)^{i} \binom{r}{i} (x-\varepsilon+i-i) = \Delta^{r} \binom{x}{r} = 1. 
\]

6. **Newton's interpolation formula and its generalizations.**

Let \( f \) be a polynomial of degree \( \leq n \). Then

\[
G(41) \quad f(x+y) = \sum_{k=0}^{n} c_{k}(y) \binom{x}{k}, 
\]

where the \( c_{k}(y) \) are determined uniquely and are given by

\[
(2) \quad c_{k}(y) = \Delta^{r} f(y) = \sum_{j=0}^{r} (-1)^{j} \Delta^{j} f(y+j). 
\]

Also, more generally,

\[
G(42) \quad f(x+y) = \sum_{k=0}^{n} d_{k}(y) \frac{x}{x+k} \binom{x+k}{k}, 
\]

\[
(3) \quad d_{k}(y) = E^{-ra} \Delta^{r} f(y) = \Delta^{r} f(y-\varepsilon a). 
\]
These formulas are special cases of the expansion theorem \( C(24) \) for basic sequences belonging to delta operators.

This works in two ways. For given \( f \), we obtain an identity \( G(41) \) when we can evaluate \( (2) \). And when a special case of \( G(41) \) is given, the relation \( (2) \) proves an identity with binomial coefficients. Examples:

\[
(1.56): \quad \sum_{k=0}^{m-1} (-2)^{m-k} \binom{x}{k} = \sum_{2k \leq m} \binom{x-2-2k}{m-2k}.
\]

The r.h.s. is a polynomial in \( x \) of degree \( \leq m \), so we have \( G(41) \) for \( y = 0 \) with

\[
\binom{C}{x} (0) = \prod_{2k \leq m} \left( \frac{x-2-2k}{m-2k} \right) = \sum_{x=0}^{\infty} \binom{-x-2-2k}{m-2k-2k} = (-1)^{m-x} \sum_{2k \leq m-2} \binom{m-x+1}{2k+1} = (-2)^{m-x},
\]

where we applied \( G(38), D(24), D(27) \) and \( (1.16) \).

We have:

\[
(3.149): \quad \sum_{k=0}^{n} (-1)^{n-k} \binom{x}{k} = \binom{x-x-1}{n-x}, \quad x \leq n,
\]

The relation \( (2) \) then gives, for \( m \leq n, x \leq n \), with \( G(22) \):

\[
\sum_{j=0}^{m} (-1)^{j} \binom{m}{j} \binom{j-1}{n-2} = (-1)^{n} \binom{m}{n-2}.
\]

At first sight this formula does not look simple. It is, however, trivial. For
For \( m > n \) both sides are zero by D(12). For \( n \leq m \) the only term in the L.H.S. that does not vanish is \( j = x \) by D(12) since \( j \leq m \leq n \). But then we have a proof of (3.49).

The identity

\[
(3.61) \sum_{3h \leq n} \binom{x}{h} \binom{-x}{n-3h} = (-1)^n \sum_{k=0}^{n} \binom{x}{k} \binom{t}{n-k}
\]

was proved by a simple generating function argument. The above relation (2) gives

\[
(3) \quad \Delta \sum_{3h \leq n} \binom{x}{h} \binom{-x}{n-3h} \bigg|_{x=0} = (-1)^n \binom{x}{n}, \quad x \leq n.
\]

With C(28a)

\[
\Delta \sum_{3h \leq n} \binom{x}{h} \binom{-x}{n-3h} = \sum_{3h \leq n} \Delta^x \binom{x}{h} \binom{-x}{n-3h} \Delta^x \binom{-x}{n-3h}.
\]

Here \( \Delta^x \binom{x}{h} = 0 \) for \( j \neq h \) when \( x = 0 \). When \( h > x \) the term \( j = h \) is absent and this is accounted for by the factor \( \binom{x}{h} \) in

\[
\Delta \sum_{3h \leq n} \binom{x}{h} \binom{-x}{n-3h} \bigg|_{x=0} = \sum_{3h \leq n} \binom{x}{h} \Delta^x \binom{-x}{n-3h} \bigg|_{x=0} =
\]
\[ \sum_{h \leq \frac{n - \varepsilon}{3}} \left( \frac{\varepsilon}{h(1)} \right) = \left( \frac{\varepsilon}{(n - \varepsilon - 2h)} \right), \]

where we used \( \xi(3b) \). So from (3b)

\[ \sum_{h \leq \frac{n - \varepsilon}{3}} (-1)^{n-h} \left( \frac{\varepsilon}{h} \right) \left( \frac{\varepsilon}{n - \varepsilon - 2h} \right) = (-1)^{n} \left( \frac{\varepsilon}{n - \varepsilon} \right), \varepsilon \leq n. \]

This is nearly identical with (3.77), except for the bound \( h \leq n/3 \). This only makes a difference when \( \varepsilon \leq n/3 \) and then both sides are zero.
7. Inverse relations. See Chapter IR.

Let the lower triangular matrices $A_{nk}$, $n, k \in \mathbb{N}_0$, and $B_{nk}$, $n, k \in \mathbb{N}_0$, be each other's inverses. If then

\[(4) \quad Y_n = \sum_{k=0}^{n} a_{nk} \times_k, \quad n \in \mathbb{N}_0,\]

for two sequences $\{x_k, n \in \mathbb{N}_0\}$ and $\{y_k, k \in \mathbb{N}_0\}$, we also have

\[(5) \quad X_n = \sum_{k=0}^{n} b_{nk} \times_k, \quad n \in \mathbb{N}_0.\]

Two (general) relations of the type (4) and (5) are called inverse relations or as also the corresponding matrices - an inverse pair. Using inverse pairs one may find new formulas as from known ones, as described on p. IR7.

Examples:

From IR (83) with $u=1$, $b=2$ and $D(24)$ we have the inverse pair

\[a_{nk} = \binom{2n+1}{n-k}, \quad b_{nk} = \frac{2n+1}{2k+1} \binom{2n+1}{n-k} = (-1)^{n-k} \frac{2n+1}{2k+1} \binom{n+k}{k},\]

$k \leq n$. Now write (1.6) in the form

\[\sum_{k=0}^{n} a_{nk} \times_k = Y_n = Y^n, \quad \text{with } x_k = 1.\]

Then from (5)

\[(1.348) \quad \sum_{k=0}^{n} (-1)^{n-k} \frac{2n+1}{2k+1} \binom{n+k}{k} y^k = 1.\]
Write (3.77) with \( D(2y) \) as
\[
\sum_{2k \leq n} \binom{-x+k-1}{k} x_{n-2k} = y_n = (-1)^n \binom{x}{n},
\]
with \( x_j = (-x) \). From the inverse pair
\( IR(71) \) with \( z = x \), \( u = -x-1 \),
\[
y_n = \sum_{2k \leq n} \binom{-x-1+k}{k} x_{n-2k},
\]
\[
x_n = \sum_{2k \leq n} (-1)^k \binom{-x}{k} y_{n-2k} = \sum_{2k \leq n} \binom{x+k-1}{k} y_{n-2k},
\]
that is written in a form different from (4) and (5), see \( IR(8), (9) \) and \( IR(65)-(69) \),
we then have
\[
\binom{-x}{n} = \sum_{2k \leq n} \binom{x+k-1}{k} (-1)^k \binom{x}{n-2k}, \quad \text{or}
\]
\[
\sum_{2k \leq n} \binom{x+k-1}{k} \binom{x}{n-2k} = \binom{x+n-1}{n}.
\]
This companion \( IR(7) \) of (3.77) is (3.360).
By considering the transposes of a pair of inverse matrices we proved, in Chapter IR, the equivalence of the relations

\begin{align*}
IR(16a) \quad & Y_n = \sum_{k=n}^{N} \binom{k}{n} x_k, \quad n = 0, \ldots, N, \\
IR(16b) \quad & x_n = \sum_{k=n}^{N} (-1)^{k-n} \binom{k}{n} Y_k, \quad n = 0, \ldots, N,
\end{align*}

for fixed \( N \). Write (3.133) as

\[ \sum_{k=n}^{N} \binom{N+1}{k} \binom{k}{n} 2^{-N} = \binom{-n}{N-n}, \quad n = 0, \ldots, N, \]

or

\[ \sum_{k=n}^{N} \binom{k}{n} x_k = Y_n = \binom{-n}{N-n}, \quad n = 0, \ldots, N, \]

with \( x_k = \binom{N+1}{2k+1} 2^{-N} \). Then from \( IR(16b) \)

\[ \sum_{k=n}^{N} (-1)^{k-n} \binom{k}{n} \binom{N-k}{k} 4^{-k} = 2^{-N} \binom{N+1}{2n+1}, \]

\[ n = 0, \ldots, N, \quad \text{which is (3.496)}. \]

8. Inclusion-exclusion. Let \( X \) be a random variable with values in \( \{0, \ldots, N\} \). Then we have the relations \( IR(17a) - (17b) \):

\begin{align*}
IR(17a) \quad & E\left\{ \binom{X}{n} \right\} = \sum_{k=n}^{N} \binom{k}{n} P(X = k), \quad n \leq N, \\
IR(17b) \quad & P(X = n) = \sum_{k=n}^{N} (-1)^{k-n} \binom{k}{n} E\left\{ \binom{X}{k} \right\}, \quad n \leq N,
\end{align*}
\[
IR(17^c) \mathbb{P}(X \geq n) = \sum_{k=n}^{-N} (-1)^{k-n} \binom{k-1}{k-n} E\left\{ \binom{X}{k} \right\}, \quad n \leq N,
\]
\[
IR(17^d) \sum_{n=0}^{N} \mathbb{P}(X=n) z^n = \sum_{n=0}^{N} E\left\{ \binom{X}{k} \right\} (2z-1)^k.
\]

Let \( A_1, \ldots, A_N \) be events on the same probability space. When \( X \) is the number of these events that occur, we have
\[
IR(17^e) E\left\{ \binom{X}{k} \right\} = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}),
\]
\[
1 \leq k \leq N, \quad E\left\{ \binom{X}{0} \right\} = 1.
\]

The relations \( IR(17^a) \) and \( IR(17^b) \) are a special case of \( IR(16^a) \) and \( IR(16^b) \) above.

When we compute \( \mathbb{P}(X=n) \) and \( E\left\{ \binom{X}{k} \right\} \) directly, the latter possibly by \( IR(17^a) \), then \( IR(17^b) \) and \( IR(17^c) \) provide us with probabilistic proofs of two identities. An example is the probabilistic proof of (3.181). The analytical proof is much simpler.

Also, let \( X \) have the hypergeometric distribution with parameters \( a, b, n \in \mathbb{N}_0 \), \( n \leq a+b \). Then, see (3.26a),
\[
\mathbb{P}(X=r) = \binom{a}{r} \binom{b}{n-r} \binom{a+b}{n}^{-1}, \quad r = 0, \ldots, n
\]
and with (3.111), for \( k \leq n \),
\[
E\left\{ \binom{X}{k} \right\} = \binom{a}{k} \frac{(a+b-k)(a+b)}{n}^{-1}.
\]
So from T(14) with $N, n$ replaced by $n, e$

$$(r)(n-r) = \sum_{k=r}^{n} (-1)^{k-r} (k)(a)(a+b-k).$$

Applying $D(14)$ to $(k)(a)$ and putting $k-e = h$, we obtain a special case of (3.365).

Similarly, from T(17c),

$$\sum_{j=r}^{n} (j)(n-j) = \sum_{k=r}^{n} (-1)^{k-r} (k-r)(k)(a+b-k) =$$

$$r(a)(a-b) \sum_{k=r}^{n} (-1)^{k-r} (k-r)(a+b-k) \frac{1}{k},$$

the r.h.s. only for $r \geq 1$.

The above identities may be extended to $a, b \in \mathbb{C}$, since all members are polynomials in $a$ and $b$.

9. Multisection of sums. This method discussed in Chapter C, pp. 63–65, sums the terms with index $j [mod d]$ of a sum ($j = 0, \ldots, d-1$). Most applications are to $d = 2$, i.e., summing the terms with even index and the terms with odd index. We mainly apply the method to power series.

When $\sum_{k=0}^{n} a(n, k) = f(n)$ and $\sum_{k=0}^{n} (-1)^{k} a(n, k) = g(n)$, we have

$G(n) = \sum_{2h \leq n} a(n, 2h) = f(n) + g(n)$.
\[ G(n) : \sum_{2h+1 \leq n} \left( a(n, 2h+1) = f(n) = g(n) \right). \]

When \( \sum_{k=0}^{\infty} c_k z^k = F(z) \), with absolute convergence, then also with absolute convergence,

\[ G(8) : \sum_{h=0}^{\infty} c_{2h} z^{2h} = F(z) + F(-z), \]

\[ G(9) : \sum_{h=0}^{\infty} c_{2h+1} z^{2h+1} = F(z) - F(-z). \]

Examples. From the binomial formula

\[ (6) \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k} = (u+v)^n \]

we obtain with \( G(10) \) and \( G(11) \)

\[ (150a) \sum_{2j \leq n} \binom{n}{2j} u^{2j} v^{n-2j} = (u+v)^n + (v-u)^n, \]

\[ (150b) \sum_{2j+1 \leq n} \binom{n}{2j+1} u^{2j+1} v^{n-2j-1} = (u+v)^n - (v-u)^n. \]

Differentiating \((150a)\) and \((150b)\) w.r. to \( u \)

and then putting \( u=v=1 \) one derives

\[ (172) \sum_{2j \leq n} \binom{n}{2j} j = n 2^{n-3}, \quad n \geq 2, \]

\[ (174) \sum_{2j+1 \leq n} \binom{n}{2j+1} j = (n-2) 2^{n-3}, \quad n \geq 2, \]

which also may be derived by differentiating

\( (6) \) and then bisecting. For \((174)\) also \((1.16)\) needed.
From (1.157) with \( x = 1 \):

\[
\sum_{k=1}^{n} \binom{n}{k} \frac{z^k}{k} = \sum_{j=1}^{n} \frac{1}{j} \left\{ (1+z)^j - 1 \right\},
\]

we obtain with (10):

\[
(1.156) \sum_{1 \leq h \leq \frac{1}{2} n} \binom{n}{2h} z^{2h} / (2h) =
\sum_{j=2}^{n} \frac{1}{2^j} \left\{ (1+z)^j + (1-z)^j - 2 \right\}, \quad n \geq 2.
\]

From Vandermonde's convolution (26):

\[
\sum_{k=0}^{n} \binom{x}{k} \binom{2n-x}{n-k} = \binom{2n}{n},
\]

and

\[
(3.70) \sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{2n-x}{n-k} = (-1)^n \binom{1/2, x - 1/2}{n}
\]

we obtain (3.58) and (3.59).

For another example see p. M 36.

10. Expansion of a factor in the summand.

We then apply a known formula to a factor in the summand, so that we have a double sum. The only thing left then is changing the summation order. Example:

\[
(3.181) \sum_{h=0}^{M} \binom{M}{h} \binom{x}{h} z^{M-h} = \sum_{\xi=0}^{M} \binom{M}{\xi} (\xi + x) (z-1)^\xi.
\]
We write the r.h.s. as
\[ \sum_{k=0}^{M} \binom{M}{k} \binom{k+x}{k} \binom{-1}{n-x} (z-1)^{M-k} \]

If here \(k+x\) could be simplified to \(k\), we could proceed. We therefore apply Vandermonde's convolution \(\mathcal{D}(26)\) to the second factor. The r.h.s. then equals
\[ \sum_{k=0}^{M} \binom{M}{k} \sum_{i=0}^{n} \binom{k}{i} (n-i) (z-1)^{M-k} = \]
\[ \sum_{i=0}^{n} \binom{x}{n-i} \sum_{k=0}^{M} \binom{M}{k} \binom{k}{i} (z-1)^{M-k} = \]
\[ \sum_{i=0}^{n} \binom{x}{n-i} \binom{-1}{i} (z-1)^{M-i} \text{ with } (3.176) \]

Another example is \((3.189)\) where the l.h.s. contains \((y-1)^i\) and the r.h.s. powers of \(y\). Our proof expands \((y-1)^i\) by the binomial theorem and interchanges summations.

11. The Beta integral. This integral,
\[ \mathcal{D}(33) \int_{0}^{1} t^{\alpha-1} (1-t)^{\beta-1} \, dt = \Gamma(\alpha) \Gamma(\beta) / \Gamma(\alpha+\beta) \]
Re \(\alpha > 0\), Re \(\beta > 0\), together with \(\mathcal{D}(16)\), \(z! = \Gamma(z+1)\), and the definition \(\mathcal{D}(14)\) of Binomial coefficients, will be used mainly to represent inverses of binomial coefficients, but also sometimes of special binomial coefficients themselves.
(35) \( D(35) \binom{2n}{n} = 4^n \pi^{-1} \int_0^1 t^{n-\frac{1}{2}} (1-t)^{-\frac{3}{2}} dt \).

Examples:

(2.43): \( (1+x)^{-1} \sum_{n=0}^{\infty} \left( \frac{x}{1+x} \right)^n \binom{a+n+m}{n+m} \)

\[ = a (1+x)^{-1} \sum_{n=0}^{\infty} \left( \frac{x}{1+x} \right)^n \int_0^1 t^{n+m} (1-t)^{a-1} dt = \]

\[ = a \int_0^1 t^m (1+x-tx)^{-1} (1-t)^{a-1} dt = \]

\[ = a \sum_{k=0}^{\infty} (-1)^k x^k \int_0^1 t^m (1-t)^{a+k-1} dt = \]

\[ = \sum_{k=0}^{\infty} (-1)^k x^k \frac{a}{a+k} \left[ \binom{a+k+m}{m} \right]^{-1}, \]

first for \( \text{Re} a > 0 \) and \( x \) so small that twice we may interchange sum and integral. The l.h.s. converges absolutely for \( \text{Re} x > -\frac{1}{2} \) and the r.h.s. for \( |x| < 1 \). We may extend (2.43) by analyticity to the common domain of absolute convergence.

Similarly, we may extend the \( a \)-domain.

For \( x \notin \mathbb{Z} \), \( \text{Re} x > \varepsilon - 1 \), see (4.4)

\[ \sum_{k=0}^{n} \binom{n}{k} \binom{n+x}{n+k}^{-1} = \]

\[ = (n+x+1) \sum_{k=0}^{n} \binom{n}{k} \int_0^1 t^{x+k} (1-t)^{n+x+\varepsilon-k} dt = \]
\[ (n+x+1) \int_0^1 t^r (1-t)^{x-r} \, dt = (n+x+1) \frac{\Gamma(r+1) \Gamma(x+r+1)}{\Gamma(x+2)} = \frac{n+x+1}{r+1} \frac{(x+r)!}{(x+1)!} = \frac{n+x+1}{r+1} \frac{(x+1)^r}{(x+1)!} \]

This identity may be extended by analyticity, but one has to be careful when \( x \in \mathbb{Z} \).

We may work the other way round in \( D(33) \) by expanding \((-t)^{x-1}\). See (1.131).


A systematic approach to evaluating sums with binomial coefficients is taking the generating function w.r. to some parameter with values in \( \mathbb{N}_0 \), mostly the number of terms. This is called the Snake Oil method by Wilf (1990) and just like snake oil it sometimes is unsuccessful.

The idea is as follows. A sequence \( a_n \), \( n \in \mathbb{N}_0 \), determines the function

\[ A(z) = \sum_{n=0}^{\infty} a_n z^n, \]

called the generating function of the sequence \( a_n \) and from the function \( A \) we may recover the sequence \( a_n \), e.g. by...
(8) \[ a_n = \frac{D^n A(0)}{n!}. \]

So when there is an unknown sequence and we somehow find its generating function, we may, in principle, determine the sequence. By our use of the term ‘function’ of the complex variable \( z \) we tacitly assumed that the power series (7) has a positive convergence radius. As seen by (8), it is not needed to know this radius, as long as it is positive. Our computations with generating functions have to hold "for sufficiently small \( z \)."

Studying generating functions may be done without any thought of convergence; we then are dealing with so-called formal power series. See Niven (1969), Wilf (1990). Strictly, what is done there is beautifully illustrated manipulation of sequences.

Formal power series are not used here. All our generating functions will have positive convergence radius. Convergence proofs often will be given, in many cases combined with justification of changing summations (see p. 1131 below). Easy proofs may be omitted.

When operating with generating functions we apply simple standard properties of power series and analytic functions. Two of these we mention here:

Since a power series may be differentiated term by term inside its convergence...
circle, we conclude from (7) with \( D(g) \) and \( D(h) \):

\[
(9) \quad \frac{1}{m!} D^m A(z) = \frac{1}{m!} \sum_{k=m}^{\infty} a_k (k)_m z^{k-m} = \\
\sum_{h=0}^{\infty} a_{h+m} \binom{h+m}{m} z^h = \sum_{h=0}^{\infty} a_{h+m} \binom{h}{h} z^h.
\]

Very often we will need the theorem on the product of two power series:

**Theorem 1.** Let \( A(z) = \sum_{k=0}^{\infty} a_k z^k \) and \( B(z) = \sum_{k=0}^{\infty} b_k z^k \) be power series with convergence radii \( \alpha > 0 \) and \( \beta > 0 \), respectively. Then

\[
(10) \quad A(z) B(z) = \sum_{n=0}^{\infty} c_n z^n,
\]

with convergence radius at least \( \alpha \beta \) and

\[
(11) \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad n \in \mathbb{N}_0.
\]

**Remark 1.** This assertion also may be stated as follows: Let the sequence \( c_n \) be given by (11). Then (10) shows that its generating function is \( A(z) B(z) \).

**Remark 2.** The sequence \( c_n \) in (11) is called the convolution of the sequences \( a_n, \quad n \in \mathbb{N}_0, \) and \( b_n, \quad n \in \mathbb{N}_0, \) and is denoted \( (a*b)_n, \quad n \in \mathbb{N}_0, \) see p. 74. Theorem 1 will be called here the convolution (or product) property of generating functions (or of power series).
Our most frequent application of generating functions is in finding sums like $S_n^* = \sum_{k=0}^{\infty} a_{nk}$. We then consider the generating function of the sequence $S_n^*$:

$$\sum_{n=0}^{\infty} S_n^* z^n = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{\infty} a_{nk} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} z^n a_{nk}.$$ 

Since we do not use formal power series, the last step, interchanging summations (the only thing left), has to be justified. Interchanging is allowed when $a_{nk} > 0$ and $z \geq 0$ and also when

$$\sum_{k=0}^{\infty} \sum_{n=k}^{\infty} |z^n a_{nk}| = \sum_{n=0}^{\infty} |z|^n \sum_{k=0}^{\infty} |a_{nk}| < \infty.$$ 

This is a consequence of

**Lemma.** Let $A_1, A_2, \ldots$ be a partition of the countable set $A$ and let $x_{\alpha} \in C_{\alpha} \in A$. Then

$$\sum_{\alpha \in A} x_{\alpha} = \sum_{i=1}^{\infty} \sum_{\alpha \in A_i} x_{\alpha}.$$ 

when $x_{\alpha} \geq 0$, $\alpha \in A$, and then both sides of (13) may be $\pm \infty$ and also when

$$\sum_{\alpha \in A} |x_{\alpha}| < \infty.$$ 

One may verify (14) by using (13) with $|x_{\alpha}|$. The lemma also justifies changes.
such as

\[(15) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x^i y^j = \sum_{n=0}^{\infty} \sum_{i+j=n} x^i y^j.\]

**Examples:**

\[(1.126): S_n(a, x) = \sum_{k=0}^{n} \binom{n+x}{n-k} \frac{a^k}{k+1} = \sum_{k=0}^{n} \frac{(a+1)^{k+1} - 1}{(k+1)a} \binom{x+n-k-1}{n-k}.\]

For small \(z\) with \(D(25)\),

\[\sum_{n=0}^{\infty} z^n S_n(a, x) = \sum_{k=0}^{\infty} \frac{a^k}{k+1} \sum_{n=k}^{\infty} \binom{n+x}{n-k} z^n = \sum_{k=0}^{\infty} \frac{a^k z^k}{k+1} \sum_{m=0}^{\infty} \binom{m+k+x}{m} z^m =\]

\[= \frac{(1-z)^{-x}}{az} \log \frac{1-z}{1-(a+1)z}.\]

\[(1.127): \sum_{k=0}^{\infty} \frac{(a+1)^{k+1} - 1}{(k+1)a} z^k,\]

and \((1.126)\) follows with Theorem 1 and \(D(25)\).

Changing the summation order is justified by \((12)\) since \(x = B(50)\) and \(D(25)\)

\[\sum_{n=0}^{\infty} \frac{1}{z^n} \sum_{k=0}^{n} \frac{a^k}{k+1} \binom{n+x}{n-k} = \sum_{m=0}^{\infty} \frac{1}{z^m} \sum_{k=0}^{\infty} \binom{m+k+x}{m} z^m =\]

\[= \sum_{k=0}^{\infty} \frac{1}{z^k} \sum_{m=0}^{\infty} \binom{m+k+x}{m} z^m.\]
\[ \sum_{k=0}^{\infty} \frac{|az|^k}{k+1} \sum_{m=0}^{\infty} \frac{(m+k+1|x|)}{m} |z|^m = \]
\[ \sum_{k=0}^{\infty} \frac{|az|^k}{k+1} (1-|z|)^{-k-1} |z| < \infty \]

for sufficiently small z. Or we could take \(x \geq 0, \delta \geq 0, \gamma \geq 0\) and afterwards extend (1.126) to \(x \in \mathbb{C}, \gamma \in \mathbb{C}\) since both sides are polynomials in \(a\) and \(x\).

A shorter proof with a variant of Newton's interpolation formula is given in Chapter 1P.

(1.45.2) \[ S_{nm} = \sum_{j} \binom{n+ja}{m+jc} \]

where \(a \in \mathbb{N}_0, c \in \mathbb{N}_0, x \in \mathbb{R}^+\) and the sum is over all nonzero terms, i.e. \(0 \leq j \leq (n-m)(c-a)^{-1}\).

For \(n \leq m\) the sum is empty. For small \(z \to 0\), with \(D(25)\)

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} z^n \sum_{j=0}^{\infty} \frac{z^m}{n+m+j(c-a)} \binom{n+ja}{m+jc} = \]
\[ \sum_{j=0}^{\infty} (z+c+j(c-a)) \sum_{k=0}^{\infty} \frac{z^k}{k+m+jc} \]
\[ \sum_{j=0}^{\infty} \frac{z^{j+1}}{(1-z)^{m-jc+1}} \]
\[ z^m (1-z)^{c-m-1} \left\{ (1-z)^c - z^{c-a} \right\}^{-1}, \]

showing that only for special \(m, a, c\) the sum admits simple expressions. For a recurrence satisfied by the \(S_{nm}\) see pp. 27-29.
\[(3.155) \quad \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2k}{m} = (n)_{-m}^{2n-m} = 0, \quad n > m.\]

Here we take the generating function w.r. to the parameter \(m\):
\[
\sum_{m=0}^{\infty} z^m \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2k}{m} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1+z)^{2k} = (z + 2z)^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} z^m = \sum_{m=n}^{\infty} \binom{n}{m-n} 2^{2n-m} z^m.
\]

\[(3.550) \quad \sum_{2k \leq n} (-1)^k \binom{x+k}{k} \binom{n-k}{k} 2^{n-2k} = \sum_{2j \leq n} (-1)^j \binom{x}{j} \binom{n+1}{j+1} = \sum_{k=0}^{n} \frac{2^{x+k+1}}{k} \binom{x}{k} (-2)^{n-k}.
\]

The author saw this relation with \(x=n\) as (25) in Gould (1972). When \(x=n\) in (3.550), taking the generating function w.r. to \(n\) is of no use. In general we have to take care that:

Not too much \(n\) in a sum when taking the generating function w.r. to \(n\).

For the proof of (3.550), from D (25)
\[ \sum_{n=0}^{\infty} \sum_{2k \leq n} (-1)^k \binom{n-k}{k} \binom{n-k}{k} \frac{z^{n-2k}}{2^k} = \]
\[ \sum_{k=0}^{\infty} (-1)^k \binom{x+k}{k} \sum_{m=0}^{\infty} \binom{m+k}{m} z^m \frac{z^{m+2k}}{2^k} = \]
\[ \sum_{k=0}^{\infty} (-1)^k z^{2k} \binom{x+k}{k} \left(1 - 2z\right)^{-k-1} = \]
\[ \left(1 - 2z\right)^{-1} \left[1 + z^{-2} \left(1 - 2z\right)^{-1}\right]^{-x/x} = \left(1 - 2z\right)^x \left(1 - z\right)^{-2x-x-2}, \]
\[ \sum_{n=0}^{\infty} z^n \sum_{2j \leq n} (-1)^j \binom{x}{j} \left(\frac{n+1}{j+1}\right) = \]
\[ \sum_{j=0}^{\infty} (-1)^j \binom{x}{j} \sum_{m=0}^{\infty} \binom{m+j+1}{m} z^m \frac{z^{m+2j}}{2^j} = \]
\[ \sum_{j=0}^{\infty} (-1)^j \binom{x}{j} z^j \left(1 - z\right)^{-2j-2} = \]
\[ \left(1 - z\right)^{-2} \left[1 - z^x \left(1 - z\right)^{-2}\right] = \left(1 - z\right)^{-2x-x-2} \left(1 - 2z\right)^x. \]

The second equality follows with the product property of power series (Theorem 1 above).

Interchanging summations is justified by (17) using the same computations as above, with absolute values, and noting that \(1 \left\lfloor \frac{x+k}{k} \right\rfloor \leq \left(\frac{1}{x} + k\right)^{-1}\) by B(50).

Sometimes we take \(z^2\) as the argument of a generating function, to avoid
square roots. When given in a small z-domain, or even an interval, such generating functions still determine their coefficient sequences.

From (1.438)

\[ \sum_{k=0}^{\infty} \binom{k}{k} \frac{z^{2k}}{2k+1} = \frac{1}{2z \arcsin^2 z} \]

From (2.58)

\[ \sum_{n=0}^{\infty} \binom{n+1}{2} (\frac{2n+2}{n+1})^{-1} z^{n+1} = \left( \frac{\arcsin^2 z}{2z} \right)^2 \]

So with the convolution property of generating functions (Theorem 1) we have from (16) and (17)

\[ \sum_{k=0}^{n} \binom{k}{k} \frac{1}{2k+1} \binom{2n-2k}{n-k} \frac{1}{2n-2k+1} = \]

\[ (n+1)^{-2} (\frac{2n+2}{n+1})^{-1} z^{n+1} \]

We now finish the computation of the sum on pp. 66, 67. From (1.426) with G(8) (Bisection, cf. pp. 66-67)

\[ \sum_{i=0}^{\infty} (\frac{2i}{2}) z^{2i} = \frac{1}{2} (1-4z)^{-1/2} + \frac{1}{2} (1+4z)^{-1/2} \]

So with the convolution property (Theorem 1) of generating functions

\[ \sum_{n=0}^{\infty} z^{2n} \sum_{i=0}^{n} (\frac{2i}{2})(\frac{2n-2i}{2n-2i}) = \]
\[
\frac{1}{2} \left\{ \left( 1-y^2 \right)^{-\frac{1}{2}} + \left( 1+y^2 \right)^{-\frac{1}{2}} \right\} = \\
\frac{1}{2} \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{2n}{n-k} \binom{n-k}{k} \binom{k}{n-k} \right) y^{2n} = \\
\frac{1}{2} \sum_{n=0}^{\infty} 4^n z^{2n} + \frac{1}{2} \sum_{n=0}^{\infty} \binom{2n}{n} 4^n z^{2n}.
\]

Therefore
\[
\sum_{i=0}^{n} \binom{2n}{n-i} \binom{i}{n-i} = 2^{n-1} \left\{ 2^n + \binom{2n}{n} \right\}.
\]

A few times we take the double generating function w.r. to two parameters. We only give an example:

\[
(3.493): \sum_{j=0}^{\infty} \left( \sum_{k=0}^{n} \binom{2j}{j} \binom{j}{k} \right) y^j = \sum_{k=0}^{n-k} \binom{n-k}{k} (-y^2)^k, \quad n \geq 3.
\]

For \( n \leq 3 \) the l.h.s. is zero. We take the joint generating function w.r. to \( n \) and \( \tau \):

\[
\sum_{n=0}^{\infty} \sum_{\tau=0}^{n} t^n z^{n} \sum_{j=0}^{n} \binom{2j}{j} \binom{j}{\tau} y^j = \\
\sum_{n=0}^{\infty} t^n \sum_{j=0}^{n} \left( \binom{2j}{j} \binom{j}{\tau} \right) y^j (1+z)^j = \\
\sum_{j=0}^{\infty} \binom{2j}{j} \binom{j}{\tau} y^j (1+z)^j \sum_{n=j}^{\infty} t^n = \\
\left( 1-t \right)^{-\tau} \sum_{j=0}^{\infty} \binom{2j}{j} \binom{j}{\tau} y^j (1+z)^j t^j.
\]
\[ (1-t)^{-\frac{1}{2}} \left( 1 - yyt - yyt^2 \right) = \left( \frac{1}{(1-yyt)^{\frac{1}{2}}} \right) \sum_{r=0}^{\infty} \left( \frac{2r}{(2r)!} \right) y^{r} t^{r} z^{r} (1-yyt)^{-r} \]

\[ = \sum_{r=0}^{\infty} \left( \frac{2r}{(2r)!} \right) y^{r} t^{r} z^{r} \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)_k (yy)^k t^k \]

\[ = \sum_{r=0}^{\infty} \sum_{n=r}^{\infty} (\frac{2r}{(2r)!}) y^{r} t^{r} z^{r} (1-t)^{n-r} \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)_k (yy)^k t^k \]

13. Partial fractions. A sum with binomial coefficients may be a rational function of some parameter. Expansion of this function into partial fractions may then lead to the identity sought for. Example:

\( (4.8): \sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} = \sum_{j=0}^{n-1} \frac{(y+1)(y+1-j)}{j!(y+1-j)!} \qquad n \geq 1 \)

With \( \Pi(1) \) we may write the l.h.s. as

\[ n + \sum_{i=0}^{n-2} \binom{n}{i+2} (i+1)! y^{-1}(y-1)^{-1} \cdots (y-i)^{-1} , \]

showing the zeros of the denominator. In the proof the partial expansion is seen to be given by the r.h.s.
14. Egorychev's method. See Egorychev (1984). The very simplest description of this method is as follows. From D(20) and D(25)

\[
\binom{x}{k} = \frac{1}{2\pi i} \oint (1+z)^x z^{-k-1} \, dz,
\]

\[
\binom{y+k}{k} = \frac{1}{2\pi i} \oint (1-z)^{-y-1} z^{-k-1} \, dz,
\]

where the integral is over a closed contour inside \( |z| = 1 \) that surrounds the origin. The idea is to replace one or more of the binomial coefficients in the sum by these integrals and then exchange sum and integration(s), manipulate the integrand and evaluate the integral, or interpret it as the coefficient in a power series, as in the above integrals.

This method is a systematic one for finding and proving binomial sums. It is extended to multiple sums and made the basis of an extensive mathematical theory by Egorychev (1984), who states that most binomial sums may be treated by it.

For the sums in this PART I, with no more than two binomial coefficients, the present author prefers the methods applied here. For the examples of our PART II in Egorychev (1984) the proofs given here look simpler.
A.J. STAM

BINOMIAL IDENTITIES

WITH OLD-FASHIONED PROOFS

PART 1B
C. POLYNOMIALS OF CONVOLUTION TYPE

The sequence of functions $\psi_n : \mathbb{R} \to \mathbb{C}$, $n \in \mathbb{N}_0$, is of convolution type when it satisfies the set of relations

$$\psi_n(x+y) = \sum_{k=0}^{n} \psi_k(x) \psi_{n-k}(y),$$

$x \in \mathbb{R}$, $y \in \mathbb{R}$, $n \in \mathbb{N}_0$.

have $\psi_n(x) = e^{\beta_n x} q_n(x)$ with $q_n$ a polynomial of degree at most $n$, the $\psi_n$ also satisfying (1), so that the $q_n$ are polynomials of convolution type. The sequence $\psi_n$ is of convolution type if and only if, the sequence $q_n = n! \psi_n$ is of bi-

type because of the importance of the convolution operation in analysis, combinatorics, and probability theory and since $\left(\begin{array}{c} x \\ n \end{array}\right)$, $n \in \mathbb{N}_0$, and related sequences are of convolution type. The theory of poly-
nomials of convolution (or binomial) type and the related Sheffer sequences provides beautiful proofs of lots of formulas by giving a unifying principle and by the interplay of operators, formal generating functions, and relations like (7) below and (1). Examples useful for our purpose are $k$-fold convolutions, (78) and (80), and relations involving Abel and Gould polynomials, see pp. C33 – C51.

The elegant theory of polynomials of binomial type was developed in Mullin and Rota (1970) and in Rota et al. (1973), to which we will refer. See also Roman.

*Sheffer (1939, 1941, 1949) – also Gould 1974*

Theorem 1. The sequence $q_n$, $n \in \mathbb{N}$, of measurable complex valued polynomials is the $k$-fold convolution of the sequence $g_n$, $n \in \mathbb{N}$, and $g^{*k} = \delta_n'$, see Chapter N. $f_n$.

The idea of the proof in Di Bucchianico (1991) is as follows. From (1) for \( n = 0 \) we find that either \( \psi_0(x) = 0, \ x \in \mathbb{R} \), and then \( \psi_n(x) = 0, \ x \in \mathbb{R} \), by induction on \( n \), or that \( \psi_1(x) = \exp(ax) \). Then \( q(x) = \exp(-ax) \psi_1(x) \) again satisfies (1). Since \( \int q(x) \ dx \), we find \( q(x) = qx \) for some \( q \in \mathbb{C} \). By induction on \( n \) it then can be

far solution \( q^* \) of \( \psi^{(n-1)}(1) \) is (3) — besides, this shows that (2) is sufficient for (1) — where we may take any fixed value for \( q \). The solution of the homogeneous equation is \( cx \), showing that \( q_n \), given \( q, \ldots, q_{n-1} \), maybe

domain of the \( \psi \) is \( \mathbb{R} \), taking \( c \) for the domain, however, does not lead to (2) but to a theory of functions of convolution type in two real variables. But when we restrict the \( \psi \) to be polynomials in a single complex variable, the proof of (3) still
with \( q = 0 \) is the standard form of a sequence of polynomials of convolution type (not all identically zero) in a complex variable.

**Remark 2.** From (1) it follows that a sequence of convolution type polynomials (4) or (3) is a convolution semigroup of sequences indexed by \( \mathbb{C} \) or \( \mathbb{R} \). The unit element is the sequence

\[
(5) \quad \delta (n) = \delta \quad n \in \mathbb{N}.
\]

**Remark 3.** The sequence \( q \) in (3) or (4) is called the coefficient sequence of the sequence \( q \) of polynomials of convolution type. The connection between the two sequences is bijective since

\[
(6) \quad g_n = \text{coeff. of } x \text{ in } q_n = \lim_{x \to 0} x^{-n} q(x), \quad n \geq 1.
\]

Often, when dealing with a sequence \( q \) of polynomials of convolution type, we will write \( q \) for its coefficient sequence, without explanation.
Remark 4. Since \( q_{n+k} = q_n \), degree \( (q) \) = \( n \) when \( q \neq 0 \) and \( q \) degree \( (q_{n+k}) \) \( \leq n \), \( n \geq 1 \), when \( q = 0 \).

Writing

\[ (q) \sum_{n=0}^{\infty} q_n z^n = \sum_{n=k}^{\infty} q_{n+k} z^n = q^k(z)/k! \]

By manipulation of formal generating functions it is shown that \((q)\) is also equivalent with

Remark. Let \((q)\) hold and let \( |q| < 1 \) be \( \sum_{n=0}^{\infty} q_n z^n \). Then \((10)\) holds, the left-hand side being absolutely convergent. This follows by substituting \((q)\) into the left-hand side of \((10)\) and interchange the summations, which is justified by the estimate \( |q_{n+k}| \leq a_{n+k} \) where
3.4.10 in Di Bucchianico (1991),

\[(12) \quad R = 0 \Rightarrow p = 0, \quad x \in \mathbb{C} - \{0\},\]

Remark 6. The relation \((10)\) suggests that the sequence \(q_n, \ n \in \mathbb{N}\), is determined uniquely by \(n\) by the \(q(c)\) for some fixed \(c \neq 0\). Theorem 1.1.13 in Di Bucchianico (1991) shows that more is true: Let \(x \in \mathbb{C}\), \(x_n \rightarrow 0, \ n \in \mathbb{N}\), and \(q_n \in \mathbb{C}^0, \ n \in \mathbb{N}\), \(q_0 = 1\) be given. Then there is a unique sequence \(q_n\) of polynomials of convolution type such that \(q_n(x_n) = q_n, \ n \in \mathbb{N}\). Uniqueness follows since \(q_n(kx_n), \ k \geq 2\), is determined by \((1)\). The \(q_n\) in \((4)\) may be constructed recursively since \(q_n\) with \(k \geq 2\) is given by an expression containing only \(q_0, \ldots, q_{n-1}\). This construction and also Remark 8 show that the coefficient sequence \(q_n\) is determined uniquely by the \(q_n(x_n)\).
(14) \( g_n(x) = \binom{x}{n}, \ x \in \mathbb{R}, \ n \in \mathbb{N}_0. \)

Here (1) is the Vandermonde convolution \( D(2b), \) and (10) is the Binomial series, \( g(z) = \log(1+z), \) so that \( g_k = (-1)^{k-1}/k \)
which also follows from \( g_k = \frac{1}{k} \) as well from the theory of the Binomial series, see Knopp (1924). We have
\( p_x = 1 \) when \( x \notin \mathbb{N} \) and \( p_x = \infty \) when \( x \in \mathbb{N}. \) This illustrates (11) and (13).
From (4) and \( D(28) \) we see that we have here

\( \star \) and e.g. Worrie (1951)
\[ E^a f(x) = f(x + a), \quad x \in C, \quad I = E^0, \quad I f = f. \]

Mark the notation \( Af(x) = (Af)(x) = (A(f))(x) \) for operators on \( \mathbb{H} \). The linear operator \( T : \mathbb{H} \to \mathbb{H} \) is called shift-invariant when

\[ x \to x + a. \]

The product of operators \( \mathbb{H} \to \mathbb{H} \) is composition.

The linear operator \( Q : \mathbb{H} \to \mathbb{H} \) is called

The following theorems are taken from Rota, e.a. (1973). Also in Di Bucchianico (1991).
(19) \( Q_0 q_n = q_{n-1}, \; n \geq 1, \; q_0(0) = 0, \; n \geq 1, \; q_0(x) = 1, \; x \in C. \)

This sequence is called the basic sequence of \( Q. \) It is a sequence of convolution type.

Remark 1. In the theory of polynomials of binomial type, e.g. in Rota e.a. (1973) it is required that \( Q_0 q^*_n = n q^*_n, \) with \( q^*_n = n! q_n. \)

Remark 2. From (18) it follows that \( \text{deg}(q_n) = n, \) so that \( q_n, \; n \in N, \) is a basis for \( \Pi. \) So from (19) and \( QQ_0 = 0, \) we see

(4) and (10). Then \( y, \; n \in W, \) is the basic sequence of a delta operator.

In the classes of basic sequences and of

with basic sequences consisting of classical polynomials, see Rota e.a. (1973) and Roman (1984). For our purpose
the most important are \( D = d/dx, \Delta = E - I, I - E^{-1} \) and, cf. Theorem 8 below, 
\( E^{-a}D \) and \( E^{-a}\Delta \). The following

(21) For \( R = \Delta \) we have \( q_n(x) = \binom{x}{n} \), \( n \in \mathbb{N} \).

(22) For \( R = I - E^{-1} \) we have \( \tilde{q}_n(x) = \binom{x+n-1}{n} \), \( n \in \mathbb{N} \).

Let \( q_n, n \in \mathbb{N} \), be the basic sequence of the \( \tau \)-delta-operator \( R \). Then, for \( f \in \mathbb{P} \) with degree \( \deg(f) \leq n \) we have the unique expansion

(23) \[ f(x) = \sum_{k=0}^{n} c_k q_k(x), \quad c_k = \langle R^k f(0) \rangle, \]

(24) \[ f(x+y) = \sum_{k=0}^{n} c_k(y) q_k(x), \quad c_k(y) = \langle R^k f(y) \rangle. \]

Proof. Since \( q_0, q_1, \ldots, q_n \) is a basis for the set of \( \mathbb{P}_n \)-polynomials of degree \( n \).

\[ \lambda_{\xi} = (R^\xi E^\chi f(0) = E^\chi R^\xi f(0) = \langle Q^\xi f(y) \rangle. \]
From (24) for \( y = 0 \) we find (23).

For \( Q = \Delta \) and \( Q = \nabla \) we find with (20) and (21), respectively, Taylor's...
shift-invariant. This means that there is a bijection between shift-invariant operators and formal power series, the upshift product to the product of series. In particular, as already follows from (27),

\[ \text{depends on } \varphi. \text{ For every delta operator } \varphi \text{ there is a different isomorphism } (28) \iff (29) \]

\[ \text{Proof. The first assertion is a direct consequence of the above isomorphism, the inverse then corresponding to the formal power series } 1/\varphi(z). \text{ A diffe-} \]

Recent proof:
\[ S q_n = \sum_{k=0}^{n} c_k q_{n-k} \]

Or, since \( \omega \to \omega - 1 \) we have \( \omega \to \omega - 1 \)
that \( S \) has a nontrivial null space.
The shift-invariance follows from:
\[ S^{-1} E^a = S^{-1} E^a S S^{-1} = S^{-1} S E^a S^{-1} = E^a \]

The most important isomorphism between.

Remark on notation. Given a delta operator \( \mathcal{D} \), it often will be understood that \( q_n, n \in \mathbb{N}_0 \), denotes its basic sequence and that \( \mathcal{D} q = q(D) \) denotes its expansion (31), where \( q(z) = \sum_k a_k z^k \).
Also \( q_k, k \in \mathbb{N}_0 \), with \( q_0 = 0 \) denotes
Proof. Note that the formal power series
\[ \sum_k b_k z^k \] with \( b_0 = 0 \), does form a group
with respect to composition.

and the theorem follows from the iso-
morphism between (28) and (29).

Alternatively, with \( q(z) = \sum_k a_k z^k \), we
have with (31) and (4) and (5).

\[ q(q(z)) = \sum_{k=1}^\infty a_k q^k(z) = \sum_{k=1}^\infty a_k \sum_{n=k}^\infty q^{k+n} z^n \]

\[ = \sum_{n=1}^\infty \sum_{k=1}^\infty a_k q^{k+n} z^n \]

We have the following extension of Theo-
rem 1 and (4), see Di Bucchianico (1991),
Theorem 1.3.9  (= Th. 2.3.10).

Theorem 1. Let \( y_n, n \in \mathbb{N} \), be a sequence of polynomials with degree \( (y_n) \leq n \) and let \( Q \) be a delta operator with basic sequence \( g_n, n \in \mathbb{N} \). Then the sequence \( y_n \) is of convolution type if and only if there is a sequence \( f_n \in \mathbb{C}, n \in \mathbb{N} \), with \( f_0 = 0 \) so that

\[
\sum_{n=0}^{\infty} y_n z^n = \left( \sum_{n=0}^{\infty} f_n z^n \right) \cdot \left( \sum_{n=0}^{\infty} g_n z^n \right)
\]

With (10) and (8) we have from (34) \( x = 1 \) where \( I \) is shift-invariant and invertible. The basic sequence \( g_n \) of \( Q \) may be expressed in terms of \( T \) by the following important
\[
\left(3) \sum_{n=0}^{\infty} \frac{\alpha^n}{(x+n)a} \right)^{2} \left. \frac{x}{y} \right|_{y=0} = \sum_{n=0}^{\infty} \frac{\alpha^n}{(x+n)a} \quad \text{for} \quad a > 0.
\]

Theorem 3 and 8. When \( a \neq 0 \), the assertion follows from

When \( a = 0 \), the assertion fails. Therefore, by (38) is of convolution type.

\( (a) \) is of convolution type. Then the sequence of polynomials of convolution type, not

Basic sequence

\( \underbrace{\text{sequence of } a. \text{ In the case } E \text{ or } G \text{ has the}} \)

\( \underbrace{\text{formula, see Rota e.a. (1973), Section 4.1}} \)

\( (a) \)
Sheffer sequences.

Our treatment of this subject is more in line with Di Bucchianico (1991). Part of the results is new.

Definition 3. The sequence \( s_n, n \in \mathbb{N}_0, \) of polynomials is a wide-sense Sheffer sequence (or Sheffer set) for the delta operator \( Q \) when

\[
(39) \quad s_0 \text{ is a constant, } Q s_n = s_{n-1}, n \geq 1.
\]

When \( s_0 \neq 0, \) the sequence \( s_n \) is called a strict Sheffer set for \( Q. \)

Remark. When not all \( s_n \) are identically zero, there is \( c \in \mathbb{N}_0 \) such that \( s_n = 0, n < c, \) \( s_c = c \neq 0, \) degree \( (s_{n+j}) = j. \) This follows from (18). When the sequence is strict Sheffer, degree \( (s_n) = n. \) In the theory of polynomials of binomial type the condition \( Q s_n = s_{n-1} \) is replaced by \( Q s_n = n s_{n-1}, n \geq 1. \) So \( s_n \) is Sheffer in our sense if and only if \( n \) \( s_n \) is Sheffer in the 'binomial' sense.

Theorem 10. Let \( Q \) be delta with basic sequence \( q. \) Then the following assertions are equivalent:

a) \( s_n \) is wide-sense Sheffer for \( Q. \)
\[ s_n(x+y) = \sum_{k=0}^{n} q_k(x) s_{n-k}(y), \ x, y \in \mathbb{C}, \ n \in \mathbb{N}_0, \]
\[ s_n(x+y) = \sum_{k=0}^{n} s_k(x) q_{n-k}(y), \ x, y \in \mathbb{C}, \ n \in \mathbb{N}_0, \]
\[ s_n(x) = \sum_{k=0}^{n} s_k(0) q_{n-k}(x), \ x \in \mathbb{C}, \ n \in \mathbb{N}_0, \]

There is a sequence \( c \in \mathbb{C}, \ n \in \mathbb{N}_0 \), such that
\[ s_n(x) = \sum_{k=0}^{n} c_k q_{n-k}(x), \ x \in \mathbb{C}, \ n \in \mathbb{N}_0, \]

There is a shift-invariant operator \( A \) such that \( s_n = A q_n, \ n \in \mathbb{N}_0 \).

The sequence is strict Sheffer if and only if \( c_0 \neq 0 \) in (e), or, equivalently, \( A \) is invertible.

**Proof.** \( a \Rightarrow b \): From (24) and (39). Note that the condition \( s_0 = \text{const.} \) is essential. If not, then \( s \) is a polynomial of degree \( > n \) and (b) does not follow.

\( b \Rightarrow c \): Interchange \( x \) and \( y \) in (b) and put \( n-k = h, \)

\( c \Rightarrow d \): Take \( x = 0 \) in (c).

\( d \Rightarrow e \): Take \( s_1 = s_k(0). \)

\( e \Rightarrow f \): Take \( A = \sum_{k=0}^{\infty} c_k Q^k \).

\( f \Rightarrow a \): \( A q_n \) is constant since \( q_0 = 1. \) Also by (30)
\[ D^n s_n = RA^n q_n = AQ^n q_n = A q_n = s_n, \ n \geq 1. \]

For the strict Sheffer property: \( s_0 = q_0 \). For \( A \): Theorem 5. Note that \( A \) is unique since the \( q_n \) form a basis.
Remark. The Sheffer property still can be weakened. When the $s$ are given by (e) and the $q$ are of convolution type, not necessarily basic, the relations (8) and (c) still hold:

$$\sum_{k=0}^{n} s_k(x) q_{n-k}(y) = \sum_{k=0}^{n} \sum_{j=0}^{k} c_{j} q_{n-j}(x) q_{k-j}(y)$$

$$\sum_{j=0}^{n} c_{j} q_{n-j}(x+y) = q_{n}(x+y).$$

This is equivalent with

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n-k}{j} x^j q_{n-k}^{j*} = \sum_{j=0}^{n} (c^{\ast} q^{j*}) \frac{x^j}{j!}.$$
\( u_n(x) = n(x+nb)^{-1} q_n(x+nb), \quad n \in \mathbb{N}, \)

\[
T^{1-n} \frac{D x^{n-1}}{(n-1)!} = T^{1-n} x^{n-2} / (n-2)! = (n-1)x^{n-1}q_{n-1} = s_{n-1}.
\]

Also \( w_0(x) = q \neq 0 \) by (4) and \( Q w_n = \)

\[
Q s_{n+1} = s_n = w_{n-1}.
\]

We may not extend (43) to \( (n+1)x^{-1}q_{n+1} \) with
\[ E^{(n-1)} q_{n-1} = t_{n-1}, \quad n \geq 1. \]

When we apply (43) and \((45^a)\) to the delta operator \(E^{-b}Q\), we obtain (45) and \((45^a)\), see Theorem 8.

From (46) and \((46^a)\) with (6) for \(x \to 0\)

\[ n! \to \gamma (1 - 5^n) \to a (1) \text{ as } (46^a) \]

Here \((46^a)\) and \((47^a)\) are just (46) and (47) with \(n\) replaced by \(n+1\).
From (44), Theorem 8 and Theorem 10

\[(49) \quad n^2 (x+y+n^2) \quad q_n (x+y+n^2) = \sum_{k=1}^{n} k (x+k) \quad q_1 (x+k) \quad y (y+n^2-k^2) \quad q_n (y+n^2-k^2).\]

Times (49) is the following:

(1) for the basic sequence \(x (x+n^2) \quad q_n (x+n^2),\) see Theorem 8.

The relation (48) for \(n \geq 1\) may be written

\[q_n (z) + \sum_{k=1}^{n} \frac{x}{x+k} \quad q_1 (x+k) \quad q_{n-k} (z-k) = q_n (x+z)\]

by taking \(y = z-n^2.\) Taking \(z = 0,\) dividing by \(x\) and letting \(x \to b\) gives by (6)

\[(50) \quad \sum_{k=1}^{n} \quad k^{-1} \quad q_1 (kb) \quad q_{n-k} (-kb) = b q_n, \quad n \geq 1.\]

The relations (46) - (50) still hold when the
sequence $a$ is of convolution type. This may be shown by a limit argument similar to the proof of the remark to Theorem 8.

**Theorem 11.** Let $s_n$, $n \in \mathbb{N}$, be a Sheffer,

**Proof.** From (23), (18) and Theorem 10c

$$
\sum_{r=0}^{\infty} s_n(x+y) Q(r) f(x).
$$

near from $f$ to arbitrary linear operators $Q$.

**Remark.** In Theorem 11 let
\[ Q^tf(y) = E^{+\kappa} E^{-\kappa} Q^\kappa \nu_n(y) = F^{\kappa} \Psi \cap \cap = \Psi L(y+k\kappa) \]

(53) \[
\sum_{k=0}^{n} s_k(x) t_{n-k}(y) = \sum_{k=0}^{n} (a*b)_k q_{n-k}(x+y)
\]
\[
= \sum_{k=0}^{n} s_k(0) t_{n-k}(x+y) = \sum_{k=0}^{n} s_k(x+y) t_{n-k}(0).
\]

Proof. The theorem follows from the associativity of the convolution product, together with (i) and Theorem 10. The left hand side of (53) is \((a*b*c*d)_n\) with \(c_m = q_m(x)\), \(d_m = q_m(y)\).

Replacing \(Q\) by \(E^{-\delta}Q\) in Theorem 12, and, according to (46) and (44), taking \(s_n(x) = n(x+n\delta)^{-1} q_n(x+n\delta)\) and \(t_n(x) = \Psi_n(x+n\delta)\), we find the relation...
Note that here, $s_0 = 0$.

(55) \[ q_n(x, a) = x(x+na)^{-1} q_n(x+na), \]

for the basic sequence of $E^{-a}$ see a Sheffer sequence, not identically zero, for $E^{-Q}$ and let $s \in C$, $s \neq 0$. Then

if and only if

(57) \[ \Psi(s^{-1} D) = \exp(\lambda D) \Psi(D). \]

Then necessarily

(58) \[ \Psi = s^{-1}, \quad \lambda = a_1 a_2 (s^{-1} - 1). \]
Proof. The sequence $q_n(x,u)$ is basic for the delta operator \( \exp(-u \partial_x^D) q(x^D) \) as is seen by writing $q_n(x,u) =$

\[
\begin{align*}
\gamma^k E_{\lambda+\gamma-u}^{\lambda+\gamma-u} s_{n-k}(y) = \\
\gamma^k \cdot (\gamma + k (\gamma + \lambda - \gamma^k u)).
\end{align*}
\]

So $\exp(\lambda D) Q s_n(y) = q(x^D) s_n(y)$, $n \geq 1$. This relation also holds for $n = 0$. From the remark to Definition 9, it follows that the $s_n$ form a basis for $\Pi$. So (37) holds.
We find (58) by equating the coefficients of $z$ and also those of $z^2$ in the formal power series that correspond to both sides of (57), according to (31) and the remarks to Theorem 4.

**Remark 1.** The condition (57) is always satisfied when $p = 1$, with $q = 1$, $\lambda = 0$, and then (56) follows immediately from (24).

When $Q = 0$, $D \exp (\beta D)$ the relation (57) holds for any $q \in \mathbb{C}$, $s \neq 0$, with $y = q^{-1}$, $\lambda = \beta (s^{-1})$. So (56) holds for $y$ an Abel polynomial (75). This relation then is essentially the same for all $p$ and follows from (24).

**Remark 2.** When $Q = \Delta = e^D - I$, the condition (57) is satisfied for $p = 1 = \lambda = -1$, so (56) holds for the Gould polynomials (84), see (98) - (102).

**Remark 3.** From (58) we see that (57) is

\begin{equation}
\varphi (D) = \exp (-a_1 a_2 D) Q(D) = \sum_{k=1}^{\infty} b_k D^k,
\end{equation}

with $b_1 = a_1$ and $b_2 = 0$. Then (57) is equivalent with
\[ q(D) = q(D) \exp (\beta D) \] and then (56) holds for any \( s \in C, s \neq 0 \), as was seen in Remark 1. Otherwise, when no \( s \) exists such that \( \beta_k = 0 \) for all \( k \geq 3 \) with \( k \neq 1 \) modulo \( s \), the relation (56) cannot be satisfied when \( s \neq 1 \). When such \( s \) exist, let \( m \) be maximum of these \( s \). Then we see from (63) that (56) holds if and only if \( \rho \) is a \( m \)th root of unity and (58) is satisfied.

Remark 4: When \( Q = \Delta = e^D - I \), the condition (57) is satisfied for \( \rho^m = 1 = -1 \) and (56) holds for the Gould polynomials, see (98)-(101).
\[ \sum_{n=0}^{\infty} \hat{p}_n = 2 \quad \text{or} \quad z + k (\lambda - \rho m)^{n-1} \]

Applications of the theory of Sheffer sets
by Niederhausen, 1980.
Umbral composition.

Let

\[ (66) \, \alpha_n(x) = \sum_{k=0}^{n} \alpha_{nk} \frac{x^k}{k!}, \quad \beta_n(x) = \sum_{k=0}^{n} \beta_{nk} \frac{x^k}{k!}, \quad n \in \mathbb{N}_0, \]

be two sequences of polynomials. The umbral composition of the sequences \( \alpha_n \) and \( \beta_n \), in this order, is defined as the sequence

\[ (67) \, \gamma_n(x) = \sum_{k=0}^{n} \gamma_{nk} \frac{x^k}{k!} = \sum_{j=0}^{n} \alpha_{nj} \beta_{j} \, (x), \quad n \in \mathbb{N}_0. \]

We have

\[ \gamma_n(x) = \sum_{j=0}^{n} \alpha_{nj} \left( \sum_{k=0}^{j} \beta_{jk} \frac{x^k}{k!} \right) = \sum_{k=0}^{n} \left( \sum_{j=k}^{n} \alpha_{nj} \beta_{jk} \right) \frac{x^k}{k!}, \]

\[ (68) \, \gamma_{nk} = \sum_{j=k}^{n} \alpha_{nj} \beta_{jk}. \]

And \( \left( \beta_{nk} \right) \), in this order.

We have \( \gamma_n(x) = \frac{x^n}{n!}, \quad n \in \mathbb{N}_0, \) if and only if \( \gamma_{nk} = \delta_{nk} \), i.e., if and only if
The importance of umbral composition in our theory is when $\alpha_n$ and $\beta_n$ are basic or Sheffer sequences. We have to be careful with notation, for in the theory of polynomials of binomial type the definition of umbral composition looks different from ours. Put

$$
\sum_{j=0}^{n} \alpha_{nj}^* \beta_j^*(x) = \sum_{j=0}^{n} \alpha_{nj}^* \sum_{k=0}^{j} \beta_{jk}^* \times x^k = \\
\sum_{k=0}^{n} \left\{ \sum_{j=k}^{n} \alpha_{nj}^* \beta_{jk}^* \right\} \times x^k,
$$

so that the coefficient matrix of the umbral composition is again the product of the coefficient matrices of $\alpha_n^*$ and $\beta_n^*$. We have from (70)

$$
\sum_{j=k}^{n} \alpha_{nj}^* \beta_{jk}^* = \sum_{j=k}^{n} n! \alpha_{nj} \beta_{jk} / k! = n! \gamma_{nk} / k!,
$$
with $Y^k_n$ given by (68). So the \textit{binomial composition} of $x^*$ and $p^*$ is equal to $Y^*(x) = n! f(x)$ with coefficients

$$\sum_{i=0}^{n} a_i p_{n-i}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k / k!,$$

$$\sum_{i=0}^{n} b_i q_{n-i}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k / k!,$$

with $f^0 = g^0 = 0$. Let $f(z), g(z), a(z), b(z)$ be the formal generating functions of the sequences $f_k, g_k, a_k, b_k$, respectively.

**Theorem 14.** Under (71), let $r_n$ be the umbral composition of $p_n$ and $g_n$ and let $u_n$ be the umbral composition of $s_n$ and $t_n$. Then $r_n$ is of convolution

$$r_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k, x \in \mathbb{K}, n \in \mathbb{N}_0.$$
\begin{equation}
(73) \quad u_n(x) = \sum_{i=0}^{n} c_i r_{n-i}(x) = \sum_{k=0}^{n} (c \ast h_k^*)_n x^k / k!, \quad n \in \mathbb{N}_0 \times \mathbb{C},
\end{equation}

for \( R = \beta (q(D)) \) and \( u_n \) is Sheffer for \( R \). It is strict Sheffer then if and only if \( a_0 b_0 \neq 0 \).

Proof: We give a proof with formal generating functions, showing that rela-

\begin{equation}
\sum_{n=0}^{\infty} r_n(x) z^n = \sum_{n=0}^{\infty} \frac{f_n q_j(x)}{z^n} = \sum_{j=0}^{\infty} f_j z^j \sum_{n=0}^{\infty} q_n(x) f^j(z)
\end{equation}

and (72) follows, since the \( r_n \) satisfy a relation similar to (10). When \( p_n \) and \( q \) are basic, we have \( f \cdot q \neq 0 \), so \( h_n \neq 0 \) and \( r_n \) is basic. (This follows from Theorem 3).

From (67), (71c), (71d), (41) and (42)
and (73) follows with (74). When \( p \) and \( q \) are basic, it follows from Theorem 10 and the assertion on \( r \) that \( u_n \) is Sheffer for \( R \). Theorem 10 shows that then \( u_n \) is strict Sheffer if and only if \( c_0 \neq 0 \). But \( q = q_0 \).

The connection between "convolutional" and "binomial" umbral composition as discussed on...

\[
\rho_n(x) = \pi \vdash \rho_n(x) = \sum_{k=0}^{\infty} \Pi_{n}^{k} \beta_{k}(x)
\]

where \( \rho, n \in \mathcal{W} \), is a "convolutional" basic sequence and \( \rho^* = n! \rho \) the corresponding "binomial" basic sequence. When the two types of umbral composition are defined as

\[
\sum_{j=0}^{n} \alpha_{nj} \beta_{j}(x) \quad \text{and} \quad \sum_{j=0}^{n} \alpha_{nj}^* \beta_{j}^*(x)
\]

respectively, a conclusion analogous to the one on \( \pi \). C30 holds. There also is an analogue of Theorem 14.
The Abel convolution semigroups

From Theorem 8 or directly from Theorem 2 we see that the delta operator $E^{-a}D$ has the basic sequence

$$q_n(x) = \frac{x(x+na)^{n-1}}{n!}, \quad n \in \mathbb{N}.$$  

Application of the preceding theory to these polyno-

(1902), Jensen (1902), Salie (1951) and Kaučík (1968).

The convolution equation (1) gives

$$\sum_{k=0}^{n} \frac{x(x+ka)^{k-1}}{k!} \frac{y(y+na-ka)^{n-k-1}}{(n-k)!} =$$

$$(x+y)(x+y+na)^{n-1}/n!, \quad x \in \mathbb{C}, \quad y \in \mathbb{C}, \quad n \in \mathbb{N}.$$  

By (6) the coefficient sequence of the Abel polynomials is

$$g_n = (\frac{1}{n!})^{n-1} \quad n \geq 1.$$
By the binomial theorem,
\[
x(x+na)^{n-1}/n! = \sum_{k=1}^{n} \frac{k}{n} \frac{(na)^{n-k}}{(n-k)!} \frac{x^{k}}{k!}, \quad n \geq 1.
\]

With absolute convergence for \( |aez| < 1 \) by Remark 5 to Theorem 1, as is seen by applying Stirling's formula to \( \lambda (n) \).

From (74) with \( \Omega = E^{-a}D \), for a polynomial \( f \) of degree at most \( n \),
\[
(80) \quad f(x+y) = \sum_{k=0}^{n} \frac{x(x+ka)^{k-1}}{k!} E^{-ka} D^{k} f(y) = \sum_{k=0}^{n} \frac{x}{(x+ka)^{k-1} f(k) / (n-ka)}
\]

\[
E^{-ka} D^{k} f(y) = E^{k(b-a)} E^{-k(b-a)} D^{k} f(y) = \frac{E^{k(b-a)}}{\sqrt{1+n+nb-kb}^{n-k-1} / (n-k)!}.
\]
\[
\sum_{k=0}^{n} \frac{x(x+ka)^{k-1}}{k!} \frac{(y+kb-ka)(y+n b-ka)^{n-k-1}}{(n-k)!}.
\]

The relation (48) with \( g \) \( (x) \) given by (75) leads to a variant of \( c n(81) \).

\[
(83a) \sum_{k=0}^{n} \frac{(x^2-kb^2)(x+kb)^{k-2}}{k!} \frac{(y+(n-k)b)}{(n-k)!} =
\]
The Abel polynomials satisfy (57) in Theorem 13 with $\lambda = 0$, $y = 1$ and $C = 10$. But (56) does not lead to new formulas.
These polynomials were studied by Gould (1956, 1960a, 1961a, 1962b) who derived a number of the identities given below. They are called Gould polynomials, see e.g. Roman (1984) but mind the factor n! as on p. C1. Applications to probability theory with references are given by Charalambides (1986). Our relations (93a) and (86) show that the probabilities in his quasi Pólya distributions I and II sum.

and Takács (1962), Mohanty (1974), Gessel (1986) where they occur as "ballot numbers".
(87) \[ \sum_{n=0}^{\infty} g_n(x,a)z^n = \exp \left\{ x \sum_{k=1}^{\infty} \frac{z^k}{ka}(ka) \right\}. \]

To study absolute convergence here, cf. Remark 5 to Theorem 1, we apply the asymptotic formula for \( \Gamma(w) \) with \( |\arg w| < \frac{1}{2} \pi - \varepsilon \) in §30 of Copson (1965). We find:

For \( \text{Re} a > 1 \): \( g_n \sim C(a) n^{-3/2} a^a (a-1)^{1-a} z^n. \)

For \( 0 < \text{Re} a < 1 \): \( g_n \sim C(a) n \left( a(-a) \right)^{1/2} \sin(n(1-a)\pi). \)
From (24), for a polynomial of degree \( \leq n \),
\[
(\text{30}) \quad f(x+y) = \sum_{l=1}^{n} \mathcal{C}_l(y) \frac{x}{\ldots(x+ka)},
\]
\[
\mathcal{C}_l(y) = \sum_{r=0}^{n} y^r \mathcal{C}_l^{(r)}(y),
\]
\[
\mathcal{C}_l^{(r)}(y) = \frac{1}{(y+kb-ka, b)},
\]
\[
(\text{30}) \quad \frac{x+y}{x+y+nb} \binom{x+y+nb}{n} = \ldots.
\]

In particular, for \( b=0 \),
\[
(\text{31}) \quad (x+y) = \sum_{l=1}^{n} \frac{x}{\ldots(x+ka)}(y-ka).
\]
\[
(\text{32}) \quad l_n = 0, n \leq \kappa, \quad l_n = \frac{\kappa}{n+\kappa} \left( \frac{\kappa}{n-k} \right)^{n+\kappa}.
\]
\[ (q^3 a) (x+y+n b) (x+y+n b+n a)^{-1} \begin{pmatrix} x+y+n b+n a \\ n \end{pmatrix} \]

\[ \sum_{k=0}^{n} \frac{x+1}{x+1+k(a-1)} \binom{x+ka}{k} \frac{y+1}{y+1+(n-k)(a-1)} \binom{y+n b-k a}{n-k} \]

\[ \sum_{k=0}^{n} \frac{x+1}{x+1+k(a-1)} \binom{x+ka}{k} \frac{y+1+k(\beta-\alpha)}{y+1+n(\beta-1)-k(a-1)} \]
\[ \kappa = 0, \ldots, \pi; (\alpha - 1) \quad \sum_{x=0}^{n} \frac{x+y+2+n(b-1)}{x+y+2+n(b+a-1)} \left( \begin{array}{c} x+y+1+n(b+a) \\ n \end{array} \right) = \sum_{k=0}^{n} \frac{x+1+k(b-1)}{x+1+k(b+a-1)} \left( \begin{array}{c} x+k(b+a) \\ k \end{array} \right). \]

Of (95) is equal to...
\[
(-1)^n \sum_{k=0}^{n} \frac{-x-1}{-x-1+k(l-a)} \binom{-x-1+k(l-a)}{k} \cdot
-\gamma-1+k(1-b)-k(l-a) / -\gamma-1+n(1-b)-k(l-a)
\]

be given. The fundamental cause of (94) - (96) is the fact that by (92) and

\[
\left(\frac{b+a}{nab}\right)^{n}, \quad n \geq 1, \quad ab \neq 0.
\]
The delta operator \( E^{-a} \Delta = e^{-a \Delta} (e^{\Delta} - I) \) satisfies (57) with \( \lambda = 0 = -1 \) and \( \lambda = 2a - 1 \).
So (56) for \( a = 0 \) with \( s_n(x) = q_n(x, u) \) given by (84) becomes

\[
(98) \quad (x+y)(x+y+nu)^{-1} \begin{pmatrix} x+y+nu \end{pmatrix}_n = \]

This is the relation (5.5) in Gould (1962\textsuperscript{b}) with \( t_b = u \), \( (1-t)b = v \). For \( a \neq 0 \) we still obtain (98). Taking \( u = 1 \) gives

\[
(99) \quad (x+y)(x+y+nu)^{-1} \begin{pmatrix} x+y+nu \end{pmatrix}_n = \\
\sum_{k=0}^{n} \binom{x}{k} \frac{y+ku}{y+nu} \binom{y+nu}{n-k}.
\]

This relation may be derived directly from (26) with (84). It is equal to (90) with \( a = 0 \).

Taking \( q(x, a) \), given by (84), for \( q(x) \)

\[
\sum_{k=0}^{n} \binom{x}{k} \frac{y+ku}{y+nu} \binom{y+nu}{n-k} \]

A similar application of (64) gives (98).
\[(x+nb)(x+na+nb)^{-1} \begin{pmatrix} x+na+nb \end{pmatrix}_n\] are

cluster invariants \[\Delta = E-b-a\Lambda \]

depends only on \(x+y\).
for the delta operator $E^{-\Delta} = g(D) = \exp\{-(aD)\} - \exp\{-aD\}$. By Theorem 6, the formal power series $g(z) = \sum_{k=1}^{\infty} a_k z^k$ of the coefficient sequence is the composition inverse of $g$, i.e., it is the solution with $g(0) = 0$ in the unknown $t$. For $a = 0$ and $a = 1$ this equation has a simple solution. We then are considering the delta operators $\Delta$ and $I - E^{-1} I$ with basic sequences $(x^n)$ and $(x+n-1) = (-1)^n \binom{-x}{n}$, respectively.

We must have $a = x$, $a = -1$ or $a = \frac{1}{2}$. These special cases will be considered now.

The Catalan convolution semigroup $(a = x)$

$$I = 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad E^{-a}$$

An application of the numbers (104) to ran-

where the series converge absolutely for $|z| \leq \frac{1}{4}$ and diverge for $|z| > \frac{1}{4}$. The log in (107) is the principal value, in fact it will be shown that
\[(109) \quad \left| 2z \left( 1 - \sqrt{1 - yz} \right) - 1 \right| < 1, \quad |z| < \frac{1}{4}. \]

According to (10) we have to interpret the convergence of the series in (107) and (108), see Remark 5 to Theorem 1. One also might estimate \( f(x, 2) \) by Stirling's for-

\[ (111) \quad \gamma (x+1) - \gamma (x-1), \quad |x| < 14. \]

By (111), (10) and (106) we have for \(|z| < \frac{1}{4}\)

\[
\lim_{n \to \infty} \frac{1}{n} \left( 1 - n \right) \exp \left( g(1z) - 1 \right) \left( 2|z| \left( 1 - \sqrt{1 - 4|z|} \right) - 1 \right) < 1,
\]
principal value for the log.

By (44) and (61) the sequence \((\binom{x+2n}{n})\) is Sheffer for \(E^{-2}G\). So with (41), (42), (108) and (13)

\[
(110) \quad \sum_{n=0}^{\infty} \binom{x+2n}{n} z^n = \left(\frac{2z}{1-\sqrt{1-4z}}\right)^x \sum_{n=0}^{\infty} \binom{2n}{n} z^n
\]

\((-1 < 1, \quad 1-1 < 1, \quad 1+1 < 1) \quad \frac{2z}{1-\sqrt{1-4z}} < 1 \quad \frac{2z}{1-\sqrt{1-4z}} < 1\]

See the remark in Chapter P on (1.427).
By extracting a factor from or adding a factor to the factorials in \( q(1,2) \) or \( q(-1,2) \) and related quantities, we may write them in different forms as the example (106) shows. In this way we may write some of the relations (85), (91), (93), (94) and (96) for \( x=\pm 1 \) and \( y=\pm 1 \) in a disguised form. We give some examples, mostly for \( x=y=1 \). Taking \( x=-1 \) and/or \( y=-1 \) usually gives a relation for \( x=y=1 \) with \( n \) replaced by \( n-1 \) or \( n-2 \). We use the following relations besides (106).

\[
\binom{2m+1}{m} = \frac{1}{2} \binom{2m+2}{m+1}, \quad m \in \mathbb{N},
\]

\[
\frac{2}{m+2} \binom{2m+1}{m} = \frac{1}{m+1} \binom{2m+2}{m}, \quad m \in \mathbb{N},
\]

\[
\binom{m}{m} = \frac{1}{2} \binom{m-1}{m-1}, \quad m \in \mathbb{N},
\]
From (85) with \( x = y = 1, \alpha = 2 \) and (106)

\[
\sum_{k=0}^{n} \binom{k+1}{n-k+1}^{3} \binom{2k}{n-k}^{3} = \binom{n+1}{n+2}, \quad n \in \mathbb{N}_0.
\]

This is the well-known convolution property of the Catalan numbers.

From (91) with \( x = y = 1, \alpha = 2 \) and (106) and D(14)

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{2k}{k} \binom{n+k-2}{n-k} = \binom{2}{n}, \quad n \in \mathbb{N}_0.
\]

From (93) with \( x = y = 1, \alpha = 2 \), (112) and (106)

\[
\sum_{k=0}^{n} \binom{2k}{k} \frac{1}{n-k+1} \binom{2n-2k}{n-k} = \frac{n}{2n+2} \binom{2n+2}{n+1}, \quad n \in \mathbb{N}_0.
\]

From (94) with \( x = y = 1, \alpha = 2 \) and (114)

\[
\sum_{k=0}^{n} \frac{1}{k+1} \binom{2k+2}{k} \frac{1}{n-k+1} \binom{2(n-k)+2}{n-k} = \frac{2}{n+4} \binom{2n+4}{n+1}, \quad n \in \mathbb{N}_0.
\]

From (96) with \( x = y = 1, \alpha = 2 \), (114) and (115)

\[
\sum_{k=0}^{n} \binom{2k+1}{k+2} \frac{1}{n-k+1} \binom{2n-2k+2}{n-k} = \frac{n}{n+4} \binom{2n+3}{n}, \quad n \in \mathbb{N}_0.
\]
\[ (123) \sum_{k=1}^{n} \binom{2k-1}{k-1} \frac{1}{n-k+1} (\binom{2n-2k+2}{n-k}) = \binom{2n+1}{n-1}. \]

From \((q_{3})\) with \(a=0, b=2, x=-1, y=1\),

\[ (2n+1)^{n} (1+2n) + \sum_{k=1}^{n} \frac{1}{2} \binom{2k}{k} \frac{1}{n-k+1} (\binom{2n-2k}{n-k}) = \binom{2n}{n}, \]

\[ (126) \sum_{k=0}^{n} \binom{2k}{k} \frac{1}{n-k+1} (\binom{2n-2k}{n-k}) = \binom{2n+1}{n}, \quad n \in \mathbb{N}. \]
\[ \sum_{h=0}^{m} \binom{2h+x}{h} \frac{1}{m+1-h} \binom{2m-2h+x}{m-h} = \binom{2m+x+2}{m}. \]

This is (92a) with \( a = 0, b = 2, y = 2 \), providing an example of the different ways in which binomial formulas may be derived.

\[ \left( \frac{x}{x-n} \binom{x-n}{n} \right) = (-1)^n \frac{x}{x-n} \binom{2n-x-1}{n} = \]

from those for \( q'(x, z) \). E.g., we have from (108) and

\[ \sum_{\substack{x \in \mathbb{N} \setminus \{n\} \setminus \{0\} \setminus \{-1\} \setminus \{\infty\} \setminus \{-x\}}} \]

\[ (1+yz)^{-1/2} \left\{ (-2z)^{-1} \left( 1 - \sqrt{1 + yz} \right) \right\}^{-x-1} = \]

\[ \frac{1}{x+1} \]
The Gould convolution semigroup with $a = \frac{1}{2}$.

Here the delta operator is $E^\frac{1}{2} - E^{-\frac{1}{2}}$ and

The polynomials $n! g_n(x, \frac{1}{2})$ are known as the central factorial polynomials, see Roman (1984), Ch.4.1.4, and the extensive discussion in Butzer e.a. (1989). The coefficient sequence is given by (86):

$$g_n = \frac{2}{n} \left( \frac{n/2}{n} \right), \ n \geq 1.$$  

where the series converge absolutely for $|z| \leq 2$ and diverge for $|z| > 2$. The log

$$\text{Re} \left( w + \sqrt{1 + w^2} \right) > 0, \ |w| < 1.$$  

Note that by (138) the function $g(z)$ is odd.
From (130), the relation \( \Gamma(t) \Gamma(1-t) = \pi / \sin \pi t \) and Stirling's formula we have

\[
(134) \quad |g_{2m+1}| \sim C_m \, m^{-3/2} \, 2^{-2m}, \ m \to \infty.
\]

This proves the assertions on convergence.

Note Remark 5 to Theorem 1. One also might estimate \( g(x, \frac{1}{2}) \) in the same way.

The equation (103) for \( \alpha = \frac{1}{2} \) is

\[ w + \sqrt{1+w^2} = \alpha \]i with \( \alpha \in \mathbb{R} \), for some \( w \), with \( |w| < 1 \). This implies \( w = 0 \i \) with \( \alpha \in \mathbb{R} \), \( |\alpha| \leq 1 \), contradicting the relation \( w + \sqrt{1+w^2} = \alpha \).
By (44) the sequence $t_n(x) = \left(\frac{x + \frac{1}{2}n}{n}\right)$ is shelffer for $E^{-\frac{1}{2}\Delta}$. So with (41), (42) and (132)
\[
\sum_{n=0}^{\infty} \binom{x + \frac{1}{2}n}{n} z^n = \left(\frac{1}{2} z + \sqrt{1 + \frac{z^2}{4}}\right)^x \sum_{k=0}^{\infty} t_k(0) z^k.
\]

a relation equivalent with B, (25). So
\[
\sum_{k=0}^{\infty} t_k(0) z^k = \left(1 + \frac{1}{4} z^2\right)^{-\frac{1}{2}} \left(\frac{1}{2} z + \sqrt{1 + \frac{z^2}{4}}\right)^x,
\]
and
\[
\left(\frac{1}{2} z + \sqrt{1 + \frac{z^2}{4}}\right)^x = 1 - \frac{1}{2} z.
\]
The official definition by the Fibonacci Association is $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, $F_0 = 0$, $F_1 = 1$.

\[(137) \quad c_1 = \frac{1}{2} + \frac{1}{2} \sqrt{5}, \quad c_2 = \frac{1}{2} - \frac{1}{2} \sqrt{5}, \quad \]

of polynomials of convolution type such that $d_n(x) = F_n$, $n \in \mathbb{N}$. In Remark 6 to Theorem 1 we saw that this
For $|z| < |c_2|$ we have

$$|z + z^2| < |c_2| + c_2^2 = -c_2 + c_2^2 = 1,$$

so that we take the principal value for the log in (139). The log in (139) has a singularity in $-c_2$. So the series in (139) converges absolutely for $|z| < c_2$ and diverges for $|z| > c_2$. By (10)

$$-x \notin \mathbb{N},$$

since then the right-hand side of (140) has a singularity in $-c_2$. From (138), (139) and (140) we see that $q_1$ and $q_2$ defined in this way fulfill our requirements.

From (139), (138) and (135)

$$\sum_{k=1}^{\infty} \frac{F}{k-1} z^k + \sum_{k=2}^{\infty} \frac{2^k F}{k-2} z^k,$$

(141) $\frac{F}{k-1} + 2 \frac{F}{k-2} = F + \frac{F}{k-2} = L_k, k \geq 2,$

with $L_k$ a Lucas number, see F(3), F(24). Also for $k=1$ with $L_1=1$. 
From (140) and D(25)

\[ \sum_{k=0}^{\infty} \left( \begin{array}{c} n \vspace{1pt} \\ k \end{array} \right) \zeta_{k=0}^{(n)} (j) \zeta \nu = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \left( \begin{array}{c} n-j \vspace{1pt} \\ j \end{array} \right) \left( x+n-j-1 \right) , \]

where interchanging the order of summation is justified for small \( z \).

\[ \sum_{k=0}^{\infty} \left| \frac{x+k-1}{n} \right| (121 + 121^2)^r \zeta < \infty , \]

by the binomial series D(20), (25). So

\[ q_n (x) = \sum_{2j \leq n} \left( \begin{array}{c} n-j \vspace{1pt} \\ j \end{array} \right) \left( x+n-j-1 \right) = \sum_{n/2 \leq k \leq n} \left( \begin{array}{c} n-k \vspace{1pt} \\ k \end{array} \right) \left( x+k-1 \right) . \]

From (6) and (142) with

\[ \text{we find for } n \leq x, \quad \text{and } \left| x \right| \leq \rho \ldots \left( x-n \right) \ldots \left( x \right) . \]
\[(143) \quad g_n = \sum_{j \leq n} \frac{1}{n-j} \binom{n-j}{j}, \quad n \geq 1.\]

factorials in the binomial coefficients, with \(D(14)\) we derive an alternative form for \(g(x)\).

(Note that \(g(0) = 0\).)
Some additions to Chapter C.

Let \( q \) be the basic sequence and \( s_n \) a (wide sense) Sheffer sequence for \( Q \). Then \( x (x + n a) q (x + n a) \) is basic for \( E^{-a} Q \).

By Theorem 8 and \( s_n (x + n a) \) is wide.

As in the remark on notation on p. 12 let \( Q = q(D) \) be a delta operator with basic sequence \( q \) and generating function

\[ g(t) = q(t)^n \]

of the coefficient sequence.

where \( n \) is small and the remains to showing

\[ T^{-m} = \exp(-mg(D)) = \sum_{i=0}^{\infty} q_i (-m) D^i. \]

So for \( m > 1 \)

\[ p_m(x) = x \sum_{i=0}^{m-1} q_i (-m) D^i x^{m-1}/m! = \]
\[
\frac{x}{m!} \sum_{i=0}^{m-1} \frac{\Gamma(-m)}{(m-1-i)!} x^{m-1-i} = \\
\sum_{\varepsilon=1}^{m} \frac{\varepsilon}{m} \frac{\Gamma(-m)}{\varepsilon!} x^\varepsilon \varepsilon!
\]

Comparing this with (4), or noting Theorem 7, we see that there is a sequence \( f_m \) of complex numbers, with \( f_0 = 0 \), so that

\begin{equation}
(146) \quad f_m = \frac{\varepsilon}{m} \Gamma(-m), \quad m \geq \varepsilon \geq 1,
\end{equation}

and \( f_m = 0 \), \( m < \varepsilon \). In particular

\begin{equation}
(147) \quad f_m = \frac{1}{m} \Gamma(-m), \quad m \geq 1.
\end{equation}

\[\text{Note to Step 2: Using the convolution equation (1) for the basic sequence (38) with } \alpha = -1.\]
\[(n_1 a - q_1 (n_1 a), \ldots, n_n a - q_n (n_1 a)) = \sum_{k=1}^{\infty} g_k(a) z^k = \sum_{n=1}^{\infty} (n_1 a - q_1 (n_1 a)) z^n.\] 

Our examples' positive convergence radii justify our computations analytically for small \(z\).
and by application of (48) this gives

\[ (151) \quad \frac{d q(z,a)}{dz} = \frac{d'}{d(z,z,a)} \left\{ q'(q(z,a)) - a q(q(z,a)) \right\} . \]

\[ (156) \quad g(z,a) = \sum_{k=1}^{\infty} \frac{1}{ka} \binom{k}{a} z^k, \quad a \neq 0. \]

The right-hand sides in relations like (10) and (132) become simple when we substitute \( z = q(t) \) or \( z = \exp(-at) q(t) \) since
\[
\sum_{n=0}^{\infty} x^{n} a^{-n} =\]

and the definition of \( g(z,a) \) becomes

\[
\sum_{k=-\infty}^{\infty} \frac{z^k}{k!}.
\]

Causes is (157), holding only for small \( |t| \). For a generalization of (161) - (163) to more variables, see Egbrychev (1984), Theorem 1.5.1 and Carlitz (1974).

For \( \Omega = \Delta \), \( \varphi(x) = (x) \), \( g(t) = e^t - 1 \), the relations (158) - (160) become
\[(164) \sum_{n=0}^{\infty} \frac{x}{x+na} \binom{x+na}{n} \left( e^{(i-a)t} - e^{-at} \right)^n = e^{xt}, \]

We now derive some formulas resembling (59). We use the same notation as above, e.g. on p. C58.

From the convolution equation (1) for the

\[ q_n(x+y,a) - q_n(y,a), \]

Also, from (1) for the \( q_n(x,a), \) for \( n \geq 2 \).
\begin{equation}
\sum_{k=1}^{n-1} \frac{1}{ka} q_k(ka) \frac{1}{na-ka} q_{n-k}(na-ka) = q_n'(0,a).
\end{equation}

Note that \((xy)^{-1/2}(x+y)^{t-1} - x^t - y^t\) \rightarrow 0 as \((x,y) \rightarrow 0\) when \(t > 2\).

From (48) we obtain, for \(n \geq 1\),

\begin{equation}
\sum_{k=1}^{n-1} \frac{x}{\sqrt{k!} a^k} q_k(x+ka) q_{n-k}(y+(n-k)a) = 
\end{equation}

Taking \(Q = D\), \(q_n(x) = x^n/n!\), \(q_n(x,a) = x^n/\sqrt{n!}\) we find from (167) - (169)
\[(171) \sum_{k=1}^{n-1} \frac{k^{k-1}}{k!} \frac{(n-k)^{n-k-1}}{(n-k)!} = \frac{2n^{n-3}}{(n-2)!}, \quad n \geq 2,\]

and noting that \(n!\):

\[d''(0, a) = \frac{2}{n!} \left( \frac{na}{n} \right) \sum_{i=1}^{n-1} \frac{1}{na-i},\]

\[(175) \sum_{k=1}^{n} \frac{1}{k \binom{k}{k} \binom{n-k}{r-n+1/a}} = \binom{y+na}{n} \sum_{i=0}^{n-1} \frac{1}{y+na-i}, \quad n \geq 1.\]
The Lah numbers, see Lah (1955), Riordan (1958), Ch.2, Problem 16. The Lah numbers \( L_n \), \( n \in \mathbb{N} \) are defined as the coeffi-

\[ L_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k!} (n-k). \]
Some relations for permanency of convolution type by generating functions.

In what follows, $g_0$ is a sequence of polynomials of the convolution type with coefficient sequence $g_k$ where $g_0 = 0$ and $g_k(x) = \sum g_k z^k$ has a positive radius of convergence. The same then is true for

$$\sum_{n=0}^{\infty} g_n(x) z^n = \exp \{ x q(z) \},$$

see Theorem 1 and the remarks to it. We then have the following identities, for small $z$:

$$\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k} g_k(x) \lambda^{n-k} =$$

$$\sum_{k=0}^{\infty} g_k(x) \sum_{n=k}^{\infty} \binom{n}{k} z^n \lambda^{n-k} =$$

$$\sum_{k=0}^{\infty} g_k(x) \sum_{m=0}^{\infty} \binom{m+k}{m} z^{m+k} \lambda^m =$$

$$\sum_{k=0}^{\infty} g_k(x) z^k (1-\lambda z)^{-k-1} = (1-\lambda z)^{-k} \exp \{ x q(\frac{z}{1-\lambda z}) \}.$$

Changing the order of summation is allowed since for sufficiently small $z$

$$\sum_{n=0}^{\infty} \binom{n}{k} g_n(x) \lambda^n =$$

$$\sum_{k=0}^{\infty} \binom{m+k}{m} z^{m+k} \lambda^m =$$
\[
\sum_{k=0}^{\infty} \left| q_k(x) \right| |z|^k (1-|\lambda z|)^{-k-1} < \infty.
\]

\[
\sum_{k=0}^{\infty} q_{k+1}(x) \sum_{m=0}^{\infty} \binom{m+k}{m} z^{k+m} \lambda^m = \\
\sum_{k=0}^{\infty} q_k(x) z^k (1-\lambda z)^{-k-1} = (1-\lambda z)^{-1} \exp \left\{ \frac{1}{1-\lambda z} \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} (1-\lambda z)^{-n} \right) \right\}.
\]
\[
(184) \quad \sum_{n=1}^{\infty} z^n \sum_{2k+1 \leq n} \left( \begin{array}{c} n \\ 2k+1 \end{array} \right) \frac{q_k(x)}{q_k(x)} \chi^{n-2k-1} =
\]

\[
= \sum_{k=0}^{\infty} q_k(x) \sum_{n=2k+1}^{\infty} \left( \begin{array}{c} n \\ 2k+1 \end{array} \right) z^{n-2k-1} \chi^n
\]

\[
= \sum_{k=0}^{\infty} q_k(x) \sum_{m=0}^{\infty} \left( \begin{array}{c} m+2k+1 \\ m \end{array} \right) z^{m+2k+1} \chi^m
\]

\[
= \sum_{k=0}^{\infty} q_k(x) z^{2k+1} \frac{(1-\lambda z)^{-2k-2}}{(1-\lambda z)^{-2}}
\]

\[
= \frac{z}{1-\lambda z} \exp \left\{ x \cdot q \left( \frac{z^2}{(1-\lambda z)^{-2}} \right) \right\}
\]

Putting
\[
A_n = \sum_{2k \leq n} \left( \begin{array}{c} n \\ 2k \end{array} \right) \frac{q_k(x)}{q_k(x)} \chi^{n-2k},
\]

\[
B_n = \sum_{2k+1 \leq n} \left( \begin{array}{c} n \\ 2k+1 \end{array} \right) \frac{q_k(x)}{q_k(x)} \chi^{n-2k-1}, \quad B_0 = 0,
\]

we see from (183) and (184) that
\[
\sum_{n=0}^{\infty} B_n z^n = z (1-\lambda z)^{-1} \sum_{n=0}^{\infty} A_n z^n,
\]

implying
\[
(184^a) \quad B_{m+1} = \sum_{h=0}^{m} A_h \chi^{m-h},
\]

\[
(184^b) \quad A_n = B_{n+1} - \lambda B_n.
\]
We now specialize the sequence $q$ in order to obtain some binomial identities. First we take $q_n(x) = \binom{x}{n}$, see (14), (21). Then (10) becomes

$$\sum_{n=0}^{\infty} \binom{x}{n} z^n = \exp(x q(z)) = (1 + z)^x,$$

and from the product property, theorem

of generating functions,
\[(185) \quad \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} \lambda^{n-k} = \]
\[
\sum_{k=0}^{\infty} \left( -x \right)^{k} \binom{x}{n-k} \left( -\lambda \right)^{k} \left( 1-\lambda \right)^{n-k},
\]

This is \((3.100)\) with \(\lambda\) replaced by \(-\lambda\).

From \((182)\) with \(q_n(x) = \binom{x}{n}\),
\[
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k+1} \lambda^{n-k} = \]
\[
z^{-1} \left[ \left( 1 + z \right) \left( 1 - \lambda z \right)^{-x} \right],
\]
\[
z^{-1} \left[ \left( 1 - \lambda z \right)^{-x} \left( 1 + \left( -\lambda \right) z \right)^{-x} \right],
\]

So, with Main Theorem 1,
\[(186) \quad \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k+1} \lambda^{n-k} = \]
\[
\sum_{k=0}^{n+1} \left( -x \right)^{k} \binom{x}{n+1-k} \left( -\lambda \right)^{k} \left( 1-\lambda \right)^{n+1-k},
\]

This is \((3.101)\) with \(\lambda\) replaced by \(-\lambda\).

From \((183)\) with \(q_n(x) = \binom{x}{n}\),
\[(187) \quad \sum_{n=0}^{\infty} \frac{z^n}{2k \leq n} \binom{n}{2k} \binom{x}{k} \lambda^{n-2k} = \]
\[
\left( 1 - \lambda z \right)^{-x} \left[ 1 + z \right] \left( 1 - \lambda z \right)^{-x}
\]
Take $x = -\frac{1}{2}$ in (8.7):

\[
\sum_{k=2}^{n} \binom{n}{k} \left( \frac{-x}{k} \right)^{2k} = \left( \frac{1}{2} \right)^{2n-k}
\]

So, with $M$, Theorem 1.

\[
(1-2z)\sum_{n=0}^{\infty} \binom{n}{2k} z^{2k} = (1-2z)^{k}
\]
\[(1 - 2x^2 + (1 + x^2)z^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \left( \frac{1}{k!} \right) \left( 1 + x^2 \right)^{\frac{k}{2}} \left( -2x \right)^{k} z^k \]

\[\sum_{k=0}^{\infty} \left( \frac{-1}{2} \right)^{k} \sum_{j=0}^{k} \binom{k}{j} \left( 1 + x^2 \right)^{\frac{j}{2}} \left( -2x \right)^{j} z^{j+k} = \]

\[\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \binom{n-j}{j} \left( \frac{-1}{2} \right)^{j} \left( 1 + x^2 \right)^{j} \left( -2x \right)^{n-j} \]

So

\[(189) \sum_{2k \leq n} \binom{n}{2k} \left( \frac{-1}{2} \right)^{k} \lambda^{-2k} = \]

\[(-2)^n \sum_{j=0}^{n} \binom{n-j}{j} \left( \frac{-1}{2} \right)^{j} \left( 1 + x^2 \right)^{j} \left( -2x \right)^{n-j}, \]

or, with B(13) and D(13),

\[(190) \sum_{2k \leq n} (-1)^k \binom{n}{2k} \left( \frac{2}{k} \right) \left( 2x \right)^{-2k} = \]

\[2^n \sum_{2j \leq n} (-1)^j \binom{n-j}{j} \left( \frac{2n-2j}{n-j} \right) \left( 1 + x^2 \right)^{j} \lambda^{-2j} = \]

\[2^n \sum_{2j \leq n} (-1)^j \binom{n}{j} \left( \frac{2n-2j}{n} \right) \left( 1 + x^2 \right)^{j} \lambda^{-2j} . \]
From (184) with \( q_n(x) = \binom{n}{h} \),

\[
\sum_{n=1}^{\infty} z^n \sum_{2k+1 \leq n} \frac{(-1)^k}{k!} \binom{n}{k} x^{n-2k-1} =
\]

\[
= z \left( 1 - \lambda z \right)^{-2x} \left( 1 + z^2 \left( 1 - \lambda z \right)^{-2} \right)^x =
\]

\[
= z \left( 1 - \lambda z \right)^{-2x - 2} \left( 1 - 2\lambda z + (1 + \lambda^2)z^2 \right)^x.
\]

Again, only special cases are interesting.

\[
\sum_{n=1}^{\infty} z^n \sum_{2k+1 \leq n} \left( \frac{n}{k+1} \right) \frac{(-1)^k}{k!} \binom{n}{k} x^{n-2k-1} =
\]

\[
= z \left( 1 - \lambda z \right)^{-2x} \left( 1 - 2i\lambda z \right)^x =
\]

\[
= z \left( 1 - i\lambda z \right)^{-2x} \left( 1 - 2i\lambda z \right)^x =
\]

\[
= z \sum_{n=0}^{\infty} \sum_{m=0}^{m=n+1} \sum_{k=0}^{\infty} \frac{(-2x-2)^m}{m!} \binom{m}{k} \frac{x^k}{(n-1-k)!} \left( -2i \right)^{m-k} =
\]

So

\[
\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \left( -2x-2 \right)^k \binom{n-1-k}{x} \frac{x}{k!} \left( -i \right)^{n-1-k}, n \geq 1.
\]

\[
\sum_{k=0}^{n-1} \left( -2x-2 \right)^k \binom{n-1-k}{x} \frac{x}{k!} \left( -i \right)^{n-1-k}, n \geq 1.
\]
Take $x = -1$ in (19.1):

With $\binom{n}{k} = (-1)^k$,

$$
\sum_{n=1}^{\infty} z^n \sum_{2k+1 \leq n} (-1)^k \binom{n}{2k+1} \lambda^{n-2k-1} =
$$

$$
z \left(1 - 2\lambda z + (1 + \lambda^2) z^2\right)^{-1} = z \sum_{t=0}^{\infty} \left(2\lambda z - (1 + \lambda^2) z^2\right)^t =
$$

$$
z \sum_{t=0}^{\infty} \sum_{j=0}^{t} \binom{m}{j} (-1)^j (1 + \lambda^2)^j (2\lambda)^{t-j} z^t =
$$

$$
\sum_{m=0}^{\infty} z^{m+1} \sum_{2j \leq m} \binom{m-j}{j} (-1)^j (1 + \lambda^2)^j (2\lambda)^{m-2j}.
$$

So

(193)\[
\sum_{2j+1 \leq n} (-1)^j \binom{n-2j}{j} (1 + \lambda^2)^j (2\lambda)^{n-2j} =
\]

$$
\sum_{2k+1 \leq n} (-1)^k \binom{n}{2k+1} \lambda^{n-2k-1} =
$$

$$
\frac{1}{i} \lambda^n \sum_{2k+1 \leq n} \binom{n}{2k+1} \lambda^{2k+1} \lambda^{-2k-1} =
$$

$$
\frac{1}{2i} \left( (\lambda + i)^n - (\lambda - i)^n \right).
$$
We now take \( q_n(x) = x(x+2n)^{-1}(\frac{x+2n}{n}) \). Then
\[
\exp(xg(z)) = \sum (2z)^{-n} (1 - \sqrt{1-4z}) \}
\]
see (104) - (108).
From (181) we then have with (108)
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x(x+2k)^{-1}(\frac{x+2k}{k})(-4)^{n-k}
\]
(195)
\[
(1+yz)^{-1} \sum (2z)^{-n} (1+yz) - (2z)^{-1} \sqrt{1+yz} \}
\]
(195)
\[
(1+yz)^{-1} \sum (-2z)^{-1} (1 - \sqrt{1+yz}) \}
\]
So with (108) and M, Theorem 1
\[
\sum_{n=0}^{\infty} \binom{n}{k} x(x+2k)^{-1}(\frac{x+2k}{k})(-4)^{n-k}
\]
(196)
\[
\sum_{k=0}^{n} (-1)^k \frac{x}{x+2k}(\frac{x+2k}{k})^{\frac{k}{2n-k}}(-4)^{n-k}
\]
Taking here \( x=1 \) we see, with (1.428) or \( B(18) \) that the generating function (195) is given by

\[
(2z) \left( 1 - (4+2z)^{-1/2} \right) = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \binom{2n+2}{n+1} z^n,
\]

So from (195) and (196), noting that

\[
(1+2k)^{-1} \binom{1+2k}{k} = (k+1)^{-1} \binom{2k}{k},
\]

\[
\frac{1}{2} \binom{2n+2}{n+1} = \binom{2n+1}{n},
\]

and applying \( B(13) \) we obtain

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \binom{2k}{k} (-4)^{n-k} = \sum_{k=0}^{n} (-1)^k \frac{1}{k+1} \binom{2k}{k} (n-k) \frac{2}{n}^{n-k} = (-1)^n \binom{2n+1}{n}.
\]

For \( x=2 \) the r.h.s. of (196) only has the term \( k=n \). So

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \binom{2k+2}{k} (-4)^{n-k} = (-1)^n \frac{1}{n+1} \binom{2n+2}{n}.
\]
Now take $\lambda = -2$ in (194):

$$
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k} \frac{x}{x+2k} \left( \frac{x+2k}{k} \right) (-2)^{n-k}
$$

\begin{align*}
&\quad = (199) \left( 1+2z \right)^{-1} \left\{ (xz)^{-1} \left( 1+2z = \sqrt{1-yz^2} \right) \right\}^x \\
&\quad = (1+2z)^{\frac{x}{2} - 1} \left\{ (xz)^{-1} \left( \sqrt{1+2z} = \sqrt{1-2z} \right) \right\}^x.
\end{align*}

Taking $x=1$ in (199) we obtain

$$
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \left( \frac{x}{k} \right) (-2)^{n-k}
$$

\begin{align*}
&\quad = (xz)^{-1} \left( 1 - (1-2z)^{1/2} (1+2z)^{-1/2} \right) \\
&\quad = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n+1} (-1)^{k-1} \left( \frac{1}{k} \right) (n+1-k).
\end{align*}

With (3.68) and (3.69) with $y=-\frac{1}{2}$ this leads to

$$
(200) \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \left( \frac{x}{k} \right) (-2)^{n-k}
$$

\begin{align*}
&\quad = (-1)^n \frac{\beta(x+1/2)}{\beta(x)} \quad , \quad n = 2z \in \mathbb{Z} \\
&\quad = (-1)^n \frac{\beta(x+1)}{\beta(x+1)} \quad , \quad n = 2z+1.
\end{align*}

We may write (199) with $x=1$ also as
\[ \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \binom{2k}{k} (-1)^{n-k} = \]

\[ \left(1 + 2z\right)^{-\frac{1}{2}} (2z)^{-1} \left( \sqrt{1 + 2z} - \sqrt{1 - 2z} \right) = \]

\[ \left(1 + 2z\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n+1} 2^{2j+1} z^{2j} = \]

\[ n = 2k \quad ; \quad q = 2 \quad (k+1) \quad , \quad \nu = \alpha + 1. \]

Taking \( x = x \) in (199) we obtain

\[ \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \binom{2k+2}{k} (-1)^{n-k} = \]

\[ \frac{1}{2} z^{-\frac{\nu}{2}} (1 - \sqrt{1 - 4z^2}) = 2 \sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m+1} (-4)^m z^{2m}. \]

So, with \( B(15) \), or with (108),

\[ (202) \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \binom{2k+2}{k} (-1)^{n-k} = 0 \quad , \quad n \text{ odd} \]

\[ = 2 \binom{\frac{1}{2}}{m+1} (-4)^m \frac{1}{m+1} \binom{2m}{m}, \quad n = 2m. \]
From (182) with \( z_n(x) = x(x+2n)^{-1} \binom{x+2n}{n} \) and (108):

\[
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k} \frac{x}{x+2k+2} \binom{x+2k+2}{k+1} \lambda^{n-k} = 
\]

\[
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k} \frac{x}{x+2k+2} \binom{x+2k+2}{k+1} \lambda^{n-k} = 
\]

\[
z^{-1} \left[ -1 + \left\{ \frac{1-\lambda z}{2z} \right\} \left(1 - \sqrt{1-\frac{1-\lambda z-yz}{1-\lambda z}} \right) \right]^x 
\]

Specializing again to \( \lambda = -4 \), with (108),

\[
z^{-1} \left[ -1 + \left\{ \frac{1+y z}{2z} - \sqrt{1+y z} \right\} \right]^x 
\]

\[
z^{-1} \left[ -1 + (1+y z)^{\frac{x}{2}} \left\{ \frac{1-\sqrt{1+y z}}{2z} \right\} \right]^x 
\]

\[
\sum_{m=1}^{\infty} z^{m-1} \sum_{k=0}^{m} (-1)^{k} \frac{x}{x+2k} \binom{x+2k}{k} \binom{x/2}{m-k} 
\]

So,

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{x}{x+2k+2} \binom{x+2k+2}{k+1} (-4)^{n-k} = 
\]

\[
\sum_{k=0}^{n+1} (-1)^{k} \frac{x}{x+2k} \binom{x+2k}{k} \binom{x/2}{n+1-k} 
\]

For \( x=1 \) the generating function in (204) becomes, with (108)
\[ Z^{-1} \left\{ \frac{1 + \frac{2}{z}}{1 + \sqrt{1+z}} \right\} = \]

\[ -Z^{-1} \left\{ -1 + \left( -\frac{2}{z} \right)^{\frac{1}{2}} (1 - \sqrt{1+z}) \right\} = \]

\[ \sum_{m=1}^{\infty} \frac{1}{m+1} \binom{2m}{m} (-z)^{m-1} = \]

\[ \sum_{n=0}^{\infty} \frac{1}{n+2} \binom{2n+2}{n+1} (-1)^n z^n. \]

From (204) and (205) we then have, noting that

\[ \frac{1}{2k+3} \binom{2k+3}{k+1} = \frac{1}{k+2} \binom{2k+2}{k+1}, \]

\[ \frac{1}{2k+1} \binom{2k+1}{k} = \frac{1}{k+1} \binom{2k}{k}, \]

(206) \[ \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+2} \binom{2k+2}{k+1} (-4)^{n-k} = \]

\[ \sum_{k=0}^{n} (-1)^k \frac{1}{k+1} \binom{n}{k} \binom{\frac{1}{2}}{\frac{n+1-k}{2}} 4^{n+1-k} = \]

\[ (-1)^n \frac{1}{n+2} \binom{2n+2}{n+1}. \] This is (198).

For \( x = \infty \) the second sum in (205) only has the terms \( k = n \) and \( k = n+1 \), giving, with D(13),

(207) \[ \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+2} \binom{\frac{n+k+4}{2}}{k+1} (-4)^{n-k} = \]
\[ (-1)^n \frac{4}{n+1} \left( \begin{array}{c} 2n+2 \\ n \end{array} \right) + (-1)^{n+1} \frac{1}{n+2} \left( \begin{array}{c} 2n+4 \\ n+1 \end{array} \right) = \]

\[ 6 (-1)^n (n+2)^{-1} (n+3)^{-1} \left( \begin{array}{c} 2n+2 \\ n+1 \end{array} \right) \cdot \]

From (203) with \( \lambda = -2, x = 1 \) and (108), noting that \( \frac{1}{2k+3} \left( \begin{array}{c} 2k+3 \\ k+1 \end{array} \right) = \frac{1}{k+2} \left( \begin{array}{c} 2k+2 \\ k+1 \end{array} \right) \) and

\[ \frac{1}{(2k+1)} - \frac{1}{(2k)} \] we have

\[ = \frac{1}{m+1} \left( \begin{array}{c} 2m \\ m \end{array} \right), \hspace{1cm} n = 2m. \hspace{1cm} This \hspace{1cm} is \hspace{1cm} (202). \]

From (203) with \( \lambda = -2, x = 2 \) and (108)

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+2} \left( \begin{array}{c} 2k+2 \\ k+1 \end{array} \right) (-x)^{n-k} = \]

\[ z^{-1} \left[ -1 + \left( \frac{1+2z}{2z} \left( 1 - \sqrt{1-x^2 \frac{1+2z}{1+2z}} \right) \right)^2 \right] = \]

\[ z^{-1} \left[ -1 + (1+2z)(2z) \frac{1}{(1-\sqrt{1-yz^2})^{-1}} \right] = \]
\[
\sum_{j=1}^{\infty} \frac{1}{j+1} \binom{2j}{j} z^{2j-1} + 2 \sum_{m=0}^{\infty} \frac{1}{m+1} \binom{2m}{m} z^{2m} = 
\]

\[
\sum_{m=0}^{\infty} \frac{1}{m+2} \binom{2m+2}{m+1} z^{2m+1} + 2 \sum_{m=0}^{\infty} \frac{1}{m+1} \binom{2m}{m} z^{2m}.
\]

So
\[
(2a_q) \sum_{k=0}^{2m} \binom{2m}{k} \frac{1}{k+2} \binom{2k+4}{k+1} (-2)^{2m-k} = \frac{x}{m+1} \binom{2m}{m},
\]

\[
(2b_q) \sum_{k=0}^{2m+1} \binom{2m+1}{k} \frac{1}{k+2} \binom{2k+4}{k+1} (-2)^{2m+1-k} = \frac{1}{m+2} \binom{2m+2}{m+1}.
\]

Take \( q(x) = x (x+2n)^{-1} \binom{x+2n}{n} \) in (183). Then with (108)

\[
\sum_{n=0}^{\infty} z^n \sum_{2k \leq n} \binom{n}{2k} \frac{x}{x+2k} \binom{x+2k}{k} x^{-2k} = 
\]

\[
\left(1-\lambda z\right)^{-1} \left\{ \frac{(1-\lambda z)^{\lambda}}{(x z)^{-1} \sqrt{1-\lambda z + x (\lambda^2 - 4 z^2)}} \right\}^x 
\]

For \( x = 1 \) this becomes

\[
\sum_{n=0}^{\infty} z^n \sum_{2k \leq n} \binom{n}{2k} \frac{1}{k+1} \binom{2k}{k} x^{n-2k} = 
\]

\[
(x z^2)^{-1} \left\{ (1-\lambda z - \sqrt{1-\lambda z + (\lambda^2 - 4 z^2) z^2}) \right\}^x.
\]
\[
1 - \frac{1}{4} \lambda^2 - (2z^2)^{-1} \sum_{t=1}^{\infty} \left( \frac{1}{t} \right) (-2\lambda z + (\lambda^2 - 4) z^2)^t = 1 - \frac{1}{4} \lambda^2 - \frac{1}{2} \sum_{t=2}^{\infty} \left( \frac{1}{t} \right) \sum_{i=0}^{t} \binom{t}{i} (\lambda^2 - 4)^i (-2\lambda)^{t-i} z^{t-i+i} = 1 - \frac{1}{4} \lambda^2 - \frac{1}{2} \sum_{s=0}^{\infty} (\frac{1}{s+2}) \sum_{i=0}^{s+2} \binom{s+2}{i} (\lambda^2 - 4)^i (-2\lambda)^{s+2-i} z^{s+2-i} = 1 - \frac{1}{4} \lambda^2 - \frac{1}{2} \sum_{n=0}^{\infty} z^n \sum_{i=0}^{n} \binom{n+2-i}{i} (\frac{1}{2}) (\lambda^2 - 4)^i (-2\lambda) = n+2-2i
\]
So,

\[
-\frac{1}{2} \sum_{i=0}^{n} \binom{n+2-i}{i} (\frac{1}{2}) (\lambda^2 - 4)^i (-2\lambda), \quad n \geq 1.
\]

Now take \( \lambda = z \), \( x = 1 \) in (210):

\[
\sum_{n=0}^{\infty} z^n \sum_{2k \leq n} \binom{n}{2k} \frac{1}{k+1} \binom{2k}{k} z^{n-2k} = 1 + (2z)^{-1} \left( 1 - \sqrt{1 - 4z} \right) = \sum_{j=1}^{\infty} \frac{1}{j+1} \binom{2j}{j} z^{j-1} = \sum_{n=0}^{\infty} \frac{1}{n+2} \binom{2n+2}{n+1} z^n.
\]
So, as also seen from (211) and B(16), cf. Riordan [1973],

\[
(\frac{1}{2}) \sum_{2k \leq n} \binom{n}{2k} \frac{1}{k+1} \binom{2k}{k} z^{n-2k} = \frac{1}{n+2} \binom{2n+2}{n+1}.
\]
For $\lambda = 2$, $x = 2$ in (210), with (108) and D(13),

$\sum_{n=0}^{\infty} z^n \sum_{2k \leq n} \binom{n}{2k} \frac{1}{k+1} \left( \frac{1}{2k} \right) \frac{2^{n-2k}}{k} = (1-2z)^{-2} \left\{ \frac{(2z)^{-1}}{(1-\sqrt{1-4z}) - 1} \right\}^2 = (1-2z) \sum_{j=\lambda}^{\infty} \frac{1}{j+1} \left\{ \left( \frac{2j+2}{j} \right) - 2 \left( \frac{2j}{j} \right) \right\} Z^{j-2}.$

So,

$\sum_{2k \leq n} \binom{n}{2k} \frac{1}{k+1} \left( \frac{1}{2k} \right) \frac{2^{n-2k}}{k} = H \frac{(2h+2)!}{(h+1)! (h+4)!} \left( \frac{h}{h+n+3} \right).$
simply from (184) and (215) below. 

Taking \( q(x) = x (x+2z)^{-1} (x+2z)^n \in (184) \) we find, with (108)

\[
(214) \sum_{n=1}^{\infty} z^n \sum_{2k+1 \leq n} \frac{n!}{2k+1} \frac{x}{x+2k} \left( \frac{x+2k}{k} \right)^{n-2k-1} = \]

\[
z^{-1} \frac{1}{(2z^2)^{1}} \left( 1 - 2z - \sqrt{1-4z^2} \right) \]

For \( x = \lambda = z \) this becomes

\[
\sum_{n=1}^{\infty} z^n \sum_{2k+1 \leq n} \frac{n!}{2k+1} \frac{1}{k+1} \left( \frac{x+2k}{k} \right)^{n-2k-1} = \]

\[
z \left\{ (2z^2)^{-1} \left( 1 - 2z - \sqrt{1-4z^2} \right) \right\}^2 = \]

\[
z^{-1} \left\{ -1 + (2z^2)^{-1} \left( 1 - \sqrt{1-4z^2} \right) \right\}^2. \]

In the same way as in the proof of (213) we derive that this is equal to

\[
z^{-1} \sum_{j=1}^{\infty} \frac{(2j)!}{j! (j+2)!} \left( -1 \right)^j z. \]

\[
z \sum_{n=0}^{\infty} \frac{(2n+2)!}{(n+1)! (n+3)!} n z^n. \]

This shows that
\[ (2.15) \quad \sum_{2k+1 \leq n} \binom{n}{2k+1} \frac{1}{k+1} \left( \frac{2k+2}{k} \right) 2^{n-2k-1} = \]

\[ 2^1 \binom{n}{2n+2} \frac{1}{(n+1)!} \left( \frac{2n+3}{n+2} \right) 2^{n+1} = \]

In the notation of (184a) and (184b) the \( (184a) \) now is implicit in the proof of (213) and (184b) becomes

\[ (2.16) \quad \sum_{h=0}^{m} \frac{(2h+2)!}{(h+1)! (h+4)!} \left( \frac{h^2 + h + 3}{2} \right) 2^{m+1-h} = \]

\[ (m+1) (2m+4)! \left\{ (m+2)! (m+4)! \right\}^{-1} \]

Now take \( Q_n(x) = x (x+\frac{1}{2}n)^{n-1} \left( \frac{x+\frac{1}{2}n}{\binom{n}{n}} \right) \) in (181). Then with (132)

\[ (2.17) \quad \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k} x (x+\frac{1}{2}k)^{n-k} \left( \frac{x+\frac{1}{2}k}{\binom{k}{k}} \right) = \]

\[ (1-z)^{-x-1} \left( \frac{z}{2} + \sqrt{i - 2z - (\lambda^2 + \frac{1}{4}) z^2} \right)^{2x} \]

Here we take \( x = \frac{1}{2} \), \( \lambda = \frac{1}{2} i \). This gives
\[
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \left(\frac{1}{2} k + \frac{1}{2}\right) 2^{k-n} i^{-k} = \\
(1 - \frac{1}{2} i z)^{-2} (\frac{1}{2} z + \sqrt{1 - i z}) = \\
\left(\sum_{j=0}^{\infty} \left(j+1\right) z^j \right) \left(\frac{1}{2} z + \sum_{h=0}^{\infty} \binom{\frac{1}{2}}{h} (-1)^h i^h z^h \right)
\]

With M. Theorem 1, this is equal to

\[
\sum_{n=0}^{\infty} n z^{-n} i^{-n-1} z^n + \\
\sum_{n=0}^{\infty} z^n i^n \sum_{j=0}^{n} (-1)^{n-j} \left(j+1\right)\binom{n}{j} 2^{-j}.
\]

So

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \left(\frac{1}{2} k + \frac{1}{2}\right) 2^{k-n} i^{-k} = \\
i^{-n} z^{-n} + \sum_{j=0}^{n} (-1)^{n-j} \left(j+1\right)\binom{n}{j} 2^{-j}.
\]

The terms with \(k\) odd give the imaginary part of the l.h.s. and are zero except for \(k=1\). This leads to the identity \(n2^n = n2^n\).

From the terms with \(k\) even, the real part, we obtain

\[
\sum_{h=0}^{\frac{n}{2}} \binom{n}{2h} \frac{1}{2h+1} \left(\frac{1}{2} h + \frac{1}{2}\right) 2^{h-n} (-1)^h = \\
\sum_{j=0}^{n} (-1)^{n-j} \left(j+1\right)\binom{n}{j} 2^{-j}.
\]
With \( B(28) \) and \( B(15) \) we may write this as

\[
(219) \sum_{2h \leq n} \binom{n}{2h} \frac{1}{1-2h} \binom{2h}{h} 2^{-2h-n} = \sum_{2h \leq n} (-1)^h \binom{n}{2h} \binom{1/2}{h} = \sum_{j=0}^n (-1)^{n-j} (j+1) \binom{1/2}{n-j} 2^{-j}.
\]

When we take \( \lambda = i/2 \) and let \( x \to -i/2 \) in \((217)\) we obtain in the same way as in \((218)\):

\[
\sum_{j=0}^n (-1)^{n-j} (j+1) \binom{1/2}{n-j} 2^{-j}.
\]

This is the same relation as \((218)\): from \( B(27) \) and \( B(28) \) we have

\[
(2h+1)^{-1} \binom{h+1/2}{2h} = (1-2h)^{-1} \binom{h-1/2}{2h}.
\]

Take \( q_n(x) = x(x+\frac{1}{2}n)^{-1} \binom{x+\frac{1}{2}n}{n} \) in \((182)\).

Then with \((132)\)

\[
(220) \sum_{n=0}^\infty \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \frac{2x}{2x+k+1} \binom{x+k+1/2}{k+1} \lambda^{n-k} = \ldots
\]
\[ z^{-1} \left\{ \frac{1}{k^2 + \lambda^2} \right\} K_{k}(x) \sqrt{1-2\lambda x + (\lambda^2 + \frac{1}{4})x^2} \]

With \( \lambda = \frac{1}{2} i \), \( x = \frac{1}{\lambda} \); this becomes

\[ \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \frac{(-1)^n}{k^{k+1}} \sum_{i=0}^{n-1} i^{n-k} \]

\[ = z^{-1} \left[ -1 + \sum_{m=0}^{\infty} z^m \sum_{j=0}^{m} 2^{-j} \left( \begin{array}{c} m \end{array} \right) (-1)^{m-j} i^m \right] = \sum_{n=0}^{\infty} 2^{-n-1} i^n z^n + \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n+1} 2^{-j} \left( \begin{array}{c} n+1 \end{array} \right) (-1)^{n+1-j} i^{n+1} \]

So

\[ \sum_{k=0}^{\infty} \frac{(-1)^n}{k^{k+1}} \sum_{i=0}^{n-1} i^{n-k} \cdot \frac{1}{k^{k+1}} \left( \begin{array}{c} k+1 \end{array} \right) \left( \begin{array}{c} k-1 \end{array} \right) i^{k-1} = \]

\[ 2^{-n-1} + i \sum_{j=0}^{n+1} \frac{(-1)^{n+1-j} }{k^{k+1}} \left( \begin{array}{c} n+1 \end{array} \right) \left( \begin{array}{c} n+1-j \end{array} \right) i^{n+1} \]

The real part of the l.h.s. is the only non-zero term with \( k \) even, viz. \( k = 0 \), giving a trivial identity. From the imaginary parts (\( k \) odd) we obtain, with \( B(28) \) and \( B(15) \),
\[(221) \sum_{2h+1 \leq n} (-1)^n \binom{n}{2h+1} \frac{1}{2h+3} \binom{h+3/2}{2h+2} 2^{2h+1-n} = \]
\[\sum_{2h+1 \leq n} \binom{n}{2h+1} \frac{1}{2h+1} \binom{h+2}{h+1} 2^{-2h-n-3} = \]
\[2^{n-1} \sum_{2h+1 \leq n} (-1)^n \binom{n}{2h+1} \binom{1/2}{h+1} = \]
\[\sum_{j=0}^{n+1} (-1)^{n-j} \binom{1/2}{n+1-j} 2^{-j}. \]

Taking \( \lambda = \frac{1}{2} \) and letting \( x \to -\frac{1}{2} \) in (220) we obtain in a way similar to (221)
\[\sum_{2h+1 \leq n} (-1)^n \binom{n}{2h+1} \frac{1}{2h+1} \binom{h+1/2}{2h+2} 2^{2h+1-n} = \]
\[\sum_{j=0}^{n+1} (-1)^{n+j-1} \binom{1/2}{n+1-j} 2^{-j}. \]

This is the same relation as (221) since by B(27) and B(28) with \( m = h+1 \)
\[(2h+3)^{-1} \binom{h+3/2}{2h+2} = - (2h+1)^{-1} \binom{h+1/2}{2h+2}. \]

Taking \( q_n(x) = 2x (2x+n)^{-1} \binom{x+n/2}{n} \) in (183) or (184) does not seem to give simple formulas.

The relations (181) = (184) with \( q_n(x) = x (x-n)^{-1} \binom{x-n}{n} \) do not lead to
formulas fundamentally different from (194) - (221). See (128) and the remark following it.
Expansion of derivatives

Let $Q = q(D) = \sum_{k=1}^{\infty} q_k D^k$ be a delta operator with basic sequence $q_n$ and coefficient sequence $g_k$ with generating function $g(z) = \sum_{k=1}^{\infty} g_k z^k$.

Let $f$ be a polynomial of degree $\leq n$.

From (24), when $n \geq 1$,

$$f(y + x) - f(y) = \sum_{k=1}^{n} g_k (x) Q^k f(y).$$

Dividing by $x$ and letting $x \to 0$, we have (by 16)

$$(225) \quad f'(y) = \sum_{k=1}^{n} g_k Q^k f(y), \quad n \geq 1.$$

By induction on $r$, we then have

$$(226) \quad D^r f(y) = \sum_{j=1}^{n} g_j Q^j f(y), \quad n \geq r \geq 1.$$

For the step $r \to r+1$, since $DQ = QD$,

$$D^{r+1} f(y) = D \left( \sum_{i=1}^{n} g_i Q^i f(y) \right) =$$

$$= \sum_{i=1}^{n} g_i Q^i \left( \sum_{h=1}^{n} g_h Q^h f(y) \right) =$$

$$= \sum_{i=1}^{n} \sum_{h=1}^{n} g_i g_h Q^{i+h} f(y) =$$

$$= \sum_{j=1}^{r+1} g_j Q^j f(y).$$
We may derive (226) immediately by noting that \( D = q(q(D)) = q(R) \), see Theorem 66, so that

\[
g_k = (\alpha a)^{k-1} / k!, \quad g_j^* = e^{-j} (\alpha a)^{j-1} / (j-1)!
\]

\[
(228) f^{(r)}(y) = \sum_{j=r}^{n} \frac{e^{-y}}{j!} \left( \frac{y}{y-j} \right)^{j-r} f^{(j-r)}(y-j), \quad 1 \leq r \leq n,
\]

for polynomials \( f \) of degree \( \leq n \).

Taking \( f(y) = y^n / n! \) in (227) we obtain (172).

The delta operator \( \Delta = \Delta \) has basic sequence \( \eta(x) = (x)^n \), see (44), (21). By the results following (44) and by (15)

\[
(229) g_k = (-1)^{k-1} / k!, \quad g_j^* = e^{-j} \eta^j(j, x) / j!,
\]

\[
\text{C90}
\]
for \( k \geq 1, j \geq 5 \), where the \( s(j, \xi) \) are Stirling numbers of the first kind, see D(28)-(31). From (225), (226) we then have

\[
(230) \quad f'(y) = \sum_{k=1}^{n} (-1)^{k-1} k^{-1} \Delta^k f(y), \quad n \geq 1,
\]

and (38),

\[
(232) \quad \binom{\gamma}{n} \sum_{i=0}^{n-1} \frac{1}{i} = \sum_{k=1}^{n} (-1)^{k-1} k^{-1} \binom{\gamma}{n-k}, \quad n \geq 1.
\]

and (238) and putting \( u - y = n \),

\[
(233) \quad \binom{u}{n} \sum_{i=0}^{n-1} \frac{1}{u-i} = \sum_{k=1}^{n} k^{-1} \binom{u-k}{n-k}, \quad n \geq 1,
\]

a relation that also may be derived from (232).

In (229) we have

\[
(234) \quad q^{\ast}_{\ast n} = (-1)^{n} \sum_{k=1}^{n-1} k^{-1} (n-k)^{-1}
\]

\[
(-1)^{n} n^{-1} \sum_{k=1}^{n-1} (k^{-1} + (n-k)^{-1}) = 2n^{-1} (-1)^{n} H_{n-1}, \quad n \geq 2,
\]
(2.36) \[ f'(x) = n b (x + nb)^{-x} (x + nb) + \]
\[ x (x + nb)^{-1} (x + nb) \sum_{i=0}^{n-1} (x + nb - i)^{-1} \]
\[ = (x + nb)^{-1} (x + nb) \left\{ 1 + x \sum_{i=1}^{n-1} (x + nb - i)^{-1} \right\} , \]

\( n \geq 2 \), From (84) or C(40)

\[ \Delta k \rho / \Delta - kb = -kb \Delta \rho / \Delta \]

Then (2.30) becomes
Now let \( Q = \sum_{k=1}^{n} \text{E}^{-a} \Delta_{k-b} \), with \( a \neq 0 \), the delta operator of the Gould polynomials (84). Then \( q_{k} = (k\text{b})^{-1} (k\text{a})^{-1} \), \( k \geq 1 \), by (86). Take

\[
\sum_{k=1}^{n} \frac{(-1)^{k-1}}{(y+k\text{b})(y+n\text{b})} \left( \frac{y+n\text{b}}{n-k} \right), \ n \geq 2.
\]

\[
\text{Q}\ f(x) = \text{E}^{-a} \Delta_{k-b} \ f(x) = \text{E}^{k(b-a)} \ f(x) =
\text{E}^{k(b-a)} \ x \left( x+(n-k)\text{b} \right) \left( \frac{x+(n-k)\text{b}}{n-k} \right) ^{-1} \left( \frac{x+n\text{b}-k\text{a}}{n-k} \right) ^{-1},
\]

and from (225) and (236) we have

\[
(y+n\text{b})^n \left( \frac{y+n\text{b}}{n} \right)^{1} \left( \sum_{i=1}^{n-1} (y+n\text{b}-i)^{-1} \right) =
\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k\text{a}} \left( \frac{y+k\text{b}-k\text{a}}{y+n\text{b}-k\text{a}} \right), \ n \geq 2.
\]
the author is not very clear and the connection is easily missed... ly from his, but the connection is easily made... give the theory and a number of his examples (some- different proofs) as far as needed.

Definition 1. Let $a_{nk}$ and $b_{nk}$, $k = 0, \ldots, n$, $n \in \mathbb{N}$, be two arrays of complex numbers. Extend their $k < n$, and define

Remark 1. The infinite matrices $A$ and $B$ defi- are lower triangular. Products above are lower triangular matrices are defi-
Remark 2. The lower triangular matrix $A$ defines a linear operator $A$ on all of $C^N$ by $Ax = y$ with $y$ given by (19). So we have to interpret the relations (19) and (18) in Definition 1 as follows. For any given $y$, the set of linear equations (19) in the unknowns $x_1, x_2, \ldots$ has the unique solution (18). Or the set (18) in the unknowns $y_0, y_1, y_2, \ldots$ has the unique solution (19). Therefore the definition of a pair of inverse relations often is stated in the form that "(19) is equivalent to (18)."

Remark 3. The above definitions may be stated in terms of finite-dimensional linear algebra. Let $U = \{ u_{nk}, n, k \in N \}$ be an infinite matrix and put $U_N = \{ u_{nk}, n, k = 0, \ldots, N \}$. Then the matrices $A$ and $B$ in Definition 1 are an inverse pair if and only if the matrices $A_N$ and $B_N$ are each other's inverses for every $N \in N$. This follows from the fact that for lower triangular matrices $U$ and $V$ we have

$$(UV)_N = U_N V_N.$$
\[ y = u x \text{ if and only if } y = w, \ldots \text{ where } x^{(N)} = (x_0, \ldots, x_N), y^{(N)} = (y_0, \ldots, y_N). \]

Remark. The lower triangular matrix \( \{ a_{nk}, \; k \in \mathbb{N}, n \in \mathbb{N} \} \) has an inverse if and only \( a_{nk} \neq 0 \) for any \( n \). Then \( a_{nk} \) is at least the solution \((a^q)\) for any \( x \).

Each of the conditions \( AB = I \) and \( BA = I \) is equivalent to \( A \) and \( B \) being each other's inverses. So the arrays \( a_{nk} \) and \( b_{nk} \) as in Definition 1 are an inverse pair if and only if

\[ \sum_{i=1}^{n} a_{nj} b_{ik} = \delta_{nk}, \; k=0, \ldots, n, \; n \in \mathbb{N}. \]
(4) \( a_{nk} = \binom{n}{k} \), \( b_{nk} = (-1)^{n-k} \binom{n}{k} \),

or

We prove the above example by (2):

\[
\sum_{j=k}^{n} \binom{n}{j} (-1)^{j-k} \binom{j}{k} = \binom{n}{k} \sum_{j=k}^{n} \binom{n-k}{j-k} (-1)^{j-k} \\
= \binom{n}{k} \delta_{nk} = \delta_{nk}. \quad \text{(see (13))}
\]

A single inverse pair may be cast into different forms, the equivalence of which is not always clear at first sight. By putting \( \xi_n = (-1)^n x_n \) or \( u_n = x_n/n! \), \( v_n = y_n/n! \), we may write (5) as

(6\(\alpha\)) \( y_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \xi_k \).
\[(76)\quad u_n = \sum_{k=0}^{n} (-1)^k \frac{\varphi_k}{(n-k)!},\]
to be interpreted in the same way.

Lemma 2: When the matrices \(\{a_{nk}\}\) and \(\{b_{nk}\}\) are inverses of each other, so are

\[
\text{Proof.} \quad \sum_j a_{nj} b_{jk} = \\
\Rightarrow \quad \varphi(j) a \cdot \varphi(k) b = \varphi(k) \Rightarrow a \cdot b \cdot L
\]
and \( a^n = \text{\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet}\)  

\( b^n = \text{\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet\textbullet} \) 

Different, e.g. (28) and each of (83)-(85); (4) and Riordan (1968), Ch. 3, problem 18.

Riordan (1968) presents some inverse

\[
(x_n) \text{ with fixed } r \in \mathbb{N}. \text{ This pair is defined by (9) and (10) with the note}
\]

\[
(86) \quad x_n = \sum_{kt \leq n} \beta_{nt} y_{n-kt}.
\]

with fixed \( r \in \mathbb{N}. \) This pair is defined in (9) and (10) with the note

\[
(9) \quad a_{nj} = 0, \text{ elsewhere,}
\]

and similarly for \( b_{nj}. \)

In what follows we will derive some general properties of inverse pairs. Then we will discuss some techniques for finding inverse relations and prove a,
In fact, we proved the pair of involutions \((1)\) or \((8)\). When we succeed, for some specific sequence \(x\), to evaluate the sum \((109)\) or \((89)\), the relation \((18)\) or \((86)\) immediately gives a second formula. As an example take \((5)\). From \((G(122))\) and \((G(147))\) we have

\[
(109) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (x+k)^{-1} = \Delta^n x^{-1} = \Delta^n x^{-1}
\]

Pars of this kind of derivation are \((62)\) to \((64)\) with \((59)\) and \((\phi(125))\) from \((\phi(121))\) \(^{(x)}\).

The inverse pairs found in this chapter will be collected in the list on pp. IR.70-77 in order to make easier the use of companionship in finding binomial sums.

\(^{(x)}\) See also pp. M19-22.
General properties. First we consider two methods to derive new inverse pairs from given ones.

Lemma 3. Let \( a_{nk} \) and \( b_{nk} \) be an inverse pair. Then so are \( a'_{nk} \) and \( b'_{nk} \) with \( l, \) for fixed \( \tau \in \mathbb{N}, \)
\[
\delta'_{nk} = \delta_{nk}, \quad n < \tau,
\]
and \( \delta'_{nk} \) defined similarly from \( b_{nk}. \)

Proof. From the fact that the inverse of a two-block matrix is found by taking the inverses of the (square and

Lemma 2 and (4), with \( st \neq 0, \)
gives the inverse pair

\[
a'_{nk} = \delta'_{nk}, \quad n < \tau, \quad a'_{nk} = 0, \quad n \geq \tau, \quad k < \tau,
\]
Applying Lemma 4.5.1, we have \( a_n(k) = \delta_{nk}, \) \( k \in \mathbb{N}_0, \) \( \psi(k) = 1, \) \( k \leq \varepsilon, \) \( \psi(k) = s^{-\varepsilon}, \) \( k \geq \varepsilon, \)

leads to the inverse pair

\[
\begin{align*}
    a_{nk} &= b_{nk} = \delta_{nk}, \quad n \leq \varepsilon, \\
    a_{nk} &= b_{nk} = 0, \quad k < \varepsilon \leq n, \\
    a_{nk} &= \binom{n-\varepsilon}{k-\varepsilon} s^k t^{n-k}, \quad \varepsilon \leq k \leq n, \\
    b_{nk} &= s^{-n} \binom{n-\varepsilon}{k-\varepsilon} (-t)^{n-k}, \quad \varepsilon \leq k \leq n.
\end{align*}
\]

Riordan (1968) provides us with the

\( n \to n+1 \) principle, e.g. on p. 46 and p. 49.
$$\sum_{j=k}^{n} a'_{nj} b'_{jk} = \sum_{j=k}^{n} a_{n+p,j+p} b_{j+p,k+p} = \sum_{i=k+p}^{n+p} a_{n+p,i} b_{i,k+p} = \delta_{n+p,k+p} = \delta_{nk}$$

Remark 1. In an inverse pair of the form (8) the shift over $p$ only affects $n$ in $\alpha_{nk}$ and $\beta_{nk}$. When $(89), (86)$ is a pair of inverse relations, so is $(89), (86)$ with $\alpha_{nk}, \beta_{nk}$ replaced by $\alpha'_{nk} = \alpha_{n+p,k}, \beta'_{nk} = \beta_{n+p,k}$. For when $a'_{nj}$ and $a_{nj}$ correspond to $\alpha'_{nk}$ and $\alpha_{nk}$ in the sense of (9), we have

$$a'_{nj} = \alpha'_{n,c(n-j)} = \alpha_{n+p,c(n-j)} = a_{n+p,j+p}$$

when $c|n-j$ and $a'_{nj} = 0 = a_{n+p,j+p}$ elsewhere. Or: Theorem 14 below.
Example: Taking $a_{nk}$ and $b_{nk}$ as in (4) and writing $\binom{n}{k} = \binom{n}{n-k}$ we find

$$a_{nk} = \binom{n}{n-k}, \quad b_{nk} = \binom{n}{k}.$$ 

Since $a_{nk}$ and $b_{nk}$ are polynomials in $p$, an argument similar to Remark 2 shows that by taking $p = -x-1$ we have the inverse pair

$$b_{nk} = (-1)^n \binom{n}{n-k} = \binom{n}{n-k}.$$ 

Lemma 2 with $\gamma(m) = 1$, $\Upsilon(m) = (-1)^k$ then gives the inverse pair

$$(13^q) \quad a_{nk} = b_{nk} = (-1)^k \binom{x-k}{n-k}.$$ 

Cf. Koutras (1994), (2.4), and (2.5).

One also might verify (2) directly for (13^q).

Em. (13) is (1) in moderate and denomi...
of upper triangular matrices is defined by finite sums, so there is no difficulty over the definition of inverses. But if we

however, the following 'transposed' version of Remarks 2 and 3 to Definition 1

are equivalent in the sense that (14.6) is the unique solution of (14.9) for every y
Lemma 5. Let $v$ and $w$ be lower triangular $N \times N$ matrices. They are each other's inverses if and only if the relations

\[
\begin{align*}
v = w^t x \quad &\text{and} \quad x = w^t y. \\
\end{align*}
\]

It follows that $w^t v^t = I$, so that $vw = (w^t v^t)^t = I$.

Proof. The above relations may be written $y = v^t x$ and $x = w^t y$. It follows that $w^t v^t = I$, so that $vw = (w^t v^t)^t = I$.

(44) It follows with Lemma 5 from the relations (44) that $A_1 B_1 = I$ for every fixed $N$. Then $AB = I$ by Remark 3 to Problem 2. The solution is found below...

\[
\begin{align*}
(15a) & \quad y_n = \sum_{k=n}^{\infty} a_{kn} x_k, \quad n \in N^o, \\
(15b) & \quad x_n = \sum_{k=n}^{\infty} b_{kn} y_k, \quad n \in N^o,
\end{align*}
\]
The double sum converges absolutely, i.e.

\[ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |b_{kn} \cdot q_{ik} \cdot x_i| < \infty. \]
the principle of inclusion and exclusion or as Poincare's theorem. See Feller (1957), Ch. IV, Takacs (1967).
Comparing (17b) with (3.1) and (1.3) in Ch. IV of Feller (1957) we see

\[ (17^d) \quad E \{ (X^k) \} = \sum_{1 \leq i < j < \ldots < \ell \leq N} \sum_{1 \leq k \leq N} P(A_i, A_j, \ldots, A_\ell), \]

\[ 1 \leq k \leq N \text{ and } E \{ (X^0) \} = 1. \]

From (17b) and (17a), by interchanging summations,

\[ (17^e) \quad \sum_{n=0}^{N} P(X=n) z^n = \sum_{k=0}^{N} E \{ (X^k) \} (z-1)^k. \]

This is in fact (6).
ter C. Throughout this section \( q_n \neq \emptyset \) will denote a sequence of polynomials of convolution type with coefficient sequence \( q, q, q, \ldots \), where \( q = 0 \). When \( q \) is a basic sequence to its delta-operator is denoted \( \delta_q \). Cf. the Remark.

\[
\begin{align*}
 a_{nk} &= q_{n-k} (a), \\
 b_{nk} &= q_{n-k} (-a).
\end{align*}
\]

Proof. From C (1) and C (5)
\[
\sum_{j=k}^n q_{n-j} (a) q_{j-k} (-a) =
\sum_{i=0}^{n-k} q_i (-a) q_{n-k-i} (a) = q_n (0) = \delta_{nk},
\]
and the theorem follows with (2).

Examples. With C, (20), (21), (75) and (84) this gives the inverse pairs, for the
\[ n - k \quad n - k \quad n - k \]
(19) \[ a_{nk} = \binom{a}{n-k}, \quad b_{nk} = \binom{-a}{n-k}, \]

\[ a_{nk} = a (a+n-b-kb)^{n-k-1} / (n-k)!, \quad b_{nk} = -a (-a+n-b-kb)^{n-k-1} / (n-k)!, \quad a \neq 0, \]

\[ b_{nk} = -a (-a+n+kb-kb) \binom{-a}{n-k}. \]

**Theorem.** For \( u+n \neq 0, n \in \mathbb{N} \), we have

\[ \sum_{i=0}^{n-k} I_n (u+kb+ib)(-u-nb)(-u-nb+(n-k-i)b)^{-1}. \]
Remark 2. Let \( u \to 0 \) in (24). Noting that (2) continues to hold in the limit, we find the inverse pair (with C(16)):

\[ a_{nk} = \frac{(u + nb)(-u - kb)^{n-k}}{(n-k)!}, \]
\[ b_{nk} = \frac{(u + nb)^{n-k-1}}{(n-k)!}. \]
(24) \( b_{nk} = \binom{n-k}{n-k} \quad \text{Equal to (84)} \).

(30) \( b_{nk} = (u+nb)(u+nb+(n-k)a)^{n-k-1}/(n-k)! \),

whereas (25) and C(77) give (30) with \( u = 0 \).

From (24) and C(84), one finds the pair

\[ a_{nk} = (u+nb)(u+nb+(n-k)a)^{-1}\left(-u-kb+(n-k)a\right) \binom{n-k}{n-k} \],

(31) \[ b_{nk} = (u+nb)(u+nb+(n-k)a)^{-1}\left(u+nb+(n-k)a\right) \binom{n-k}{n-k} \],

whereas (25) and C(86) give (31) with \( u = 0 \).
\[(32) \quad q(\rho^{-1} D) = \rho^{-1} \exp(\lambda D) q(D),\]

Theorem 8. Then we have the inverse pair
\[
a_{nk} = q_{n-k}(\alpha + k z, \nu),
\]
\[
b_{nk} = q_{k-n}(\rho \alpha - kp z, \nu),
\]
\[
\sum_{j=k}^{n} a_{nj} b_{jk} = 1.
\]
\[ q_{n-k}(0,v) = \delta_{nk} \]

**Examples.** The delta operator \( R = D \) satisfies (32) for every \( p \neq 0 \) with \( \lambda = 0 \).

For \( R = D \) we have
\[ q_n(x,v) = x(x + nv)^{n-1}/n! \quad \text{see } C \ln f(z) \]

Then, with \( z = v - \frac{1}{p} u \), we have from (33) putting \( p^{-1} u = w \), the inverse pair
\[ v = \frac{1}{p} w + v \]

**Applying Lemma 2 with \( \psi(m) = \psi(m) = u + mb \)**

to (30) one sees that (34) is equivalent.

The delta operator \( x \mapsto u = x \exp(-y) - 1 \)

satisfies (32) with \( p = \lambda = -1 \). We have
\[ \alpha(x,v) = x/(x +nv)^{n-1}(x +nv) \]
\[ u \left( x + n \gamma - \frac{1}{4} \right)^4 \]

\[ n=0 \]

\[ x + n \gamma - \frac{1}{4} \]
degree \( q_n \) = \( n \) so that, for \( x \) fixed, 

\[ q_m (y + mb), \quad m = 0, \ldots, N, \]

is a basis for the polynomials in \( y \) of degree \( \leq N. \)

The same holds for \( q_m (x - y + mb), \quad m = 0, \ldots, N. \)

The above result implies that the

*own version and the new version* of the above statement follows from Remarks 3 to Definition 1.

**Example** , see Riedman (1968), p. 98, (4). For \( A = D \), \( q_n (x) = x^n/n! \), the pair (36) is

\[ a_{nk} = b_{nk} = (-1)^k (x + 2nb)(x + nb + kb)^{n-k-1}/(n-k)! \]

This pair is equivalent to (34) with \( a = x \). 


\( q_n(x) = \sum_{k=0}^{n} q_{nk} \frac{x^k}{k!}, \quad n \in \mathbb{N}, \)

be sequences of polynomials of convolution type with coefficient sequence \( f_k \) and \( g_k \), respectively, so that by \( C(4) \), noting that \( a_{0k}^{*} = \delta_{0n}\),

\( p_{nk} = f_{n}, \quad q_{nk} = g_{n} \)

The umbral composition of \( f \) and \( g \) (in this order) is the sequence \( r \)

\( r_n(x) = \sum_{j=0}^{n} p_{nj} q_j(x) = \sum_{k=0}^{n} r_{nk} \frac{x^k}{k!}, \)

\( r_{nk} = \sum_{j=k}^{n} p_{nj} q_{jk} \)

It was proved in \( C_2 \) Theorem 14, that the sequence \( r_n \) is of convolution type, so

\( r_{nk} = h_{n}^{**} \)

The sequence \( r_n \) is basic if and only if
\( h(z) = \sum h'_n z^n \). Then, see \( C \), Theorem 14,
\[
(44) \quad h(z) = g(f(z)).
\]

(40) \(-\) (44) that we have

**Theorem 10.** The pair \((40)\) is an inverse pair if and only if \( f^* \) and \( g \) are each other's compositional inverses.

**Remark.** Since \( f^* = g = 0 \), we can have \( g(f(z)) = z \) only if \( f^* f \neq 0 \), i.e., if \( \rho^*_n \) and \( \rho_n \) are basic sequences.

**Examples.** Let \( f(z) = z(1-z)^{-1} \) and \( g(z) = f^{-1}(z) = z(1+z)^{-1} \). Then
\[
\begin{align*}
f^k(z) &= \sum_{n=k}^{\infty} (n-1)\frac{1}{(n-k)} z^n, \\
g^k(z) &= \sum_{n=k}^{\infty} (-1)^{n-k}\left(\frac{1}{(n-k)}\right) z^n,
\end{align*}
\]
giving the inverse pair
\[
(45) \quad \rho_{n+k} = f^*_n = (n-1)\frac{1}{(n-k)}, \quad \rho_{n+k} = g^*_n = (-1)^{n-k}\left(\frac{1}{(n-k)}\right).
\]
which is equal to (13) with \( p = -1 \).
Written in the form (8) this pair states the equivalence of

\[
\begin{align*}
    f^k(z) &= \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i} z^{k+i} = \\
    \sum_{n=k}^{2k} \binom{k}{n-k} a^{n-k} b^{2k-n} z^n = \\
    \sum_{n=k}^{\infty} \binom{k}{n-k} a^{n-k} b^{2k-n} z^n.
\end{align*}
\]

Now let \( q = f^{-1} \). By C. Theorem 6, the sequence \( q^n \) is basic for the delta operator \( Q^n \).

\[
Q = q(D) = q^{-1}(D) = f(D) = aD + bD.
\]

By C. (37) we have \( T = bI + aD \), with \( n \geq 1 \).
inverse pair

\( p_{nk} = f_n^* = \binom{k}{n-k} a^{n-k} b^{2k-n} \),

\( q_{nk} = g_n^* = \frac{k}{n} \binom{2n-k-1}{n-1} (-a)^{n-k} b^{k-2n} \), \( n \geq 1 \),

\( q_{00} = 1 \).

We also may write \( q_{nk}, n \geq 1, \) as

\( q_{nk} = \frac{k}{2^{n-k}} \binom{2n-k}{n} (-a)^{n-k} b^{k-2n} \).

One also might obtain \( g = f^{-1} \) from \( f \) and then the series expansion of \( g^k \) from (108).
These were introduced by Cahit (1976, 1978a, 1978b), see also Kyriakoussis (1993).

Let the sequence \( y \) be of convolution type. If then...
\[ \beta(z) = \sum_{n=1}^{\infty} \beta_n z^n = z \exp(-\beta v(z)) \]

and \( \beta = 1 \). By Theorem 10 the array \( B^{k*} \)

satisfied, by taking

\[ v(z) = -B^{-1} \log(z^{-1} \beta(z)) \]

We also may prove directly that

\( a^*_{nk} = a^{k*}_n \), \( b_{nk} = \beta^{k*}_n \) by using the
we have the expansion

\[ f(x+y) = \sum_{k=0}^{n} q_k(y) q_k(x), \quad q_k(y) = Q_k^t f(y). \]

Applying this to \( E^{-u} Q \) with basic sequence \( q_n(x,u) = x(x+nu)^{-1} q_n(x+nu) \), see \( \Omega \) in Theorem 8, and \( f(x) = q_n(x,u) \) we find by noting that

\[ E^{-hu} Q_n^t q_n(x,u) = E^{-hu} Q \sum_{k=0}^{n-k} (x+ku-v) q_n(x,v), \]

\[ (x+y)(x+y+nu)^{-1} q_n(x+y+nu) = \sum_{k=0}^{n} \frac{x}{x+ku} q_k(x+ku) \frac{y+kv-ku}{y+nu-ku} q_k(y+nu-ku). \]

**Theorem 11.** We have the inverse pair

\[ a_{nk} = (x+ku-v)(x+nu-v)^{-1} q_{n-k}^t (x+nu-v), \]

\[ b_{nk} = (-x+ku-\nu)(-x+nu-\nu)^{-1} q_{n-k}^t (-x+nu-\nu). \]

**Proof.** We have \( a_{nn} = b_{nn} = 1 \). For \( 0 \leq k \leq n-1 \).
\[
\sum_{j=k}^{n} (\alpha + j \nu - j \mu) (\alpha + n \nu - j \mu) \sum_{j=1}^{n-j} (\alpha + n \nu - j \mu).
\]

\[
\sum_{i=0}^{n} \frac{\alpha + k (u - v) + i \mu}{\alpha + k (u - v) + i \mu} q_i (-\alpha + k (u - v) + i \mu).
\]

\[
\frac{\alpha + k (v - u) + i \nu - i \mu}{\alpha + k (v - u) + (n-k) \nu - i \mu} q_{n-k-i} (\alpha + k (v - u) + (n-k) \nu - i \mu).
\]

Let \( P \) be another delta operator with basic sequence \( p_n \). Then from (53) with \( y = 0 \),

\[
l_n (x) = \sum_{k=0}^{n} p_k q_k (0) p_k (x),
\]

\[
p_n (x) = \sum_{k=0}^{n} q_k p_k (0) q_k (x).
\]

Since \( p_n \), \( n=0, \ldots, N \) and \( q_n \), \( n=0, \ldots, N \) are two bases for the polynomials of degree \( \leq N \) we have by Remark 3 to Definition 1.

**Theorem 12.** Under the above assumptions we have the inverse pair

\[
a_{nk} = p_k q_n (0), \quad b_{nk} = q_k p_n (0).
\]
**Examples.** Let \( R \) be a delta operator with basic sequence \( r_n \). Take \( P = E^{-v} R, Q = E^{-w} R \), so that with \( C_2 \), Theorem 8, we have \( p_n(x) = x(x+nu)^{-1} r_n(x+nu), \)

\[ q_n(x) = x(x+nu)^{-1} r_n(x+nu). \]

Then

\[ P_{q_n(0)} = E^k(x) \quad E^{-k} R \quad q_n(0) = \]

\[ q_{n-k}(x-ku) = (x-ku)(nu-ku)^{-1} r_{n-k}(nu-ku), \]

and we have the inverse pair

\[ a_{nk} = (x-ku)(nu-ku)^{-1} r_{n-k}(nu-ku), \]

\[ b_{nk} = (x-ku)(nu-ku)^{-1} r_{n-k}(nu-ku), \]

which is equal to (55) with \( q_n \) replaced by \( r_n \) and \( \alpha = 0 \).

By taking \( P = \Delta, p_n(x) = \binom{x}{n} \), \( R = D, q_n(x) = x^n/n! \) we find the inverse pair

\[ a_{nk} = k! S(n,k)/n!, \quad b_{nk} = k! s(n,k)/n! \]

from (56), with \( S(n,k) \) and \( s(n,k) \) the Stirling numbers of the second and first kind, see D (28), (29). Or from D (18) (49) and Lemma 2.

Taking \( q_n(x) = x^n/n! \) in (55) we find

\[ q_n(x) = \binom{x}{n} \]

the inverse pair (55) becomes
(386) \[ b_{nt} = (-\alpha + nu - kv)^{1} \left( \begin{array}{c} -\alpha + nu - kv \\ n - k \end{array} \right) \]
This pair was found by Youla and A. K. (1973). A generalization to n-dimensional indices and variables was given by A. K. (1973).

\[(5q_a) \ a_{00} = 1, \ a_{nk} = (-1)^k \binom{n}{k} \prod_{i=1}^{n} (a_i + k b_i), \ n \geq 1,\]

\[(5q_b) \ b_{nk} = (-1)^k \binom{n}{k} (a_{k+1} + k b_{k+1}) \prod_{i=1}^{n} (a_i + n b_i).\]

**Proof.** We verify (2). We have \(a_{00} = b_{00} = 1,\)

\(a_{nn} = (-1)^n \prod_{i=1}^{n} (a_i + n b_i), \ b_{nn} = (-1)^n \prod_{i=1}^{n} (a_i + nb_i),\)

\(n \geq 1.\) For \(0 \leq k \leq n-1,\)

\[
\sum_{j=k}^{n} a_{nj} b_{jk} = (a_{k+1} + k b_{k+1}) \sum_{j=k}^{n} (-1)^{k+1} \binom{n}{j} (\binom{j}{k}) \psi(k, n, j),
\]

where

\[
\psi(k, n, x) = \prod_{i=k+2}^{n} (a_i + x b_i), \ k \leq n-2,
\]

\[
\psi(n-1, n, x) = 1. \quad \text{So}
\]
\[ \sum_{j=k}^{n} a_{nj} b_{jk} = \]
\[ (a_{k+1} + k b_{k+1}) \binom{n}{k} (-1)^{n-k} \sum_{h=0}^{n-k} (-1)^{n-k-h} \binom{n-k}{h} \psi(k, n, k+h) \]
\[ = (a_{k+1} + k b_{k+1}) \binom{n}{k} (-1)^{n-k} \Delta^{n-k} \psi(k, n, k+x) \bigg|_{x=0} \]

**Examples. Taking** \( a_i = a \), \( b_i = b \) **in** (59) **gives the inverse pair**

\[ (60^a) \quad a_{nk} = (-1)^k \binom{n}{k} (a + k b)^n, \]
\[ (60^b) \quad b_{nk} = (-1)^k \binom{n}{k} (a + k b)(a + nb)^{-k-1}. \]

This also follows from (26) with \( u = a \) by Lemma 2 with \( \psi(m) = (u + mb)^m/m! \) and \( \psi(m) = (-1)^m (u + mb)^m/m! \).

**Taking** \( b_i = b \), \( a_i = a - i + 1 \) **in** (59) **we find** the inverse pair

\[ a_{nk} = (-1)^k \binom{n}{k} n! \binom{a + k b}{n}, \]
\[ b_{nk} = (-1)^k \binom{n}{k} \frac{a + k b - k}{a + nb - k} \binom{a + nb}{k}^{-1}. \]
\[(0^n)_{mk} = (-1)^k \frac{a + n - 6 - k}{k} \cdot \]

This relation is equivalent to (1.5), (1.6).

\[
\binom{k}{
\begin{array}{c}
\uparrow \\
\text{up to} \\
\downarrow
\end{array}
\begin{array}{c}
\binom{n}{k} \\
\text{and} \\
\binom{n}{n-k}
\end{array}
\] \cdot \]

Taking \( b = b', a = a + 1 \) gives a variant of (61): write \( b = -b', a = -a' \).

The pair (59) gives another example of the derivation of formulas by the use of inverse pairs. This is the technique discussed on p. 77 and applied to derive (10b) from (10a). Taking \( b = b \) in (59a) and putting

\[
\begin{align*}
 f_0(x) &= 1, \\
 f_n(x) &= \prod_{i=1}^{n} \left( a_i + b x \right), \quad n \geq 1,
\end{align*}
\]

we have, with \( G(2x), (24b) \)

\[
\sum_{k=0}^{n} q_{mk} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f_n(k) =
\]
(64) \[ \sum_{k=0}^{n} \binom{n}{k} b^k \frac{a+k-b}{a+n-b-k} \binom{a+n-b}{k}^{-1} = 1. \]
Generating functions.

\[ \sum_{j=k}^{n} a_{nj} b_{jk} = \sum_{j=k}^{n} c_{n-j} d_{j-k} = \]

\[ y_n = \sum_{k=0}^{n} a_k b_{n-k} \]

(67b) \[ x_n = \sum_{k=0}^{n} d_k y_{n-k} \].

Let \( C(z), D(z), X(z), Y(z) \) be the (formal) generating functions of the sequences \( c_n, d_n, x_n, y_n \). Then (65) is equivalent with

(68) \[ C(z) D(z) = 1 \].
This corresponds to the fact that we may write (67a) and (67b) as \( Y(z) = C(z)X(z) \)
and \( X(z) = D(z)Y(z) \).

Remark. The relation (68) also implies inverse pairs of the form (8) with \( \kappa > 1 \), since it implies \( C(z)D(z^{-1}) = 1 \), giving the equivalence of

\[
C(z) = (1-z)^{-u-1} = \sum_{k=0}^{\infty} \binom{u+k}{k} z^k,
\]

\[
(\gamma_0^a) a_{nk} = \binom{u+n-k}{n-k} = (-1)^{n-k} \binom{-u-1}{n-k}.
\]
\((f_0^k) \ \beta_{nk} = (-1)^{n-k} \binom{u+1}{n-k}, \)

which is equivalent to \((19)\). By the above remark we have the equivalence of

\((19) \ \ \ N = \sum (u+1) x_n, \ \ \ \ n \in \mathbb{N}_0.\)

From \((20)\) and \(B(13), (15)\) we have

\((-yz)^{1/2} = \sum_{k=0}^{\infty} \binom{2k}{k} \left( \frac{2k}{z} \right) z^k, \)

now the inverse take in the form \((16)\)

\((g_0^k) \ \ x_n = \sum_{k=0}^{\infty} \binom{2k}{k} \left( \frac{2k}{z} \right) y_{n-k}, \ \ n \in \mathbb{N}_0. \)

From \(C(106), (108)\) we have, taking for \(C(2)\)

\(C(z) = (2z)^{-1} (1 - \sqrt{1 - 4z}) = \sum_{k=0}^{\infty} \frac{1}{k+1} \binom{2k}{k} z^k. \)

Then from the above generating functions
giving the inverse pair in the form (74a)

\[ y_n = \sum_{k=0}^{n} (k+1)^{-1} \binom{2k}{k} x_{n-k}, \quad n \in \mathbb{N} \]

\[ y_0 = x_0. \]

Taking \( \Delta(z) = 1 - z - z^2 \),

For \( C(k) = z(\pi^2 - 1) \) with \( c_j = B_j / k! \), where the \( B_j \) are the Bernoulli numbers, and \( d_k = n^{k-1} / (k+1)! \), we find the pair

\[ \text{(74c)} \quad x_n = \sum_{k \leq n} B_k x_{n-k} / k!, \]

\[ \text{(74d)} \quad x_n = \sum_{k \leq n} y_{n-k} / (k+1)! \].
**Direct verification of inverse relations of type (8).**

**Theorem 4.** The relations (8a) and (8b) form an inverse pair if and only if

\[ \sum_{n} \alpha \beta = \delta \quad \text{where } n, m \in \mathbb{N}. \]

... given by (8b) is a 'solution' of the set of equations (8a). So we must have...

\[ \sum_{ktj=m} \beta_{nk} \alpha_{n-ktj} = \delta_{om}, \quad m \leq n \in \mathbb{N}, \]

is also sufficient.

**Remark 1.** By a similar argument, or by interchanging (8a) and (8b) we see that

\[ \sum_{ktj=m} \beta_{nk} \alpha_{n-ktj} = \delta_{om}, \quad m \leq n \in \mathbb{N}, \]

is also necessary and sufficient for (8).

**Remark 2.** When (8) is a pair of inverse...
\[ \lambda^m \delta_{om} = \delta_{om}. \]

**Examples.** These are from Riordan (1968), but with different proofs: We verify (77) or (75) for \( n \geq 1 \). In all cases considered we

\[ (77a) \quad x_n = \sum_{k, \ell \leq n} \binom{n}{k} x_{n-k} - \frac{n}{k, \ell \leq n} \sum_{k, \ell \leq n} (-1)^k \frac{n}{n-k+\ell} \binom{n-k+\ell}{k} x_{n-k+\ell}, \quad n \geq 1. \]
\[ n(-1)^n \Delta^m (n-1 + (1-x) x)_{m-1} \bigg|_{x=0} / m! = 0, \]

since \((a x + b)_k\) is a polynomial of degree \(k\).

\[(78b) \quad x_n = \sum_{k \leq n} \left\{ \left( \begin{array}{c} n \\ k \end{array} \right) + (1-x) \left( \begin{array}{c} n \\ k-1 \end{array} \right) \right\} y_{n-k} \quad n \in \mathbb{N}^+,
\]

where \(\langle x \rangle = 0\).

\[ l = \sum_{k=0}^{m} (-1)^k \binom{n}{k} \binom{m-k}{m-k} = \]

\[ \sum_{k=0}^{m} \binom{n}{k} \binom{m-k-1}{m-1} = \binom{m-1}{m-1} \]

\[(1-x) \sum_{h=0}^{m-1} \binom{n}{h} \binom{m-1-h}{m-1} = (1-x) \binom{m^2-1}{m-1} = - \binom{m^2-1}{m} \]
\[ (79) \quad \binom{n-1+k}{k} - \binom{n-1+k}{k-1} = \frac{n-k \xi}{n} \binom{n-1+k}{k}, \quad k \geq 1. \]

**Proof.** For \( m \geq 1 \), with \( D, (24), (26) \),
\[
T = \sum_{h=0}^{m-1} (-1)^h \binom{n+h}{h} \binom{n-m \xi}{m-1-h} = \sum_{h=0}^{m-1} (-1)^h \binom{n-h}{h} \binom{n-m \xi}{m-1-h} = -\xi \sum_{h=0}^{m-1} \binom{m-1-h}{h} \binom{n-m \xi}{m-1-h} = \xi \sum_{h=0}^{m-1} \binom{m-1-h}{h} \binom{n-m \xi}{m-1-h} = \xi \binom{m-1}{m \xi}. \]
\[
\sum_{k \leq n} \binom{n-k}{k} \binom{n-k}{k-1} x_{n-k}, \quad n \in \mathbb{N},
\]

\[
\sum_{k=0}^{m} \binom{n+k}{k} \binom{n-m}{m-k} = (-1)^m \binom{m}{m-1} \binom{m}{m-1} = (-1)^m \binom{m}{m-1} \binom{m}{m-1} = (-1)^m \binom{m}{m-1}.
\]

For \( m=0 \) the above sum is equal to 1.

Again, these identities, i.e. (75) and (75*), for (79) and (80), are not restricted to \( n \in \mathbb{N} \) and
Some further examples and special cases.

From (23) with \( u = 1 \), \( b = 2 \), from (24), and

\[
\begin{aligned}
\hat{c}_{nk} &= (-1)^{n-k} \frac{2k+1}{2n+1} \binom{n-k}{k}, \\
\end{aligned}
\]

\[
\hat{b}_{nk} = \begin{cases} 
\frac{u+n b}{u+k b} \binom{u-k b}{n-k} & u \neq 0, \\
0 & \text{otherwise}.
\end{cases}
\]
which is (29) with \( a_{nk} \) and \( b_{nk} \) interchanged. For \( b=2 \),

\[
a_{nk} = \binom{2n}{n-k}, \quad b_{k0} = 1,
\]

\[
a_{nk} = \binom{2n+1}{n-k}, \tag{86}
\]

\[
b_{nk} = (-1)^{n-k} \frac{2n+1}{2k+1} \binom{n+k}{n-k} = (-1)^{n-k} \frac{2n+1}{n+k+1} \binom{n+k+1}{2k+1}.
\]

The next two inverse pairs derive from the Abel summation formula \( C \) (83\(a\)). They are equivalent with and proved in the same way as (2) and (26) in Riordan (1986), Ch. 3.2.
For \( k=n \) the above sum is equal to 1.

The limit for \( x \to 0 \) of \((87)^x\) is
Inverses with Euler and related numbers.

The secant coefficients $\alpha_0, \alpha_2, \alpha_4, \ldots$ are

\[(96) \quad \alpha_0 = \alpha_2 = 1, \quad \alpha_4 = 5, \quad \alpha_6 = 61,\]

Taking $z = iw$ in (95) we find, for $|z| < \pi/2$,

\[(97) \quad z(e^w + e^{-w})^{-1} = \sum_{k=0}^{\infty} (-1)^k \alpha_{2k} w^{2k}/(2k)!.\]

We have $\alpha_{2k} = (-1)^k E_{2k}$, where the Euler numbers $E_n$ and the Euler polynomials $E_n(x)$ are defined by $E_n = E_n(1/2)$,

\[(96a) \quad \sum_{n=0}^{\infty} E_n(x) z^n/n! = \frac{1}{2} (e^z + 1)^{-1} e^{xz},\]

\[(96b) \quad \sum_{n=0}^{\infty} E_n z^n/n! = 2 (e^z + e^{-z})^{-1}.\]

We refer to Nörlund (1924), Roman (1984) Note that $E_n = 0$, $n$ odd.

The Euler numbers should not be confused with the Eulerian numbers defined in Graham, Knuth and Patashnik (1988), Carlitz (1959).

We also define the numbers $\beta_0, \beta_2, \beta_4, \ldots$ by
$z = i w$ we find, for $|w| < \pi$,

$$ (99) \quad i w (e^w - e^{-w})^{-1} = \sum_{k=0}^{\infty} (-1)^k \beta_{2k} w^{2k} / (2k)! . $$

We have $\beta_{2k} = (-1)^k D_{2k}$. The coefficients $D_n$ are defined in Nörlund (1924), Ch. 1, in Con-
We will use a number of inverse pairs involving binomial coefficients in a simple way, and also the above coefficients. The

\[ n! = \sum_{2k \leq n} \binom{2k}{k}! \binom{n-2k}{n-2k}! \sum_{j=0}^{n-j} \frac{(-1)^j}{j!} (n-j)! \]

with \( c_j = 1/j! \), \( j \) even, \( c_i = 0 \), \( i \) odd.

\[ \frac{x_n}{n!} = \sum_{j=0}^{n} d_j \frac{y_{n-j}}{(n-j)!} \]

In the same way, with (95), or by considering \( i^k \times k \) and \( i^k \times k \), we find the inverse pairs.
with $\beta_{2k}$ as in (98). To see this write (105a) as

with $c_j = 1/(j+1)!$, $j$ even, $c_j = 0$, $j$ odd.
A further pair:

\[(107^a) \quad y_n = \sum_{k=0}^{n} \frac{(2n)}{2k} \times k,\]

\[
\frac{y_n}{(2n)!} = \sum_{k=0}^{n} \frac{1}{(2n-2k)! \times (2k)!} \times k.
\]

\[
\frac{x_n}{(2n)!} = \sum_{k=0}^{n} (-1)^{n-k} \frac{\alpha_{2n-2k}}{(2n-2k)! \times (2k)!} \times y_k.
\]
Proof. Write (108a) as
\[ y_n = \sum_{k=0}^{n} \frac{1}{(2n-2k+1)!} \frac{x^k}{(2^k)!} \]

We have with (97) and (104), (65) - (68), the inverse
which is (109). The final part is

\[
(110^a) \quad y_n = \sum_{k=0}^{n} \frac{(2n+2)}{(2k+1)} x_k,
\]

Proof. Write (110^a) as

\[
y_n = \sum_{k=0}^{n} \frac{1}{x_k + 1}.
\]

\[
\frac{n}{(2n+1)!} = \sum_{k=0}^{2n} \frac{(-1)^{n-k} 2n-xk! (2n-2k)! x_k}{(2k+2)!},
\]

which is (110^b).
\begin{align}
\tag{113} \gamma_n &= \sum_{2k \leq n} \binom{n}{2k} x_k, \quad n \in \mathbb{N}_0, \\
\text{and hence has a solution as is seen.}
\end{align}

\begin{align}
\tag{114} j &= \frac{j}{2^s}, \quad j \in \mathbb{Z}, \quad j \geq 0.
\end{align}

Then (113) may be written as

\begin{align}
\tag{115} \gamma_n &= \sum_{j=0}^{n} \binom{n}{j} x_j, \quad n \in \mathbb{N}_0.
\end{align}

It follows from (4) that (115) implies

\begin{align}
\tag{118} \gamma_m' &= \gamma_{2m} = \sum_{k=0}^{m} \binom{2m}{2k} x_k, \\
\tag{119} \gamma_m'' &= \gamma_{2m+1} = \sum_{k=0}^{m} \binom{2m+1}{2k} x_k.
\end{align}
and when there is a solution, it follows from (107) and (108) that it is given by
\[ X_m = \sum_{k=0}^{m} (-1)^{m-k} \binom{2m}{2k} \alpha_{2m-2k} Y_{2k}, \]
\[ 0 \leq m \leq m - k \binom{2m+1}{2k+1} \beta_{m+1} \gamma, \]

so where the existence of a solution in satisfying (113) is known, the relations (116), (117), (120) and (121) are proved.

Similar considerations apply to the set of equations in the unknowns \( x_0, x_1, \ldots \)
\[ Y_n = \sum_{2k+1 \leq n} \binom{n}{2k+1} X_k, \quad n \in \mathbb{N} \]

Obviously we should have \( Y_n = 0 \). Putting
Writing (122) as

\[(127) \quad y_{2m+1} = \sum_{k=0}^{m} \left( \frac{2m+1}{2k+1} \right) x_k, \quad m \in \mathbb{N}_0, \]

\[(128) \quad y_{2m+2} = \sum_{k=0}^{m} \left( \frac{2m+2}{2k+1} \right) x_k, \quad m \in \mathbb{N}_0, \]

relations (125), (126), (129) and (130) hold.

Generalizations of (113) and (122) are...

...in the unknowns $x_0, x_1, \ldots$, has at least one solution if and only if
(133) \( x_m = \sum_{k=0}^{2^m} \beta_{2^m,k} y_k , \ m \in \mathbb{N}_0.\)

Proof: Let (131) have the solution \( x_n, n \in \mathbb{N}_0.\)

Now let (132) hold. Define \( x_m \) by (133) and then \( x_f \) by (134). By considering \( n = 2m \) and \( n = 2m+1 \) separately, we see that

so that \( x_n \) is a solution of (131) by (134). The unicity follows from the first part of the proof.
Then the solution is unique and is given by
\[ (137) \quad x_m = \sum_{j=0}^{2m+1} \mathbf{b}_{2m+1,j} \mathbf{Y}_j, \quad m \in \mathbb{N}. \]

Proof. Let \( x_n, n \in \mathbb{N}, \) be a solution of (135). Put
\[ (138) \quad x'_j = x_{j-1}V_2, \quad j \text{ odd}, \quad x'_j = 0, \quad j \text{ even}. \]

and since \( a_{nk} \) and \( b_{nk} \) form an inverse pair we must have (136) and (137).

\[ \ln \qquad k = 0, \ldots, n \]
so that \( x_n \) is a solution of (135) by (138). The uniqueness follows from the first part of the proof.
Under (132) the unique solution \( x_n \) of (131) has to satisfy (139a) and (139b). Since the matrix \( A \) has an inverse we

We then obtain the generalization of (120), (121). When \( a_{2m+1, 2m} = 0 \) for some \( m \), the set of equations (139b) may have more than one solution.

\[
(141^a) \quad y''_m = y_{2m+2} = \sum_{k=0}^{m} a_{2m+2, 2k+1} x_k,
\]

where (140a) has a unique solution and (141a) only when \( a_{2m+2, 2m+1} \neq 0 \). We then obtain the generalization of (129), (130).
Pairs of equalities between sums.

This small piece of theory is inspired by the special case in Riordan (1968), Ch1, p.30, Prob1. 6d. In what follows $U = \{ u_{ij}; i \in N, j \in N \}$ is an invertible lower triangular matrix. This should be interpreted as in Remark to Definition 1. We may write \( (41) \)
(144) \( U V^{-1} = V U^{-1} \),

or, when \( U \) and \( V \) satisfy our conditions,

We now show that \( V^{-1}U = U^{-1}V \):

\[
(V^{-1}U)_{n,k} = \sum_{j=k}^{n} (V^{-1})_{n,j} \ u_{j,k} =
\sum_{j=k}^{n} (-1)^{n-j} \binom{n-j}{n-k} \binom{j-k}{j-k} =
(-1)^{n} \sum_{h=0}^{n-k} \binom{n-k}{h} \binom{n-k-h}{k} = (-1)^{n} \binom{-k-1}{n-k},
\]
\[
\frac{(-1)^k}{n-y} \sum_{h=0}^{n-k} (h+k-y) \binom{n-y}{n-k-h} \binom{y+1}{h} = \\
\frac{(-1)^k}{n-y} \left\{ (k-y) \binom{n+1}{n-k} + \frac{(n-k)(y+1)}{n+1} \binom{n+1}{n-k} \right\} \\
\]

By D(26) and (3.111) with \( \varepsilon = 1 \). This is seen to be equal to
\[
(-1)^k \frac{k+1}{n+1} \binom{n+1}{n-k},
\]

We have

\[
\ldots
\]
\[ V^{-1} = A^{-1} U^{-1} = U^{-1} B^{-1} \quad \text{and} \quad (143) \quad \text{and} \quad (144) \quad \text{reduce to} \quad A^{-1} = A \quad \text{and} \quad B^{-1} = B. \]

An example of an invertible lower triangular matrix \( A \) with \( A = A^{-1} \) is, by (13a) and the remark to Lemma 2,

\[ A = U V = A^{-1} = V U, \]

\[ a_{nk} = \sum_{j=k}^{n} (-1)^{n-j} \binom{n-j}{j-k} = 1 \]

\[ \sum_{i=0}^{n-k} (-1)^{i} \binom{n-k-i}{i} \binom{y-k}{i} = (-1)^{n-k} \binom{-k-1}{n-k} = (-1)^{n} \binom{n}{k}, \]

which is (149) with \( x = c = -1 \). Or see (4) - (6).

Remark: Note that when \( U, V, A \) are invertible lower triangular matrices satisfying \( V = UA \) but no other assumptions, and

\[ (4.94) \sum_{k=0}^{n} u_{nk} (a+k)^{-1} = \sum_{j=0}^{n} v_{nj} \frac{a}{a+j}, \]
When \( U \) and \( V \) satisfy (143) then also, as is easily seen, for invertible lower triangular \( C \)

\[
(C^{-1} U C \quad \text{and} \quad C^{-1} V C)
\]

so

\[
(\Psi^T U \Lambda \Psi^T - \sum^n_{j=1} \Psi(j) \Psi(j)^T \Psi(k)) \cdot u_k = 0
\]
Remark. For (153) and (154) to make sense when \( k = 0 \) we define \( a_{nk} = b_{nk} = 0, \quad k \geq 0, \) and set \( d_1 \neq 0. \) We then have

\[
(155) \quad a_{n+1,0} = c_n a_{n,0}, \quad b_{n+1,0} = -\frac{c_n}{d_0} b_{n,0}, \quad n \geq 0.
\]

This theorem is Proposition 1 in Halbeisen and Hungerbühler (2000). Note that their \( S_+(k, \hbar) \) plays the role of our \( \mathbf{b}_{nk}. \) We give a proof different from theirs. By induction on \( n \) we verify (2):

\[
\sum_{j=k}^n a_{nj} b_{jk} = \delta_{nk}, \quad 0 \leq k \leq n.
\]

For \( n = 0 \) this is (152). For \( n = 1 \) we have with (152)–(155)
\[ a_{10} \cdot b_{00} + a_{11} \cdot b_{10} = a_{0} \cdot b_{00} + (c_{0} a_{01} + d_{0} a_{00}) \cdot b_{10} = \]
\[ a_{0} \cdot b_{00} + d_{0} a_{00} \cdot b_{10} = c_{0} a_{00} \cdot b_{00} - d_{0} a_{00} \cdot b_{00} = 0, \]

\[ \sum_{j=k}^{n+1} a_{n+1, j} \cdot b_{j, n+1} = a_{n+1, n+1} \cdot b_{n+1, n+1} = \]
\[ d_{n} \cdot a_{\ldots} \cdot \frac{1}{b_{\ldots}} = a_{\ldots} \cdot b_{\ldots} = 1. \]

\[ \sum_{j=k}^{n+1} a_{n+1, j} \cdot b_{j, k} = \sum_{j=k}^{n+1} (c_{n} \cdot a_{n, j} + d_{n} \cdot a_{n, j-1}) \cdot b_{j, k} = \]
\[ d_{n} \sum_{j=k}^{n+1} a_{n, j-1} \cdot b_{j, k} = d_{n} \sum_{i=1}^{n} a_{n, i} \cdot b_{i+1, k} = \]
\[-\frac{d^{n}c_{k}}{d_{k}} \sum_{i=k}^{n} a_{ni} b_{ik} + \frac{d_{n}}{d_{k-1}} \sum_{i=k-1}^{n} a_{ni} b_{i,k-1} = 0.\]

For \(c_{n} = d_{n} = 1\) we obtain the inverse pair \((y)\):
\[a_{nk} = \binom{n}{k}, \quad b_{nk} = (-1)^{n-k} \binom{n}{k}.\]

For \(c_{n} = n, \ d_{n} = 1\) we find the pair
\[a_{1} = |s(n, k)| = (-1)^{n-k} s(n, k).\]

where the \(s(n, k)\) and \(\bar{s}(n, k)\) are the Stirling numbers of the first and second kind, see [][1],[](30) and Graham, Knuth and
\[(19) \quad a_{nk} = \binom{a}{n-k}, \quad b_{nk} = \binom{-a}{n-k}\]

\[(20) \quad a_{nk} = a (a + nb - kb)^{n-k-1} / (n-k)!, \quad a \neq 0.
\]

\[(21) \quad b_{nk} = -a (-a + nb - kb)^{n-k-1} / (n-k)! , \quad a \neq 0.
\]
\[ a_{nk} = (u+nb)(-u-kb)^{n-k}/(n-k)! , \]
\[ \beta_{nk} = (u+nb)^{n-k-1}/(n-k)! , \quad u+mb \neq 0, \quad m \in \mathbb{N} \]

\[ a_{00} = b_0 = 1, \quad a_0 = -b, \quad a_i = 0, \quad n \geq 2, \]

\[ a_{nk} = nk^{-1}(-kb)^{n-k}/(n-k)!, \quad 1 \leq k \leq n, \]
\[ \beta_{nk} = (nb)^{n-k}/(n-k)!, \quad n \geq 1. \]

\[ a_{n0} = 1, \quad a_0 = (-1)^n b, \quad n \geq 1, \quad a_{nk} = \frac{-1}{k} \left( \frac{(-1)^k}{(n-k)!} \right), \quad 1 \leq k \leq n, \]

\[ \beta_{n0} = \binom{n}{k}(n-k)^{n-k-1}, \quad u + mb \neq 0, \quad m \in \mathbb{N} \]

\[ a_{nk} = (-1)^{n-k} (u+nb)(u+kb-(n-k)a)^{n-k-1}/(n-k)!, \]
\[ \beta_{nk} = (u+nb)(u+nb+(n-k)a)^{n-k-1}/(n-k)!, \]

\[ a_{nk} = (u+nb)(u+kb-(n-k)a)^{-1} \left( -u-kb+(n-k)a \right), \]
\[ \beta_{nk} = (u+nb)(u+nb+(n-k)a)^{-1} \left( u+nb+(n-k)a \right). \]
\[
\alpha_{nk} = (\alpha + kv - kw)(\alpha + nu - kw)^{n-k-1} / (n-k)!, \\
(3y) \quad 0, \quad 1, \quad \ldots, \quad \ldots, \quad \ldots, \quad \ldots, \quad \ldots, \quad \ldots, \quad \ldots
\]

\[
\int p_{nk} = c^{n-k \left(\frac{-1}{2}n - \frac{1}{2}k\right)} a^{\frac{1}{2}(n-k)} , \quad n-k \text{ even},
\]

\[
q_{nk} = \frac{k}{n} \binom{n}{\frac{1}{2}n - \frac{1}{2}k} a^{\frac{1}{2}(n-k)} c^{-n}, \quad n-k \text{ even, } n \geq 1,
\]

\[
q_{00} = 1, \quad q_{nk} = p_{nk} = 0, \quad n-k \text{ odd, } (c \neq 0)
\]

This is identical with

\[
\gamma_n = \sum_{2j \leq n} \left(\frac{2j-n}{j}\right) c^{n-2j} a^j x_{n-2j} ,
\]

\[
(47) x_0 = \gamma_0 , \quad
\]

\[
(47) x_n = \sum_{2j \leq n} \left(1 - \frac{2j}{n}\right) \binom{n}{j} a^j c^{-n} \gamma_{n-2j} , \quad n \geq 1
\]
\((a+mb \notin \{0, 1, \ldots, m\}, m \in \mathbb{N}_0)\).

\[
\gamma_n = \sum_{2k \leq n} \binom{u+k}{k} x_{n-2k},
\]

\((\mathcal{F}1)\)
\[
\chi_n = \sum_{2k \leq n} (-1)^k \binom{u+1}{k} \gamma_{n-2k},
\]

\[
\psi_n = \sum_{k=0}^{n} \binom{2^k}{k} x_{n-k},
\]

\((\mathcal{F}2)\)
\[
\chi_n = \sum_{k=0}^{n} (1 - 2k)^{-1} \binom{2^k}{k} \gamma_{n-k}.
\]
\[ y_n = \sum_{\varepsilon k \leq n} \binom{n-\varepsilon k}{k} x_{n-\varepsilon k} \]

\[ x_0 = y_0 \quad x_n = \sum_{\varepsilon k \leq n} (-1)^k \frac{n}{n-\varepsilon k+k} \binom{n-\varepsilon k+k}{k} y_{n-\varepsilon k}, \quad n \geq 1. \]
\[(81) \quad a_{nk} = \binom{n+k}{2k}, \quad b_{nk} = (-1)^n \binom{n-k}{2n+1} \binom{2n+1}{n-k}.\]

\[(82) \quad a_{nk} = \binom{n+k+1}{2k+1}, \quad b_{nk} = (-1)^n \binom{n-k}{k+1} \binom{2n+2}{n-k} .\]

\[(83) \quad n \geq u + nb, \quad \ell \geq u + nb \quad (-u - kb) \]

\[(86) \quad a_{nk} = \binom{2n+1}{n-k} \quad n, n-k+1, n+1, \ldots, n-k, \ldots, 1, 1.\]

\[(87) \quad b_{nk} = (x-(n-k)b)^{n-k-2} / (n-k)! \quad x \neq 0.\]

\[\text{for } k < n, \quad a_{nn} = 1,\]
\[
\begin{align*}
\text{(103)} & \quad \gamma_n = \sum_{2k \leq n} \binom{n}{2k} x_{n-2k} . \\
\text{(104)} & \quad \gamma_n = \sum_{2k \leq n} (-1)^k \binom{n}{2k} x_{n-2k} . \\
\text{(105)} & \quad \gamma = \sum \binom{n}{n+1} x_{n+1} . \\
\text{(107)} & \quad \gamma_n = \sum_{k=0}^{n} \binom{2n}{2k} x_k . \\
\text{(108)} & \quad \gamma_n = \sum_{k=0}^{n} \binom{2n+1}{2k} x_k . \\
\text{(109)} & \quad \gamma_n = \sum_{k=0}^{n} \binom{2n+1}{2k+1} x_k . \\
\text{(110)} & \quad \gamma_n = \sum_{k=0}^{n} \binom{2n+2}{2k+1} x_k .
\end{align*}
\]
\[ \phi(x) = F(3x) \quad a_{00} = 1, \quad a_{n0} = 0, \quad n \geq 1 \]

\[ a_{nk} = \frac{n}{k} \binom{k}{n-k}, \quad 1 \leq k \leq n \]

\[ b_{00} = 1, \quad b_{n0} = 0, \quad n \geq 1 \]

\[ x_0 = y_0, \quad x_i = y_i - y_0, \quad x_n = y_n - y_{n-1} - y_{n-2}, \quad n \geq 2. \]

\[ (\forall \sigma, \tau) \quad \sigma < \tau \Rightarrow \sigma \times 1 \in H. \]

\[ k \leq n \]
A.J. STAM

BINOMIAL IDENTITIES

WITH OLD-FASHIONED PROOFS

PART 1c
FIBONACCI AND LUCAS NUMBERS

Our treatment of this subject is determined by our purpose: finding formulas with binomial coefficients. There are many connections between Fibonacci theory and binomial sums. Still more binomial sums may be derived from the theory in the next chapter, where we derive generalizations of the relations in this section. These generalizations are concerned with polynomials so that we may invoke the help of their variable $x$. General references on Fibonacci theory are Vorob'ev (1963), Hoggatt (1969) and Vajda (1989). A lot of information is to be found in the Fibonacci Quarterly. Many of our formulas were found in Vajda (1989) and presented as slight generalizations or with different proofs.

**Definitions.** We define the sequence of Fibonacci numbers $F_n$, $n \in \mathbb{N}$, by the recurrence and initial values

1. $F_n = F_{n-1} + F_{n-2}$, $n \ge 2$,

2. $F_0 = F_1 = 1$.

Our definition follows Riordan (1958, 1963), Wilf (1990) and Gould (1972) and differs from the definition given by the Fibonacci Association (see the Problem Section of the Fibonacci Quarterly) and by Vorob'ev.
Hoggatt (1969) and Vajda (1989). The latter definition consists of (1) together with \( F_0 = 0, F_1 = 1 \). It is easily seen that our \( F_n \) is equal to their \( F_n \). The reason for our definition is that we prefer the formulas (20) and (41) to their variants by the other definition and that we want to embed the sequence \( F_n, n \in N_0 \), into a convolution group, see pp.7-9 and 52-

We define the sequence of Lucas numbers \( L_n, n \in N_0 \), by

\[
L_n = L_{n-1} + L_{n-2}, \ n \geq 2, \ L_0 = 2, \ L_1 = 1.
\]

This definition coincides with all definitions in the literature known to the author.

Two extensions are obvious here. First, we may consider sequences \( A_n, n \in N_0 \), satisfying the recurrence

\[
A_n = A_{n-1} + A_{n-2}, \ n \geq 2,
\]

but with initial values other than \( F_0, F_1 \) or \( L_0, L_1 \). Then we may consider two-sided sequences \( A_n, n \in \mathbb{Z}, \) satisfying

\[
A_n = A_{n-1} + A_{n-2}, \ n \geq 2,
\]

It is easily seen that a sequence \( A_n \) satisfying (5) is determined uniquely when \( A_i \) and \( A_{i+1} \) are given for some \( i \in \mathbb{Z} \). We
The above definitions provide an important method to prove identities between Fibonacci-like sequences. When the sequences $X_n$, $n \in \mathbb{Z}$ and $Y_n$, $n \in \mathbb{Z}$, both satisfy (5) and $X_i = Y_i$, $X_i = Y_i$, for some $i \in \mathbb{Z}$, these sequences coincide. In this respect it is important to note that with $A_n$, $n \in \mathbb{Z}$, also the sequence $A_{n+8}$, $n \in \mathbb{Z}$, for fixed $t \in \mathbb{Z}$, satisfies (5).

E.g., when $A_0$ satisfies (5), the same is true for the sequence $(-1)^n A_{-n}$, $n \in \mathbb{Z}$:

$$(-1)^n A_{-n} = (-1)^{n-1} A_{-n-1} + (-1)^{n-2} A_{-n-2}, n \in \mathbb{Z}.$$  

This leads to the relations

$$F_{-n} = (-1)^n F_{n-2}, \quad L_{-n} = (-1)^n L_n, n \in \mathbb{Z},$$
\( A_n = a F_n + b F_{n-1} \)

with \( F = 0 \), since the initial values \( A = a \)

and \( A_1 = a + b \) may be chosen arbitrarily.

One finds Fibonacci and related numbers in many parts of pure and applied mathematics. Just some examples: digital filtering, Arce (1984); graph theory, Baron e.a. (1985); knots, Turner (1986); subsets of \( \{1,2,3,\ldots, n\} \) without unit separation on a circle, Konvalina and Liu (1991); microwave systems, Trzaska (1996); binary trees and search, Ranum (1995); probability, Taillie and Patil (1986).
Binet's formula and generating functions.

A usual method to solve the recurrence (4) or (5) is, trying $A_n = c^n$, which is a

This equation has the roots

\[(1) \quad \alpha - 1 \pm 1 \sqrt{5} \quad \beta - 1 - 1 \sqrt{5} \]

(11) $A_n = \lambda \frac{c^n}{\alpha} + \mu \frac{c^n}{\beta}$,  

since any pair of initial values $A_0, A_1$ may be obtained by (11). By adapting (11) to the initial values in (2) and (3) we find the well-known Binet formulas

(14) $\phi + \phi = 1$, $\phi - \phi = \sqrt{5}$, $\phi \phi = -1$,  

(15) $\lambda \phi^2 - \alpha \sqrt{5}$ $\mu \phi^2 = - \phi \sqrt{5}$
\( n \in \mathbb{N}^0, \) converges for \( |z| < |c_1| = |\lambda| \)
and diverges for \( |z| \geq |c_2| \), unless \( \lambda = 0 \).
and that for \( |z| < |c_2| \)

\[ (\text{9}) \sum_{n=0}^{\infty} A_n z^n = \lambda \left(1 - \sigma z\right)^{-1} + \mu \left(1 - \gamma z\right)^{-1} = \]
\[ \frac{1}{\lambda \sigma + \mu \gamma} z \left(1 - z - z^2\right)^{-1} \]

\[ (\text{21}) \sum_{n=0}^{\infty} \overline{L}_n z^n = (2 - z)(1 - z - z^2)^{-1}. \]

\[ z^{-1} \left( (2 - z)(1 - z - z^2)^{-1} - x \right) = (1 + 2z)(1 - z - z^2)^{-1} = - \frac{d}{dz} \log (1 - z - z^2), \text{ so that} \]

\[ (\text{32}) \sum_{n=1}^{\infty} n^{-1} \overline{L}_n z^n = - \log (1 - z - z^2), \ |z| < |c_2|, \]
Linear relations. We have

\[(24) \quad L_n = 2F_{n-1} - F_n = F_{n-1} + F_{n-2} = F_{n+1} - F_{n-3}, \quad n \in \mathbb{Z}.
\]

All members of (24) satisfy (5), by linearity, and have the same value for \(n = 0\) and

\[(28) \quad \sum_{i=m}^{n} A_{2i+1} = \sum_{i=m}^{n} (A_{2i+2} - A_{2i}) = A_{2n+2} - A_{2m}.
\]

One may derive the relation

\[(29) \quad (1 - \theta - \theta^2) \sum_{k=0}^{n} \theta^k A_k =
\]

\[ A_1 - \theta A_n + A_{-2} + \theta A_{-1} - \theta^{n+1} A_{n-1} - \theta A_n =
\]

\[ A_0 + \theta (A_{-1} - A_0) - \theta^{n+1} A_{n+1} - \theta A_n, \quad n \in \mathbb{N}.
\]

Once found, this relation also may be pro-
\[ A_0 + \Theta (A_1 - A_0) - \Theta^{n+2} A_{n+2} - \Theta^{n+3} A_{n+1}. \]

T. \quad \text{Integrate.}

\[ \sum_{k=0}^{n} 2^{-k} A_k = 2 A_0 + 2 A_1 - 2^{-n} A_{n+1} - 2^{-n} A_n \]
\[ = A + A_1 - 2^{-n} A_{n+1}, \quad n \in \mathbb{N}. \]

\[ \sum_{k=0}^{n} (-1)^k c_i A_{n+1} + (-1)^{n+1} c_i A_{n+1} + \cdots = 0. \]

We will show that \( B_n \) satisfies (4). Then (33) follows since \( B = A - A_0 \) and \( B = A_1, B = A_0 - A_1, B = A_0 - F_1. \) We show that \( B_n \) satisfies (4) when
\[ A_m = c_i^m, \quad i=1,2. \] The general case follows by (11) with linearity. From (14)
\[
\sum_{k=0}^{n} (-1)^k c_i^{n-2k} = (1+c_i^{-2})^{-1} (c_i^n + (-1)^n c_i^{-n-2}) =
(1+c_i^2)^{-1} (c_i^{n+2} + c_2^n),
\]

\( \theta = \pi \), apply (26) to the result. This leads to
\[
(34) \sum_{k=0}^{n} k A_k = 2 A_1 + A_0 + (n-2) A_{n+1} + (n-1) A_n, \quad n \in \mathbb{N}.
\]
One also may derive (34) by induction on \( n \), using (4).
functions of $m$ satisfy (5). They have the same value when $m=0$ and also when $m=1$.

(36) \[ F_n + F_{n+1} = F_{2n+2}, \quad n \in \mathbb{Z}. \]

Also

(13) and (14)

\[ \epsilon_i^{n+m} + (-1)^m \epsilon_i^{n-m} = \epsilon_i^n (\epsilon_i^m + \epsilon_i^{-m}) = I_m \epsilon_i^n. \]

For $A_i = \epsilon_i^k$ the proof is similar.

From the identity $A_j^2 = A_{j+1} A_j - A_j A_{j-1}$

(38) \[ \sum_{j=n}^{m} A_j^2 = A_{n+1} A_n - A_m A_{n-1}, \quad m \leq n, \quad m, n \in \mathbb{Z}. \]

Also from (5) \[ \cdot \]
\[ \sum_{j=m}^{n} A_j A_{j-1} = \hat{A}^2 - \hat{A}^2_{m-1} + \frac{1}{2} \{ (-1)^m + (-1)^n \} \int (A^2_0 - A^2_\lambda), \]

\[ m \leq n, \quad m, n \in \mathbb{Z}. \]
(42) $L_n = \sum_{2k \leq n} \frac{n}{n-k} \binom{n-k}{k}$, $n \geq 1$.

have $b_0 = a_1 = \cdots = a_n$, $b_1 = a_2 = \cdots = a_n$,

$$S_n = \sum_{k=0}^{n-1} \binom{n-1-k}{k} + \sum_{k=1}^{n-1} \binom{n-1-k}{k-1} = S_{n-1} + S_{n-2}.$$
Writing (41) as
\[ y_n = \sum_{h=0}^{n} (n-h) x_h = \sum_{j=0}^{n} (n-j) x_{n-j}, \]
with \( x_i = 1, y_i = F_i \), we find its companion formula, in the sense of \( IR_7 \), by the inverse pair \( IR_7 \) (50a), (50c)
\[ \sum_{k=0}^{n} (-1)^{n-k} \frac{k}{n-k} \binom{2n-k}{n} F_k = \quad (= (129)) \]
\[ \sum_{j=0}^{n} (-1)^{n-j} \frac{n-j}{n+j} \binom{n+j}{n} F_{n-j} = 1, \quad n \geq 1. \]

A similar argument applies to (42) and the inverse pair \( IR_7 \), (29) with \( b = -1 \). We may write (42) as
\[ (1)^{n+1} + L_n = \sum_{h=0}^{n} a_{nh} x_h, \quad n \in \mathbb{N}_0, \]
with \( x_h = 1, a_0 = 1, a_{n0} = (-1)^{n+1}, n \geq 1 \); \( a_{nh} = \frac{n}{h} \binom{h}{n-h}, \) \( h \geq 1, \) \( n \geq 1. \) Note that \( L_0 = 2. \)

\[ \sum_{k=0}^{n} (-1)^{n-k} \binom{2n-k-1}{n-k} (L_k + (-1)^{k+1}) = \]
\[ \sum_{j=0}^{n} (-1)^{j} \binom{n-j+j}{j} (L_{n-j} + (-1)^{n-j+1}). \]

With the general relation (see (1.317))
\[(1+z)^{-1}(1-z-z^2)^{-1} = (1-zz^2-z^3)^{-1} =\]
\[\sum_{k=0}^{\infty} (z^3+2z^2)^k = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} 2^j z^{2k+j} =\]
\[\sum_{n} z^n \sum_{k=0}^{n} \binom{k}{n-2k} z^{3k-n}.\]

\[\sum_{2k \leq n} \binom{k}{n-2k} z^{3k-n} = \sum_{k=0}^{n} (-1)^{n-k} F_k = (-1)^n + F_{n-1} .\]
So with \( (20) \) and \( (26) \) with \( m = 0 \) and \( A_k = F_k \),
\[
\sum_{n-2k}^n \binom{n-2k}{k} \frac{1}{2^{n-2k}} = 2^n F_n = F_{n-1}.
\]

have a list of formulas, all of similar type, with binomial coefficients. They were found in or inspired by Vajda (1986), Clark (1982). These formulas are proved (see below) in two ways. First, by an appeal to \((11)\) - (13). Alternatively, by application of operators (cf. the solution by N.J. Fine of Prob. E 1347, Monthly 66, 1959, 592). Define the operators \( E^k \), i.e. \( \mathbb{Z} \), on the set of sequences \( X, n \in \mathbb{Z} \), by \( E^k X = X_{n+k} \), more precisely \( \sum_{n} E^k X = \sum_{n} Y \) with \( Y_n = X_{n+k} \). All operators of this type commute. We put \( E^0 = \mathbb{I} \). We may apply the theory of finite differences of Chapter 6 to functions \( \mathbb{Z} \to \mathbb{C} \), in particular relations like \( G(2x) \), \((15)\).

The relations \((48)\) - \((60)\) below hold for \( n \in \mathbb{Z} \), \( \mathbb{N} \), \( n \in \mathbb{N} \). Let the sequence \( A_n \), \( n \in \mathbb{Z} \), satisfy \((5)\). Then
\[ (51) \quad (I - E) A_n = \sum_{k=0}^{\infty} (-1)^k (\frac{r}{k}) A_{n-k} = A_{n-2r} \]
\[ (52) \quad (E^2 - I) A_n = \sum_{k=0}^{r} (-1)^k (\frac{r}{k}) A_{n+2k} = A_{n+r} \]
\[ (53) \quad (I - E^2) A_n = \sum_{k=0}^{r} (-1)^k (\frac{r}{k}) A_{n-2k} = A_{n-r} \]
\[ (54) \quad (I + E^2)^m A_n = \sum_{k=0}^{2m} \binom{2m}{k} A_{n+2k} = 5^m A_{n+2m} \]
\[ (55) \quad (I + E^2)^{2m+1} F_n = \sum_{k=0}^{2m+1} \binom{2m+1}{k} F_{n+2k} = 5^m \sum_{n=0}^{m} C_n (\frac{r}{t}) F_{n+2m+2} \]
\[ (56) \quad (I + F^2)^{2m+1} F - \sum_{n=0}^{2m+1} \binom{2m+1}{k} F_{n+2m+2} \]
\[ (59) \quad (I + E^2)^{2m+1} L_n = \sum_{k=0}^{2m+1} \binom{2m+1}{k} L_{n+2k} = 5^m F_{n-2m+1} \]
\[ (60) \quad (I + F^2)^{2m+1} A - \sum_{n=0}^{2m+1} \binom{2m+1}{k} 2^k A_{n+2m+2} \]
\[(I + E^{-1})A_n = A_n + A_{n-1} = A_{n+1} = E A_n,\]

... and so on...

(56): In the same way as (55), with \[(I + E^2)L_n = 5F_n\] by (25).
(50): \[ \sum_{k=0}^{\infty} (-1)^{n-k} \binom{\kappa}{k} c_i^{n+k} = c_i^n (c_i - 1) = c_i^{n-\kappa} \]

by (49), and similarly for \( a_j \).

\[ \sum_{k=0}^{\infty} \binom{\kappa}{k} F_{n+k} = \frac{1}{\sum_{i=1}^{2m+1} (2m+1) / (c_i^{n+2k+1} - c_i^{n+2k+1})} = \]

\[ c_i^{n-2m-2} (c_i^2 + 1)^{2m+1} + c_j^{n-2m-2} (c_j^2 + 1)^{2m+1} = \]
\[ \left( m + \frac{1}{2} \right) \left( n - 2m - 1 \right) \left( n - 2m - 1 \right) \left( m + 1 \right) \]

\[ = 5^{\frac{m}{2}} \left( \lambda c_1 c_2^{2n+2} + (-1)^{\nu} \mu \bar{c}_2^{2n+2} \right) \]

\[(61) \quad \sum_{k=0}^{\infty} \left( \begin{array}{c} k \\ m \end{array} \right) F_{n+k} = 5^{-m-1} L_{2n+2m+2}, \quad n \in \mathbb{Z}, \quad m \geq 1,\]

\[(62) \quad \sum_{k=0}^{\infty} \left( \begin{array}{c} k \\ 2m \end{array} \right) F_{n+k} = 5^m L_{n+1}, \quad n \in \mathbb{Z}, \quad m \geq 1.\]

Even and odd index. The Fibonacci and Lucas numbers with even and odd index lead to particular binomial sums. E.g. (41) and (42) take the following form that we found in Jennings (1993):

\[ a_n = \sum_{j=0}^{m} (m-j) \binom{m+j}{2j}, \ m \geq 1, \]

(69) \[ L_{2m+1} = \sum_{k=0}^{m} \frac{2m+1}{2m+1-k} \binom{2m+1-k}{k} = \sum_{j=0}^{m} \frac{2m+1}{m+j+1} \binom{m+j+1}{2j+1}, \ m \geq 0. \]

with \( x_i = 1, i \in \mathbb{N}^* \), we see from the inverse pair IR (81), in the way discussed on p. IR7, that we have

(70) \[ \sum_{k=0}^{m} \frac{2k+1}{2m+1} \binom{2m+1}{m-k} (-1)^{m-k} \frac{F_k}{2k} = 1, \ m \in \mathbb{N}^*. \]
\[ \gamma_0 = 1 = a_{00} x_0 , \quad \gamma_m = L_{2m} = \sum_{j=0}^{m} a_{mj} x_j , \quad m \geq 1 \]

with \( x_j = 1, j \in \mathbb{N}_0 \) and \( a_{-1j} = 0 \) for \( j \in \mathbb{N}_0 \) and \( a_{m-1j} = 1 \) for \( j \in \mathbb{N}_0 \). This inverse pair then gives

\[ \sum_{k=0}^{m} (-1)^{m-k} \binom{2m+1}{m-k} L_{2k+1} = 1 , \quad m \geq 0 . \]

Thus \( C_1 \) satisfies the form: \( \gamma_j = a_{00} x_0 , \gamma_m = L_{2m} = \sum_{j=0}^{m} a_{mj} x_j \quad m \geq 1 \)

with \( x_j = 1, j \in \mathbb{N}_0 \) and \( a_{-1j} = 0 \) for \( j \in \mathbb{N}_0 \) and \( a_{m-1j} = 1 \) for \( j \in \mathbb{N}_0 \).

\[ \left( A_0 + (A_1 - 2A_0)z \right) \left( 1 - 3z + z^2 \right) , \]

and
\[ \sum_{m=0}^{\infty} L_{2m} z^m = \frac{(2-3Z)(1-3Z+z^2)}{1-z}, \]

\[ (1-3Z+z^2) = \sum_{k=0}^{\infty} (3Z-z^2)^k = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \binom{k}{i} (-1)^i 3^{k-i} z^{k+i} = \sum_{m=0}^{\infty} z^m \sum_{\lambda\leq m} \binom{m-i}{i} (-1)^i 3^{m-2i}, \]
\[\sum_{k=0}^{n} \binom{y-n-2-k}{n-k} F_k = \sum_{j=0}^{n} \binom{y-j}{j} F_{2k+1}\]

The second equality is Problem 4 in Chapter 4 of Riordan (1968). Also note that \(2n - x \leq m\) and \(2n + 1 - x \leq m\) and noting \(D(2x)\), we see that (81) is equivalent with the two relations

\[\sum_{j=0}^{n} \binom{2n-x-j}{j} = \sum_{h=0}^{n} (-1)^h \binom{x}{h} F_{2n-2h}\]

\[
e_{nh} = \sum_{j=h}^{\infty} \Delta^h \binom{h-x-j}{j} \bigg|_{x=0}
\]

Now, by (45), (46), for \(\varepsilon \leq m\),

\[
\Delta \binom{a-x}{m} = -(a-x-1) \binom{a-x}{m-1}, \quad \Delta^e \binom{a-x}{m} = (-1)^e \binom{a-x-\varepsilon}{m-\varepsilon}
\]
For \( b = 2n \) and \( b = 2n+1 \) this gives (82) and (83), respectively, by (41).

By a similar method we derive

\[
(84^a) \binom{y-n-1}{n} + \sum_{k=1}^{n} \binom{y-n-1-k}{n-k} L_{2k} = \sum_{j=0}^{n} \frac{y}{y-j} \binom{y-j}{j}, \quad n \geq 1,
\]

valent with

\[
(85) \sum_{j=0}^{n} \frac{2n-x}{2n-x-j} \binom{2n-x-j}{j} = \sum_{h=0}^{n-1} (-1)^h \binom{x}{h} L_{2n-2h}
\]

\[+ (-1)^n \binom{x}{n}, \quad n \geq 1,
\]
We may rewrite this as \( T_n \).

Then by iteration, for \( k \leq m \)

\[
\Delta^k \frac{b-x}{b-x+m} \left( \frac{b-x-m}{m} \right) = (-1)^k \frac{b-x-2k}{b-x-k-m} \left( \frac{b-x-2k-m}{m-k} \right).
\]

So

\[
\sum_{k=1}^{n} \left( b-2k \right) \left( b-h-i \right)
\]

Here we want to take \( b=2h \) and \( b = \sum_{i=0}^{\infty} \left( 2h-i \right) \). This is valid for

\[
C_{nn} = (-1)^n.
\]

And (85) and (86) follow.
The Fibonacci convolution semigroup. C.f. p. 452-
Hoggatt and Bicknell-Johnson (1977), Horadam

\[ \sum_{k=0}^{\infty} \psi_k(s) \psi_{n-k}(t) = \psi_n(s+t), \quad n \in \mathbb{N}, \quad s, t \in C. \]

This means that we have to apply the theory
of Chapter C: polynomials of convolution
type. Theorem 1 in C states that \( \psi(t) = e_t \)
\( \psi(t) = \exp(\alpha t) q_n(t) \), where \( q \) is a polynomial
of degree \( \leq n \). Since \( \psi_0(t) = F_0 = 1 \), we have
\( \alpha = 2\pi \xi i \) with \( \xi \in \mathbb{Z} \). We take \( \xi = 0 \) and
then the \( q \) are determined uniquely, see
Remark 6\( \text{^n} \) to C, Theorem 1. We might denote \( q_n(t) \) as \( F^{\text{\tiny 1}} \), the \( t \)-fold convolution of \( F \). See Chapter N.

Comparison of (20) and C (10) suggests to
determine the coefficient sequence \( q_0, q_1, \ldots \)
of the \( q \) (see Remark 3 to C, Theorem 6) by
(88) \[ g(z) = \sum_{k=1}^{\infty} q_k z^k = - \log(1-z-z^2). \]

For \( |z| < |c_2| \) we have \( |z+z^2| < |c_2|+c_2^2 =
-1+c_2^2 = 1 \), so that we may, and do, take
the principal value of the \log\ in (88). The sin-
gularity of \( \log(1-z-z^2) \) nearest to 0 is \( z =
-c_2 \), so that the series in (88) converges
absolutely for \( |z| < -c_2 \) and diverges for
\( |z| > -c_2 \). This also follows from (32),
By (10) the $q_n$ are determined by
\[
q_n(t) = e^{tq(z)} = (1-z-z^2)^t,
\]
where the series converges absolutely for $|z| < 1$.

From (90) we see that these $q_n$ satisfy our requirements.
Since $q \neq 0$ the sequence $q$ is a basic

From (90) we have for small $z$, with the binomial series
\[
\sum_{n=0}^{\infty} q_n(t) z^n = \sum_{i=0}^{\infty} (-i)^i (t^i) (z^2 + z)^i = \\
\sum_{i=0}^{\infty} (-1)^i (-t)^i \sum_{i=0}^{\infty} (i^i) z^{i+i^i} =
\]
where changing the order of summation is justified since by the binomial series

\[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{t}{1-t} \right)^n \]

(91) \[ q_n(t) = \sum_{\sum j \leq n} (-1)^{n-j} \binom{n-j}{j} \binom{-t}{n-j} = \]

\[ \sum_{\sum j \leq n} \binom{n-j}{j} \frac{t^{n-j-1}}{n-j} = \sum_{\sum j \leq n} \binom{t+j-1}{j} \frac{t^{n-j-1}}{n-j} \]

From (91), (14) and the relation (3.94):

\[ \sum_{\sum j \leq n} \binom{n-j}{j} \binom{x}{n-j} (uv)^j (u+v)^{n-j} = \]

\[ \sum_{k=0}^{n} \binom{x}{k} \binom{x}{n-k} u^k v^{n-k} \]
we have from the inverse pair \( IR, (50^a), (50^c) \) with \( a = b = 1 \), cf. the discussion on \( p. IR7 \),

\[
(93) \binom{t+n-1}{n} = \sum_{k=0}^{n} \frac{1}{2^{n-k}} \binom{2n-k}{n} (-1)^{n-k} q_k(t) = \\
\sum_{k=0}^{n} (-1)^k \binom{n-k}{n} \binom{n+k}{n} q_{n-k}(t), \ n \geq 1.
\]

presses \( q_n(t) \) for \( t = 2, 3, y \) in terms of Fibonacci numbers, by the application of inverse pairs. We prove this formula

\[
\frac{1}{(1+2z) \frac{d}{dz} (1-z-z^2)^{-1} + y (1-z-z^2)^{-2}} = 5 (1-z-z^2)^{-2},
\]

and (94) follows from the convolution property Theorem M1 of generating functions. Cf. Probl. B.586, Fib. Q. 26(1), 1988, \# 86. From (91) and (92):

\[
(95) \ q_n(\lambda) = \sum_{\lambda j \leq n} (n-j+1) \binom{n-j}{j} = \\
\sum_{\lambda j \leq n} (j+1) \binom{n-j+1}{j+1} = \sum_{\lambda i \leq n+2} i \binom{n+2-i}{i},
\]
\begin{equation}
(40) \sum_{k=0}^{n} \binom{n}{k} \Phi(k) \cdot k = \frac{(2n+4)F_n}{n}, \quad n \in \mathbb{N}_0.
\end{equation}

Application of the Fibonacci convolutions.

Let \( U(m, n) \) be the number of paths from \((0, 0)\) to \((m, n)\) consisting of steps 1 or 2 in the positive \(X\)-direction or 1 in the positive \(Y\)-direction. Replacing in a path a step 2 by two steps 1 in the \(X\)-direction (or vice versa) gives a different path. Specification

if we define \( U(0,0) = 1 \). Since \( U(1, 0) = 1 \) we have \( U(m, 0) = F_m \). By specifying the point at which the \(m\) path leaves the
X-axis we have

\[(101) \quad U(m, n) = \sum_{k=0}^{m} U(k, 0) U(m-k, n-1) = \sum_{k=0}^{m} \binom{m}{k} U(m-k, n-1), \quad m \geq 0, \quad n \geq 1.\]

\[\sum_{2j \leq m} (m+n-j)!/(j!(m-2j)!(n)!).\]

The last sum has a combinatorial interpretation. When there are \(j\) steps 2, there are \(m-2j\) and \(n\) steps 1 in the X- and Y-direction, respectively. The total set of \(m+n-j\) steps then has to be divided into \(j\), \(m-2j\) and \(n\) steps, respectively, of the three kinds.

The relations (99) and (102) suggest that the \(q_m(t)\) satisfy

This follows from the generating function (90):

\[\sum_{m=0}^{\infty} \left[ q_m(t+1) - q_m(t) \right] \left\{ z^m = (z+z^2) \left( 1-z-z^2 \right)^{-t-1} \right\}.\]
Derivatives of the generating function.

Following Riordan (1968), Problem 17 in

\[(106) \quad f(z) = (1 - z - z^2)^{-1}, \quad z \neq -c_1, z \neq -c_2.\]

By (20) we have \(f(z) = \sum_{k=0}^{\infty} \frac{F_k z^k}{k!}, \quad |z| < -c_2,\)
so that \(d_n(0) = F_n.\)

There is a number of different formulas for \(d_n(z),\) giving rise to binomial identities. We have for small \(w\)

\[(107) \quad f(z+w) = (1 - z - z^2 - (1+z)w - w^2)^{-1} = f(z)\left\{ 1 - (w^2 + (1+z)w)f(z) \right\}^{-1} = \]

so that by comparison with the Taylor series for \(f^\circ \) at \(z\) we have
$$P(\zeta) - \frac{1}{5} (1 - 5 (1 + \zeta)) S - 7 \zeta = 0 \quad \ldots$$

(110) \quad d(z) = 2^{n+2} \sum (2^k \binom{n}{k} (1+2z)^{2z-n-3z-1}

$$\langle \chi = 0 \rangle \binom{h+1}{n} \quad n+1 \quad 121 < -c_2$$

Expansion of (108) into powers of (1+z) with 
(109) and comparison with (110) gives (3.425) and 
(3.497) but not easily. Expansion of (110) 
into powers of \( z \) and comparison with (112) 
gives (111) with \( n \) replaced by \( n+h \). 
From (108) with \( n \geq 2 \) we find the recurrence 

(112) \quad d(z) = \sum \binom{n-k}{n-2k} \binom{n+1-k}{n-k} f(z) = \ldots$$
\[
\sum_{k=1}^{n-1} \left( \frac{n-1-k}{k-1} \right) (1+2z)^{n-2k} f^{n+1-k}(z) = (1+2z)^{n-2} f(z) d_{n-2}(z) + \sum_{h=0}^{n-2} \left( \frac{n-2-h}{h} \right) (1+2z)^{n-2-k} f(z) d_{n-2}(z), \quad n \geq 2.
\]

\[
5^{-k/2} \left\{ \left( z + \frac{1}{2} + \frac{i}{2} \sqrt{5} \right)^{-k} - \left( z + \frac{1}{2} - \frac{i}{2} \sqrt{5} \right)^{-k} \right\},
\]

(114) \quad d_n(z) = (-1)^n 5^{-k/2} \left\{ \left( z + \frac{1}{2} + \frac{i}{2} \sqrt{5} \right)^{-n-1} - \left( z + \frac{1}{2} - \frac{i}{2} \sqrt{5} \right)^{-n-1} \right\}.

So, for \(|2z+1| > \sqrt{5}\), in particular for

\[
(-1)^{n+1} 2^{n+2} \sum_{h=0}^{\infty} \binom{n+1+2h}{n} 5^h (2z+1)^{-n-2h-2}.
\]

We have with you.
and from the third member of (107)

\[ \left( \begin{array}{c} n \\ m \end{array} \right) \leq n \left( \begin{array}{c} n+1 \\ m+1 \end{array} \right) \left( 1+2z \right) \left( 1-2z \right) \]

\[ 5^{-\frac{m}{2}} 2^{-n-1} \left( \left( 1+2z+\sqrt{5} \right)^{n+1} - \left( 1+2z-\sqrt{5} \right)^{n+1} \right) \]
This is $\phi(86)$ with $x = w + \sqrt{5}$, $y = w - \sqrt{5}$.
To us, or were suggested by the literature, after the preceding part of Chapter F was written. For generalizations to polynomials

\begin{equation}
\sum_{k=0}^{\infty} \binom{2k+1}{2k+1} m^m = \sum_{m=1}^{\infty} \binom{2m+1}{2m+1} m^m.
\end{equation}

\begin{equation}
L_{m-1} = \sum_{m=1}^{\infty} \frac{2m+1}{m+1}. \tag{126}
\end{equation}

\begin{equation}
(E^4 + I)^x A_n = \sum_{k=0}^{\infty} \binom{x}{k} A_{n+4k} = 3^x A_{n+2x}.
\end{equation}

\text{(Iteration of } (E^3 + I) A_n = 2E^2 A_n \text{ and } (E^4 + I) A_n = 3 E^2 A_n \text{, notation as in (98) - (60)).}
To obtain a similar relation from (42) we apply Lemma 2 in IR to the pair IR, 
(50a) - (50c) with \( a = b = 1 \), taking

\[
\sum_{k=0}^{n} \frac{k}{2n-k} \binom{2n-k}{n} (-1)^{n-k} f_k = 1, \quad n \geq 1. \tag{44}
\]

\[
\alpha^0_0 = 1, \quad \alpha^j_n = 0, \quad n \geq 1, \quad j = 1, \ldots, n / k.
\]
\[ \sum_{k=0}^{2m} \frac{k}{2m-k} \binom{2m-k}{m} 3^k F_{2k+1} = 3^{2m}, \quad m \geq 1. \]
a_{n} \text{ solution. Therefore, IR. (116), (117), (120), (121) hold, giving}
\begin{equation}
\sum_{j=0}^{m} \binom{j}{m} (-2)^{j} j_{j} = 0,
\end{equation}
\begin{equation}
\sum_{i=0}^{2m} \binom{2m+1}{i+1} (-2)^{i} F_{i} = 5^{m},
\end{equation}

with \alpha_{i} and \beta_{i} defined by IR. (95) - (98). Similarly, from (23a) and IR. (122) - (130),
The relations (134), (135), (137) and (138) also may be proved by (13) and (12)
We want to extend the definition of Fibonacci and Lucas numbers to sequences of polynomials \( \Phi \) and \( \Lambda \), satisfying a recurrence generalizing "\( F(n) \)" and such that \( \Phi(1) = F_1, \Lambda(1) = L_1 \). The presence of a "variable" might help us to derive more Binomial formulas. The Fibonacci Association defines the sequences \( F(x) \) and \( L(x) \), called Fibonacci and Lucas polynomials, respectively, by

\[
\begin{align*}
(2) & \quad f_0(x) = 0, \quad f_1(x) = 1, \quad x \in \mathbb{C}, \\
(3) & \quad l_n(x) = x l_{n-1}(x) + l_{n-2}(x), \quad x \in \mathbb{C}, \quad n \geq 2, \\
(4) & \quad f_0(x) = 2, \quad f_1(x) = x, \quad x \in \mathbb{C},
\end{align*}
\]

see e.g. Webb and Farbary (1969), Bicknell (1976), Hoggatt and Bicknell (1973), Bicknell and Hoggatt (1973), Burrage (1990), Filipponi and Horadam (1993), and the Fibonacci Quarterly.

We start from definitions with a recurrence different from (1) and (3). So our polynomials differ from those in (1)-(4) by more than a shift, as in \( F(1), (2) \). Therefore we do not use the notations \( F \) and \( L \). We define
\[ \phi_n(x) = \phi_{n-1}(x) + x \phi_{n-2}(x), \quad x \in \mathbb{C}, \quad n \geq 2, \]
and
\[ \phi_n(x) = \phi_{n-1}(x) + x \phi_{n-2}(x), \quad x \in \mathbb{C}, \quad n \geq 2. \]

In [1984] and [1990], Jacob's first polynomials, see Bergum et al. [1989] and Burdage [1990]. The \( \phi_n(x) \) in the form (68) were defined by Kremeras [1970] and \( \phi_n(x) \) and \( \Lambda_n(x) \) were also defined by Doman and Williams [1987]. Our \( \phi_n(x) \) coincides with \( \phi_n(x) \) in Riordan [1958], Ch. 7.8 and 8.2 and with \( u_n(x) \) in Riordan [1968], Ch. 7, Problem 7 and Ch. 4, Problem 7, where we found some identities.

Applications of the above polynomials are to be found in Hopkins and Staton [1984, graphs], Filippone [1991, number theory], Horadam [1996, curves], Flajolet et al. [1999, computer science].

We note that \( \phi_n(1) \) is the number of words of length \( n \) from the alphabet \( \{0, 1, \ldots, k\} \) without neighboring zeroes. A recurrence is found by considering the first letter, see (70).

As in Chapter F we may extend the definition of \( \phi_n(x) \) and \( \Lambda_n(x) \) to \( n \in \mathbb{Z} \). For \( x > 0 \), this is done by keeping (6) and (8) and requiring (5) and (7) for \( n \in \mathbb{Z} \). In this way \( \phi_n(x) \) and \( \Lambda_n(x) \) are defined for \( n < 0 \) as
it follows that $\Phi_n$ and $\Lambda_n$ for $n \geq 0$ are polynomials with

\[ \Phi_{n+1}(x) = \frac{1}{n+1} \left( \Phi_n(x) - \lambda_n \right) \]

in $\mathbb{R}_x$, satisfying

\[ \Phi_n(0) = 0, \quad \Phi_n'(0) = 1 \]

The $\Phi$ are the sequence of root polynomials of a "staircase" chess board, see Riordan (1958), Ch. 7.8 and 8.2.

References on Jacobsthal polynomials are in Horadam (1994). His polynomials are $J_n(x) = \Phi_n(2x)$ and $J_n(x) = \Lambda_n(2x)$. His paper contains a number of the relations given below. So ours are not (all) new.

For a connection with Dickson polynomials, see Filippone, Menicocci, and Horadam (1994) and Filippone (1997). See also...
and $A_n(x)$ to be defined and finite for $x \in \mathbb{C}$. The common domain of the $A_n$, $n \leq 0$, is $\mathbb{C} - \{0\}$, and for the $A_n$, $n \geq 0$, it is $\mathbb{C}$. We have $A_n(0) = A_n(0)$, $n \geq 1$.

The sequence $A_n(x)$ is determined uniquely by (10) and $A_0(x)$, $A_i(x)$ for some $i$. The

(12) $A_n(x) = a(x)$, $A_i(x) = a(x) + b(x)$.

There is a connection between our definition (5)-(8) and the definition (1)-(4). Let $A_n(x)$, $n \in \mathbb{Z}$, satisfy (10). Then $B_n(x) = x^n A_n(x^{-2})$ satisfies

(13) $B_n(x) = x B_{n-1}(x) + B_{n-2}(x)$, $n \in \mathbb{Z}$,

as is verified by substitution. In particular, since (14) holds for $n = 0$ and $n = 1$,

(14) $F_n(x) = x^{n-1} \Phi_{n-1}(x^{-2})$, $I_n(x) = x^n \Lambda_n(x^{-2})$, $x \neq 0$, $n \in \mathbb{Z}$.
<table>
<thead>
<tr>
<th>$n$</th>
<th>$\phi_n(x)$</th>
<th>$\Lambda_n(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-7$</td>
<td>$-x^{-6} - 4x^{-5} - 3x^{-4}$</td>
<td>$-x^{-7} - 7x^{-6} - 14x^{-5} - 7x^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$-5 - x - x^2$</td>
<td>$-5 - 5x - 5x^2$</td>
</tr>
<tr>
<td>$-4$</td>
<td>$x^{-3} + x^{-2}$</td>
<td>$x^{-4} + 4x^{-3} + 2x^{-2}$</td>
</tr>
<tr>
<td>$-3$</td>
<td>$-x^{-2}$</td>
<td>$-x^{-3} - 3x^{-2}$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$x^{-1}$</td>
<td>$x^{-2} + 2x^{-1}$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$0$</td>
<td>$-x^{-1}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$1$</td>
<td>$2$</td>
</tr>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$2$</td>
<td>$1 + x$</td>
<td>$1 + 2x$</td>
</tr>
<tr>
<td>$3$</td>
<td>$1 + 2x$</td>
<td>$1 + 3x$</td>
</tr>
<tr>
<td>$4$</td>
<td>$1 + 3x + x^2$</td>
<td>$1 + 4x + 2x^2$</td>
</tr>
<tr>
<td>$5$</td>
<td>$1 + 6x + 10x^2 + 4x^3$</td>
<td>$1 + 7x + 14x^2 + 7x^3$</td>
</tr>
<tr>
<td></td>
<td>$\ldots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$7$</td>
<td>$1 + 6x + 10x^2 + 4x^3$</td>
<td>$1 + 7x + 14x^2 + 7x^3$</td>
</tr>
</tbody>
</table>
We will try to find generalizations of the relations in Chapter F. The list on p. \( \Phi_4 \) suggests that

\[
(15) \quad \phi_{-n}(x) = (-1)^n x^{-n} \phi_n(x), \quad \Lambda_{-n}(x) = (-x)^{-n} \Lambda_n(x),
\]

for \( n \in \mathbb{Z} \), \( x \neq 0 \), cf. F (7). This relation holds for \( n = 0 \) and \( n = 1 \) and we have the analogue of F (6): When \( A(x) \) satisfies (10), the same

Binet's formula and generating functions. As an analogue of F (9) - (18) we determine a general solution of (10) by trying \( A(x) \)

\[
(18) \quad \phi_1(x) = \frac{1}{2} + \frac{1}{2} \sqrt{1+4x}, \quad \phi_2(x) = \frac{1}{2} - \frac{1}{2} \sqrt{1+4x},
\]

and the general solution of (10) is

\[
(19) \quad A_n(x) = \lambda(x) \phi_1^n(x) + \mu(x) \phi_2^n(x), \quad n \in \mathbb{Z}.
\]

When \( x = -\frac{1}{4} \) the equation (17) has the double root \( c = \frac{1}{2} \) and (10) has the general solution
\[ (20) \ A_n \left( -\frac{1}{4} \right) = (\alpha + \beta n) 2^{-n}, \ n \in \mathbb{Z}. \]

In (18) we take the principal value of \( \sqrt{1 + 4x} \) when \( \Re x > -\frac{1}{4} \). This is sufficient to derive the relations below, most of which are polynomial. However, work and all our relations continue to hold when \( x \) varies we might have some difficulty with e.g. (19), but not in formulas symmetric in \( c_1(x) \) and \( c_2(x) \). We have

\[ (21) \ c_1(x) + c_2(x) = 1, \ c_1(x) c_2(x) = -x, \]
\[ (22) \ c_1(x) - c_2(x) = \sqrt{1 + 4x}, \]
\[ (23) \ x + c_1^2(x) = c_1(x) \sqrt{1 + 4x}, \ x + c_2^2(x) = -c_2(x) \sqrt{1 + 4x}, \]
\[ (24) \ c_i^3(x) = (i+1) c_i(x) + x = (i+1) c_i^2(x) - x^2, \ i = 1, 2, \]
\[ (25) \ c_1^3(x) + c_2^3(x) = 4 + x. \]

From (19), (20), and the initial values (8) and (8), with (21) and (22)

\[ (26) \ \Phi_n(x) = \left\{ c_1(x) - c_2(x) \right\} \left\{ c_1^{n+1}(x) - c_2^{n+1}(x) \right\} = (1 + 4x)^{-\frac{1}{2}} \left\{ c_1^{n+1}(x) - c_2^{n+1}(x) \right\}, \ n \in \mathbb{Z}, x \neq -\frac{1}{4}. \]

\[ (27) \ \Lambda_n(x) = c_1^n(x) + c_2^n(x), \ n \in \mathbb{Z}, x \in \mathbb{C}, \]

(and \( x \neq 0 \) in (26) and (27) when \( n \leq 0 \)).
(28) \( \Phi_n(-i'y) = (n+1)x^{-n} \), \( \Lambda_n(-i'y) = 2^{-n}, n \in \mathbb{Z} \).

The following relation combines (26) and (28) for \( \Phi_n \) with \( n \geq 0 \):

(29) \( \Phi_n(x) = \sum_{k=0}^{n} c_k(x) c_{n-k}^{(2)} \), \( n \in \mathbb{N}_0, x \in \mathbb{C} \).

From (26), by continuity also for \( x = -i'y \), and from (27), by bisection of the binomial formula,

(30) \( 2^n \Phi_n(x) = \sum_{2i \leq n} \binom{n+1}{2i+1} (i'y)^i, n \geq 0, x \in \mathbb{C} \).

(31) \( 2^{n-1} \Lambda_n(x) = \sum_{2i \leq n} \binom{n}{2i} (i'y)^i, n \geq 0, x \in \mathbb{C} \).

When \( x = (u-u^{-1})^{-2} \) we have \( i'y = (u+u^{-1})^2 (u-u^{-1})^{-2} \), so for \( u^2 \neq 1 \), \( u \neq 0 \).

(32) \( c_1(x) = u^2 (u^{-1})^{-1} \), \( c_2(x) = (1-u^2)^{-1} \),

where we have to note the remark following (29). In any case (32) gives two different roots of (17) when \( u^2 \neq 1 \).

When \( x = -(u+u^{-1})^{-2} \) we have \( i'y = (u-u^{-1})^2 (u+u^{-1})^{-2} \). So for \( u^2 \neq -1, u \neq 0 \),

(33) \( c_1(x) = u^2 (u^2+1)^{-1} \), \( c_2(x) = (u^2+1)^{-1} \),

with the same remark as above, when \( u^2 \neq 1 \).

The relations (32) and (33), in connection with (121) - (124) below, are used in Jennings (1993).
(36) \[
\frac{d}{dx} \Phi_n (x) = -2 \left( 1 + 4x \right)^{-\frac{3}{2}} \left\{ \phi_1^{n+1} (x) - \phi_2^{n+1} (x) \right\}
\]

which may be extended to \( x \in \mathbb{C} \), with \( x \neq -\frac{1}{4y} \) in (36).

In the same way from (29) and (34)

(37) \[
\frac{d}{dx} I_n (x) = \left( 1 + 4x \right)^{-\frac{1}{2}} \sum_{k=1}^{n} k \phi_1^k (x) \phi_2^{n-k} (x)
\]

\[-\left( 1 + 4x \right)^{-\frac{1}{2}} \sum_{k=0}^{n-1} (n-k) \phi_1^k (x) \phi_2^{n-k-1} (x) =
\]

\[
\left( 1 + 4x \right)^{-\frac{1}{2}} \sum_{k=0}^{n-1} \left( 2h+1-n \right) \phi_1^h (x) \phi_2^{n-1-h} (x),
\]

\( h \in \mathbb{N} \), \( x > 0 \).
To find the analogues of \( F (19)-(22) \), let \( A_n, n \geq 0, \) satisfy \( (10) \) for \( n \geq 2 \). From \( (19) \) and \( (20) \) we see that \( \sum_{n=0}^{\infty} A_n(x) z^n \) converges absolutely at least for

\[
|z| < \min \left\{ \frac{|c_1(x)|}{|c_2(x)|} \right\} = |x| \min \{ |c_1(x)|, |c_2(x)| \}.
\]

\[
(41) \quad \sum_{n=0}^{\infty} A_n(x) z^n = (2-z)(1-z-xz) .
\]

Also, since by \( (41) \) both sides have the same derivative with respect to \( z \) and have the same value for \( z = 0 \),

\[
(42) \quad -1 + \sum_{n=0}^{\infty} a_n(t) n \quad 0 \leq t \leq x.
\]
\[ (44) \ \Phi_n(x) = x \psi_n(x) - \psi_{n-1}(x) = n \in \mathbb{Z}, x \in \mathbb{C}, \]
\[ \Phi_n(x) + x \Phi_{n-2}(x) = \Phi_{n+1}(x) - x^2 \Phi_{n-3}(x), \quad x \neq 0. \]

All members of (44) satisfy (10) and have the same value for \( n = 0 \) and for \( n = 1 \).

By a similar argument, for \( n \in \mathbb{Z}, x \in \mathbb{C}, \)
\[ (45) \ \Lambda_{n+1}(x) + x \Lambda_{n-1}(x) = (1 + xy) \Lambda_{n-1}(x). \]

Let \( A_n, n \in \mathbb{Z}, \) satisfy (10). Then
\[ (46) \ \times \sum_{i=1}^{n} A_i(x) = \sum_{i=m}^{n} \left( A_{2i}(x) - A_{i-1}(x) \right) \]

\[ (1-x) \sum_{i=m}^{n} A_{2i+1} + x \sum_{i=0}^{n} A_{2i+1} = \sum_{i=m}^{n} (A_{2i+2} - x A_{2i}) = \]

\[ \sum_{i=m+1}^{n} A_{2i} - x \sum_{i=m}^{n} A_{2i} = (1-x) \sum_{i=m}^{n} A_{2i} + A_{2n+2} - A_{2m}. \]
From these two relations \( \sum_{i=m}^{n} A_{2i} \) and \( \sum_{i=m}^{n} A_{2i+1} \) may be solved to give

\[
(48) \quad (x-x^2) \sum_{i=m}^{n} A_{2i+1}(x) = (x-x^2) A_{2n+1}(x) \\
+ A_{2n+2}(x) - (x-x^2) A_{2m-1}(x) - A_{2m}(x),
\]

for \( m \leq n, m, n \in \mathbb{Z} \), \( x \) in domain of the \( A_n \).

For \( x=0 \) the left-hand sides of (47) and (48) vanish. So the same should hold for the right-hand sides. This may be verified

\[
\Phi_1 \text{, (10) noting that } \Phi_1(0) = 0 \Rightarrow \Phi_1(-1).
\]
For \( \theta = x^{-1} \) and \( \theta = -x^{-1} \), the relation (49) becomes
\[
(59) \quad (x-2) \sum_{k=0}^{n} A_{n}(\theta) x^{n-k} = x^{n+1} A_{n+1}(\theta) +
\]

Cf. Doman and Williams (1981), (18)–(20). Note that (52) and (53) also hold when \( x = 0 \), identically for \( n = 0 \) and, since \( \Lambda_{n}(0) = \Lambda_{n}(0) \), also for \( n \geq 1 \).

In (49) we have, for \( x \neq 0 \),
\[
1 - \theta - x \theta^2 = -x (\theta - \theta_2) (\theta - \theta_1),
\]
\[
\theta_1 = \theta_1(x) = (2x)^{-1} (-1 + \sqrt{1 + 4x}) = -x^{-1} \hat{g}_2(x) = 1/\hat{g}_1(x),
\]
\[
(54) \quad \theta_2 = \theta_2(x) = (2x)^{-1} (-1 - \sqrt{1 + 4x}) = -x^{-1} \hat{g}_1(x) = 1/\hat{g}_2(x).
\]
Dividing (49) by \( \theta - \theta \) and letting \( \theta \to \theta \), with De l’Hospital’s rule we find for \( x \neq 0, \ x \neq -\frac{1}{4} \)

\[(n+1) \theta_1(x) A_{n+1}(x) + (n+2) x \theta_1(x) A_n(x).\]

In the same way, dividing by \( \theta - \theta \) and

\[-(n+1) \theta_2^n(x) A_{n+1}(x) - (n+2) x \theta_2^n(x) A_n(x).\]

These relations also hold for \( x = 0 \) and \( x = -\frac{1}{4} \), by continuity, when \( A \) and \( A \) are continuous.

Differentiating both sides of (49) with respect to \( \theta \), putting \( \theta = 1 \) and applying

\[((n-1)x-1) A_{n+1}(x) + (nx^2-x) A_n(x), \ n \in \mathbb{N}.\]
for \( n=1 \), it is sufficient to show that \( B_n \)
satisfies (10). By (19) we only have to
consider \( A_k(x) = \phi_k(x) \), \( k=1, 2 \), when

\[
\left\{ 1 + \frac{x^2}{\phi_1(x)} \right\}^{-\lambda} \left\{ \frac{\phi_1(x)}{\phi_2(x)} \right\} = \frac{\phi_1^n(x) - (-x)^{n+1}}{\phi_2^{n+2}(x)} = -1, \quad n \geq 0.
\]

The denominator is nonzero by (23).

For \( x = -\frac{1}{2} \), we have from (20) and (28)

\[
\sum_{k=0}^{n} \frac{\phi_1^{-k} (\alpha + \beta (n-2k)) 2^{-(n-2k)}}{\alpha (n+1) 2^{-n}} = A_0 (-\frac{1}{2}) \phi_1 (-\frac{1}{2}).
\]

For \( x = 0 \), it is safer to consider the
limiting behaviour for \( x \to 0 \) of (58).

Some quadratic relations. Here we look
for extensions of \( F \) (35) - (40). Let \( A_n, n \in \mathbb{Z} \),
satisfy (10). Then, for \( m, n \in \mathbb{Z} \),

\[
(60) \quad \phi_m(x) A_n(x) + x \phi_{m-1}(x) A_{n-1}(x) = A_{n+m}(x).
\]
ticular, for \( A_k = \Phi_k \), with \( m, n \in \mathbb{Z} \),

\[
(61) \; \Phi_m(x) \Phi_n(x) + x \Phi_{m-1}(x) \Phi_{n-1}(x) = \Phi_{m+n}(x).
\]

Also, for \( m, n \in \mathbb{Z} \),

\[
(62) \; A_{m+m}(x) + (-x)^m A_{n-m}(x) = \Lambda_m(x) A_n(x).
\]

Proof. For \( x \neq -1 \) we apply (19) and (22). It is sufficient to consider \( A_k(x) = c_i^k(x) \), \( i=1,2 \). We have with (21)

\[
c_i^{n+m}(x) + (-x)^m c_i^{n-m}(x) = \\
c_i^n(x) (c_i^m(x) + c_i^m(x)) = \Lambda_m(x) c_i^n(x).
\]

and with \( n=m+1 \) one finds

\[
(63. a) \; \Phi_n(x) \Lambda_{n+1}(x) = \Phi_{2n+1}(x),
\]

\[
(63. b) \; \Phi_n(x) \Lambda_n(x) = \Phi_{2n}(x) + (-x)^n,
\]

\[
(63. c) \; \Phi_{m+1}(x) \Lambda_m(x) = \Phi_{2m+1}(x) + (-x)^m,
\]

and with \( A_k = \Lambda_k \), \( n=m \) and \( n=m+1 \),

\[
(63. d) \; \Lambda_n(x) = \Lambda_{2n}(x) + x (-x)^n,
\]

\[
(63. e) \; \Lambda_m(x) \Lambda_{m+1}(x) = \Lambda_{2m+1}(x) + (-x)^m.
\]
Also, from the relation
\[ x^{-j} A_j A_{j+1} (x) = x^{-j} A_{j+1} (x) A_j (x) - x^{1-j} A_j (x) A_{j-1} (x), \]
\[ (64) \sum_{j=m}^{\infty} x^{-j} A_j^2 (x) = x^{-n} A_n A_{n+1} (x) A_n (x) \]
\[ \Delta_m (n) \Delta (x) \Delta (x) \Rightarrow n = \infty \]
\[
A_{n+1} A_{n-1} - A_n^2 = A_n A_{n-1} + x A_{n-1}^2 - A_n^2 =
-x (A_n A_{n+1} - A_{n-1}^2), \quad \text{so for } n \in \mathbb{Z}
\]
\[ (65) A_{n+1} (x) A_{n-1} (x) - A_n^2 (x) = (-x)^n \{ A_n (x) A_{n-1} (x) - A_{n+1}^2 (x) \}. \]

From (10) and (65)

\[ x^{-j} A_j A_{j-1} = x^{-j} A_j A_{j-1} = x^{1-j} A_{j-1}^2 + x^{j} (A_{j+1} A_{j-1} - A_j^2) \]
\[ = x^{-j} A_j A_{j-1} + x^{1-j} A_{j-1}^2 + (-1)^j (A_j A_{j-1} - A_j^2), \]
Binomial sums for $\Phi_n$ and $\Lambda_n$. We have the following generalizations of $F(y_1)$ and $F(y_2)$.

Relations with binomial coefficients may be derived from (68) or (69) in three ways. Substituting (68) or (69) into the preceding formulas and equating coefficients of $x^n$ on both sides leads to a binomial relation, mostly not an interesting one. See e.g. (77)-(84) below. Finding $\Phi_n(x)$ or $\Lambda_n(x)$ for some special $x$, e.g. from (10), (16), (17), or (28), gives a binomial sum, see e.g. (96). Still another way is illustrated by (72).

Proof of (68). The second and third member of (68) are equal by (12). Denote their common value by $S_n$. Then $S_0 = \Phi_0(x) = 1$, $S_1 = \Phi_1(x) = 1$ and for $n \geq 2$

$$S_n = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k = \sum_{k=0}^{n-1} \binom{n-1}{k-1} x^k + \sum_{k=1}^{n-1} \binom{n-1}{k-1} x^k = S_{n-1} + x S_{n-2}$$

So $S_n$ satisfies (10) with the same initial values as $\Phi_n(x)$. 

[Interpr.: Sved (1983)]
Proof of (67). For $n=1$ it holds trivially. For $n \geq 2$ we have, by B(31) with $a_t = t^n (68)$ and $\Phi_1 (44)$,
\[ a_n (\varepsilon) \equiv n \left( n-k \right) + k \equiv (n-k)k. \]

The interpretation. Let $a_n (\varepsilon), n \geq 1, \varepsilon \geq 1$ denote the number of words $(x_1, \ldots, x_n)$ from the alphabet $\{0, 1, \ldots, \varepsilon\}$ such that for no $i \in \{1, \ldots, n-1\}$ we have $x_i = x_{i+1} = 0$. Then
\[ a_n (\varepsilon) = \varepsilon^n \Phi_1 (\varepsilon^{-1}) = \sum_{k=0}^{n-1} \binom{n+1-1}{k} \varepsilon^{n-k}. \]

By specifying whether $x_i = 0$ or $x_i = 0$ we have, which we define $a_0 (\varepsilon) = 1$,
\[ a_n (\varepsilon) = \varepsilon a_{n-1} (\varepsilon) + \varepsilon a_{n-2} (\varepsilon), \quad n \geq 2. \]

From (10) we see that $\varepsilon^n \Phi_1 (\varepsilon^{-1})$ satisfies the same recurrence. The initial values are the same: $a_0 (\varepsilon) = \Phi_1 (\varepsilon^{-1}) = 1$, $a_1 (\varepsilon) = \varepsilon \Phi_1 (\varepsilon^{-1}) = \varepsilon + 1$. The third member of (70) reflects the construction of a word. When there are $k$ zeroes, consider these as the partition-walls between $k+1$ cells, in which $n-k$ balls, at
first indistinguishable, should be placed so that the \( k-1 \) inner cells are not empty. This can be done in \( (n+1-k) \) ways.

\[
\sum_{2i \leq n} \left( \frac{n+1}{i} \right) \cdot \sum_{x \leq n} x^{n/2} \cdot 2^{x-n} = \sum_{x \leq n} \left( \frac{n+1}{x} \right) , \quad n \in \mathbb{N}.
\]

\[
(72) \quad \sum_{i=0}^{\lfloor \sqrt{n} \rfloor} (2i+1)(n) = 2 \quad (\mathcal{E}), \quad 2\mathcal{E} \leq n \in \mathbb{N}.
\]
\begin{align*}
2^{-n} \sum_{2i \leq n} \binom{n}{2i} \sum_{k=0}^{i} \binom{i}{k} 4^k x^k &= \\
\leq \left[ \frac{n}{2} \right] \left[ \frac{n+1}{2} \right] \left[ \frac{n+2}{2} \right] &\leq \frac{\binom{n}{n/2}}{n/n/1111111111111111}. 
\end{align*}

For the relations \( (\frac{a}{a}) \), see also \( (8.133) \) and \( (3.132) \).

From \( (68) \) and \( (36) \); for \( n \in \mathbb{N}_0 \), \( x \in \mathbb{C} \),

Since \( \frac{1}{n} \frac{1}{d} = \frac{1}{n} \frac{1}{d} \), \( x = 1 \), etc., from \( (69) \) and \( (35) \):

Substitution of \( (68) \) into \( (49) \) gives, after some rearrangement, a trivial relation.
From (6.1) and (6.8) we have, for \( m \geq 1, n \geq 1, \)
\[
\sum_{2k \leq m+n} \binom{n+m-k}{k} x^k = T + \tilde{T},
\]

\[
T = \sum_{2i \leq m} \binom{m-i}{i} x^i \cdot \sum_{2j \leq n} \binom{n-j}{j} x^j =
\]
\[
\sum_{\tau \leq \lceil n/2 \rceil} x^\tau \sum_{\tau \leq \lceil n/2 \rceil} \binom{n-j}{j} \binom{m-\tau+j}{\tau-j},
\]

\[
\tilde{T} = \sum_{\tau > \lceil m/2 \rceil} \binom{m-1-i}{i} x^i \cdot \sum_{\tau > \lceil n/2 \rceil} \binom{n-1-j}{j} x^j =
\]
\[
\sum_{s \leq \lceil m/2 \rceil + \lceil n/2 \rceil} x^{s+1} \sum_{s \leq \lceil n/2 \rceil} \binom{n-1-j}{j} \binom{m-1-s+j}{s-j} x^j,
\]

\[
\sum_{1 \leq \tau \leq 1 + \lceil m/2 \rceil + \lceil n/2 \rceil} x^\tau \sum_{\tau \leq \lceil n-1/2 \rceil} \binom{n-1-j}{j} \binom{m-\tau+j}{\tau-1-j},
\]

Equating coefficients of \( x^\tau \) shows that
\[
\sum_{\tau \leq \lceil n/2 \rceil} \binom{n-j}{j} \binom{m-\tau+j}{\tau-j} + \binom{\tau}{\lceil n/2 \rceil} \binom{\tau}{\lceil m/2 \rceil} = \binom{n+m-\tau}{\tau}, \quad 1 \leq \tau \leq \lceil (n+m)/2 \rceil.
\]
This relation, together with (64) was given in Riordan (1968), Ch. 2, Problem 7. We now derive a seemingly more general formula:

\((78)\) \[ \sum_{j=0}^{\kappa} \binom{x-j}{j} \binom{y-\varepsilon+j}{\kappa-j} + \sum_{j=0}^{\kappa-1} \binom{x-1-j}{j} \binom{y-\varepsilon+j}{\kappa-1-j} = \binom{x+y-\varepsilon}{\kappa}, \]

\(\varepsilon \geq 1, \ x, y \in \mathbb{C}.\)

Proof. We apply the theory of Chapter C. The sequences \(\frac{x}{x-n} \binom{x-n}{n}\) and \(\binom{x-n}{n}\), \(n \in \mathbb{N}\), are basic and Sheffer, respectively, for the delta operator \(E \Delta\), see C, Theorem 8, and C, (44), (84). By C, (48) or C, (91), and C, Theorem 12,

\[ \sum_{j=0}^{\kappa} \binom{x-j}{j} \binom{y-\varepsilon+j}{\kappa-j} = \]

\[ \sum_{j=0}^{\kappa} \binom{x-j}{j} \frac{y}{y-\varepsilon+j} \binom{y-\varepsilon+j}{\kappa-j} - \sum_{j=0}^{\kappa} \binom{x-j}{j} \frac{\varepsilon-j}{y-\varepsilon+j} \binom{y-\varepsilon+j}{\kappa-j} = \]

\[ \binom{x+y-\varepsilon}{\kappa} - \sum_{j=0}^{\kappa-1} \binom{x-1-j}{j} \binom{y-1-\varepsilon+j}{\kappa-1-j} = \]

\[ \binom{x+y-\varepsilon}{\kappa} - \sum_{j=0}^{\kappa-1} \binom{x-1-j}{j} \binom{y-\varepsilon+j}{\varepsilon-1-j} . \]
For a more general formula see (3.527), (3.528). There seems to be a difference between (77) and (78). When \( x = n \) and \( \epsilon > n \) there are nonzero terms in (78) that do not occur in (77), viz. the terms with \( n < j \leq \epsilon \).

Similar formulas are found by taking, \( A = \)

\[
\phantom{\sum_{i}^{m} \sum_{j}^{n} (m-i) (n-j) x^{i} j^{j}}
\sum_{x \leq m+n} (m-i) x^{i} \sum_{j \leq n} \frac{n}{n-j} (n-j) x^{j} =
\]

\[
\sum_{x \leq m+n} \sum_{j \leq n} \frac{n}{n-j} (n-j) x^{j} (m-i+j) =
\]

\[
\sum_{s \leq \lceil m-\frac{1}{2} \rceil + \lceil n-\frac{1}{2} \rceil} \sum_{j=0}^{s} \frac{n-1}{n-1-j} \left(\binom{n-1-j}{j} \left(\binom{m-1}{s-j} \right)^{s} \right)
\]

\[
\sum_{1 \leq \epsilon \leq 1 + \lceil m-\frac{1}{2} \rceil + \lceil n-\frac{1}{2} \rceil} \frac{n-1}{n-1-j} \left(\binom{n-1-j}{j} \left(\binom{m-1}{\epsilon-1-j} \right)^{\epsilon} \right).
\]
Equating coefficients of \( x^\xi \) gives

\[
\sum_{j=0}^{\lfloor \frac{n}{\xi} \rfloor} \frac{n}{n-j} \binom{n-j}{j} \binom{m-\xi+j}{\xi-j} + \sum_{j=0}^{\lfloor \frac{n-1}{\xi} \rfloor} \frac{n-1}{n-1-j} \binom{n-1-j}{j} \binom{m-\xi+j}{\xi-1-j} = \frac{n+m}{n+m-\xi} \binom{n+m-\xi}{\xi}, \quad 1 \leq \xi \leq \left\lceil \frac{(n+m)\xi}{n} \right\rceil,
\]

(cf 3.625) \quad m \geq 1, \; n \geq \xi

There is a formula standing in the same relation to (79) as (78) to (77). With the same identification of basic and Sheffer sequences has in the proof of (78) we have from (48), or (9)

\[
\sum_{j=0}^{\xi} \frac{x}{x-j} \binom{x-j}{j} \binom{y-\xi+j}{\xi-j} + \sum_{j=0}^{\xi-1} \frac{x-1}{x-1-j} \binom{x-1-j}{j} \binom{y-\xi+j}{\xi-1-j} = \begin{cases} \left(\frac{t}{t-1}\right) + \left(\frac{x+y-\xi-(t-1)}{t-1}\right) = \frac{x+y}{x+y-\xi} \left(\frac{x+y-\xi}{\xi}\right) & (t \geq 1) \end{cases}
\]

The author did not find a formula for \( \Lambda_m^k \) and \( \Lambda_k^m \) analogous to (61). However, we may take \( \Lambda_k^m = \Lambda_m^k \) in (62). Then from (69),

for \( m+n \geq 1 \) (cf 3.624)
\[
\Lambda_{n+m}(x) = \sum_{2t \leq n+m} \frac{n+m-t}{n+m-t} \binom{n+m-t}{t} x^t,
\]

and for \(0 \leq m < n\)

\[
(-x)^m \Lambda_{n-m}(x) = (-1)^m \sum_{2i \leq n-m} \frac{n-m-(n-m-i)}{n-m-i} \binom{n-m-i}{i} x^{m+i}
\]

\[
= (-1)^m \sum_{m \leq r \leq (n+m)/2} \frac{n-m}{n-m} \binom{n-r}{r-m} x^r.
\]

In the same way as in the proof of (77) and (79), for \(m \geq 1, n \geq 1\),

\[
\Lambda_m(x) \Lambda_n(x) = \sum_{\varepsilon \leq [m/2]} \sum_{j=0}^{\varepsilon} \Delta \left[ \frac{\varepsilon}{2} \right] \frac{n}{n-j} \frac{m}{j} \frac{m-r+j}{j-r}.
\]

So by (62) for \(1 \leq m < n\), \(m \leq \varepsilon \leq \frac{m+n}{2}\),

\[
\sum_{j=0}^{\varepsilon} \frac{n}{n-j} \frac{m}{j} \frac{m-r+j}{j-r} = -1,
\]

(81)

\[
\frac{m+n}{m+n-t} \binom{m+n-t}{t} + (-1)^m \frac{n-m}{n-m} \binom{n-r}{r-m}.
\]

This relation is definitely different from (85) with \(a = -1\):

(82)

\[
\sum_{j=0}^{\varepsilon} \frac{x}{x-j} \binom{x-j}{j} \frac{y}{y-r+j} \binom{y-r+j}{r-j} = \frac{(x+y)(x+y-r)}{(x+y-r)} ^\varepsilon \binom{x+y-r}{r}.
\]
From (62) with $A_k = \xi_k$, one may derive by (68) and (69), in the same way as (81)

(83) \[ \sum_{j=0}^{m} \left( \begin{array}{c} n-j \\ j \end{array} \right) \frac{m}{m-\varepsilon+j} \left( \frac{m-\varepsilon+j}{\varepsilon-j} \right) = \]

Again, this relation is definitely different from its 'general' counterpart

(84) \[ \sum_{j=0}^{\varepsilon} \left( \begin{array}{c} x-j \\ j \end{array} \right) \frac{y}{y-\varepsilon+j} \left( \frac{y-\varepsilon+j}{\varepsilon-j} \right) = \left( \frac{x+y-\varepsilon}{\varepsilon} \right) \]

which follows from C. (91), with $a = -1$, or from C. (48) with $f_n(x) = \left( \frac{x}{n} \right)$ and $b = -1$.

For versions of (81) and (83) that are slightly more general, see pp. 995-105.
Alternative forms of (68) and (69). We have

\( \sum_{2k \leq n} (-1)^k \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = \)

\( (x+y)^n \sum_{n \in \mathbb{N}_0} (-xy (x+y)^{-1}) = \)

\( (x-y)^{-1} (x^{n+1} - y^{n+1}) = \sum_{j=0}^{n} x^j y^{n-j}, \)

with \( x+y \) for the third member, and

\( \sum_{2k \leq n} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = \)

\( (x+y)^n \sum_{n \geq 1} (-xy (x+y)^{-2}) = x^n + y^n, \quad n \geq 1, \)

where the second equality holds also for \( n=0. \)

The equality of first and third members in (66) and in (87) is known as Waring's formula. See Gould (1999), Lucas (1961), Ch. 18, p. 67-94 below.

Proof of (86) and (87). We have, by (69) and (69), only to prove the second equalities. For (86) we apply induction on \( n. \) It holds for \( n=0 \) and \( n=1. \) By (69), assuming (86) for \( n \leq m \), we have, by (69),

\( (x+y)^m \sum_{m \in \mathbb{N}_0} (-xy (x+y)^{-1}) = (x+y)^m \sum_{m \in \mathbb{N}_0} (-xy (x+y)^{-1}) = \)

\( (x+y)^{m-2} xy \sum_{m \in \mathbb{N}_0} (-xy (x+y)^{-1}) = \)

\( (x+y)(x-y)^{-1} (x^m - y^m) - xy (x-y)^{-1} (x^{m-1} - y^{m-1}) = \)
\[(x - y)^{-1} (x^{m+1} - y^{m+1}).\]

Then for \(n \geq 1\) from (44) and (86)
\[(x+y)^{-1} \sum_{n} (-xy(x+y)^{-2}) = 2(x-y)^{-1} (x^{n+1} - y^{n+1})
- (x+y)(x-y)^{-1} (x^n - y^n) = x^n + y^n.\]

**Remark 1.** We might prove (86) also by the generating functions identity (cf. (40))
\[
(1 - (x+y)z + x y z^2)^{-1} = (1 - x z)^{-1} (1 - y z)^{-1}.
\]
And (86), (87) also with (28), (29), (18).

**Remark 2.** The second equalities in (86) and (87) also hold for \(-n \in \mathbb{N}_0\). This may be proved with (15).

For \(y = 1\) we obtain from (86) the relation
\[(88) \sum_{2k \leq n} \binom{n-k}{k} (-x)^k (4x)^{n-2k} = \sum_{j=0}^{n} x^j.\]

This relation was studied by Sury (1993) who also derived its consequences given below.

For \(x = t - 1\) the formula (88) becomes
\[(89) \sum_{2k \leq n} \binom{n-k}{k} (1-t)^k t^{n-2k} = (2-t)^{-1} (1 - (t-1)^{n+1}).\]

A probabilistic interpretation of (89) is given by Horibe (1990). For (88) cf. Wolf (1990), Ch. 4.3.
\[
(2u-1)^{-1}(u^{n+1} - (1-u)^{n+1}) = \sum_{j=0}^{n} u^j (1-u)^{n-j},
\]

By Remark 2 to (86), (87) the second equalities in (90) and (91) also hold for \( n < 0. \) This also follows since we may take \( c_1(u^2-u) = u, \ c_2(u^2-u) = 1-u \) in (17), (18).

Applying the Binomial Theorem to \((1+x)^{n-2k} \) in (88) and equating coefficients of \( x^k \) does not

\[
\frac{1}{2} \leq n \quad \text{and} \quad \frac{1}{2} \leq k \leq n
\]

\[
(92) \quad \sum_{2k \leq n} (-1)^k \binom{n}{k} \binom{n-k}{k} = \ldots
\]

Putting \( u = \frac{1}{2} + v \). \]
\[ (93) \sum_{2k \leq n} (-1)^k \left( \frac{n+1}{2k+1} \right) \left( \frac{1}{k} \right)^{-1} = \sum_{j=0}^{n} \left( \frac{n}{j} \right)^{-1} \]

In the same way, from (91)
\[ (94) \sum_{2k \leq n} (-1)^k \frac{n}{n-k} \left( \frac{n-k}{k} \right)^{-1} \frac{1}{2k+1} \left( \frac{2k}{k} \right)^{-1} = \int_0^1 \Lambda_n (u^2-u) \, du = \frac{2}{n+1}, \quad n \geq 1, \]
\[ (95) \sum_{2k \leq n} (-1)^k \left( \frac{n+1}{2k+1} \right) \left( \frac{n-1}{k} \right)^{-1} = 2, \quad n \geq 1. \]

From (90) and (91) with \( u = -x \) and the subsequent remark we obtain
\[ (95a) \quad \Phi_n (x^2+x) + (x^2+x) \Phi_{n-1} (x^2+x) = \]
\[ (x+1)^n + (-1)^n x^n, \quad n \in \mathbb{Z}, \]
\[ (95b) \quad \Lambda_n (x^2+x) + (x^2+x) \Lambda_{n-1} (x^2+x) = \]
\[ (2x+1) \left\{ (x+1)^n + (-1)^n x^{n-1} \right\}, \quad n \in \mathbb{Z}. \]

Of Riordan (1968), Ch. 4, Exercise 7, in connection with (121), (122) below.
the binomial sum formulas. We mention some cases, the binomial sum formulas are listed in Table 1.

\[ \Phi_n(-\frac{1}{2}) = (n+1) 2^{-n}, \quad \Lambda_n(-\frac{1}{2}) = 2^{-n}, \quad n \in \mathbb{Z}. \]

For a combinatorial interpretation see Sved (1983).

We have, for \( n \in \mathbb{Z} \),

\[
\Phi_n(-1) = \frac{\pi}{\sqrt{3}} \sin \left( \frac{(n+1)\pi}{3} \right), \quad \Lambda_n(-1) = 2 \cos \frac{n\pi}{3}.
\]

\[ \left( \begin{array}{c}
\Gamma(n/2) \sin \left( \frac{(n+1)\pi}{3} \right) \\
\Gamma(-n/2) \cos \left( \frac{(n+1)\pi}{3} \right)
\end{array} \right) \quad \text{and both have the correct initial values. C.F. Egorychev}
\]

When \( x = -\frac{1}{2} \), we have in (18), (1984), (1,32)

\[ (-n/2)^{-1/2} = a \to \infty. \]
These relations also may be derived from (26), (27). The numbers $\Phi(n, 2)$ and $\Lambda(n, 2)$ are known as Jacobsthal and Jacobsthal-Lucas numbers, see Horadam (1996).
Some more binomial sums. We have the following extensions of (46), (47).

\[(101) \quad (2-x) \sum_{2k \leq n} \binom{k}{n-2k} x^{n-2k} (1+x)^{3k-n} = (-1)^n + \Phi_{n+1}(x) - x \Phi_n(x), \quad x \in \mathcal{N}_0,\]

\[(102) \quad (2-x) \sum_{2k \leq n} \binom{n-2k}{k} (-1)^k x^k (1+x)^{n-3k} = \Phi_{n+1}(x) + \Phi_n(x) - x^{n+1}, \quad x \in \mathcal{N}_0.\]

The summations in (101) and (102) may be restricted to \( \frac{1}{3} n \leq k \leq \frac{1}{2} n \) and \( 3k \leq n \), respectively.

Proof of (101). We have for small \( z \)

\[
(1+z)^{-1} (1-z-xz^2)^{-1} = (1-(1+x)z^2-xz^3)^{-1} = \sum_{k=0}^{\infty} (xz^3+(1+x)z^2)^k = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} x^j (1+x)^{k-j} z^{2k+j} = \sum_{n=0}^{\infty} z^n \sum_{2k \leq n} \binom{k}{n-2k} x^{n-2k} (1+x)^{3k-n},
\]

So from (40) and (50) with \( \lambda_k = \Phi_k \)

\[
(2-x)^{-1} (-1)^n + \Phi_{n+1}(x) - x \Phi_n(x) = \sum_{2k \leq n} \binom{k}{n-2k} x^{n-2k} (1+x)^{3k-n} = \sum_{k=0}^{n} (-1)^{n-k} \Phi_k(x) = (2-x)^{-1} \left( (-1)^n + \Phi_{n+1}(x) - x \Phi_n(x) \right).
\]
\[ \sum_{2k \leq n} \binom{n-2k}{k} (-1)^k x^k (1+x)^{n-3k} = \sum_{k=0}^{n-k} \frac{x^{n-k}}{k!} \]

\[ = (2-x)^{-1} \left( \Phi_{n+1}(x) + \Phi_n(x) - x^{n+1} \right) \]

We now will derive generalizations of \( F_3(48) - (60^a) \). Proofs are similar to those in Chapter 3. We define operators \( E^i \) in \( \mathbb{Z} \) that operate on the index \( n+1 \), but here we have to deal with \( x = -1 \) separately, by (28). Or we might use a continuity argument.
\[(108)\quad (I - x^2) A_n(x) = \sum_{k=0}^{\infty} \binom{n}{k} \cdot (-1)^k \cdot A_{n-2k}(x) = A_{n-r}(x),\]

\[(1+4x)^m \Delta_{n+2m+2}(x),\]

\[(111)\quad (E + xI)^{2m+1} \Delta_n(x) = \sum_{k=0}^{2m+1} \binom{2m+1}{k} \cdot x^{2m+1-k} \cdot \Delta_{n+2k}(x) = (1+4x)^{m+1} \Delta_{n+2m}(x).\]
(112) \((I + x E^{-x}) A_n(x) = \sum_{k=0}^{2m} \binom{2m}{k} x^k A_{n-2k}(x) = (1 + x)^m A_{n-2m}(x)\),

(113) \((I + x E^{-x})^{2m+1} \Phi_n(x) = \sum_{k=0}^{2m+1} \binom{2m+1}{k} x^k \Phi_{n-2k}(x) = \)

(115) \((x I + (1+x) E) A_n(x) = \sum_{k=0}^{n+1} \binom{n+1}{k} (1+x) (-x^2)^{-k} A_{n+k}(x) = A_{n+k}(x)\),

(115a) \((1+x) E - x^2 I)^r A_n(x) = \sum_{k=0}^{\infty} \binom{r}{k} (1+x)(-x^2)^{-k} A_{n+2k}(x) = A_{n+2k}(x)\).

Proofs of (113) - (115a). All proofs here are simp-
\[(E^n + xI)^{m+1} \Delta_n(x) = (1+yx)(E^n + xI)^{m+1} \Phi_n(x) = (1+yx)^{m+1} \Phi_{n+2m}(x),\]

(115g): From
Substitution of (68) and (69) into (103) and equating coefficients of \(x^i\) in both sides gives varieties of Vandermonde's convolution \(D(26)\) and of (3.145). By the same method we obtain special cases of (3.395) and (3.464) from (109) – (114).

Taking \(\alpha = 2m\), \(\beta = 2m+1\), \(\lambda(x) = (1+yx)\frac{\sqrt{3}}{2^m} c_2(x)\), \(\mu(x) = - (1+yx)^{-\frac{1}{2}} c_2(x)\) and \(\lambda(x) = \mu(x) = 1\), we obtain by (26) and (27), respectively,

\[
\sum_{k=0}^{2m} \binom{2m}{k} x^{2m-k} \frac{\sqrt{3}}{2^{n+k}} \Lambda_{n+k}(x) = (1+yx)^{m-1} \Lambda_{2n+2m+2}(x), \quad m \geq 1,
\]

\[
\sum_{k=0}^{2m} \binom{2m}{k} x^{2m-k} \Lambda_{n+k}^2(x) = (1+yx)^m \Lambda_{2n+2m}(x), \quad m \geq 1.
\]
\[(118) \quad \sum_{k=0}^{2m+1} \binom{2m+1}{k} x^{2m+1-k} \phi_n^2(x) = \]
\[\phi_{2n+2m+2}(x), \quad m \geq 0,\]

\[(119) \quad \sum_{k=0}^{2m+1} \binom{2m+1}{k} x^{2m+1-k} \Lambda_{n+k}^2(x) = \]
\[(1+yx)^m \phi_{2n+2m}(x), \quad m \geq 0,\]

Since both sides of (118)-(119) are polynomials in \(x\), these relations hold for all \(x \in \mathbb{C}\).
Even and odd index. The sequences $\Phi_{2m}(x)$, $\Lambda_{2m}(x)$, $\Phi_{2m+1}(x)$ and $\Lambda_{2m+1}(x)$, $m \in \mathbb{N}_0$, have some properties, though derivable from those of $\Phi$ and $\Lambda$, that look different. We have from (68) and (69)

\[(121) \Phi_{2m}(x) = \sum_{k=0}^{m} \binom{2m-k}{k} x^k = \sum_{h=0}^{m} \binom{m+h}{2h} x^{m-h},\]

\[(122) \Phi_{2m+1}(x) = \sum_{k=0}^{m} \binom{2m+1-k}{k} x^k = \sum_{h=0}^{m} \binom{m+h+1}{2h+1} x^{m-h},\]

\[(123) \Lambda_{2m}(x) = \sum_{k=0}^{m} \frac{2m}{2m-k} \binom{2m-k}{k} x^k = \sum_{h=0}^{m} \frac{2m}{m+h} \binom{m+h}{2h} x^{m-h}, \quad m \geq 1,\]

\[(124) \Lambda_{2m+1}(x) = \sum_{k=0}^{m} \frac{2m+1}{2m+1-k} \binom{2m+1-k}{k} x^k = \sum_{h=0}^{m} \frac{2m+1}{m+1+h} \binom{m+1+h}{2h+1} x^{m-h}.\]

An application of (122) is to be found in Ferri et al. (1991, 1992). Their $F_m(x)$ is equal to $x^m \Phi_{2m+1}(x)$. The above relations have companions in the sense of the remark on p. 197.

Writing (121) as

\[Y_m = \sum_{h=0}^{m} \binom{m+h}{2h} x^h, \quad m \in \mathbb{N}_0,\]
With \( x^h = x^{-h} \), \( y_m = x^{-m} \Phi_{2m+1}^+(x) \), we have, with the inverse pair \( IR(81) \),

\[
(125) \quad x^{-m} = \sum_{k=0}^{m} (-1)^{m-k} 2^{h+1} \left( \frac{2m+1}{m-k} \right) x^{-k} \Phi_{2h}^+(x),
\]

In the same way, from (123) and \( IR(82) \),

\[
(126) \quad x^{-m} = \sum_{k=0}^{m} (-1)^{m-k} 2^{h+1} \left( \frac{2m+2}{m-k} \right) x^{-k} \Phi_{2h+1}^+(x),
\]

We may write (123) as

\[
x^{-m} U_m = \sum_{k=0}^{m} (-1)^{m-k} b_{mk} x^{-k}, \quad m \in \mathbb{N},
\]

with \( b_{mk} \) as in \( IR(85) \) and \( U_0 = 1 \).

\[
U_m = \Lambda_{2m}^+(x), \quad m \geq 1. \text{ Then, as above, with the inverse pair \( IR(85) \)}
\]

\[
(127) \quad x^{-m} = (-1)^{m} \binom{2m}{m} + \sum_{k=1}^{m} (-1)^{m-k} \left( \frac{2m}{m-k} \right) x^{-k} \Lambda_{2k}^+(x), \quad m \geq 1.
\]

Similarly, from (124) and \( IR(86) \),

\[
(128) \quad x^{-m} = \sum_{k=0}^{m} (-1)^{m-k} \left( \frac{2m+1}{m-k} \right) x^{-k} \Delta_{2k+1}^+(x).
\]

By bisection of the series (40), see G(3)-19, we have for \( |z| < |x| \min\{ |c_1(x)|, |c_2(x)| \} \), see (38):

\[
2 \sum_{m=0}^{\infty} \Phi_{2m}^+(x) z^{2m} = \left( I + z - xz^2 \right)^{-1} + \left( 1 + z - xz^2 \right)^{-1} = (2 - 2x z^2)(1 - (1 + 8x) z^2 + x^2 z^4),
\]
For \( z = y^{1/2} \), \( y \geq 0 \), this becomes

\[
(129) \sum_{m=0}^{\infty} \Phi_{2m}(x) y^m = \frac{1}{(1-xy)(1-(1+2x)y + x^2y^2)^{-1}},
\]

with absolute convergence for \( |y| < |x|^{-2} \min \{ |c_1(x)|^2, |c_2(x)|^2 \} \). By analyticity we have \((129)\) with absolute convergence for \( y \in \mathbb{C} \) with

\[
|y| < |x|^{-2} \min \{ |c_1(x)|^2, |c_2(x)|^2 \}.
\]

Also, by Bisection of \((40)\)

\[
2 \sum_{m=0}^{\infty} \Phi_{2m+1}(x) z^{2m+1} = (1-z-xz^2)^{-1} = (1+z-xz^2)^{-1} = 2z \left(1-(1+2x)z^2+x^2z^4\right)^{-1}
\]

So, again under \((130)\)

\[
(131) \sum_{m=0}^{\infty} \Phi_{2m+1}(x) y^m = \left(1-(1+2x)y + x^2y^2\right)^{-1}
\]

and in the same way from \((41)\), under \((130)\), or from \((129)\), \((131)\) and \((44)\),

\[
(132) \sum_{m=0}^{\infty} \Lambda_{2m}(x) y^m = \frac{1}{(1-xy)(1-(1+2x)y + x^2y^2)^{-1}}
\]

\[
(133) \sum_{m=0}^{\infty} \Lambda_{2m+1}(x) y^m = \frac{1}{(1+xy)(1-(1+2x)y + x^2y^2)^{-1}}.
\]
From (129), (131), and (61) we obtain the generating functions of the sequences \( \Phi(x) \) and \( \Phi(x) \Phi(x^2) \). Taking \( m = n \) in (61), multiplying by \( \frac{z^n}{n!} \) and summing over \( n \) we obtain:

\[
(134) \quad \sum_{n=0}^{\infty} \Phi_n(x) z^n = (1+xz)^{-1} \sum_{n=0}^{\infty} \Phi_n(x) z^n = (1-xz)(1+xz)^{-1}(1-(1+2x)z+x^2z^2)^{-1}.
\]

Similarly, from (61) with \( m = n+1 \),

\[
(135) \quad \sum_{n=0}^{\infty} \Phi_n(x) \Phi_{n+1}(x) z^n = (1+xz)^{-1} \sum_{n=0}^{\infty} \Phi_n(x) z^n = (1+xz)^{-1}(1-(1+2x)z+x^2z^2)^{-1}.
\]

From (26) and (21) we see that the convergence in (134) and (135) is absolute for

\[
|z| < \min \left( \left| c_1(x) \right|^{-2}, \left| c_2(x) \right|^{-2} \right) = \left| x \right|^{-2} \min \left( \left| c_2(x) \right|^2, \left| c_1(x) \right|^2 \right).
\]

Therefore, the singularity \( z = -x^{-1} \) in (134) and (135) must lie outside this domain, i.e., \( \min \left( \left| c_2(x) \right|, \left| c_1(x) \right| \right) \leq \left| x \right|^{-1} \).

This follows from (21). There may be equality, e.g., when \( x \in \mathbb{R} \), \( x < -14 \). Then (17) has two complex conjugate roots.

Relations analogous to (134) and (135) may be obtained from (60) and (62) for \( \Phi_n^2(x), \Phi_n(x) \Phi_{n+1}(x) \), etc., but here the
relations \((63^a) - (63^e)\) are more direct. From \((134)\) and \((135)\) we obtain the "inversions" of \((61)\) for \(m=n\) and \(m=n+1\):

\[
(136) \quad \sum_{k=0}^{n} (-x)^{n-k} \Phi_k(x) = \Phi_n^2(x), \quad n \in \mathbb{N}_0,
\]

\[
(137) \quad \sum_{k=0}^{n} (-x)^{n-k} \Phi_{k+1}(x) = \Phi(x)\Phi_n(x), \quad n \in \mathbb{N}_0.
\]

Let \(A_n(x), \ n \in \mathbb{Z}\), satisfy \((10)\). Then we have the following relations, for \(m \geq 0\), when \(A_0(x) = 1\),

\[
(138) \quad \sum_{j=0}^{m} x^{m-j} A_{2j+1}(x) = A_{2m+2}(x) - x^{m+1}.
\]

When \(A_0(x) = A_1(x)\),

\[
(139) \quad \sum_{j=0}^{m} x^{m-j} A_{2j}(x) = A_{2m+1}(x).
\]

When \(A_0(x) = 2\),

\[
(140) \quad \sum_{j=0}^{m} x^{m-j} A_{2j+1}(x) = A_{2m+2}(x) - 2x^{m+1}.
\]

When \(A_0(x) = 1 + A_1(x)\),

\[
(141) \quad \sum_{j=0}^{m} x^{m-j} A_{2j}(x) = A_{2m+1}(x) + x^{m}.
\]

In particular, \((138)\) and \((139)\) hold when \(A_1(x) = \Phi_1(x)\), and \((140)\) and \((141)\) hold when \(A_1(x) = \Delta_1(x)\).

Proofs. We prove \((138)\) - \((141)\) by induction...
\[ \sum_{j=0}^\infty x^{2j+1} A_{j+1}(x) = x \sum_{j=0}^\infty x^{2j} A_{j+1}(x) + A_{2m+3}(x) = x A_{2m+2}(x) - x^{m+2} + A_{2m+3}(x) = \]

The induction steps for \( H_j \) are similarly similar.

From (131) we have for small \( y \)

\[
\sum_{m=0}^\infty \Phi_{2m} (x) y^m = \sum_{m=0}^\infty \sum_{i=0}^m (-1)^i \binom{m}{i} x^i (1+2x)^{m-i} y^i = \\
\sum_{n=0}^\infty \sum_{j=0}^n (-1)^j \binom{n}{j} x^j (1+2x)^{n-j} y^j = \\
\sum_{m=0}^\infty y^m \sum_{j=0}^m (-1)^j \binom{m-j}{j} x^j (1+2x)^{m-2j}.
\]

So

\[
(142) \quad \Phi_{2m+1} (x) = \sum_{i=0}^m (-1)^i \binom{m-j}{i} x^i (1+2x)^{m-2j}.
\]

cf. Cassidy and Hodgson (1994). In particular, for \( x=1 \) with \( \Phi(t) = F \) and (68) we obtain \( F(\beta_0) \), cf. Prob. B-880, nF16, nR 38/2, 2000, 182-183:

\[
\sum_{2k \leq m} (-1)^k \binom{m-k}{k} 3^{m-2k} = F_{2m+1}, \quad m \in A_{\alpha}.
\]
For $x = i(1 - x i)^{-1}$ and $x = -i (1 + x i)^{-1}$ we have from (14.2)

$\Phi_{46}$

cf. Pla (1996). For $x = -\frac{1}{2}$ and $x = -1$ we find

(143) $\Phi_{2m+1} (-\frac{1}{2}) = \Phi_m (-1)$, $\Phi_{2m} (-1) = (-1)^m \Phi_m (-1)$

in agreement with (96).

We also have the following relations. cf.
Derivatives of the generating function.

Here we generalize $F(105) - (122)$. We define, noting (40), for $|z - x| < 1$,

(149) \[ c_n(z, x) = \frac{1}{n^!} \frac{d^n}{dz^n} f(z, x), \]

(150) \[ f(z, x) = (1 - z - x z^2)^{-1} = \sum_{k=0}^{\infty} \Phi_k(x) z^k, \]

the last equality holding under (38). We have, for sufficiently small $w$,

(151) \[ f(z + w, x) = (1 - z - x z^2 - (1 + 2 x z) w - x w^2)^{-1} = f(x, z) \left\{ 1 - (x w^2 + (1 + 2 x z) w) f(z, x) \right\}^{-1} = \sum_{k=0}^{\infty} (x w^2 + (1 + 2 x z) w)^k f^k(z, x) = \sum_{k=0}^{\infty} f^k(z, x) \sum_{r=0}^{\infty} \binom{r}{k} x^r (1 + 2 x z)^{r-k} w^{r+k} = \sum_{n=0}^{\infty} w^n \sum_{2k \leq n} \binom{n-k}{k} x^k (1 + 2 x z)^{n-2k} f^{n+1-k}(z, x), \]

where the last step is justified since the double series converges absolutely for sufficiently small $w$. Comparison with (149) then gives

(152) \[ c_n(z, x) = \sum_{2k \leq n} \binom{n-k}{k} x^k (1 + 2 x z)^{n-2k} f^{n+1-k}(z). \]
From

\[(153) \quad 1 - z^n x^2 = \frac{1}{y} \left(1 + y x - (1 + 2 x z)^2 \right)\]

we have for

\[(154) \quad |1 + 2 x z|^2 < |1 + y x|,\]

in particular for

\[(155) \quad x > 0 \text{ and } |z| < -x^{-1} c_2(x),\]

\[f(z, x) = 4 x (1 + y x) \sum_{\ell=0}^{\infty} \frac{(1 + y x)^{-\ell - 1}}{(1 + 2 x z)^{2\ell}} = 4 x \sum_{\ell=0}^{\infty} \frac{(1 + y x)^{-\ell - 1}}{(1 + 2 x z)^{2\ell}},\]

\[(156) \quad d_n(z, x) = 2^{n+2} x^{n+1} \sum_{2\ell \geq n} \frac{(\ell)!}{(n)!} (1 + y x)^{-\ell - 1} (1 + 2 x z)^{2\ell - n},\]

so with (149) and (150) for \(|1 + y x| > 1\)

\[(157) \quad \Phi_n(x) = 2^{n+2} x^{n+1} \sum_{2\ell \geq n} \frac{(\ell)!}{(n)!} (1 + y x)^{-\ell - 1}\]

From (149) and (150), under (38)

\[(158) \quad d_n(z, x) = \sum_{h=0}^{\infty} \binom{n+h}{n} \Phi_{n+h}(x) z^h,\]

from (152) with \(n \geq 2\) we find the recurrence

\[(159) \quad d_n(z, x) = \sum_{k=0}^{n-1} \binom{n-k}{k} x^k (1 + 2 x z)^{n-2k} n^{n+k} (z, x).\]
\[
\sum_{k=0}^{n-1} \binom{n-1-k}{k} x^k (1+2xz)^{n-2k} f^{n+k} (z, x) + \sum_{k=1}^{n-1} \binom{n-1-k}{k-1} x^k (1+2xz)^{n-2k} f^{n+k} (z, x) = \\
(1+2xz) f^0 (z, x) d'_{n-1} (z, x) + \\
\sum_{h=0}^{n-2} \binom{n-2-h}{h} x^{h+1} (1+2xz)^{n-2-2h} f^{n-h} (z, x) = \\
(1+2xz) f^0 (z, x) d'_{n-1} (z, x) + x f^0 (z, x) d'_{n-\alpha} (z, x),
\]

From (150), (18), and (21),

\[f^0 (z, x) = x (\xi (x) - \xi^2 (x)) \left[ (xz + \xi (x))^{-1} - (xz + \xi^2 (x))^{-1} \right],\]

\[d'_{n} (z, x) = (-1)^{n+1} x^{n+1} (\xi (x) - \xi^2 (x))^{-1},\]

\[
\left\{ (xz + \xi (x))^{-n-1} - (xz + \xi^2 (x))^{-n-1} \right\} = \\
(-1)^n x^{n+1} (1+yx)^{-1/2} (xz)^{-n-1}.
\]

\[
\left\{ (1+(1+2xz)^{1/2})^{-n-1} - (1-(1+2xz)^{-1/2})^{-n-1} \right\}.
\]

So for \(|1+2xz|^2 > |1+yx|\), with \(D(25)\)

\[
(160) \ d'_{n} (z, x) = (-1)^n x^{n+1} z^{n+2} (1+yx)^{-1/2} (1+2xz)^{-n-1}.
\]

\[\sum_{i \text{ odd}} (-1)^i \binom{n+1}{i/2} (1+yx)^{i/2} (1+2xz)^{-i} =
\]

\[(-x)^{n+1} z^{n+2} \sum_{h=0}^{\alpha} \binom{n+2h+1}{h} (1+yx)^{h} (1+2xz)^{-2h-2}.
\]
Put

\[ w_i = w_i(z, x) = (xz + c_i(x)) f(z, x), \quad i = 1, 2. \]

Then we have from (151) and the relations

\[ w_1 + w_2 = (1 + z x) f(z, x), \quad w_1 w_2 = -x f(z, x), \]

\[ f(z + w, x) = (1 + z x)^{-1/2} \left( w_1 (1 - w, w)^{-1} - w_2 (1 - w, w)^{-1} \right) = (1 + z x)^{-1/2} \sum_{n=0}^{\infty} (w_1^{n+1} - w_2^{n+1}) w^n. \]

So with (159)

\[ d_1(z, x) = (1 + z x)^{-1/2} \left( w_1^{n+1} - w_2^{n+1} \right) = 2^{-n-1} (1 + z x)^{-1/2} f^{n+1}(z, x), \]

\[ \{ (1 + 2xz + \sqrt{1 + 4x})^{n+1} - (1 + 2xz - \sqrt{1 + 4x})^{n+1} \} = (161) \]

\[ 2^{-n} (1 + z x)^{-1/2} f^{n+1}(z, x). \]

\[ \sum_{j \text{ odd}} (\binom{n+1}{j}) (1 + z x)^{j/2} (1 + 2xz)^{n+1-j} f^j = 2^{-n} f^n(z, x) \sum_{2k \leq n} (2k+1)(1 + 4x)^k (1 + 2xz)^{n-2k}. \]

Equating the right-hand sides of (152) and (161) leads to (86) with \( x \) and \( y \) replaced by \( xz + c_1(x) \) and \( xz + c_2(x) \).
Factorizations. The polynomials $\Phi_n$ and $\Delta_n$ admit interesting factorizations, to be derived from the following elegant result in Carroll and Yanosko (1991).

Let

$$\sum_{i \leq n} \frac{(-1)^i (n-i)}{i!} x^{n-2i}, \quad n \in \mathbb{N}^*.$$ 

We have $f_n(x) = f_{n+1}(x)$ in the notation of Carroll and Yanosko. From (168)

$$f_n(x) = x^n \Phi_n (-x^{-2}).$$

From (164) and (10), or directly from (163),

$$f_n(x) = x^n \Phi_{n-1} (-x^{-2}) - x^{n-2} \Phi_{n-2} (-x^{-2}) =$$

$$x f_{n-1} (x) - f_{n-2} (x), \quad n \geq 2.$$  

Carroll and Yanosko showed that

$$f_n (2 \cos \alpha) = (\sin \alpha)^{n+1} \sin (n+1) \alpha, \quad \alpha \pi \notin \mathbb{Z}, n \geq 0,$$

$$f_n (x) = \prod_{j=1}^{n} \left( x - 2 \cos \frac{j \pi}{n+1} \right), \quad n \geq 1.$$  

Both sides of (166) coincide for $n=0$ and for $n=1$ and satisfy the same recurrence:

$$f_n (2 \cos \alpha) = 2 \cos \alpha f_{n-1} (2 \cos \alpha) - f_{n-2} (2 \cos \alpha),$$

$$\sin (n+1) \alpha = 2 \cos \alpha \sin n \alpha - \sin (n-1) \alpha.$$
It follows from (166) that \(2\cos((n+1)^{-1}j\pi), j=1, \ldots, n\), are zeroes of \(f_n\). Since these zeroes are different and the degree of \(f\) is \(n\), whereas the coefficient of \(x^n\) in \(f_n\) is 1, the relation (167) follows.

From (164) and (167)

\[
\Phi_n(-u^2) = \prod_{j=1}^{n} \left(1 - 2u \cos((n+1)^{-1}j\pi)\right), n \geq 1.
\]

In this product the factor with \(j = \frac{1}{2}(n+1)\), when present, is equal to 1. The factors with \(j = h + \frac{1}{2}\) and \(j = n+1-h\) give

\[
(1 - 2u \cos((n+1)^{-1}h\pi))(1 + 2u \cos((n+1)^{-1}h\pi)) = 1 - 4u^2 \cos^2((n+1)^{-1}h\pi),
\]

so that (cf. Prob. \(\text{E}^{[n/2]}_{2737}\), Monthly 87, 1980, 304-306)

\[
(168) \quad \Phi_n(x) = \prod_{h=1}^{n} \left(1 + 4x \cos^2((n+1)^{-1}h\pi)\right), n \geq 2.
\]

From (168) and (167), for \(n \geq 3\)

\[
(169) \quad \Lambda_n(x) = \frac{\Phi_{2n-1}(x)}{\Phi_{n-1}(x)} = \prod_{h=1}^{n-1} \left(1 + 4x \cos^2\left(\frac{h\pi}{2n}\right)\right) / \prod_{h=1}^{[n/2]} \left(1 + 4x \cos^2\left(\frac{h\pi}{n}\right)\right) = \prod_{1 \leq h \leq n-1, h \text{ odd}} \left(1 + 4x \cos^2\left(\frac{2h+1}{2n}\pi\right)\right).
\]
This relation also holds for \( n = 2 \).

For these factorizations see also Problems H-466, Fib. Q. 30(2), 1992, 188-199 and B-742, Fib. Q. 32(3), 1994, 470-471.
A generalization. Many generalizations of Fibonacci numbers and polynomials, satisfying recurrences of order greater than 2, are to be found in the literature. See e.g. Charalambides (1991), Klein (1991), G.N. Philippou (1988).

Here we consider the following one, since it is a direct generalization of (68) and (69) and leads to some binomial sums. We define

\[(172) \quad \Lambda_{n,k}(x) = \sum_{k=0}^{n} \binom{n-k}{k} x^k, \quad x \in \mathbb{N}, \quad n \in \mathbb{N}.\]

For a suitable definition of \( \Lambda_{n,k}(x) \) see (178) below. These polynomials and coefficients occurred before in the literature, with applications, e.g. Curtiss (1971, 1972), Frame (1957), Hod and McQuistan (1958), Riordan (1958), Ch.8, Exercises 1, 2. For the corresponding generalized Fibonacci numbers see Bicknell-Johnson and Spears (1996), for the connection with Fibonacci polynomials of order \( k \), Philippou et al. (1985).
\[ \Phi_{n \varepsilon} (x) + \varepsilon \sum_{0 \leq h \leq (n-\varepsilon-1)/(k+1)-1} \binom{n-\varepsilon-1-\varepsilon h}{h} x^{h+1} = \]

\[ \sum_{(k+1)h \leq n-1} \binom{n-1-\varepsilon h}{k} x^k + \sum_{0 \leq h \leq (n-\varepsilon-1)/(k+1)-1} \binom{n-\varepsilon-1-c n}{h} x^h \]

\[ (175) \quad \Phi_{n \varepsilon} (x) = 1, \; n = 0, \ldots , \varepsilon, \]

that follow from (171), determine the sequence \( \Phi_{n \varepsilon} (x), \; n \in \mathbb{N}, \) uniquely.

From (173) and (174) we find the generalization

for \( n \geq \varepsilon + 1. \) From (175) and (172) we see that

(176) also holds for \( 1 \leq n \leq \varepsilon. \)

From (176) and (174) it follows that

(177) \[ \Lambda_{n \varepsilon} (x) = \Lambda_{n-1, \varepsilon} (x) + \varepsilon \Lambda_{n-\varepsilon-1, \varepsilon} (x), \]
for \( n \geq \varepsilon + 2 \). If we want (177) to hold for \( n = \varepsilon + 1 \), we have to define

\[
(178) \quad \Lambda_{\varepsilon}^{(\varepsilon+1)}(x) = \varepsilon + 1,
\]

which compares well with (8). This relation follows since \( \Lambda_{\varepsilon+1}^{(\varepsilon+2)}(x) = 1 + (\varepsilon+1)x \) and \( \Lambda_{\varepsilon}^{(\varepsilon+1)}(x) = 16y \) (172). 

The recurrence (177) with the initial values (178) and \( \Lambda_{\varepsilon}^{(\varepsilon+1)}(x) = 1, n = 1, \ldots, \varepsilon \), determines the sequence \( \Lambda_{\varepsilon}^{(\varepsilon+1)}(x) \), \( n \in \mathbb{N} \), uniquely.

Let the sequence \( \Lambda_{n}^{(\varepsilon+1)}(x) \), \( n \in \mathbb{N} \), satisfy a recurrence similar to (174). Then \( \sum_{n=0}^{\infty} \Lambda_{n}^{(\varepsilon+1)}(x) z^{n} \) converges for sufficiently small \( z \) since \( |A_{n}^{(\varepsilon+1)}(x)| \) does not increase faster than exponentially. Multiplying both sides of the recurrence with \( z^{n} \) and summing over \( n \) with \( n \geq \varepsilon + 1 \) we obtain

\[
\sum_{n=\varepsilon+1}^{\infty} A_{n}^{(\varepsilon+1)}(x) z^{n} = \sum_{n=\varepsilon+1}^{\infty} A_{n-1}^{(\varepsilon+1)}(x) z^{n+1} z^{n} + z^{n} \sum_{n=\varepsilon+1}^{\infty} A_{n-1}^{(\varepsilon+1)}(x) z^{n},
\]

(179) \( (1 - z - x z^{\varepsilon+1}) \sum_{n=0}^{\infty} A_{n}^{(\varepsilon+1)}(x) z^{n} = \sum_{n=0}^{\infty} A_{n}^{(\varepsilon+1)}(x) z^{n} - \sum_{m=0}^{n} A_{m}^{(\varepsilon+1)}(x) z^{m+1} - A_{0}^{(\varepsilon+1)}(x) + \sum_{n=1}^{\infty} (A_{n}^{(\varepsilon+1)}(x) - A_{n-1}^{(\varepsilon+1)}(x)) z^{n} 
\)

From (175), (178) and since \( \Lambda_{\varepsilon}^{(\varepsilon+1)}(x) = 1 \), \( 1 \leq n \leq \varepsilon \), we obtain

\[
(180) \sum_{n=0}^{\infty} \frac{\Phi_{n}^{(\varepsilon+1)}(x)}{\Lambda_{\varepsilon}^{(\varepsilon+1)}(x)} z^{n} = (1 - z - x z^{\varepsilon+1})^{-1}.
\]
\[
\sum_{n=0}^{\infty} \Delta_n(x) z^n = \frac{(1+2-2z)(1-z-xz^{n+1})}{},
\]
Some special values of \( S_n(x) \) and \( \Lambda_n(x) \).

First we consider the relations (4.10), (4.11) and (4.12) in Mahon and Horadam (1990).

In our notation these relations are

\[
(183) \quad S_n = \sum_{3i \leq n} (-1)^i \binom{n-xi}{i} 2^{n-3i} = F_{n+2} - 1,
\]

\[
(184) \quad \sum_{3i \leq n} (-1)^i \frac{n-xi}{n-3i} \binom{n-xi}{i} 2^{n-3i} = F_n, \quad n \geq 1,
\]

\[
(185) \quad T_n = \sum_{3i \leq n} (-1)^i \binom{n-xi}{i} 2^{n-3i} = L_n + 1, \quad n \geq 1,
\]

with \( F \) and \( L_n \) as in \( F \), (1), (2), (3). We found (183) before: see \( F \) (47). Here we prove it by induction using (174). For \( n = 0, 1, 2 \) we may verify (183) directly. Then for \( n \geq 3 \)

\[
S_n = 2^n \Phi_{n+2}(-1/2) = 2^n \Phi_{n+1,2}(-1/2) = \frac{2^n}{2} \Phi_{n-3,2}(-1/2) = 2 \cdot S_{n-1} - S_{n-3} = 2F_{n+1} - F_{n-1} - 1 = F_{n+2} - 1,
\]

where we applied \( F \) (1). With (176)

\[
T_n = 2^n \Lambda_{n+2}(-1/2) = 2^3 \Phi_{n+2}(-1/2) = 2^2 \Phi_{n+1,2}(-1/2) = 3 \cdot S_n - 4 \cdot S_{n-1} = 1 + 3F_{n+2} - 4F_{n+1} = 1 + 3F_n - F_{n+1} = 1 + L_n, \quad \text{with } F \text{ (1)} \text{ and } F \text{ (2y}).
\]

The left-hand side of (184) is equal to

\[
\frac{1}{4} S_n + \frac{1}{4} T_n \quad \text{and from (183), (185) and } F \text{ (1), (2y)}.
\]
\[ S_n + T_n = F_{n+2} + L_n = F_{n+2} + 2F_n - F_{n-1} = 4F_n. \]

Note that (184) and (185) hold for \( n = 0 \) when we express \( S_n \) and \( T_n \) in terms of \( \Phi_n \) and \( \Lambda_n \), as above.

We now give two examples of solving the recurrences (174) and (177). First take \( x = -2 \). Then the characteristic equation \( c^3 - c^2 + 2 = 0 \) of the recurrence has the roots \(-1, 2^{\frac{1}{3}} \exp((\pi i/4)) \) and \( 2^{\frac{2}{3}} \exp(-\pi i/4) \). So the general solution of the recurrence may be written as

\[
A(-1)^n + B 2^{\frac{n}{3}} \sin \frac{n\pi}{4} + C 2^{\frac{n}{3}} \cos \frac{n\pi}{4}.
\]

Computing \( A, B \) and \( C \) from the initial values \( \Phi(175) \), with \( k = 2 \), we find

\[
(186) \quad \Phi_{n_2}(-2) = \frac{1}{5} (-1)^n + \frac{2}{5} 2^{\frac{n}{3}} \sin \frac{n\pi}{4} + \frac{1}{5} 2^{\frac{n}{3}} \cos \frac{n\pi}{4}.
\]

In the same way for \( \Lambda_{n_2}(-2) \), with the initial values 3, 1 and 1 for \( n = 0, 1, 2 \), see (178), we find

\[
(187) \quad \Lambda_{n_2}(-2) = (-1)^n + 2^{\frac{n}{3}} \cos \frac{n\pi}{4}.
\]

Then we take \( x = -\frac{3}{8} \). The characteristic equation \( c^3 - c^2 + \frac{3}{8} = 0 \) of the recurrence has the roots \(-\frac{1}{8}, \frac{1}{2}\sqrt{3} \exp(\pi i/6) \) and \( \frac{1}{2}\sqrt{3} \exp(-\pi i/6) \). So the general solution of the recurrence may be written as
a \left(-\frac{1}{2}\right)^n + b \left(\frac{1}{2}\sqrt{3}\right)^n \sin \frac{n\pi}{6} + c \left(\frac{1}{2}\sqrt{3}\right)^n \cos \frac{n\pi}{6}.

One may compute \(a, b,\) and \(c\) from the initial values (175) of \(\Phi_{\frac{1}{2}}(-\frac{3}{18})\) and one finds:

\[(188) \; \Phi_{\frac{1}{2}}(-\frac{3}{18}) = \frac{1}{7} \left(-\frac{1}{2}\right)^n + \frac{12}{7} \cdot 3^{-\frac{1}{2}} \left(\frac{1}{2}\sqrt{3}\right)^n \sin \frac{n\pi}{6} + \frac{6}{7} \left(\frac{1}{2}\sqrt{3}\right)^n \cos \frac{n\pi}{6}.

Similarly from \(\Lambda_{\frac{1}{2}}(x) = 3, \; \Lambda_{\frac{1}{2}}(x) = \Lambda_{\frac{2}{2}}(x) = 1,\)

\[(189) \; \Lambda_{\frac{1}{2}}(-\frac{3}{18}) = \left(-\frac{1}{2}\right)^n + 2 \left(\frac{1}{2}\sqrt{3}\right)^n \cos \frac{n\pi}{6}.

Curious values of \(\Phi_{\frac{1}{2}}\) were given without proof in Fib. R. 36(4), 1998, 375.

curred to us or were suggested by the literature after the preceding part of Chapter 9 was written.

Using \( D_{\lambda \mu} \), we may write (123) and
\begin{equation}
\Lambda_{2m}(x) = 2x^m + \sum_{h=1}^{m} \frac{m}{h} \frac{(m+h-1)!}{(2h-1)!(m-h)!} x^{m-h} = 2x^m + \sum_{k=0}^{m-1} \frac{m}{k+1} \left( \frac{m+k}{2k+1} \right) x^{m-1-k}, \quad m \geq 1,
\end{equation}
(Cf. Problem H-508, Fib. Q. 35(2), 1997, 188-190, (8)).

Replacing \( m \) by \( m+1 \) in (195) and applying the inverse pair IR1 (82), with

Lemma IR2, we find a companion in the sense of IR7, of (196). With
\begin{equation}
\sum_{k=0}^{m} \frac{m}{k+1} \left( \frac{m+k}{2k+1} \right) i^{m-k} \frac{x-2i}{x} = \frac{(x-2i)^m}{(x-2i)^m} \sum_{k=0}^{m} \frac{m}{k+1} \left( \frac{m+k}{2k+1} \right) i^{m-k} \frac{x-2i}{x} \quad \text{for} \quad \phi 
\end{equation}
1. Theorem 11-13, in contact with (11).
2. Q1-192, Theorem 1, in connection with (14).
One might start a similar derivation from (121), (123) or (124), but the result will be less simple, since the analogue of (142) for $\Phi_m$, $\Delta_m$ and $\Delta_m^{1+1}$ is less simple, as is suggested by (129), (132), (133) in compar-
\[(201) \sum_{k=0}^{n} \frac{k}{2n-k} \binom{2n-k}{n} (-x)^n \phi_k(x) = 1, \quad n \geq 1.\]

To obtain a similar relation from (6g) we apply Lemma 2 in IR to the pair \( IR, \)
\((50^a)-(50^c)\) with \( a = b = 1, \) taking
\[ \psi(0) = g(0) = 1, \quad \psi(m) = g(m) = m^{-1}, \quad m \geq 1. \]

\[ a_n^k = \frac{n}{k} \binom{k}{n-k}, \quad 1 \leq k \leq n, \]
\[ b_n^k = \frac{1}{2n-k} \binom{2n-k}{n-k}, \quad 1 \leq k \leq \infty. \]

We may write (6e) as
Write the first identity in (142) as

\[(1 + 2x)^{2m}, \quad m \geq 1.\]
\[
\sum_{n=1}^{\infty} n^{-\frac{1}{n}}(x)^{z^n} = -\log (1 - z - xz) = \\
-\log \frac{1+xy}{x} \left(1 - (1+xy)^{-1}(1+2xz)^2\right) = \\
\text{separately.}
\]

Now consider (73) and (71):

\[
\left(2k + 1 \leq n \leq 2k + 1 + \cdots + n\right) \implies \Phi \to 0
\]

(note that \(\Phi(x) = 0\)), and let us apply to them the considerations of IR, (115) - (130).
So the equations $IR_\nu(n)$ with $y = 2^n x$ have a solution. Therefore $IR_\nu(n)$ (116), (117), (120), (121) hold. Here they become

\[
\sum_{j=0}^{2m} \binom{2m}{j} (-2)^j \Lambda_j(x) = x (1 + yx)^m,
\]

\[
\sum_{k=0}^{m} (-1)^{m-k} \binom{2m+1}{2k+1} \alpha_{2m-2k} 2^{2k} \Phi_k(x) = \]

$\Phi_66$
We now list the papers Carlitz (1978b) and Xinrong Ma (1998) where a number of identities is given from which combinatorial sums follow, e.g. (3.583) - (3.586), (4.28).


\[(216) \quad \alpha + \beta + \gamma = 0, \ (217) \quad \alpha / \beta = 1, \]
\[(218) \quad \beta \gamma + \gamma \alpha + \alpha \beta = t. \]

The relations (216)-(218) are equivalent with
\[(219) \quad \frac{1}{\beta} \gamma + \frac{1}{\gamma} \alpha + \frac{1}{\alpha} \beta = 0, \quad (220) \quad \frac{1}{\beta} \frac{1}{\gamma} = 1. \]
(221) \[ \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = t. \]

Then we have from (217) only

\[ \beta \gamma \gamma (\sigma(m-n)-\alpha^{\cdots\cdots}) + \int \alpha (\sigma(m-n)-\beta^{\cdots\cdots}) = \]

\[ = \cdots \cdot \cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \]
(229) \( \sigma(x) = (x + \beta + y)^2 - 2x\beta - 2\beta y - 2\alpha x = -2t \),

(233) \( \sigma(-m) - \sigma(-(m-1)) - \sigma(-(m-3)) = 0, \ m \in \mathbb{Z} \).

This follows from (224) since \( \alpha, \beta, y \) are roots of the characteristic equation of (232) by (223) and \( \alpha^{-1}, \beta^{-1}, y^{-1} \) are roots of the characteristic equation of (222).
\((230)\), \((231)\) and \((170)\), \((172)\). Here we assume the definition of \(\Lambda^m(x)\), and also of \(\Phi^m(x)\), by extension of the recurrences \((174)\) and \((177)\) to \(n \in \mathbb{Z}\).

From these recurrences, and the initial values, it follows that \(t^n \Lambda^m(t^{-3})\) and \(t^n \Phi^m(t^{-3})\) are polynomials in \(t\) for all \(m \in \mathbb{Z}\).

We want a relation similar to \((234)\) with \(\Phi^m\). So we define the sequence \(r(n), n \in \mathbb{Z}\), by
\[(235)\quad r(-m) = t^m \Phi^m(t^{-3}) \quad m \in \mathbb{Z}.\]

\[(239)\quad r(n) + t \cdot r(n-2) - r(n-3) = 0 \quad n \in \mathbb{Z},\]
\[(240)\quad r(-m) + t \cdot r(-(m-1)) - r(-(m-3)) = 0 \quad m \in \mathbb{Z}.

We now show that
\[(242) \quad A = (1 + 2\alpha^2)^{-1} = (3 - 2\alpha t)^{-1}, \]
\[(243) \quad B = (1 + 2\beta^3)^{-1} = (3 - 2\beta t)^{-1}, \]

Now we prove the second equality in (242) by noting that with (216),\(=\)(218),
\[
\alpha^2 + \alpha\beta + \beta^{-1} = \alpha (\alpha + \beta + \alpha\beta^{-1}) = \alpha (\alpha + \beta + \alpha) = 0,
\]
(245) \quad \[ t = \beta^2 \beta + \alpha + \alpha\beta = \alpha^{-1} + \beta^{-1} + \alpha\beta = \alpha^{-1} - \alpha^2. \]

We then have
\[
\[ t + 3\alpha^2 = \alpha^{-1} + 2\alpha^2 = \alpha^{-1}(1 + 2\alpha^2), \]
inition of the $\tau(n)$ with $n \geq 0$.

Remark With (216), (217) we find from (241)

So we have to avoid $t$-values such that $\mu_3 + \eta - \nu_0$. However, since $\tau(n)$ and
easily since every term satisfies the same third-order linear recurrence, see (232) and (239). We only have to verify the relation

\[ \sigma(-2) = t^2, \quad \sigma(-1) = t, \quad \sigma(0) = 3, \]

for (248) we take $n = 0, 1, 2$ and need $\tau(-2) = t^2, \tau(-1) = t, \tau(0) = 3$. 

\( \sigma(0) = 0, \quad \sigma(x) = -x^t, \quad \sigma_{(-x)} = t^2, \quad \sigma(-1) = t, \quad \sigma(x) = 1, \quad \sigma(1) = 0, \quad \sigma(2) = 0. \)

The quadratic formula

(249) \( \epsilon^2(n) - \epsilon(n+1) \epsilon(n-1) = \epsilon(-n+3), \quad n \in \mathbb{Z}, \)

\[ + 2a^3 \beta^2 + 4a^3 \beta y + 2a^3 y^3 = \]

\[ + a^3 \beta (x \cdots y^3 \cdot 2x \cdots y^3) = \]

\[ + a^3 \beta (x \cdots y^3 \cdot 2x \cdots y^3) = \]

\[ + a^3 \beta (x \cdots y^3 \cdot 2x \cdots y^3) = \]
\[ \begin{aligned}
\chi^2(n) = \chi(n+1) \chi(n-1) &= -A B \alpha^{n-1} \beta^{n-1} (\alpha - \beta)^2 \\
&- B C \beta^{n-1} \gamma^{n-1} (\beta - \gamma)^2 - C A \gamma^{n-1} \alpha^{n-1} (\gamma - \alpha)^2.
\end{aligned} \]

So with (250), (217) and (241)

\[ \begin{aligned}
+ A \alpha \beta \chi + B \beta \chi \alpha &= \\
C \gamma^{3-n} + A \alpha^{3-n} + B \beta^{3-n} &= \chi(-n+3).
\end{aligned} \]

(251) restricts by \( n-1 \), and with (250),

\[ \begin{aligned}
(251) &. \quad \sigma^2(n) = \sigma(n+1) \sigma(n-1) = \\
\end{aligned} \]
\begin{equation}
\sum_{n=0}^{\infty} e(n) z^n = (1 + t z^2)(1 + t z^2 - z^3)^{-1},
\end{equation}

From (224), (223) and (219) - (221)
\begin{equation}
\sum_{m=0}^{\infty} \sigma(-m) z^m = (1 - a^{-1} z)^{-1} + (1 - \beta^{-1} z)^{-1} + (1 - \gamma^{-1} z)^{-1} = (1 - t z - z^2)^{-1} \cdot (1 - \beta^{-1} z)(1 - \gamma^{-1} z) + (1 - \gamma^{-1} z)(1 - a^{-1} z) + (1 - a^{-1} z)(1 - \beta^{-1} z)
\end{equation}
\[ \sum_{n=0}^{\infty} c(n) z^n = A (1-\alpha z)^{-1} + B (1-\beta z)^{-1} + C (1-\gamma z)^{-1} = \]
\[ (1-tz-z^3)^{-1} A + B + C + A\alpha z + B\beta z + C\gamma z \]

\[ (1-tz-z^3)^{-1} \left\{ c(0) + c(1) z + c(-1) z^2 \right\} = \]
\[ (1-tz-z^3)^{-1} (1-tz-z^3)^{-1} \]

\[ -B (t-\beta^{-1}) z + B B z^2 - C (t-\gamma^{-1}) z + \gamma c z^2 \]
\[ = \]
\[ (1-tz-z^3)^{-1} \left\{ c(0) - t c(0) z + c(-1) z + c(1) z^2 \right\} = \]
\[ (1-tz-z^3)^{-1} . \]
We now derive explicit polynomials in $t$
for $\tau(n), \tau(-m), \sigma(n), \sigma(-m), m, n \geq 0$.

From (25v)
\[
\sum_{n=0}^{\infty} \sigma(n) z^n =
(3 + tz^2) \sum_{n=0}^{\infty} z^{2n} \sum_{j=0}^{n} \binom{n}{j} z^j (-t)^{n-j} =
\]
\[
m < \infty \quad m \geq \frac{3t-3m}{3m-3t}
\]

\[
- \sum_{\frac{1}{2}(n+1) \leq k \leq \frac{1}{2}n} \binom{k-1}{n-2k} (-1)^{n-k} t^{3k-n} =
\]
For \( n = 1 \) the relation (256) holds by (228) when an empty sum is defined as zero.

In the same way from (253) or more

\[ \pi \leq m \]

From (254)

\[ \frac{1}{3} (n+1) \leq k \leq \frac{5}{2} n \]

\[ \ldots \sim \]
From (258), putting \( m-k = h \),

\[
(259) \quad r(2m) = \sum_{1 \leq h \leq \frac{1}{3}m} \binom{m-h-1}{2h-1} (-1)^{m-h} t^{m-3h}, \quad m \geq 1,
\]

\[
(260) \quad r(2m+1) = \sum_{\frac{3h}{2} \leq m-1} \binom{m-h-1}{2h} (-1)^{m-1-h} t^{m-1-3h}, \quad m \geq 1.
\]

\[
\sum_{j=0}^{n} 3j+2h = n \quad \text{if} \quad h \text{ and } j \text{ are integers.}
\]

So with (258)
\[(261) \quad \sum_{3j+2h = n} (j + h - 1) \binom{j}{h} (-t)^h = \mathfrak{c}(n) = \]
\[
\sum_{\frac{1}{3}n \leq k \leq \frac{1}{2}(n-1)} \binom{k-1}{n-2k-1} (-1)^{n-k} t^{3k-n}, \quad n \geq 2.
\]

\[(265) \quad \mathfrak{c}(-35-2) = \sum_{h=0}^{s} \binom{s+2h+2}{3h+2} t^{3h+2}.
\]
(266) \( v^2 w + wu + uv = 0 \),

or, equivalently, as in Xinrong Ma (1990),

Then \( \lambda, \beta, \gamma \) satisfy (219) and (220) and from (264), (234), (221) we obtain

Since both sides of (269) are polynomials, it holds for \( u, \alpha, w \) in \( \mathbb{C} \). One also may

\[
(271) \quad v^n + w^n = (-\beta)^n \Lambda_n (-\alpha/\beta).
\]
(Note that the second equality in (87) holds for \( n \in \mathbb{Z} \), by the recurrence (10) or by (87)).
From (270) and (271), for \( n \in \mathbb{Z} \),

\[
\binom{c}{n} + \sum_{2k \leq n} \binom{n-k}{k} \binom{-\ell}{k} (-1)^k, \quad n \geq 1,
\]

\[
\sum_{3k \leq n} (-1)^k \frac{n}{n-2k} \binom{n-2k}{k} \binom{\ell}{k} (\ell+1)^{n-3k} = \]

\[
y^n + \sum_{k=1}^{\infty} \frac{n}{n-k} \binom{n-k}{k} \ell^k, \quad n \geq 1.
\]

Special values of \( A \) are known, e.g. by (96) - (99). They are known by the name of Girard-Waring, see Gould (1999).
\[ \frac{3n}{2} \leq k \leq \frac{3n}{2} \]

\( n \geq 2 \) and for \( n = 1 \) with zero empty sum.

\[ \sum_{k=\frac{3n}{2}}^{\frac{3n}{2}} = 2^n \]

\[ (\nu \nu) + (\nu \nu) + (\nu \nu) \]

\[ n \geq 1. \]

\[ (-1)^{2k} \sum_{k=\frac{n-k}{2}}^{n-k} (k) / \]

\[ = (-1)^{2k} \sum_{k=\frac{n-k}{2}}^{n-k} (k) / \]

\[ \cdots \cdots \]
Proof. First take \( uvw > 0 \) and define \( \alpha, \beta, \gamma \) by (268). Then \( x^\alpha \beta^\gamma \) satisfy (219)-(220).

\[ (283) \quad (u + v + w) \Phi_{lnw}(uvw(u + v + w)^w) = A u^n + B v^n + C w^n, \quad n \in \mathbb{Z}, \]

and similarly for \( B v^n \) and \( C w^n \). Finally we apply continuity. For \( n > 0 \) we may write (282) as
\[
\sum_{i+j=M} v^i w^j = (-b)^M \Phi_M (-c/b).
\]

So with (52), since \( \Phi_0(x) = \Phi_1(x) = 1 \),

or, putting \( c = -yb \),
With (171) and (68) we may write (282), (287) and (288) as
\[(292) \quad u^{-m} (u^{-1} v^{-1}) (u^{-1} w^{-1})^* + v^{-m} (v^{-1} w^{-1}) (v^{-1} u^{-1})^* \]

With (86) the r.h.s. here is equal to

\[
\sum_{h=0}^{m-3} u^{-h} \sum_{i+j=m-3-h} v^{-i} w^{-j} = \sum_{i=1}^{m-3} u^{-h} (v + w)^{m-3-h} \Theta \left( -v w (v + w)^{-2} \right).
\]

\[
\sum_{h=0}^{m-3} e^{-h} (-1)^{m-3-h} e^{h+3-m} \Phi \left( \frac{-v w}{e} \right) = \]

\[(293) \quad (c - b)^{-m} \phi_{-m, 2} \left( c^2 b (c - b)^{-3} \right) = \]
\[c^{1-m} \ldots m-2 \ldots m+1 \cdot (m-1) \cdot (m-1) \]
\[
\frac{1+2 (-1) \psi_{m-2} (-c/b) + (-1) \psi_{m-1} (-b)}{2b+c}
\]
\[m \geq 3,\]

These relations also hold for \( m = 0, 1, 2 \) as may be verified directly. This raises:

\[
\sum_{\frac{1}{3} m \leq k \leq \frac{1}{2} (m-1)} (-1)^{m-k} \left( \frac{k-1}{m-2k-1} \right) (w^k w^{k-1} w^{3k-m}) =
\]
\[( -1)^{m-2} \sum_{2k \leq m-2} \binom{m-2-k}{k} y^k + \sum_{2k \leq m-1} \binom{m-1-k}{k} y^k, \]

\[m \geq 2.\]

We now give some examples of binomial identities that may be derived from the above formulas.

From (226) for \( n \geq 1 \) and (256), (257)

\[
\left\{ \sum_{\frac{1}{n} < k < \frac{1}{n}} \frac{n}{k} \begin{pmatrix} k \\ n-2k \end{pmatrix} \begin{pmatrix} n-1 \end{pmatrix} (-1)^{n-k} t^{3k-n} \right\}^2
\]
\begin{align*}
\phi_{90} &= 2 \sum_{3k \leq n} \frac{n}{n-2k} \binom{n-2k}{k} t^{n-3k} \\
&\quad + 2 \sum_{\frac{2}{3} \leq m \leq n} \frac{n}{2m-n} \binom{2m-n}{n-m} t^{3m-2n} \\
&\quad \left\{ \sum_{3k \leq m} \frac{m}{m-2k} \binom{m-2k}{k} t^{m-3k} \right\} = \\
&\quad \leq 2m \left( 2m \cdot 2m \right)^\frac{1}{2m-3k}
\end{align*}
since \((\frac{m-\zeta}{2m-3\zeta})=0\) for \(\zeta \leq m/2\).

So by equating coefficients of \(t^{2m-3\zeta}\) on both sides

\[
(300) \quad \sum_{\substack{h+k\leq\zeta \\ h,k \leq m/3}} \frac{m}{m-2k} \begin{pmatrix} m-2k \\ k \end{pmatrix} \frac{m}{m-2h} \begin{pmatrix} m-2h \\ h \end{pmatrix} = \]
\[
\Phi_{99}
\]
\[
\left\{ \sum_{\frac{n+1}{3} \leq k \leq \frac{n}{2}} \binom{k-1}{n-2k} (-1)^{n+1-k} t^{3k-n-1} \right\} =
\left\{ \sum_{\frac{n-1}{3} \leq h \leq \frac{n-2}{2}} \binom{h-1}{n-2h-2} (-1)^{n-1-h} t^{3h-n+1} \right\}
\]
\[
\sum_{3k \leq h-3} \binom{n-3-2k}{k} t^{n-3-3k} =
\sum_{\frac{2}{3} n \leq m \leq n-1} \binom{2m-n-1}{n-m-1} t^{3m-2n}
\]

or, with \(D(12)\) and \(D(27)\),
\[
\left\{ \sum_{3k \geq n} \binom{k-1}{3k-n} (-1)^k t^{3k} \right\} =
\left\{ \sum_{3k \geq n+1} \binom{k-1}{3k-n-1} (-1)^k t^{3k} \right\}. \left\{ \sum_{3h \geq n-1} \binom{h-1}{3h-n+1} (-1)^h t^{3h} \right\} =
\sum_{3m \geq 2n} \binom{2m-n-1}{3m-2n} t^{3m}, \quad n \geq 3.
\]

By equating coefficients of \(t^{3m}\) on both sides we find
\[(301) \sum_{h+k=m} \binom{k-1}{3k-n} \binom{h-1}{3h-n} - \sum_{h+k=m} \binom{k-1}{3k-n-1} \binom{h-1}{3h-n+1} = \]

\[
\left\{ \sum_{3k \leq m} \binom{m-2k}{k} t^{m-3k} \right\}^z -
\]

\[
\sum_{\frac{1}{3}m \leq j \leq \frac{1}{3}m} \binom{j}{m-2j} (-1)^{m-j} t^{3j-m} =
\]

\[
\sum_{\frac{1}{3}m \leq \varepsilon \leq \frac{2}{3}m} \binom{m-\varepsilon}{2\varepsilon-m} (-1)^\varepsilon t^{2m-3\varepsilon} =
\]

\[
\sum_{3\varepsilon \leq 2m} \binom{m-\varepsilon}{2m-3\varepsilon} (-1)^\varepsilon t^{2m-3\varepsilon} .
\]
\[ \sum_{\substack{h+k=m \geq 3 \\ 3k \leq m-1, 3h \leq m+1 \\ 3s \leq 2m, m \geq 1}} \binom{\frac{m-1}{2}}{k} \binom{n}{h} = (-1)^{m-1} (2m-3s). \]
A recurrence and its consequences.

Theorem. For fixed $x \in \mathbb{C}, x \neq 0$, let

$$
(3.11) \quad B_n(x) = \Lambda_1(x) B_{n-1}(x) - (-x) B_{n-2}(x), \quad n \in \mathbb{Z}.
$$

Proof. For fixed $x \neq -\frac{1}{2}$ consider the dif-
so that $B_n(x)$ is a solution of (312).

For $x = -\frac{1}{4}$ the theorem follows by continuity. Start from (311) for all $x \neq -\frac{1}{4}$ in a neighborhood of $-\frac{1}{4}$ where the initial values $A_0(x)$ and $A_1(x)$ are equal to the initial values assumed by $A_n(-\frac{1}{4})$.

$$
Y_n(y) = \beta \Lambda_1(y) Y_{n-1}(y) - \beta^2 (-y)^k Y_{n-2}(y)
$$

The same recurrence, viz.

$$(315) \quad u_n = \alpha \Lambda_1(x) u_{n-1} - \alpha^2 (-x)^k u_{n-2}, \quad n \in \mathbb{Z}.$$
Therefore, when (313) and (314) hold and

$$\text{(316)} \quad \text{ad}^n \Upsilon_{2n+1} (x) = b \beta^n V_{n+m} (y),$$

for \( n = m \) and \( n = m+1 \) and some \( m \in \mathbb{Z} \),

the identity (316) holds for all \( n \in \mathbb{Z} \).

[We assume \( \alpha \beta \neq 0 \), \( xy \neq 0 \).]

In this way we may obtain binomial identities, when \( \Upsilon_{2n+1} (x) \) or \( V_{n+m} (y) \) are given by formulas such as (68), (69), (121)–(124), (90) or have simple values such as a Fibonacci number or as in (96)–(99). When trying to find examples of this, we first look for solutions of (313), (314) and then hope to find \( a \) and \( b \) in (316) by exa-.
The author does not have a method to find all solutions to (318), i.e. values of i, j, a, b and x so that it holds for

\[
\phi_{25(n+1)}(x) = \prod_{k=0}^{n} \left( \frac{n+k+1}{2k+1} \right) x^{s(n-k)} (A_{25}(x) - 2x^s)^k
\]

(And also for s = 0). With (122) this becomes

\[
\phi_{25-1}(x) \sum_{k=0}^{n} \left( \frac{n+k+1}{2k+1} \right) x^{s(n-k)} (A_{25}(x) - 2x^s)^k
\]

We note that the sequence \( A_{25}(x) \) is the Fibonacci sequence, and we refer to Döbner (2000) who has

\[
F_0 = 0, \quad F_1 = 1 \quad \text{in his definition of the Fibonacci numbers.}
\]
Now let $U_n(x) = \phi_n(x)$, $V_n(x) = \Lambda_n(x)$, $i = j = 1$.

Then, there is equality in (318) for $n = 0$.

\(\Phi_{n+1}(x) = (1+yx) \Lambda_{2n+1}(x^2 (1+yx))\),

For $x = 1$ this is (5.7) in Dilcher (2000).

A second set of solutions of (317) starts from $y = -\alpha \beta^{-1} x^s$ and then
\[\alpha \Lambda (\alpha + 2) \sqrt{s} = 1\].
As in (3.19) there is a solution when
\[ U_n(x) = V_n(x) = \phi(x) \] and \( i = 1, j = 2s-1 \). There is equality in (3.23) for \( n = 0, n = -1 \), with
\[ A = a \frac{\phi}{2s-1} (x) \] and all \( x \in \mathbb{C} \) with \( A(x) + 2x \neq 0 \).

So we proved

\[ \Phi_{2s(n+1)-1} (x), \ x \in \mathbb{C}, s \in \mathbb{N}_0, \ n \in \mathbb{N}_0, \]

For \( x = 1, s = 2 \) this is (5.11) in Dilcher (2000).
From (326) and (68) we obtain for $n \geq 0, r \geq 0$

$$\Psi_{r-1}(x) \leq (-1)^{k} \binom{r-1}{k} (-x)^{k} \psi_{k}(x), \quad 2k \leq n$$

**Example 3.** The identities obtained above in examples 1 and 2 hold for all $x \in \mathbb{C}$. For many values of $k, j, r, p, i$ identities

$$H_{r-j} \psi_{j}(x) = \sum_{n=0}^{\infty} \frac{(r-j)!}{n!} \left( \frac{x}{2} \right)^{n}$$

hold for $n = 0, x \in \mathbb{C}$ if and only if $a = b$, and for $n = -1$ if and only if
\[ \Lambda_{6n+1}(-1) = y^n \Lambda_{2n+1}(-iy), \]

\[ 6x^3 + iy x^2 - 7x + i = 0. \]
Now consider (311) with \( A_n = \Phi_n \), \( n \geq 2 \):
\[
(330) \quad \Phi_{L_m + j}(x) = \Lambda(x) \sum_{t=1}^{\infty} \phi_{L_{m-t} + j}(x) + (-1)^{t+1} x^t \phi_{L_{m-t} + j}(x).
\]

Here we substitute.

\[
(332) \quad b(n, i) = \frac{n-i}{n-i} \binom{n-i}{i}, \quad n \geq 1, \quad i \leq n-1.
\]

and \( a(m, i) = b(m, i) = 0 \), \( i > m \). Note that we may not use (331) or (332) for \( i > m \).

In order that we may apply (331) and (332) here to all terms, we must have:
\[
m \leq \Theta n + j, \quad h \leq n-1, \quad h \geq [m-(n-1)\Theta-1]_+,
\]
\[
m \leq (n-1)\Theta+j.
\]

This may be simplified to \( 0 \leq h \leq n-1, \quad m \leq (n-1)\Theta+j \).
\[ (-1)^{\ell - 1} \frac{n + \xi + j - m}{m - \xi} \]

But this may be extended immediately to

\[ (333) \sum_{\frac{m - h}{h \leq \xi}} \frac{c}{\xi - h} \left( \frac{\xi - h}{h} \right) \left( u - \xi - m - h \right) = \]

Equating coefficients of \( x^m \) on both sides
\[ b(\varepsilon n + j, m) = \sum_{h=0}^{m} b(\varepsilon, h) b(\varepsilon n - \varepsilon + j, m - h) + (-1)^{h+j} b(\varepsilon n - 2\varepsilon + j, m - 2) \]

To apply (331) and (332) to all terms, we must have 
\[ m \leq \varepsilon n + j - 1, \; h \leq \varepsilon - 1, \; [h \geq m - (n - 1) \varepsilon + j + 1], \]
\[ m \leq \varepsilon (n - 1) + j - 1, \]
which may be simplified to 
\[ 0 \leq h \leq \varepsilon/2, \; m \leq \varepsilon n - \varepsilon + j - 1. \]
So
\[ b(\varepsilon n + j, m) \overset{m \to \varepsilon}{=} b(\varepsilon - h) \]

\[ \frac{u}{u-m} (u-m) + (-1)^{c} \frac{u-2c}{u-\varepsilon-m} \left( \frac{u-\varepsilon-m}{m-\varepsilon} \right), \; u \in \mathbb{C}, \; 1 \leq \varepsilon \leq m. \]

This is an improved version of (81).
For \( m < \varepsilon \) the above argument leads to a special case of C(85) with \( a = -1 \).
A.J. STAM

BINOMIAL IDENTITIES

WITH OLD-FASHIONED PROOFS

PART I \(D\)
INTRODUCTION TO TABLES

The ordering of these tables is similar to the one applied by H. W. Gould, Combinatorial Identities, Morgantown, W. Va. 1972. There are five tables:

Table 1, sums with a single binomial coefficient ( ).
Table 2, sums with a single binomial coefficient ( )'.
Table 3, sums with two binomial coefficients ( )( ).
Table 4, sums with two binomial coefficients ( )( )'.
Table 5, sums with two binomial coefficients ( )' .

Entries in a table have a formula number, e.g. (3, 145) for formula (145) in Table 3. To indicate the arrangement of entries in a table (although not always followed consistently) we give, here a immediately by formula (3.351).

Proofs of identities and also remarks and references are in Chapter P, each one under the same number as the corresponding entry in the tables.
Examples of entries

(1.1) \[ \sum_{k=0}^{n} \binom{n}{k} \]  

(1.3) \[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \]

(1.132) \[ \sum_{k=0}^{\infty} (-1)^{k} \binom{n}{k} (u+k)(v+k) \]

(1.137) \[ \sum_{k=0}^{\infty} \binom{k}{k} z^k (k+1)^{-1} \]
(1.141) \[ \sum_{1 \leq k \leq n/2} \binom{n}{2k} k^{-a} \]

(1.171) \[ \sum_{k=0}^{n} \binom{n}{k} (x+ka)^{k-1} y (y+na-ka)^{n-k-1} \]

(1.199) \[ \sum_{j=0}^{m} \binom{m}{m-j} z^j / j! \]

(1.201) \[ \sum_{j=0}^{m} \binom{m}{m-j} \leq n \]

\[ \text{Fibonacci recurrence} \]

(1.215) \[ \sum_{k=0}^{\infty} \binom{n}{k} x^{r-k} A_{n+k}(x) \] (\( A_n(x) \) solution of \( A_n(x) = A_{n-1}(x) + x A_{n-2}(x) \))
\[(z \cdot x) \leq k=0 \left( \sum \right) k \] 
\[(z \cdot y) \leq k=0 \left( \sum \right) k \times \]

\[\Gamma (n) \leq \sum_{k=1}^{n} \left( \frac{1}{k} \right) \left( \Gamma (n) \leq \sum_{k=1}^{n} \left( \frac{1}{k} \right)^{-1} \right) \]

\[(3.1) \leq k=0 \left( \sum \right) k \] 
\[(3.7) \leq k=0 \left( \sum \right) k \]

\[\sum_{k=0}^{\infty} \left( \sum \right) k \leq \sum_{k=0}^{n} \left( \sum \right) k \]

\[(z) \leq k=0 \left( \sum \right) k \] 
\[(z) \leq k=0 \left( \sum \right) k \]
(3.130) \[ \sum_{k=0}^{n} \binom{n}{k} \binom{k}{4} \]

(3.137) \[ \sum_{2k \leq n+1} \binom{n+1}{2k} \binom{x+k}{n} \]
(3.365) \[ \sum_{k=0}^{n} (-1)^k \binom{k}{x} \binom{y-k}{n-k} \]
$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} k^x \leq \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} x^k \leq \sum_{k=0}^{\lfloor \frac{n-m}{2} \rfloor} \binom{n-m}{k} x^{k-1}$$
\[(3.557) \sum_{k=0}^{n} \frac{1}{2k-1} \left( \begin{array}{c} 2k \\k \end{array} \right) \left( \begin{array}{c} 2n-2k \\n-k \end{array} \right)\]

\[(3.633) \geq_{k=0}^{n} \left( \begin{array}{c} x+\infty \\k \end{array} \right) \left( \begin{array}{c} y+\infty-2\pi \\n-k \end{array} \right)\]

\[(4.40) \sum_{k=0}^{n} k \left( \begin{array}{c} z \\k \end{array} \right) \left( \begin{array}{c} x \\k \end{array} \right)^{-1}\]
\[ T_x \]
TABLE 1.

(1.1) \( \sum_{k=0}^{n} \binom{n}{k} = 2^n. \quad \text{From } D \text{ (19).} \)

(1.2) \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0, \quad n \geq 1. \quad \text{From } D \text{ (19).} \)

(1.3) \( \sum_{k=0}^{n} (-1)^k \binom{x}{k} = (-1)^n (x-1)^n. \quad \text{From } D \text{ (19).} \)

(1.5) \( \sum_{k=0}^{m} \binom{x^m}{k} = 2^{x-1} + \frac{1}{2} \binom{x^m}{m}. \quad \text{From } (1.3). \)

(1.6) \( \sum_{k=0}^{m} \binom{2m+1}{k} = 2^{2m}. \quad \text{From } (1.3). \)

(1.8) \( \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} = (-1)^m \binom{2m}{m}. \quad \text{From } (1.3). \)

(1.9) \( \sum_{k=n+1}^{\infty} (-1)^{k-1} \binom{x}{k} = (-1)^n \binom{x-1}{n}, \quad Re > 0. \quad \text{From } (1.4) \text{ and } (1.3). \)

(1.10) \( \sum_{k} \binom{n}{k} (-1)^k \frac{(n-2k+2)/5}{(n-2k+2)/5} = \frac{F_{n-k}}{n}, \quad n \geq 0. \)

Sum over \( 0 \leq k \leq n, \ n-2k \neq 0 \ [\text{mod} 5]. \) For \( F: \) Chapter F, Proble. H-444, Fib. Q. 30(1), 1992, "

(1.11) \( \sum_{k=0}^{m} \binom{2m}{m+k} = \sum_{k=0}^{m} \binom{2m}{m-k} = 2^{m-1} + \frac{1}{2} \binom{2m}{m}. \)
\[ (1.13) \sum_{k=0}^{m} \binom{2m}{m+k}(-1)^k = \sum_{k=0}^{m} \binom{2m}{m-k}(-1)^k = \binom{2m-1}{m}, \quad (= \frac{1}{2} \binom{2m}{m} \text{ for } m \geq 1) \]

\[ (1.14) \sum_{k=0}^{m-1} 2m \binom{2m-1}{k} = \sum_{k=0}^{m-1} 2m \binom{2m-1}{k} \]
\[ 2k \leq m \quad (2k) - 2 \cdots - 2 \cdot 0 \]

\[ 2^{n/2} \cos \frac{1}{4} n\pi. \quad \text{From (1.50) with } u = i, v = 1. \]

\[(1.19) \sum_{2k \leq m} \binom{2m}{2k} = 2 \quad + \quad \frac{1 - (-1)^m}{4} \binom{2m}{m}, \quad m \geq 1.\]

\[(1.20) \sum_{2k \leq m-1} \binom{2m}{2k+1} = 2^{2m-2} \quad + \quad \frac{1 - (-1)^m}{4} \binom{2m}{m}, \quad m \geq 1.\]

\[(1.21) \sum_{2k \leq m} \binom{2m+1}{2k} = 2^{2m-1} \quad + \quad \frac{1}{2} (-1)^m \binom{2m}{m}.\]

\[(1.22) \sum_{2k \leq m-1} \binom{2m+1}{2k+1} = 2^{2m-1} \quad + \quad \frac{1}{2} (-1)^{m+1} \binom{2m}{m}, \quad m \geq 1.\]

(for \( m = 0 \) with empty sum zero).\)
\[ 2^{k m - 2} + \frac{1}{2} \binom{2m}{m}, \quad m \geq 1; \text{ Cf. (1.35)} \]

\[ (1.26) \sum_{2k \leq m-1} \binom{2m}{m+2k+1} = \sum_{2k \leq m-1} \binom{2m}{m-2k-1} = 2^{k m - 2}, \quad m \geq 1. \]

\[ (1.27^a) \sum_{3k \leq n} \binom{n}{3k} = 2^n + 2 \cos \frac{n \pi}{3}, \]

\[ (1.27^b) \sum_{3k+1 \leq n} \binom{n}{3k+1} = 2^n + 2 \cos \frac{(n-2) \pi}{3}, \]

\[ (1.27^c) \sum_{3k+2 \leq n} \binom{n}{3k+2} = 2^n + 2 \cos \frac{(n+2) \pi}{3}. \]

\[ (1.28^a) \sum_{4k \leq n} \binom{n}{4k} = 2^n + 2 (n+2)^{\frac{1}{2}} \cos \frac{n \pi}{4}, \quad n \geq 1, \]

\[ (1.28^b) \sum_{4k+1 \leq n} \binom{n}{4k+1} = 2^n + 2 (n+2)^{\frac{1}{2}} \sin \frac{n \pi}{4}, \quad n \geq 1, \]
\( (1.28^c) \ \frac{1}{4} \sum_{y_{k+2} \leq n} \binom{n}{y_{k+2}} = 2^n - 2^{(n+2)/2} \cos \frac{n\pi}{4}, n \geq 1, \)

\( (1.28^d) \ \frac{1}{4} \sum_{y_{k+3} \leq n} \binom{n}{y_{k+3}} = 2^n - 2^{(n+2)/2} \sin \frac{n\pi}{4}, n \geq 1. \)

\[ A_n(1) = \frac{1}{6} (4^n - 1) + \frac{1}{2} \sum_{k=0}^{n} \binom{2k}{k}, n \geq 1, \]

\[ A_n(2) = \frac{1}{6} (4^n - 1) - \frac{1}{2} \sum_{k=0}^{n-1} \binom{2k}{k}, n \geq 1. \]

\( (1.31) \ B_n(j) = \sum_{3k+j \leq n} \binom{2n+1}{n-3k-j}, j = 0, 1, 2, \)

\[ B_n(0) = \frac{1}{6} + \frac{1}{3} \cdot 4^n + \frac{1}{2} \sum_{k=0}^{n} \binom{2k}{k}, \]
\[ B_n(1) = \frac{1}{3} (4^n - 1), \quad B_n(2) = \frac{1}{6} + \frac{1}{3} 4^n - \frac{1}{2} \sum_{k=0}^n \binom{2k}{k} \cdot \quad (1.32) \]

\[ U_n(j) = \sum_{3k+j \leq n} \binom{2n}{n-3k-j} (-1)^j k^j, \quad j = 0, 1, 2, \]

\[ V_n(0) = 3^{n-1} + (2n-1), \quad n > 1. \]

\[ V_n(1) = 3^{n-1} - 2 \sum_{k=0}^{n-1} 3^{n-1-k} \binom{2k}{k}, \quad n > 1, \]

\[ V_n(2) = \binom{2n-1}{n} - 2.3^{n-1} + \sum_{k=0}^{n-1} 3^{n-1-k} \binom{2k}{k}, \quad n > 1. \]

(1.34) \[ \sum_{k \geq 1} \left( \frac{n}{\log k} \right)^k = 3^n, \quad Knuth (1992). \]

(1.35) \[ \sum_{\lambda \leq m} (-1)^{\lambda} \binom{2m}{m-\lambda} = 2^{m-1} + \frac{1}{2} \binom{2m}{m}, \]

\[ \text{see } P, (1.77) \rightarrow (1.84). \quad \text{Similarly} \]

(1.36) \[ \sum_{k=0}^{m} \binom{2m+1}{m-k} \cos(2k+1) \frac{\pi}{2} = 2^{m-1/2}. \]
\[ a \leq \frac{(2m+1)}{m} \leq b \]
\[(1.40) = D(19) \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k} = (u+v)^n.\]

\[(1.41) = n \text{ (20)} \sum_{k=0}^{\infty} \binom{x}{k} z^k = (1+z)^x, |z| < 1.\]

\[(1.43) \sum_{k=0}^{n} \binom{2n+1}{k} u^k (1-u)^{n-k} = \sum_{k=0}^{n} \binom{n+k}{k} u^k.\]

\[(1.44) \sum_{k=0}^{n} \binom{n+a}{k} u^k v^{n-k} = \sum_{k=0}^{n} \binom{-a}{k} (-u)^k (u+v)^{n-k}.\]

(Cf. Riordan (1968), p47, line 7).

\[(1.45) \sum_{k=0}^{n} \binom{x}{k} z^{n-k} = \sum_{k=0}^{n} \binom{n-x}{k} (-1)^k (1+z)^k.\]

\[(1.46) \sum_{h=\tau}^{m} \binom{m}{h} u^{m-h} v^h = \sum_{i=\tau}^{m} \binom{i-1}{\tau-1} u^i v^i (u+v)^{m-i}, \tau \geq 1.\]

\[(1.47) \sum_{j=\tau}^{n} \binom{n-a+1}{n-j} (1-x)^{n-j} x^j = \sum_{j=\tau}^{n} \binom{j-a}{j-\tau} (1-x)^{j-\tau} x^\tau = \sum_{h=\tau}^{n} (-1)^h \binom{h-a}{h-\tau} \binom{n-a+1}{n-h} x^h = (n-a+1) \binom{n-a}{n-\tau} \sum_{i=0}^{n-\tau} (-1)^i \binom{h-\tau}{i} x^{i+1} \tau + a + 1.\]

(- D(14) and h-\tau = i in 3rd member)
\[(1.49) \sum_{k=0}^{n-1} \binom{2n}{k} x^k y^{2n-k} + \frac{1}{2} \binom{2n}{n} x^n y^n = \]

\[(1.50a) \sum_{\sum j \leq n} \binom{n}{j} u^j v^{n-2j} = (u + v)^n + (v - u)^n, \]

\[(1.50b) \sum_{\sum j+1 \leq n} \binom{n}{j+1} u^{j+1} v^{n-2j-1} = (u + v)^n - (v - u)^n. \]

Bisection, see G (10), (11), of \(2(19). \)

\[(1.51a) \sum_{i=0}^{\binom{n}{2i}} z^{2i} = (1 + z) - (1 - z)^n, \quad |z| < 1. \]

Bisection, see G (8), (9) of \(D \)

\[(1.52a) = F(23a) \sum_{2i \leq n} \binom{n}{2i} 5^i = 2^{n-1} \text{L}_n, \]

\[(1.52b) = F(23a) \sum_{2i \leq n} \binom{n+1}{2i+1} 5^i = 2^n \text{F}_n. \]

\[(1.53a) = \Phi(31) \sum_{2i \leq n} \binom{n}{2i} (1 + 4x)^i = 2^{n-1} \text{L}_n(x), \]
\[ (1.53) = \Phi(3b) \geq \sum_{2i \leq n} \left( \binom{n+1}{2i+1} (1+y)^i \right) = 2^n \Phi_n(x). \]

From (1.40), \[ \frac{2^n \exp\left(\frac{1}{2} \sin x\right) (\cos \frac{1}{2}x)^n}{(1.60)} \]

\[ \sum_{k=0}^{m} \binom{2m+1}{k} (-1)^k \sin(2m+1-2k)x = \]

\[ (-4)^m (\sin x)^{2m+1} \]

See also (1.36), (1.37).
\( m \sum_{k=0}^{m} \binom{2m+1}{2k} e^{2ikx} = 2^m e^{i(m+\frac{1}{2})x} \left\{ \left( \cos \frac{1}{2}x \right)^{2m+1} + \left( \sin \frac{1}{2}x \right)^{2m+1} \right\} \) \\

\( \sum_{k=0}^{m-1} \binom{2m}{2k+1} e^{(2k+1)ix} = \) \\

\( \sum_{k=0}^{m} \binom{m-k}{k} (u + u^-) = u \left( 1 + u^2 \right)^{(m)} \) \\

\( \sum_{k=0}^{m} (-1)^{m-k} \binom{2m+1}{m-k} (u^{k+1} - u^{-k}) = u^{-m} (u - 1)^{2m+1} \) \\

\( \sum_{k=0}^{m} \binom{X}{k} (-2)^k. \) See (3.73),
\[(1.66) \quad \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} r_k = \frac{n!}{(n-r)!} 2^{n-r}, \quad r = 0, \ldots, n,\]
\[= 0, \quad r > n.\]

\[(1.67) \quad \sum_{k=0}^{n} (-1)^k \binom{x}{k} r_k = (-1)^r r! \binom{n}{r} \frac{(n-x)}{r},\]
\[r = 0, \ldots, n, \quad = 0, \quad r > n,\]

For \(k_r\) instead of \(k_r\), see D (29). Asymptotic Behaviour: Prob. 4551, Monthly. 1953, 482.

\[(1.68) \quad \sum_{k=0}^{n} \binom{x}{k} \frac{1}{k!} x-k = \frac{1}{2^x} (n+1) \binom{x}{n+1}.\]

Induction on \(n\) with D (14). Cf. Graham, Knuth and Patashnik (5.18).

\[(1.69) \quad \sum_{k=0}^{n} (-1)^n-k \binom{n}{k} (x+k)^m = \Delta^n x^m,\]
\[\Delta^n x^m = 0, \quad m < n, \quad \Delta^n x^n = n! ,\]
\[x = 0: \quad \Delta^n x^n = n! S(m, n), \quad \text{Stirling number of the second kind, see e.g. Graham, Knuth and Patashnik (1984), Comtet (1974), Charalambides and Singh (1988)}.
\]
\[\Delta^n x^{n+1} = (n+1)! \left(x + \frac{1}{2} n\right),\]
\[\Delta^n x^{n+2} = \frac{1}{2^x} (n+2)! \left(3(2x+n)^2 + n\right),\]

\( (1.70) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^\alpha = \Delta^n x^\alpha \bigg|_{x=0} = \frac{n!}{S'(\alpha, n)} \text{ with } S'(\alpha, n), \text{ Stirling function} \)

\( (1.71) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (k)^{\alpha-n} = \)


\( (1.72) \sum_{j=0}^{i} (-1)^{j} \binom{i+1}{j} (i+1-j)^{h} = (-1)^{i-h} S_{i}^{h}(\epsilon, i), \)

\( S_{i}^{h}(\epsilon, i) : \text{ pseudo-Eulerian coefficient, } \)

Carroll et al. (1991)

\( (1.74) \sum_{2k+1 \leq n} (2k+1)^{\kappa} = (n-2) \zeta , \quad n \geq 2 . \)

\( \kappa = 1 \ldots k \in \mathbb{N}, \quad k \leq n-1, \quad n \in \mathbb{N} \)

\( (1.76) \sum_{2k+1 \leq n} (-1)^{k} \binom{n}{2k+1} k = (n-1) 2^{(n-3)/2} \cos \frac{1}{4} (n-1) \pi 

- 2^{(n-3)/2} \sin \frac{1}{4} (n-1) \pi . \)
and $k \equiv 2, k \equiv 3 \mod 4$ in $\Sigma$.

\[(1.80) \sum (-1)^j (2i+1) (2m+2) = (m+1) 2^m.\]

\[
\begin{aligned}
\sum_{j \leq m} & \sum_{k \geq 3j+1} \sum_{\nu \leq m-k} \sum_{\mu \leq m-\nu} \sum_{\gamma \leq m-\nu-\mu} (-1)^{\nu+\mu+\gamma} \frac{(2m+2)}{4^{k+1}} \\
\end{aligned}
\]

\[(1.81) \sum_{k=0}^{m} (-2)^{m-k} \frac{1}{(k+1) \sqrt{2m+2} \sqrt{4^{k+1}-1}} = 3m+3.\]
\[(1.86) \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k} (k)_e = (n)_e u^e (u+v)^{n-e}.
\]

Differentiation (e times) of (1.40) w.r. to u:
\[(1.87) \sum_{k=0}^{\infty} \binom{x}{k} z^k (k)_e = (x)_e z^e (1+z)^{x-e}, |z|<1.
\]

\[(1.89) \sum_{k=0}^{m} \binom{n}{k} \beta^k (1-\beta)^{n-k} = x(j+1) \binom{n}{j+1} \beta^j (1-\beta)^{n-j}, j = \lfloor n/\beta \rfloor, 0 \leq \beta \leq 1.
\]

\[(1.90) \sum_{k=0}^{m} k \binom{x+1}{k+1} x^{-k-1} = 1 - \binom{x}{m+1} x^{-m-1}.
\]

Special case \(a=-x\) of (1.160), or induction on \(m\):
\[(1.91a) \sum_{2j \leq n} \binom{n}{2j} u^{2j} v^{n-2j} (2j)_e = (n)_e u^e \left\{ (v+u)^{n-e} + (-1)^e (v-u)^{n-e} \right\}.
\]

\[(1.91b) \sum_{2j+1 \leq n} \binom{n}{2j+1} u^{2j+1} v^{n-2j-1} (2j+1)_e = (n)_e u^e \left\{ (v+u)^{n-e} + (-1)^{e+1} (v-u)^{n-e} \right\}.
\]
\( \sum_{i=0}^{\infty} (2^i) \mathcal{Z} (2^i) = (\mathcal{Z}) (L^2 + 2) + (L - 2) \int \infty \cdot e^{-x} dx \)
\[ (1.100) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^{-\ell} = (-1)^n \Delta^n x^{-\ell} = n! (x-1)! / (x+n)! = x^{-\ell} \left( \frac{x+n}{n} \right)^{-\ell}. \]

\[ (1.101) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+n)^{-\ell} = x \left( \sum_{j=0}^{n} J = (-1)^n x^{-\ell} \left( \frac{x-1}{n} \right)^{-\ell}, \quad x \not\in \{ 0, \ldots, n \}. \right. \]

\[ (1.103) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{y^{1-k}} = \frac{(-1)^n}{y+1} \binom{n}{y}, \quad y \not\in \{ -1, 0, \ldots, n-1 \}. \]

\[ (1.104) \quad \mathcal{A}_n = \sum_{k=1}^{n} (2n\binom{n}{k})^{-1} = \frac{1}{2} \binom{2n}{n} H_n, \quad n \geq 1, \quad H_n = \sum_{k=1}^{n} k^{-1}. \]

\[ (1.105) \quad \mathcal{B}_n = \sum_{k=1}^{n} (2n+1)\binom{n+1}{n-k}^{-1} = \frac{1}{2} \binom{2n+1}{n} H_n - \left( n+1 \right)^{-1} y^n + \left( n+1 \right)^{-1} \binom{2n+1}{n}, \quad n \geq 1. \]

\[ (1.106) \quad \mathcal{C}_n = \sum_{k=1}^{n-1} (2n\binom{n}{k})^{-1} = \frac{1}{2} \binom{2n}{n+1} H_n - \left( n+1 \right)^{-1} y^n + \left( n+1 \right)^{-1} \binom{2n+1}{n}, \quad n \geq 2. \]
\[ \sum_{k=0}^{n} \binom{2n}{n-k} \frac{1}{2k+1} = \]
\[ (4n+2) \binom{2n}{n} - (4n+2) \binom{2n}{n}^{-1} \left( \binom{2n}{n} \right)^{-1} \left( \binom{2n}{n} \right)^{-1} (2n+2) \left( \frac{1}{2} \right)^{2n-2k} \left( \binom{2n}{n} \right)^{-1} (2n+1) \left( \binom{2n}{n} \right)^{-1} (2n)^{-1} H_{2n} \]

\[ + (4n+2) \binom{2n}{n} \]

\[ \sum_{k=0}^{n} \binom{2n+1}{n-k} \frac{(-1)^k}{2k+1} = (2n+1) \binom{2n}{n}^{-1} (2n)^{-1} H_{2n} \]

\[ \sum_{k=0}^{n} \binom{2n+3}{n-k} \frac{1}{2k+1} = \]
\[ 2^{2n+1} \binom{2n}{n} + \sum_{k=0}^{n} \frac{2k+1}{2k+1} \left( \frac{2k}{k} \right)^2 2^{n-2k} \]

\[ (4n+2) \binom{2n}{n} \sum_{k=0}^{n} \binom{2n}{n-k} \frac{1}{2k+1} = -H_{2n} \binom{2n}{n} + \sum_{k=0}^{n} \frac{2k+1}{2k+1} \left( \frac{2k}{k} \right)^2 2^{n-2k} \]
\[(1.114) \sum_{k=0}^{n} \binom{2n+1}{k} \frac{(-1)^k}{2k+1} = -3 \cdot 4^{2n} \]
\[+ 2 \sum_{k=1}^{n} \frac{1}{2k-1} \binom{2n}{k} 4^{2n-2k}, \quad n \geq 1.\]

\[(1.115) \sum_{k=0}^{n} \binom{2n}{k} (-1)^k (2k-1)^{-1} = \frac{2n}{n}.\]

\[(2n)! - 3 \cdot 4^{2n} + 2 \sum_{k=1}^{n} \binom{2n}{k-1} 4^{2n-2k} (2k-1)^2, \quad n \geq 1.\]

\[(1.116) \sum_{k=0}^{n} \binom{2n+1}{n-k} \frac{1}{k+1} = \]

\[(2n+1) \left\{ 1 + \sum_{k=1}^{n} \frac{1}{2k} - \sum_{k=1}^{n} \frac{4k}{2k(2k+1)} (2k)^{-1} \right\}, \quad n \geq 1.\]

\[(1.117) \sum_{k=0}^{n} \binom{2n}{n-k} \frac{1}{k+1} = \frac{4^n}{2n+1} + \]

\[(2n) \left\{ 1 + \sum_{k=1}^{n} \frac{1}{2k} - \sum_{k=1}^{n} \frac{4k}{2k(2k+1)} (2k)^{-1} \right\}, \quad n \geq 1.\]

\[(1.118) \sum_{k=0}^{n} \binom{2n+1}{n-k} \frac{(-1)^k}{k+1} = \binom{2n+1}{n} \left[ 1 - \sum_{k=1}^{n} \frac{1}{2k(2k+1)} \right], \quad n \geq 1.\]

\[(1.119) \sum_{k=0}^{n} \binom{2n}{n-k} \frac{(-1)^k}{k+1} = \frac{1}{2n+1} \binom{2n}{n} + \]

\[\binom{2n}{n+1} \left[ 1 - \sum_{k=1}^{n} \frac{1}{2k(2k+1)} \right], \quad n \geq 1.\]
\[(1.120) \sum_{2k \leq n} \binom{n+1}{2k+1} \frac{1}{2k+1} = \sum_{j=0}^{\frac{n}{2}} \frac{2^j}{j+1}. \]

From (1.155).

From (1.155) with \( z = i \):

\[(1.123) \sum_{1 \leq k \leq \frac{n}{2}} \binom{n}{2k} \binom{-1}{k}^{2k} \frac{1}{2k} = \sum_{j=2}^{n} \frac{1}{j} \left( -1 + 2^{i \frac{j}{2}} \cos \frac{j\pi}{4} \right), \quad n \geq 2. \]

From (1.156) with \( z = i \):

\[(1.124) \sum_{k=0}^{n} \binom{n+x-1}{n-k} \frac{(-1)^k}{x+k} = \frac{1}{x+n} \binom{x+n-1}{n}, \quad x \notin \mathbb{N}. \]

\[(1.125) \sum_{k=0}^{n} \binom{n+x}{n-k} \frac{(-1)^k}{x+k} = \sum_{j=0}^{n} \binom{x+j-1}{j} \frac{1}{x+j}, \quad x \notin \mathbb{N}. \]

\[(1.126) S_n(a, x) = \sum_{k=0}^{n} a^k \binom{n+x}{n-k} \frac{1}{k+1} = \sum_{k=0}^{n} \frac{1}{k+1} \left( \frac{(a+1)^{k+1}}{a} \right) \left( \frac{x+n-k-1}{n-k} \right), \]
\[(1.127) \sum_{k=0}^{n} \binom{n+x}{n-k} \frac{(-1)^k}{k+1} = \sum_{k=0}^{n} \frac{1}{k+1} \binom{x+n-k-1}{n-k}.\]

From (1.126) with \(a = -1\),
\[(1.128) \sum_{k=0}^{n} \binom{n+x}{n-k} \frac{(-2)^k}{k+1} = \sum_{j=1}^{x} \frac{1}{2^j+1} \binom{x+n-2j-1}{n-2j}.\]

\[(1.129) \sum_{k=1}^{\infty} \frac{(-1)^k}{n-k} \left( \binom{n-1}{n-k} \right) = -2 \cdot \frac{1}{\sqrt{2}} \quad n \geq 1.\]

\[(1.130) \sum_{k=0}^{\infty} \frac{x^k}{z+k} = \Gamma(z) \Gamma(x+1) / \Gamma(x+z+1),\quad \text{Re} \cdot x > -1, \quad -z \notin \mathbb{N}, \quad -(x+1) \notin \mathbb{N},\]

with absolute convergence.

\[(1.132) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (u+k)^{-1}/(v+k)^{-1} = (y^2 - u)^{-1} \left\{ u^{-1} \left( u+n \right)^{-1} - v^{-1} \left[ \frac{u+n}{n} \right]^{-1} \right\}, \quad u \neq y^2.\]

\[(1.133) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^{-2} = (-1)^n \Delta^n x^{-2} = x^{-1} \left( \frac{x+n}{n} \right)^{-1} \sum_{j=0}^{n} \frac{1}{x+j}.\]
\[ (1.134) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^{-3} = x^{-1} \left( \frac{1}{x^n} \right)^{-1} \left\{ \left( \sum_{j=0}^{n} \frac{1}{x+j} \right)^2 + \sum_{j=0}^{n} \frac{1}{(x+j)^2} \right\}, \]

\[ (1.135) \quad \sum_{k=0}^{\infty} (-1)^k \binom{k}{x} z^k \frac{1}{(z+k)(z+k+1) \cdots (z+k+x)} = \frac{1}{\Gamma(z) \Gamma(x+1)} \Gamma(z+x+1), \]

\( \text{Re} x > -1, \) where \( \zeta = -1 \) is Euler's constant. The series converges absolutely.

\[ (1.137) \quad \sum_{k=0}^{\infty} \binom{x}{k} \frac{z^k}{k+1} = \frac{(1+z)^{x+1} - 1}{z(x+1)}, \quad |z| < 1. \]

\[ (1.138) \quad \sum_{k=0}^{n} \binom{n}{k} \frac{z^k}{k+1} = \frac{(1+z)^{n+1} - 1}{z(n+1)}, \quad z \in \mathbb{C}. \]

\[ (1.139) \quad \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^k}{k(k+1)} = -\sum_{k=0}^{n+1} \frac{k^{-1}}{z}, \quad n \geq 1. \]

\[ (1.140) \quad \sum \binom{n+1}{2k+1} (2k+1)^2 = \sum_{k=1}^{n} \frac{2^j}{z} \sum_{k=0}^{n} \frac{1}{k^j}. \]
\[ \sum_{1 \leq k \leq \frac{n}{2}} \binom{n}{2k} \frac{1}{y^{2k}} = \sum_{j=2}^{n} \frac{x^{j-1}}{j} \sum_{k=j}^{n} \frac{1}{k} , \]

\[ n \geq 2. \text{ From } (1.162), \]

\[ \sum_{k=1}^{n} \binom{n+1}{2k+1} \frac{(-1)^k}{(2k+1)^2} = \sum_{j=0}^{n} \left( \frac{1}{j+1} \cdot \frac{2j^{(j+1)/2}}{\sin \left( \frac{(j+1)\pi}{4} \right)} \right) \sum_{k=j}^{n} \frac{1}{k+1}, \]

\[ \text{From } (1.161) \text{ with } \ z = i. \]

\[ \sum_{1 \leq k \leq \frac{n}{2}} \binom{n}{2k} \frac{(-1)^k}{y^{2k}} = \sum_{j=2}^{n} \frac{1}{j} \left( -1 + 2 \cdot j^{(j+1)/2} \cos \left( \frac{j\pi}{4} \right) \right) \sum_{k=j}^{n} \frac{1}{k} , \]

\[ n \geq 2. \text{ From } (1.162) \text{ with } \ z = i. \]

\[ \sum_{i=0}^{m-1} \binom{m-1}{i} (-1)^i (i+1)^{-n} = (-1)^m \Delta_{x=0}^{m-1} (-1)^{m-1} x^m \]

\[ = P \left( \mathcal{U}_n - \mathcal{U}_{n-1} \geq m \right), \quad m \geq 1, \quad n \geq 1, \]

\[ \sum_{j=0}^{h-1} \binom{h-1}{j} (-1)^j (j+2)^{-n} = (-1)^{h-1} \Delta_{x=0}^{h-1} (-1)^{h-1} x^h \]

\[ = P \left( \mathcal{U}_n - \mathcal{U}_{n-1} = h \right), \quad h \geq 1, \quad n \geq 1. \]

\[ \mathcal{U}_n = n^{th} \text{ record epoch in a sequence of independent identically distributed random} \]

\[ \text{(exponentially)} \]

\(\sum_{j=0}^{N} (-1)^j \binom{N}{j} (j+1)^{-p-1} = \ldots \)

\[2 < n = 1 \cdots \cdots \cdots \]

And see (1.134).

\[\frac{\sum_{k=0}^{n} (-1)^k \binom{k}{2} (n+k+1)}{2n+1} = \frac{1}{n} \left( \frac{n}{2n+1} \right) .\]

\(k < -1/2\) \quad \beta \quad n (2n+1)^{-1} \)
\begin{align*}
(1.150) \quad \sum_{k=0}^{n} \binom{x}{k} \gamma^{n+1-k} / (n+1-k) = \\
\sum_{h=0}^{n} \binom{n-h}{h} (-1)^{n-h} \left( (1+\gamma)^{h+1} - 1 \right) / (h+1).
\end{align*}

\begin{align*}
(1.151) \quad \sum_{k=0}^{n} \binom{n}{k} \frac{x^{\epsilon+k}}{\epsilon+k} = \sum_{j=0}^{n} \binom{n-j}{j} (-1)^{n-j} \frac{(x+1)^{n+j}}{n+1+j},
\end{align*}

\begin{align*}
(1.158), (1.159).
\end{align*}

\begin{align*}
\frac{1}{a} \begin{pmatrix} a+n \end{pmatrix}^{-1} \sum_{j=0}^{n} \binom{a+n}{n-j} c^j (-c)^{n-j}, -a \notin \{0, 1, \ldots, n\}.
\end{align*}

\begin{align*}
(1.153) \quad \sum_{2k \leq n} \binom{n}{2k} \frac{z^{2k+1}}{2k+1} = \frac{(1+z)^{n+1} - (1-z)^{n+1}}{2n+2}.
\end{align*}

\begin{align*}
(1.155) \quad \sum_{2k \leq n} \binom{n+1}{2k+1} \frac{z^{2k+1}}{(2k+1)} = \frac{1}{2} \left\{ (1+z)^{n+1} \left[ (1-z)^{n+1} - 2 \right] \right\}, \quad n \geq 2.
\end{align*}
\[ \sum_{j=0}^{n} \binom{j}{m+1} \frac{j+1}{1 + (-1)^{m} \binom{m+x}{n-j+1} a^{-m+j}} \]
(1.163) \[ \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} (a+k)^x (x+k)^{-z} = \]

\[ (a-x)^x x^{-1} (1 + \frac{x+n}{x})^{-1}, \quad z \leq n, \]

\[ \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} (a+k)^x \log(a+k). \]


\[ n=0, (a+1) \ldots (a+10+1), \]

\[ |y| \leq 1, \quad \text{Re} \alpha > -1, \quad \text{Re} \beta > 1. \]
\[
(1.168) \sum_{k=0}^{m} (-1)^{m-k} \frac{1}{m+1-k} \binom{x}{k} = \\
\binom{x}{m+1} \sum_{i=0}^{m} \frac{1}{x-i}.
\]

\[
\binom{x}{m+1} \sum_{i=0}^{m} \frac{1}{u+k-1}.
\]
(1.171) = \binom{176}{76} \sum_{k=0}^{n} \binom{n}{k} (x + ka)^{k-1} y(y + na - ka)^{n-k-1} \\
= (x+y)(x+y+na)^{n-1}.

\sum_{k=0}^{n} \binom{n}{k} k(x+ka)^{k-1}(y+na-ka)^{n-k-1}.

(1.173) \sum_{k=0}^{n} \binom{n}{k} \binom{k}{x} (x+ka)^{k-1} (y+na-ka)^{n-k-1}.

(1.175) = \binom{183}{183} n(x+y+na)^{n-1}.

(1.176) = \binom{179}{170} n(a+y)(y+na)^{n-2}.

(1.177) = \binom{171}{171} \sum_{k=1}^{n-1} \binom{n}{k} k^{-1} (n-k)^{n-k-1}.

2(n-1) n^{n-2}, \quad n \geq 2.
From (1.178) with \( y = 0 \):

\[
(1.180) \quad \sum_{k=1}^{n} \binom{n}{k} \frac{k!}{k} n^{-k-1} = 1, \quad n \geq 1.
\]

From (1.183) with \( x = n \) or (1.191) with \( x = 0 \).

(1.183) \( \sum_{k=0}^{n} \binom{x}{k} k! k x^{-k-1} = \sum_{k=1}^{n} \binom{x-1}{k-1} k! x^{-k} = 
\]

\[
1 - \binom{x}{n+1} (n+1)! x^{-n-1} = 1 - n! \binom{x-1}{n} x^{-n}, \quad n \geq 1, \quad x \neq 0.
\]

\[ x = y = n, \quad \text{Kauzkiy (1973), § 6.6.B, Austin (1976)}. \]

\[ (1.185) \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} (k+1)^{k-1} (y+k)^{n-k} = (y-1)^n. \]

From (1.173) with \( x = a = -1, \)

\[ n! \sum_{k=0}^{n} \binom{n}{k} (k+1)^{k-1} (y+k)^{n-k} = n^{-1} n^n \quad n > 0. \]

From (1.173) with \( a = -a, \ y = 0, \)

\[ n! \sum_{k=0}^{n} \binom{n}{k} (k+1)^{k-1} (y+k)^{n-k} = \]

\[ (1.190) \sum_{k=0}^{n} \binom{n}{k} (x+k a)^{k} (y-k a)^{n-k} = n! \sum_{k=0}^{n} \frac{(x+y)^k a^{n-k}}{k!} \]
(1.194) \[ \sum_{k=0}^{n} \binom{n}{k} (a+kb)(a+nb)^{k-1} k! b^k = 1. \]

(1.196) \[ \sum_{k=1}^{N} \binom{N}{k} (-1)^{k-1} \theta^k (1 - \theta^k)^{-1} = \]
\[ \sum_{h=0}^{N-1} (-1)^h \Delta^h \theta^x (1 - \theta^x)^{-1} \bigg|_{x=1} = E Z_N, \]

\[ N \geq 1, \text{ last equality for } 0 < \theta < 1. \] Here \( Z_N = \max (X_1, \ldots, X_N) \) with \( X_1, \ldots, X_N \) independent and \( \mathbb{P} (X_i = j) = \theta^j (1-\theta), \) \( j \in \mathcal{N}, i = \)
(1.199) \[ \sum_{j=0}^{m} \binom{x}{m-j} \frac{z^j}{j!} = \]

\[ e^{-k} \sum_{k=0}^{\infty} \frac{x+k+ma}{m} \binom{m}{n-k} \frac{z^k}{k!}, \text{ z e C.} \]

(1.201) \[ \sum_{k=0}^{n} (-1)^{n-k} \binom{n-k}{n-k} \frac{z^k}{k!} \]

Pseudo-Laquerre polynomial, Boas and

\[ \frac{n!}{n!(n+u)!} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{n+u}{n-k}. \]

From (1.14). Now see (3.65) - (3.75).
(1.203) \[ \sum_{h=0}^{N-j} (-1)^h \binom{N-j}{h} (mn-j-h)! m^h, \]

(1.204) \[ \sum_{h=0}^{N-j} (-1)^h \frac{1}{h+j} \binom{N-j}{h} (mn-j-h)! m^h, \quad j \geq 1. \]

(1.205) \[ \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} e^{ak} (u+k)! / (v+k)! . \]

(1.205a) \[ \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} e^{ak} (u-k)! / (v-k)! . \]

(1.206) \[ \sum_{h=1}^{n} (-1)^h \binom{n}{h} \sum_{i=1}^{h} i^{-1} = \]

\[ 2 (-1)^n \sum_{2h+1 \leq n} (2h+1)^{-1}, \quad n \geq 1. \]

(1.207) \[ \sum_{h=0}^{n} (-1)^h \binom{n}{h} \sum_{i=1}^{h+1} i^{-1} = \]

\[ 2 (-1)^n \sum_{2h+1 \leq n} (2h+1)^{-1} + (1+(e^{-1}))^{(2n+2)} - 1. \]
\[(1.210) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{j=0}^{k} (x+j)^{-1} = \frac{1}{x+n} \left( \begin{array}{c} n \cr x \end{array} \right) \sum_{j=0}^{x} (x+j)^{-1}, \quad n \geq 1, \quad -x \notin \{ 0, 1, \ldots, n \}.\]

\[(1.213) \quad \lim_{x \to 0} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{j=0}^{k} (x+j)^{-1} = -n^{-1}, \quad n \geq 1.\]

\[(1.215) \quad \sum_{k=1}^{n} (-1)^k \binom{n+x}{n-k} \sum_{j=0}^{k} \frac{1}{x+j} = \frac{1}{x+n} \left( \begin{array}{c} n \cr x \end{array} \right) \left( \begin{array}{c} n \cr x \end{array} \right) = \frac{1}{2n} \left( \begin{array}{c} 2n \cr n \end{array} \right), \quad n \geq 1.\]

\[(1.216) \quad \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \sum_{j=1}^{k} j^{-1} = \frac{1}{2n} + \frac{H^n}{2n} \left( \begin{array}{c} 2n \cr n \end{array} \right)^{-1}, \quad n \geq 1.\]

\[(1.217) \quad \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \sum_{j=1}^{x} (x+j)^{-1} = \sum_{i=0}^{x-1} \left( \begin{array}{c} n \cr n-i \end{array} \right) \frac{1}{x+i} \left( \begin{array}{c} n \cr n-i \end{array} \right), \quad n \geq 1, \quad x \geq 1.\]
$$\sum_{k=1}^{n} (-1)^{k-1} k^{-1} \binom{n}{k} \sum_{j=1}^{k} j^{-1} H_j = \sum_{k=1}^{n} k^{-3}, \quad n \geq 1. \text{ With generalization:} \quad H_m = \sum_{i=1}^{m} i^{-1}. \quad (1.2.18)$$

$$\sum_{k=1}^{n} (-1)^{k-1} k^{-1} \binom{n}{k} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{-1} = \sum_{i=1}^{n} i^{-1} \left(\frac{h}{j}\right)^{-1} \quad (1.2.19)$$
(1.221) = F_5(53) \sum_{k=0}^{n} (-1)^k \binom{n}{k} A_{n-2k} = A_{n-\varepsilon},

(1.222) = F_5(58) \sum_{k=0}^{2m+1} \binom{2m+1}{k} F_{n-2k} = 5^m \lambda_{n-2m},

\vdots

in F_5(48) = (60^u), and F_5(127).

(1.225) = F_5(134) \sum_{j=0}^{2m} \binom{2m}{j} (-2)^j L_{j} = 2^2 5^m.

(1.226) = F_5(135) \sum_{j=0}^{2m+1} \binom{2m+1}{j} (-2)^j L_{j} = 0.
\[ (1.228) = F(1.38) \sum_{i=0}^{2m-1} \left( \frac{2m}{i+1} \right)(-2)^i F_i = 0. \]

\[ (1.229) = F(\ell 0) \sum_{k=0}^{m-1} \frac{2k+1}{2m+1} \left( \frac{2m+1}{m-k} \right)(-1)^{m-k} F_{\ell k} = 1. \]

\[ (1.230) = F(\ell 1) \sum_{k=1}^{m} \left( \frac{k+1}{2m+1} \right)(-1)^{m-k} F_{\ell k} = 1. \]

\[ (-1)^{m+1} \left( \frac{2m}{m-k} \right) \left( \frac{m-k}{2m+1} \right), \quad m \geq 1. \]

\[ (1.233) = F(81) \sum_{k=0}^{n} \left( y-n-1-k \right) F_{\ell k} = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{y-j} \left( \begin{array}{c} y-j \\ j \end{array} \right), \quad n \geq 1. \]

\[ (1.235) = F(10, 6) \sum_{k=1}^{n} (y-n-2-k)^1. \]
(1.236) \[ \sum_{k=0}^{m} \frac{2k+1}{2m+1} \binom{2m+1}{m-k} L_{2k+1} = 5^m, \]

From \( \Phi \) (125) and \( \Phi \) (144).

From \( \Phi \) (126) and \( \Phi \) (145).

(1.238) \[ \sum_{k=1}^{m} \binom{2m}{m-k} L_{2k} + \binom{2m}{m} = 5^m, \quad m \geq 1. \]

From \( \Phi \) (127) and \( \Phi \) (146).

(1.243) \[ F_{(64)} \sum_{k=0}^{m} \binom{2m+1}{k} L_{n+k}^2 = 5^{m+1} F_{2n+2m}. \]
\[(1+4x)^m \Lambda_{n+2m+2}(x).\]
\[(1.248) = \Phi(111) \sum_{k=0}^{2m+1} \binom{2m+1}{k} x^{2m+1-k} \Lambda_{n+2k}(x) =\]
\[(1+11x)^{m+1} \Phi(111)(x).\]

A number of similar relations is to be found in \(\Phi\), (103) - (115), and \(\Phi_2\), (198), (199).
Special values of \(\Phi(x), \Lambda_n(x)\) are in \(\Phi\), (28), (96) - (99), \((\frac{1+4}{x})\) - (97), leading to Binomial sums in (1.245) - (1.267).
\[ (1 + 4x) \sum_{2n+2m+\lambda} (x) > m \leq \lambda \]
\[
(1.259) \quad \sum_{k=0}^{2m} \binom{2m}{k} x^{2m-k} \Lambda_{n+k}^2 (x) = \\
\Phi (117) \quad (1+4x)^m \Lambda_{n+1} (x), \quad m \geq 1.
\]

\[
(1+4x)^m \Phi \phi_{2n+2m+2} (x).
\]

\[
(1.261) = \Phi (119) \quad \sum_{k=0}^{2m+1} \binom{2m+1}{k} x^{2m+1-k} \Lambda_{n+k}^2 (x) = \\
(1+4x)^{m+1} \Phi \phi_{2n+2m} (x).
\]
\begin{align*}
(1.370) \quad & \sum_{2k \leq n} (-1)^k (2k) \alpha_{2k} 2^k \cos \left(\frac{1}{4} n - \frac{1}{2} k \right) \pi = 1, \\
(1.373) \quad & \sum_{k=0}^{m} (-1)^k \binom{2m+1}{2k+1} \alpha_{2m-2k} 2^{k+\frac{1}{2}} \sin \frac{\pi}{4} (2k+1) \pi = 1.
\end{align*}
\[
\sum_{k=0}^{n} (-1)^k \binom{n+a}{n-k} L^{(a)}_{n-k}(x) = x^n / n!,
\]
\[
(1.281) \quad \sum_{k=m}^{n} (x+k) = (n+1)x \sum_{k=m}^{n} - (m+1)x \sum_{k=m}^{n}, \quad r \geq 0
\]

\[
(1.282) \quad \sum_{k=m}^{n} (x-k) = (x+1-m) \sum_{k=m}^{n} - (x-m-1) \sum_{k=m}^{n}, \quad m \leq n
\]

\[
(1.283) \quad \sum_{k=0}^{n} (-1)^{k} = \sum_{k=0}^{n} (-1)^{k} = (n+1)
\]
\[ \sum_{2k \leq n} \left\{ (y+3n+2-k) - (y-k-1) \right\} \left( \frac{1}{3} \right)^k, \]

\[ (1, 292) \quad \sum_{k=0}^{n} (y+k) \left( \begin{array}{c} n \vspace{1pt} \\ k \end{array} \right) t^k = \]

\[ \frac{1}{1-t} \sum_{h=0}^{\infty} \left\{ (y) - t^{n+1} (y+n+1) \right\} \left( \frac{t}{1-t} \right)^{e-h}. \]

\[ n \leq \infty (y+k) \left( \begin{array}{c} k \vspace{1pt} \\ a \end{array} \right) = \sum_{e} (y) \left( \begin{array}{c} t \vspace{1pt} \\ e \end{array} \right) \left( \frac{1}{1-t} \right)^{e-h}. \]

\[ (1, 264) \quad \sum_{k=0}^{n} (r+k) \left( \begin{array}{c} n \vspace{1pt} \\ k \end{array} \right) \vspace{1pt} \quad a \in \{ 0, 1, \ldots, \sigma \}. \]
\[(1.295) \sum_{k=0}^{n} \left( \binom{2n-k}{n} \right) 2^k = 2^{2n}. \]

\[\sum_{k=0}^{\infty} (2n+k)_{-k} = 2^{2n}.\]

**Lucas number, see F (133).**

\[(12.9) \sum_{k=1}^{n-1} \binom{k}{k} = (n) \sum_{k=1}^{n-1} k^{-1}.\]
\[(1.303) \sum_{k=0}^{n} \frac{\frac{x+k}{2r+y+k}}{e^{\frac{2r+y+k}{2}}} = \]

\[n+y-1 \mid n + 2r + y + 1 \mid y - 2 \mid 2r + y\]

\[f(x) = \binom{x}{m} \text{ and } G(38), \]

\[(1.305) \sum_{k=1}^{\infty} \frac{x+k}{z^k} = e^z \sum_{m=0}^{m} \binom{x}{m} \frac{z^d}{d!}, \]

\[(1.307) = \left(1, \frac{131}{1} \right) \left[ \frac{1}{n-2k} \left( \frac{n}{4k} - 4k \right) \right] \cup 4k = \left\{ n \right\}, \]

\[n \geq 1. \]
\( \Phi_n, A_n \): Fibonacci- and Lucas-like polynomials, see Ch. 5, especially p. \( \Phi_1, \Phi_2 \).

\[
(1.3/1) = \mathcal{O}(204) \sum_{m}^{n} \left( \begin{array}{c} 2m-k \end{array} \right) x^{2m-2k} / (1+2x)^k \delta, (x)
\]

\[
(1.3/2) \sum_{k=0}^{\infty} \left( \begin{array}{c} x+k \end{array} \right) z^k / k!, \text{ see (1.199)}.
\]

In connection with the above and the following entries note that
\[ (-\infty \ldots L \downarrow 1 \ldots -n-1) \]

\[ (a+1) \left[ \begin{array}{c} \nu + \eta + 1 \\ \eta \end{array} \right] \int_0^\eta (\nu + \eta - 1) \nu \, d\nu, \quad \nu \geq -1 \]

\[ \kappa = 0, 1, \ldots, n. \]
The text appears to be a mathematical expression or formula, but it's not clearly visible due to the quality of the image. It seems to be related to summations and products involving variables and indices. Without clearer visibility, it's challenging to transcribe accurately. If this is a question or a statement in a mathematical context, it would typically involve algebraic expressions, series, or integrals. If you have a clearer image or can provide more context, I would be better equipped to help.
\[(1.333) \sum_{k=0}^{n} \binom{k}{n} \frac{1}{x+k} \binom{x+k}{k} = \frac{(x-1+\epsilon)!}{x!} \binom{x+n}{n-\epsilon} \]
\[= (n)_x \frac{1}{x+\epsilon} \binom{x+n}{n}, \quad \epsilon = 0, 1, \ldots, n. \]

\[(1.334) - (1.337): \text{Fibonacci and Lucas numbers}, \]

\[(1.336) = F(68) \quad \sum_{j=0}^{\infty} \frac{2^m}{m+j} \binom{m+j}{2j} = L_{2m}, \quad m \geq 1. \]

\[(1.337) = F(69) \quad \sum_{j=0}^{\infty} \frac{2^m}{m+j+1} \binom{m+j+1}{2j+1} = L_{2m+1}. \]

\[(1.338) - (1.341): \text{Fibonacci and Lucas-type polynomials, Chapter } \Phi, \text{ especially } \Phi_1, \Phi_2. \text{ Special values for } x = \frac{1}{14}, -1, -\frac{1}{2}, 2, 6 \text{ and } -\frac{1}{5} \text{ are in } \Phi, \text{ (28), (96), (99), (144) - (147).} \]

\[(1.338) = \Phi(64) \quad \sum_{h=0}^{m} \binom{m+h}{2h} x^{m-h} = \Phi_{2m}(x). \]
\[ \sum_{h=0}^{m-1} \left( \frac{m-1+h}{2h} \right) \left( x+x \right)^{m-h} = (x+1)^{2m} + x^{2m}, \quad m \geq 1. \]

\[ \sum_{h=0}^{m-1} \frac{2m-2}{m-1+h} \left( \frac{m-1+h}{2h} \right) \left( x+x \right)^{m-h} = \]

\[ (2x+1)/(x+1)^{2m-1} \quad m \geq 2. \]

\[ \sum_{h=0}^{m-1} \frac{2m-2}{m+h} \left( \frac{m+h}{2h+1} \right) \left( x+x \right)^{m-h} = \]
\[
\sum_{h=0}^{m} (-1)^{m-h} \frac{m+1}{h+1} \frac{m}{m+h} \left( \frac{m+h}{2h} \right)^h = (1+(-1)^m) \left( 2-2m \right)^{-1} \quad m \geq 2,
\]
\[
(1.346b) \sum_{h=0}^{m} (-1)^{m-h} \frac{m+1}{h+1} \left( \frac{m+h}{2h} \right)^h = 1 + (2m)^{-1} (1-(-1)^m),
\]
\[
(1.346c) \sum_{h=1}^{m} (-1)^{m-h} \frac{m+1}{h+1} \left( \frac{m+h-1}{2h-1} \right)^{2h-1} = m(m-1)^{-1}, \quad m \text{ even} \quad \Rightarrow = 1 + m^{-1}, \quad m \text{ odd}.
\]
\[
(1.346d) \sum_{h=0}^{m} (-1)^{m-h} \frac{m+1}{h+1} \left( \frac{m+h+1}{2h+1} \right)^h = (-1)^m + (2m)^{-1} (1-(-1)^m), \quad m \geq 1.
\]

See (1)–(4) in the proofs of (4.22)–(4.25).
and Fibonacci-type polynomials, see Chapter $\Phi$. Special values for $x = -\frac{1}{4}, -1, -\frac{1}{2}, 2, 6$ and $-\frac{1}{5}$ are in $\Phi$, (28), (96)-(99), (144)-(147).

\[(1.351) = \Phi(197) \sum_{k=0}^{m} \binom{m+k+1}{2k+1} i^{m-k} (x-2i)^k = x^m \Phi(x^2)\]


\[(1.352) \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{x+k}{2k}\right) z^{2k+1} (1+z)^{-k} =
\]

\[\left(2x+1\right)^{-1} \left\{ \left(1+z\right)^{x+1} - \left(1+z\right)^{-x} \right\}, \quad |z|^2 |1+z|^{-1} \leq \frac{1}{4}, \]

\[\left(2x+2\right)^{-1} \left\{ \left(1+z\right)^{x+2} + \left(1+z\right)^{-x} - 2 - 2z \right\}, \quad |z|^2 |1+z|^{-1} \leq \frac{1}{4}, \quad x \neq -1\]
\[
\sum_{k=0}^{\infty} \binom{2k}{k} z^k (1+z) = (1-z)^2 \cdot \frac{1}{1-z} - 1
\]

(1.357) \quad \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{k + \frac{1}{2}}{k} \right) z^{k+1} / (1-z)^{k+1} + \]

\[
\sum_{k=0}^{\infty} \binom{n-k}{k} z^k (1-z)^{n-k} = (1-z) \quad .
\]

(1.358) \quad (1-x)^{n+1} \sum_{k=0}^{n} \binom{n+1-k}{k} \sum_{i=0}^{n} \frac{i!}{i+1} \binom{2i}{i} x^i (1-x)^i =
\]

\[
1 - \sum_{i=0}^{n} \frac{1}{i+1} \binom{2i}{i} x^i (1-x)^i \quad .
\]
\[(1.360) \sum_{k=0}^{j} \left( \frac{j}{y} - k \right) \left( \frac{S + k - 1}{k} \right) (1-y)^k = y^{-1} (j+1) \left( \frac{S + j}{j+1} \right) (1-y)^{j+1}.\]

\[(1.361) \sum_{k=0}^{\infty} \left| k - \frac{j}{p} \right| \beta^k \left( \frac{S + k - 1}{k} \right) = \frac{(-1)^{n-r} x^{2-n-z}}{n!} \frac{d^n}{dz^n} \left( 1 - z - z^2 \right).\]

\[(1.303) = (1 + F) \sum_{j=0}^{\infty} \frac{(-1)^j \prod_{l=0}^{j-1} \frac{1}{n+l}}{n-j} = I, \quad n \geq 1,\]

\[(126m) = F(n \overline{r}) \sum_{k=0}^{n} \left( \begin{array}{c} n+j-1 \\ j \end{array} \right) t^{n-j}.\]

\[J \left( t + n - 1 \right), \quad n \geq 1. \text{ The sequence } \left( \begin{array}{c} t \\ n \end{array} \right), \quad n \in \mathbb{N},\]

is the "\(t\)-fold convolution" of the Fibonacci sequence, see p. C 52-55, p. F 27-32.
\[
(1.366) = F(n12) \sum_{h=0}^{n} \binom{n+h}{h} F_{n+h} z^h = \frac{1}{n!} \frac{d^n}{dz^n} (1 - z - z^2)^{-1}.
\]

\[
(1.367) = \Phi(201) \sum_{h=0}^{n} \frac{n-h}{n+h} \binom{n+h}{h} (-x)^h \Phi_{n-h}(x) = 1, \quad n \geq 1.
\]

\[
(1.368) = \Phi(203) \sum_{h=0}^{n-1} \frac{n-h}{n+h} \binom{n+h}{h} (-x)^h \Lambda_{n-h}(x) = 1, \quad n \geq 1.
\]

For \( \Phi \) and \( \Lambda \), see Chapter 5, especially pp. 114, 121. Special values of \( \Phi(x) \) and \( \Lambda(x) \) for some \( x \) are in (5.28), (96)-(99), (144)-(147).

\[
(1.369) = (1.373) F_n \text{ and } L_n \text{ are Fibonacci and Lucas numbers, Chapter F, especially } \beta F_1, 2.
\]

\[
(1.369) = F(41) \sum_{k=0}^{n} (-1)^{n-k} \binom{n-k}{k} = \sum_{2k \leq n} \binom{n-k}{k} = F_n.
\]

\[
(1.370) = F(42) \sum_{k=0}^{n} \frac{n}{n-k} \binom{n-k}{k} = \sum_{2k \leq n} \frac{n}{n-k} \binom{n-k}{k} = L_n, \quad n \geq 1.
\]
(1.373) \( F(q) = \sum_{i=0}^{n} \binom{n-i}{i} \). \\
(\therefore) \( F_n = (n-1) F_{n-1} + 2n F_{n-2}, \ n \in \mathbb{N}_0. \) \\

(1.374)-(1.375) \( \Phi_n(x) \) and \( \Lambda_n(x) \) are Fibonacci - and Lucas-type polynomials, defined (p. 218) and studied in Chapter \( \Phi. \) Values for special \( x \) are in \( \Phi_{18}, \Phi(96)-(99), \Phi(144)-(147). \) Note that \( \Phi_n(1) = F_n, \ \Lambda_n(1) = L_n. \) \\

(1.376) \( \Phi(75) \sum_{k=0}^{n} \binom{n-k}{k} x^k = \sum_{2k \leq n} \binom{n-k}{k} x^k = \Phi_n(x). \) \\

(1.377) \( \Phi(76) \sum_{i=0}^{n} \frac{n}{n-i} \binom{n-i}{i} i x^{i-1} = \) \\
\( n \begin{pmatrix} \Phi_n(x), \ n \geq 1 \end{pmatrix} \)
\[(1.378) \sum_{2k \leq n} \binom{n-k}{k} \frac{w^{k+1}}{k+1} = \frac{1}{n+2} \left( \Lambda_{n+2}(w) - 1 \right). \]

\[(1.379) = \Phi(86) \sum_{2k \leq n} (-1)^k \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = \sum_{j=0}^{n} x^j y^{n-j}. \text{ (Waring, cf. Gould (1999))} \]

\[(1.380) = \Phi(87) \sum_{2k \leq n} (-1)^k \binom{n}{n-k} (xy)^k (x+y)^{n-2k} = x^n + y^n, \text{ } n \geq 1. \text{ (Waring, see Gould (1999))} \]

\[(1.381) = \Phi(95^a) \Phi_n(x^2+x) + (x^2+x) \Phi_{n-1}(x^2+x) = (x+1)^n + (-1)^n x^n, \text{ } n \in \mathbb{Z}. \text{ (cf. (1.375)).} \]

\[(1.382) = \Phi(95^b) \Lambda_n(x^2+x) + (x^2+x) \Lambda_{n-1}(x^2+x) = (x+1) \{ (x+1)^{n-1} + (-1)^n x^{n-1} \}, \text{ } n \in \mathbb{Z}. \text{ (cf. (1.375)).} \]

\[(1.383) = \Phi(142) \Phi_{2m+1}(x) = (1+2x)^m \Phi_m \left( -x^2 (1+2x)^{-2} \right). \text{ (Cf. (1.374)).} \]

\[(1.384) = \Phi(143) (1+2i)^m \Phi_{2m+1} \left[ -i (1+2i)^{-1} \right] = \frac{F}{m}. \text{ Take } x = -i (1+2i)^{-1} \text{ in } \Phi(142). \text{ For } x = -\frac{1}{3}, \text{ with } \Phi(96), \text{ see } \Phi(143^a), \]

\[(1.385) \Phi_{2m+1} \left( -\frac{1}{3} \right) = 2, 3^{-m-\frac{1}{2}} \leq \sin (m+1) \pi/3. \]
(3.386) \( \Phi_n(x) = \sum_{(r+1)k \leq n} \left( \frac{n-rk}{k} \right) x^k, \quad n \in \mathbb{N}, \quad r \in \mathbb{N} \)

(3.387) \( \Lambda_n(x) = \sum_{(r+1)k \leq n} \frac{n}{n-rk} \left( \frac{n-rk}{k} \right), \quad n \in \mathbb{N}, \quad r \in \mathbb{N} \)

Generalizations of \( \Phi_n, \Lambda_n \), see Chapter 9, p. 54-60. Special values for \( x = -1/2, -3/8 \) of \( \Phi_n(x) \) and \( \Lambda_n(x) \) are in \( \Phi_n(183), (185), (186)-(189) \)

(3.388) \( \Phi_n(184) \sum_{3i \leq n} (-1)^{n-i} \left( \frac{n-3i}{n-2i} \right) x^{n-3i} \)

(3.389) \( \Phi(102) \frac{(2-x)}{2k \leq n} \left( \frac{n-2k}{k} \right) (-1)^k x^{2k} (1+x)^{n-3k} \)

(3.390) \( \Phi(101) \frac{(2-x)}{2k \leq n} \left( \frac{k}{n-2k} \right) x^{n-2k} (1+x)^{3k-n} \)

(3.391) \( F(46) \sum_{2k \leq n} \left( \frac{k}{n-2k} \right) x^{3k-n} = (-1)^n F_{n_{-1}} \)

(3.392) \( \sum_{3i \leq m-1} \left( \frac{m-i}{2i+1} \right) x^{m-1-3i} = F_{2m-1}, \quad m \geq 1 \)
(1.393) \[ \sum_{\beta i \leq m} \binom{m-i}{2i} 2^{m-3i} = \frac{F}{2^{m-1}} + 1. \]

(1.394) \[ \sum_{\beta i \leq m} \frac{m+i}{m-i} \binom{m-i}{2i} 2^{m-3i} = \frac{F}{2^{m-2}}, m \geq 1. \]

(1.395) \[ \sum_{\beta i \leq m-1} \frac{m+1+i}{m-i} \binom{m-i}{2i+1} 2^{m-3i} = \frac{F}{2^{m-1}}, m \geq 1. \]

(1.396) \[ \sum_{\beta i \leq m-1} \frac{2m+1}{m-i} \binom{m-i}{2i+1} 2^{m-3i} = \frac{F}{2^{m+1}}, m \geq 1. \]

(1.397) \[ \sum_{\beta i \leq m} \frac{2m}{m-i} \binom{m-i}{2i} 2^{m-3i} = \frac{F}{2^{m+1}}, m \geq 1. \]

(1.398) \[ \chi_n (\lambda, \mu) = \sum_{k=0}^{n} \binom{n+k+\lambda}{k} \frac{\mu^{k+1}}{n-k}, \]

\[ \sum_{k=0}^{n} \binom{n+k-\lambda}{k} \left( \frac{\mu^{n-k}}{n-k} + \frac{\mu^{n+1-k}}{n+1-k} \right) \frac{1}{n+1-k}, \]

\[ y_i = \frac{1}{2} (2 + \mu + \sqrt{4\mu + \mu^2}) = \mu C_i (u^{-1}), \]

\[ y_2 = \frac{1}{2} (2 + \mu - \sqrt{4\mu + \mu^2}) = \mu C_2 (u^{-1}), \]

\[ C_1 (x) = \frac{1}{2} + \frac{1}{2} \sqrt{4 + 4 \mu}, \]

(1.399) \[ \chi_n (\lambda, \mu) = \]
\[(1.400) \quad \sum_n (1, u) = \frac{1}{n+1} \left( u^{n+1} \Lambda_{2n+2} (u^{-1}) - 2 \right) \]

\[(1.401) \quad \sum_n (0, u) = \frac{1}{n+1} \left( u^{n+1} \Lambda_{2n+2} (u^{-1}) - \frac{4}{n} u^n \Lambda_{2n} (u^{-1}) \right) + 2n^{-1} (n+1)^{-1}, \quad n \geq 1.\]

\[(1.402) \quad \sum_n (\lambda, -4) = -4 \sum_{2i \leq n} \left( \lambda + n - 2i - 2 \right) \frac{1}{n - 2i + 1}.\]

\[(1.403) \quad \frac{d}{du} \sum_n (\lambda, u) = \sum_{k=0}^{n} \left( \frac{n+k+\lambda}{n-k} \right) u^k = \frac{yu+u^2}{y+u} \sum_{k=0}^{n} \left( \frac{\lambda+k-2}{k} \right) \left( \frac{n+1-k}{2} - \frac{n+1-k}{2} \right),
\quad u \neq 0, \quad u \neq -4.\]

\[(1.404) \quad \frac{d}{du} \sum_n (\lambda, u) = \sum_{k=0}^{n} \left( \frac{\lambda+k-2}{k} \right) u^n \Lambda_{2n-2k+1} (u^{-1}).\]

\[(1.405) \quad \sum_{k=0}^{n} \left( \frac{n+k+\lambda}{n-k} \right) (-4)^k = \sum_{k=0}^{n} (-1)^{n-k} \left( \frac{\lambda+k-2}{k} \right) (n-k+1).\]
The relations (1.408) - (1.409) deal with the (general) Gould polynomials $x(x+na)^{n-1}$, some special cases of them, and some related quantities. The theory of Gould polynomials is in Chapter C, as a special case of the theory of polynomials of convolution type, see e.g. C.36-42, C58-63. Some formulas containing two binomial coefficients and belonging to List 3, have special cases here. By application of e.g.

$$\binom{-1}{k} = (-1)^k, \quad \binom{-2}{k} = (-1)^k(k+1),$$

see D(24).

(1.408) $\sum_{n=0}^{\infty} x(x+na)^{n-1} \frac{z^n}{n!} = \exp(xg(z,a))$,

(1.409) $g(z,a) = \sum_{k=1}^{\infty} \frac{z^k}{ka} \binom{ka}{k}$, $a \neq 0$.

see C(86), (87), also for convergence. The function $g$ is the composition inverse of $q(t) = e^{-at} (e^t - 1)$, see C, Theorem 6.

(1.410) $C(155) \sum_{n=0}^{\infty} \frac{(x+na)^n}{n!} z^n = \exp((x+1)g(z,a))(a+(1-a)\exp(g(z,a)))^{-1}$,

with $g(z,a)$ as in (1.409). Cf. C(86), (87) for convergence.

(1.411) $C(164) \sum_{n=0}^{\infty} \frac{x}{x+na} \binom{x+na}{n}(e^{t-at} - e^{-at})^n = e^t$. 


The text contains mathematical equations related to series expansions and exponential functions. The specific equations are as follows:

\[(1.4.12) = \sum_{k=1}^{\infty} \frac{1}{ka} \left( \frac{ka}{k} \right) \left( e^{-at} - e^{-at} \right)^{k-1} \]

\[(1.4.13) = \sum_{n=0}^{\infty} \left( \frac{x+na}{n} \right) \left( e^{-at} - e^{-at} \right)^{n} \]

\[= \exp \left( t+xt \right) \{ a + (1-a) e^{t} \}^{-1} \]

\[(1.4.14) \sum_{k=0}^{n} \frac{(-1)^{k} (k+1)}{y+n} \frac{y+k}{y+n} = \frac{y-x}{y+2+n} \]

Take: \( a = 0, \ x = -2 \) in \( C \) \( (90) \).

\[(1.4.15) \sum_{k=0}^{n} \frac{(-1)^{k} (y+1+k \lambda)}{y+n} \left( \frac{y+n}{n-k} \right) = \frac{y+1}{y+n} \left( \frac{y+1+n \lambda}{n} \right) \]

Take: \( a = 0, \ x = -2 \) in \( C \) \( (95) \).

\[(1.4.16) \sum_{k=0}^{n} \frac{y+k \nu-k}{y+n \nu-k} \left( \frac{y+n \nu-k}{n} \right) = \frac{y+1}{y+n \nu} \left( \frac{y+1+n \nu}{n} \right) \]

Take: \( u = 0, \ x = 1 \) in \( C \) \( (98) \).

\[(1.4.17) \sum_{k=0}^{n} \frac{(k+1)}{y+n \nu-k} \left( \frac{y+n \nu-k}{n} \right) = \frac{y+2}{y+2+n \nu} \left( \frac{y+2+n \nu}{n} \right) \]

Take: \( u = 0, \ x = 2 \) in \( C \) \( (98) \).

\[(1.4.18) \sum_{k=1}^{\infty} \frac{a+k \lambda-k}{a+n \lambda-k} \left( \frac{a+n \lambda-k}{n} \right) \lambda^{k} \]

\[= \left( \frac{a+n \lambda-k}{n} \right) \lambda^{k}, \ k = 0, 1, \ldots, n \]
(1.419) \[
\sum_{k=0}^{n} \binom{k+1}{a+kb-k} \binom{a+nb-k}{n-k} b^k = \sum_{r=0}^{n} \binom{a+nb-r}{n-r} b^r.
\]

Help may then come from (1.326) or (1.317) - (1.321).

In (1.420) - (1.424) the above could polynomials are specialized to \(a = -1\), see p.C49, and \(a = \frac{1}{2}\), see pp.C50 - 51.

(1.420) \[
C \left(128^a\right) \sum_{n=0}^{\infty} \frac{x}{x-n} \left(\frac{x-n}{n}\right) z^n = \left(\frac{1}{2} + \frac{1}{2} \sqrt{1+yz}\right)^x.
\]

(1.421) \[
C \left(128^b\right) \sum_{n=0}^{\infty} \left(\frac{x-n}{n}\right) z^n = \left(1+yz\right)^{-\frac{1}{2}} \left(\frac{1}{2} + \frac{1}{2} \sqrt{1+yz}\right)^{x+1}.
\]

(1.422) \[
C \left(132\right) \sum_{n=0}^{\infty} \frac{2x}{2x+n} \left(\frac{x+n}{n}\right) z^n = \exp\left(g_1(z, \frac{1}{2})\right) = \left(\frac{1}{2} z + \sqrt{1+2z^2}\right)^{2z}, |z| \leq 2.
\]

(1.423) \[
C \left(130\right), \left(131\right) \quad g_1(z, \frac{1}{2}) = \sum_{k=1}^{\infty} \frac{2}{k} \left(\frac{k/2}{k}\right) = 2 \log \left(\frac{1}{2} z + \sqrt{1+z^2}\right), |z| \leq 2.
\]

(See also C (133)).
\[(1.424) = C \binom{134}{a} \sum_{n=0}^{\infty} \left( \frac{x+\frac{1}{2}a}{n} \right) z^n = \left(1 + \frac{1}{4} z^2 \right)^{-\frac{1}{2}} \left( \frac{1}{2} z + \left(1 + \frac{1}{4} z^2 \right)^{\frac{1}{2}} \right)^{2x+1}, |z| < 2. \]

The next formulas, \[(1.425) - (1.446),\] contain the Gould polynomials \[q_n(x, z) = x(x+2n)^{-1} \binom{x+2n}{n}, \text{ see the notation in } C \{84\}, \{86\}, \{104\}, \{105\}. \] We have \[q_n(x, z) = \frac{(2n+1)^{-1} \binom{2n+1}{n}}{(n+1)^{-1} \binom{2n}{n}}, \text{ the } n^{th} \]
Catalan number (references on p. C44). We therefore called the convolution group \[q_n(x, z), n \in \mathbb{N}, x \in \mathbb{C}, \text{ the Catalan convolution group. \text{ The formulas } (1.425) - (1.446) \text{ also contain related quantities, in particular } \binom{2k}{k}. \text{ That so many relations exist here is caused partly by the richness of the Catalan convolution group (see e.g. pp. C43 - C49) \text{ and also by } B(13) \binom{2k}{k} = (-4)^k \binom{-12}{k} \text{ and related identities. A number of formulas with } \binom{2k}{k} \text{ with proofs, sometimes different from ours, is in Lehmer (1985) and Zucker (1985).}\]
(1.425) = C (105), (107) \sum_{k=1}^{\infty} \frac{1}{2k} \binom{2k}{k} z^k =

\log \left( \frac{2z}{1 - \sqrt{1 - 4z}} \right), |z| \leq \frac{1}{4}.

(1.426) = C (108) \sum_{n=0}^{\infty} \frac{x}{x+2n} \binom{x+2n}{n} z^n =

\left( \frac{2z}{1 - \sqrt{1 - 4z}} \right)^x, |z| \leq \frac{1}{4}.

(1.427) = C (111^a) \sum_{n=0}^{\infty} \binom{x+2n}{n} z^n =

\left( 1 - 4z \right)^{-\frac{1}{2}} \left( \frac{2z}{1 - \sqrt{1 - 4z}} \right)^x, |z| < \frac{1}{4}.

(1.428) \sum_{k=0}^{\infty} \binom{2k}{k} z^k = \left( 1 - 4z \right)^{-\frac{1}{2}}, |z| < \frac{1}{4}.

From B (13), and D (20).

(1.429) \sum_{k=0}^{\infty} \binom{2k}{k} 4^{-k} = \left( 2n+1 \right) 4^{-n} \binom{2n}{n}.

From B (13), (1.3) and B (14), or induction.

(1.430) \sum_{k=0}^{\infty} \binom{2k+1}{k} z^k = \left( 1 - 4z \right)^{-\frac{3}{2}}, |z| < \frac{1}{4}

From B (14) and D (20). Or by differentiation of (1.428).

(1.431) \sum_{k=0}^{n} \binom{2k+1}{k} 4^{-k} = \frac{1}{3} (2n+1)(2n+3) \binom{2n}{n} 4^{-n}.
\[ (1.4.32) \sum_{k=0}^{\infty} \binom{k}{r} \left( \frac{2^k}{k} \right) z^k = \frac{(2r)!}{\pi^r} z^r (1-yz)^{-\frac{r}{2}-\frac{1}{r}}, \quad |z| < \frac{1}{4}. \]

\[ (1.4.33) \sum_{k=0}^{n} \binom{k}{r} \left( \frac{2^k}{k} \right) y^{n-k} = \frac{(2r)!}{\pi^r} \left( \frac{n+\frac{1}{2}}{n-\frac{1}{2}} \right) y^{-\frac{r}{2}}, \quad r = 0, 1, \ldots, n. \]

\[ (1.4.34) \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^{k+1}}{k+1} = \frac{1}{2} - \frac{1}{\pi} \sqrt{1-yz}, \quad |z| \leq \frac{1}{4}. \]

For the sequence \((k+1)^{-1} \binom{2k}{k}\), see C (106). They are Catalan numbers. From \((1.4.26)\), with \(x=1\), or both sides have the same derivative by \((1.4.28)\), and the same value for \(z=0\).

\[ (1.4.35) \sum_{k=0}^{\infty} (-2k)^{-1} \binom{2k}{k} z^k = (1-yz)^{\frac{1}{2}}, \quad |z| \leq \frac{1}{4}. \]


\[ (1.4.36) \sum_{k=0}^{n} (-2k)^{-1} \binom{2k}{k} y^{-k} = y^{-n} \binom{2n}{n}. \]

\[ (1.4.37) \sum_{k=n+1}^{\infty} (2k-1)^{-1} \binom{2k}{k} y^{-k} = y^{-n} \binom{2n}{n}. \]

\[ (1.4.38) \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^{2k+1}}{2k+1} = \frac{1}{\pi} \arcsin 2z, \quad |z| \leq \frac{1}{\sqrt{2}}. \]

\[ (1.4.39) \sum_{k=0}^{\infty} \binom{2k}{k} \frac{y^{-k}}{2k+1} = \frac{\pi}{2}. \]

See \((1.4.43)\).

From \((1.4.38)\) for \(z = \frac{1}{2}\).
(1.440) \[ \sum_{k=0}^{\infty} (-1)^{k} \left( \frac{2}{k} \right) \frac{z^{2k+1}}{2k+1} = \frac{1}{2} \log \left( 2z + \sqrt{1 + yz^{2}} \right), \quad |z| \leq \frac{1}{2}. \]


(1.444) \[ \sum_{k=0}^{n} \left( \frac{2}{k} \right) x^k (1-x)^k, \quad \sum_{k=0}^{n} \frac{1}{2k+1} \left( \frac{2}{k} \right) x^k (1-x)^k. \]

See (1.359) and (1.328) \(- (1.330) \).

(1.445) \[ \sum_{k=0}^{\infty} \frac{(-1)^{k}}{2k+1} \left( \frac{2}{k} \right) y^{-2k} z^{2k+1} (1+z)^{-k}. \]

(1.446) \[ \sum_{j=0}^{n} \left( \frac{2}{j} \right) \frac{y^j z^{n-j}}{n+1-j} = \left( \frac{2}{n+1} \right) \sum_{k=0}^{n} \frac{1}{2k+1}. \]
Sums with
\[
\binom{2k-1}{k}, \binom{2k+1}{k},
\]
and related binomial coefficients might be reduced to (1.425)–(1.446) by the identities (106), (112)–(116).

(1.450) \sum_{2k \leq m} \binom{y-2k}{m-2k} = \sum_{k=0}^{m} (-2)^{m-k} \binom{y+2}{k}.

See (1.56).

(1.451) \sum_{2k \leq m} \binom{y-2k}{m-2k} + 2 \sum_{2k \leq m-1} \binom{y-2k}{m-1-2k} = \binom{y+2}{m}, \quad m \geq 1.

(1.452) \sum_{0 \leq j \leq (n-b)(c-a)^{-1}} \binom{n+jax}{b+jc}.

a \in \mathbb{Z}, b \in \mathbb{N}, c \in \mathbb{N}, c > a, n \geq b.

See Chapter P, (1.452), and for a recurrence pp. 627–29.
TABLE 2

(2.1) \( \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (n^{-1}) = (n+1) \sum_{j=0}^{n} x^{j-n} / (j+1). \)

(2.2) \( \sum_{k=0}^{n} \frac{(-1)^{k+1}}{(1+k)^{n+1}} = \frac{x+1}{x+2} \left\{ \frac{1 + (-1)^{n}}{(n+1)} \right\}, \)

for \( x \neq 0, 1, \ldots, n-1 \), \( x \neq -2 \).

Induction on \( n \), with \( \mathbb{D} (4) \).

(2.3) \( \sum_{k=0}^{\infty} \frac{(-1)^{k}}{(k)} \left( \begin{array}{c} x \\ k \end{array} \right)^{-1} = \frac{x+1}{x+2}, \quad \text{Re} x \leq -2, \)

with absolute convergence.

(2.4) \( S_{n}(x) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) x^{k} = (n+1) \sum_{j=0}^{n-1} \frac{1}{j+1} x^{-j} (1+x)^{j-n-1} (1+x^{-1}), \)

(2.5) \( (1+x) S_{n+1}(x) = (1+x)^{n+2} + x (n+2) (n+1) \sum_{j=0}^{n-1} (j+1)_{n-j+1} S_{j}(x). \)

(2.6) \( \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) x^{k} y^{n-k} = \sum_{2k \leq n} \frac{n+1}{n+1-k} \left( \begin{array}{c} n-k \\ k \end{array} \right) (-xy)^{k} (x+y)^{n-2k}. \)

(2.7) \( \sum_{n=0}^{\infty} \left( \begin{array}{c} y \\ n \end{array} \right) x^{n+1} = \sum_{k=0}^{\infty} \frac{y+1}{y+1-k} \left( \frac{x}{1+x} \right)^{k+1}, \)

\( y+1 \neq 0, \quad x \in \mathbb{D}, \quad z \in \mathbb{C}: \quad |z| < 1, \quad \text{Re} z > -\frac{1}{2}. \)
\[ (2.8) \sum_{k=0}^{n} (-1)^k \binom{x}{k}^{-1} = -(x+1)(x+2)(x+3)^{-1} \]

\[ (2.9) \sum_{k=0}^{\infty} (-1)^k \binom{x}{k}^{-1} = -\frac{x+1}{(x+2)(x+3)}, \quad \text{Re} x < -3, \]

\[ (2.10) \sum_{k=0}^{r} (n-2k) \binom{n}{k}^{-1} = (n+1)\left\{ 1 - \binom{n+1}{r+1}^{-1} \right\}, \]

\[ (2.11) \sum_{k=0}^{n} (n-2k) \binom{n}{k}^{-1} = 0, \]

\[ (2.12) \sum_{k=1}^{n} (-1)^k k^{-1} \binom{x}{k}^{-1} = (x+1)^{-\frac{1}{2}} - 1 + (-1)^n \binom{x}{n}^{-\frac{1}{2}}, \quad x \notin \{-1, 0, \ldots, n-1\}, \quad n \geq 1. \]

\[ (2.13) \sum_{k=1}^{\infty} (-1)^k k^{-1} \binom{x}{k}^{-1} = -(x+1)^{-1}, \quad \text{Re} x < -1. \]

\[ (2.14) \sum_{k=0}^{n} (-1)^k \binom{k+1}{k}^{-1} \binom{n}{k}^{-1} = \]

\[ (h+1) \sum_{k=0}^{n} (-1 + \alpha (-1)^k) \binom{k+1}{k}^{-2}. \]

\[ (2.15) \sum_{k=0}^{2m} (-1)^k \binom{k+1}{k}^{-1} \binom{2m}{k}^{-1} = \]

\[ (2m+1) \sum_{k=1}^{m+1} \binom{k+m}{k}^{-2}. \]
\begin{align*}
(2.16) \quad & \sum_{k=0}^{2m+1} (-1)^k (k+1)^{-1} \left( \frac{2m+1}{k} \right) = \\
& (2m+2) \sum_{k=1}^{m+1} (k+m+1)^{-2}.
\end{align*}

\begin{align*}
(2.17) \quad & \sum_{k=1}^{n} (-1)^k \left( \frac{x}{k} \right)^{-1} H_k = - (x+1)(x+2)^{-2} \\
& + (-1)^{n} \frac{x+1}{x+2} \left( \frac{x+1}{n+1} \right)^{-1} H_n + (-1)^{n} \frac{x+1}{(x+2)^2} \left( \frac{x+1}{n} \right)^{-1},
\end{align*}

\begin{align*}
H_n &= \sum_{j=1}^{n} j^{-1},
\end{align*}

\begin{align*}
(2.18) \quad & \sum_{k=1}^{2m} (-1)^k \left( \frac{2m}{k} \right)^{-1} H_k = (2m+1)(2m+2)^{-1} H_{2m} \\
& - \frac{1}{2} m (m+1)^{-2}.
\end{align*}

\begin{align*}
(2.19) \quad & \sum_{k=1}^{2m-1} (-1)^k \left( \frac{2m-1}{k} \right)^{-1} H_k = -2 m (2m+1)^{-1} H_{2m-1},
\end{align*}

\text{for } m \geq 1.

Formulas with \( \left( \frac{1}{k} \right)^{-1} \) may be changed by \( D(x) \) into formulas with \( \left( -1 \frac{1}{k+1} \right)^{-1} \),
see (2.33) - (2.47).
(2.24) \( \sum_{k=c}^{\infty} \binom{\kappa}{k} = \frac{c}{c-1} \), \( c \geq 2 \).

(2.25) \( \sum_{k=n}^{\infty} \binom{k}{c}^{-1} = \frac{n}{c-1} \binom{n}{c}^{-1} \), \( 2 \leq c \leq n \).

(2.26) \( \sum_{k=1}^{\infty} \binom{km}{m}^{-1} = m \int_0^1 (1-t^m)^{-1} (1-t)^m \, dt \),

\( m \geq 2 \).

(2.27) \( \frac{m}{m+1} \sum_{k=0}^{n} \binom{y-k}{m+1} = \binom{y-n-1}{m} - \binom{y}{m} \),

\( y \notin \{0, 1, \ldots, m+n\} \).

(2.28) \( \frac{m}{m+1} \sum_{k=0}^{n} \binom{x+k}{m+1} = \binom{x-1}{m} - \binom{x+n}{m} \),

\( x \notin \{ -n, \ldots, 0, \ldots, m \} \).
The next formulas contain \((y + k)^{-1}\). They may be reduced to preceding ones by

\[
(y + k)^{-1} = (-1)^k (-y)^{-1},
\]

We only list some of the formulas of this type.

\[\sum_{k=0}^{n} \binom{y + k}{k}^{-1} = \frac{y}{y - 1} \left\{ 1 - \left(\frac{y + n}{n+1}\right)^{-1} \right\},\]

\(-y \notin \{0, 1, \ldots, n\}, \ y \neq 1\). From D (24) and (2.2).

\[\sum_{k=0}^{\infty} \binom{y + k}{k}^{-1} = \frac{y}{y - 1}, \ \text{Re} \ y > 1,
\]

From D (24) and (2.3).

\[\sum_{k=1}^{n} k^{-1} \binom{y + k}{k}^{-1} = y^{-1} \left\{ 1 - \left(\frac{y + n}{n+1}\right)^{-1} \right\},\]

\(-y \notin \{0, 1, \ldots, n\}, \ n \geq 1\).

From D (24) and (2.18).

\[\sum_{k=1}^{\infty} k^{-1} \binom{y + k}{k}^{-1} = y^{-1}, \ \text{Re} \ y > 0.
\]


\[\sum_{k=1}^{\infty} k^{-1} z^k \binom{x + k}{k}^{-1} = z \int_0^1 (1-t)^x (1-zt)^{-1} dt,
\]

\(\text{Re} \ x > 0, \ |z| < 1\).
\[(2.38) \sum_{k=1}^{\infty} (-1)^k k^{-1} z^k \left( \binom{x+k}{k} \right)^{-1} = -\sum_{k=0}^{\infty} (x+1+k)^{-1} \left( \frac{z}{1+z} \right)^{k+1}, \]

\[-x+1 \notin \mathbb{N}_0, \ z \in \{ w \in \mathbb{C} : |w| < 1, \text{Re } w > -\frac{1}{2} \}.\]

\[(2.39) \sum_{n=0}^{\infty} z^{-n-1} \left( \binom{a+n}{n} \right)^{-1} = \sum_{k=0}^{\infty} (-1)^k \frac{a}{a+k}, \quad a > 0.\]

\[(2.40) \sum_{n=0}^{\infty} z^{-n-1} \left( \binom{rn+n}{n} \right)^{-1} = (-1)^{r-1} z^{r-1} \log z + \]

\[r \sum_{h=1}^{\infty} (-1)^{r-1-h} h^{-1}, \quad r \in \mathbb{N}, \]

and \(r = 1\) for empty sum zero.

\[(2.41) \sum_{n=0}^{\infty} (-1)^n \left( \binom{a+n}{n} \right)^{-1} = \sum_{k=0}^{\infty} z^{k-1} \frac{a}{a+k}, \quad a > 0.\]

\[(2.42) \sum_{n=0}^{\infty} (-1)^n \left( \binom{rn+n}{n} \right)^{-1} = r z^{r-1} \log z - \]

\[r \sum_{h=1}^{\infty} h^{-1} z^{r-1-h}, \quad r \in \mathbb{N}, \]

and \(r = 1\) for empty sum zero.

\[(2.43) (1+x)^{-1} \sum_{n=0}^{\infty} \left( \frac{x}{1+x} \right)^n \left( \binom{a+n+m}{n+m} \right)^{-1} = \]

\[\sum_{k=0}^{\infty} (-1)^k \frac{a}{a+k} \left( \binom{a+k+m}{m} \right)^{-1}, \quad -a \notin \mathbb{N}_0, \]

with l.h.s. absolute convergent for \(\text{Re } x > \frac{1}{2} \).
and r.h.s. for \(|x| < 1\).

\[
(2.44) \quad \sum_{k=1}^{\infty} k^{-2} \left( \frac{k}{k+n} \right)^{-1} = \sum_{k=n+1}^{\infty} k^{-2}.
\]

\[
(2.45) \quad S(n, z) = S(z, n), \quad n \geq 1, \quad z \geq 1, \quad \text{with}
\]

\[
S(n, z) = \sum_{k=1}^{n} k^{-2} \left\{ 1 - \left( \frac{k}{k+n} \right)^{z} \right\}.
\]

\[
(2.46) \quad \sum_{k=0}^{\infty} \frac{z^{k}}{k!} \left( \frac{z}{k} \right)^{-1} = e^{z} \sum_{j=0}^{\infty} \left( \frac{z}{j} \right)^{j} \frac{v^{j}}{v+j},
\]

\[ z \in \mathbb{C}, \quad v \notin \mathbb{N}. \quad \text{For } v = 0 : \text{limit in r.h.s.}
\]

\[
(2.47) \quad \sum_{k=0}^{\infty} \left( \frac{z+k}{k} \right)^{-1} z^{k} (1+z)^{-k-1} \sum_{n=0}^{\infty} (-z)^{n} \frac{v^{n}}{v+n},
\]

\[ -v \notin \mathbb{N}, \quad \text{limit for } v \to 0 \quad \text{in r.h.s.}\]

The l.h.s. converges absolutely for \(\text{Re} \ z > -\frac{1}{2}\),

the r.h.s. for \(|z| < 1\).
\[ (2.53) \sum_{k=1}^{\infty} \left( \frac{2}{k} \right)^{-1} \left( \frac{2k}{2x} \right)^k = x^2 \left( 1-x^2 \right)^{-1} + \]

\[ -2x \left( 1+x^2 \right)^{-\frac{3}{2}} \log \left( x + \sqrt{1+x^2} \right), \quad |x| < 1. \]

\[ (2.55) \sum_{k=1}^{\infty} (-1)^k \left( \frac{2k}{k} \right)^{-1} \left( \frac{2x}{2x} \right)^k = -x^2 \left( 1+x^2 \right)^{-1} \]

\[ -x \left( 1+x^2 \right)^{-3/2} \log \left( x + \sqrt{1+x^2} \right), \quad |x| < 1 \]

By differentiation of (2.54).
By differentiation of (2.54),
\[ (2.56) \sum_{k=0}^{\infty} (2k+1)^{-1} \binom{2k}{k}^{-1} (2x)^{2k} = x^{-1} (1-x^2)^{-1/2} \arcsin x, \quad |x| < 1. \]

\[ (2.57) \sum_{k=0}^{\infty} (2k+1)^{-1} \binom{2k}{k}^{-1} (-1)^k (2x)^{2k} = x^{-1} (1+x^2)^{-1/2} \log (x + (1+x^2)^{1/2}), \quad |x| < 1. \]

\[ (2.58) \sum_{k=1}^{\infty} k^{-2} \binom{2k}{k}^{-1} (2y)^{2k} = 2 (\arcsin y)^2, \quad |y| \leq 1. \]

\[ (2.59) \sum_{k=1}^{\infty} (-1)^k k^{-2} \binom{2k}{k}^{-1} (2y)^{2k} = -2k \log (y + (1+y^2)^{1/2})^2, \quad |y| \leq 1. \]

\[ (2.60) \sum_{k=0}^{\infty} (2k+1)^{-2} \binom{2k}{k}^{-1} (2y)^{2k+1} = \]

\[ \int_{0}^{y} x^{-1} (1-x^2)^{-1/2} \arcsin x \, dx = \int_{0}^{\arcsin y} \frac{t}{\sin t} \, dt, \quad |y| \leq 1. \]

(2.61) \[ \sum_{k=0}^{\infty} \frac{(2k+1)^{-3}}{(2k)!} (-1)^k y^{2k+1} = \]

\[ \int_0^{1+y^2} \frac{x^{-1} (1+x^2)^{-\frac{3}{2}}} \log (1+(x^2)^{\frac{1}{2}}) dx = \]

\[ \int_0^{1+y^2} \frac{t dt}{\sinh t}, \quad |y| \leq 1. \]

Integrate (2.57) over \([0, y]\).

(2.62) \[ \sum_{k=1}^{\infty} k (2k+1)^{-3} (\frac{2k}{k})! y^{2k+1} = \]

2y + 2(1-y^2)^{\frac{1}{2}} \arcsin y, \quad |y| \leq 1.

(2.63) \[ \sum_{k=1}^{\infty} (-1)^k k^{-3} (2k+1)^{-3} (\frac{2k}{k})! y^{2k+1} = \]

2y + 2(1+y^2)^{\frac{1}{2}} \log (1+(1+y^2)^{\frac{1}{2}}), \quad |y| \leq 1.

(2.64) \[ \sum_{n=1}^{\infty} y^n \left\{ \frac{2n}{(2n+1)} \left( \frac{2n}{n} \right)^{-1} \right\} \frac{1}{w^{2n}} = \]

\[ 1 - (1-w^2)^{\frac{1}{2}} \frac{w^{-1}} w, \quad |w| < 1, \quad n \geq 1. \]

(2.65) \[ \sum_{k=1}^{\infty} k^{-1} (\frac{2k}{k})! y^{-1} z^k = \]

\[ \sum_{j=0}^{\infty} (-1)^j (\frac{j+\frac{1}{2}}{2})^{-1} \left( \frac{z}{1-z} \right)^{j+1}, \quad |z| < 1, \quad \text{Re} z < \frac{1}{2}. \]

cf (2.52)
(2.66) \[ \sum_{k=1}^{\infty} \frac{(2k-1)^{-1}}{k!} z^{k-1} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left( \frac{z}{1-z} \right)^j (1-z)^{-2} \]

\[ |z| < 1, \quad \text{Re} \ z < \frac{1}{2} \text{.} \quad \text{Differentiate (2.65) w.r. to } z. \]
\[(3.3) \sum_{2h \leq n} (\binom{n}{2h})^2 = \frac{1}{2} \binom{2n}{n} + \frac{1}{4} (1+(-1)^{n-1}) \binom{n}{\frac{n}{2}}.\]

\[(3.6) \sum \binom{2m}{k} = \frac{1}{n} \binom{4m}{2m} - \frac{1}{n} (-1)^m \binom{2m}{2m}.\]

From (3.07) and \((3.7)\),

\[(2.8) \sum_{k=1}^{m} \binom{2m}{k} (-1)^k \binom{2m}{2m} + (2m)^2.\]
\begin{align*}
(3.9) \quad (1-z)^{2u-1} & = \sum_{j=0}^{\infty} \binom{u}{j} z^j \\
& = \sum_{m=0}^{\infty} (m+u)^2 z^m, \quad u \in \mathbb{C}, \quad |z| < 1; \\
(3.10) \quad \sum_{k=0}^{n} \binom{n}{k} z^k & , \quad \text{See (3.89), (3.100), (3.158), (3.198), (3.24). Legendre pol. Ch.5, Rainville (1960).} \\
(3.11) \quad \sum_{k=0}^{n} \binom{x}{k} \binom{y}{k} & = \sum_{k=0}^{n} \frac{(-1)^k \binom{n-x}{k} \binom{y+n-k}{n}}{k!}, \\
(3.12) \quad \sum_{k=0}^{\infty} \frac{(-\alpha)}{k^\beta} z^k & = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_{0}^{1} t^{\alpha-1}(1-t)^{-\beta}(1-zt)^{-\alpha} dt, \\
& = \frac{\pi}{\sin \pi \beta} \int_{0}^{\pi/2} (\tan u)^{\beta-1} (1-z \sin^2 u)^{-\alpha} du, \\
& \quad |z| < 1, \quad 0 < \Re \beta < 1, \quad \text{and also for } |z| = 1, \quad z \neq 1, \quad 0 < \Re \beta < 1, \quad \Re(\alpha+\beta) < 1.
\end{align*}
\[ (3.14) \sum_{k=0}^{\infty} \binom{-\alpha}{k} \binom{-\beta}{k} = \frac{\Gamma(-\alpha-\beta)}{(\Gamma(1-\alpha) \Gamma(1-\beta))^2}, \]

Re(\alpha+\beta) < 1, when \( 1/\Gamma(\alpha) = 0 \) for \( -\alpha \in \mathbb{N} \).

\[ (3.15) \sum_{k=0}^{\infty} \binom{\beta}{k} (-1)^k = \left\{ \frac{\Gamma(1+\beta)}{(1+\beta) \Gamma(1-\beta)} \right\}^{-1} \]

\( (\pi \beta)^{1/2} \sin \pi \beta, \quad \beta \in \mathbb{C} \),

with \( 1/\Gamma(\gamma) = 0, \quad -\gamma \in \mathbb{N} \). For \( \sum_{k=0}^{\infty} \), see (3.17) (3.28).

\[ (3.16) \sum_{k=0}^{\infty} \binom{-1/2}{k} z^k = 2\pi^{-1} \int_0^{\pi/2} (1 - z \sin^2 u)^{-1/2} du, \]

\( |z| < 1 \). From (3.12), (3.13) with \( \alpha = \beta = \frac{1}{2} \).

\[ (3.17) \sum_{k=0}^{\infty} \binom{1/2}{k} (-1/2)^k z^k = 2\pi^{-1} \int_0^{\pi/2} (1 - z \sin^2 u)^{1/2} du, \]

\( |z| \leq 1 \). From (3.12), (3.13) with \( \alpha = -1/2, \beta = 1/2 \).

\[ (3.18) \sum_{k=0}^{\infty} (-1)^k \binom{x}{k} \binom{-x}{k} = \]

\[ \pi^{-1} \sin \pi x \int_0^{1} t^{-x-1} (1-t)^{-x} (1+t)^x \ dt, \]

\[ 2\pi^{-1} \sin \pi x \int_0^{\pi/2} (\tan u)^{2x-1} (1 + \sin^2 u)^{x} \ du, \]

\( 0 < \text{Re} x < 1 \).

From (3.13) with \( \alpha = -x, \beta = x, \ z = -1 \).
\[
(3.19) \sum_{k=0}^{\infty} \left( -\beta \right)^{2k} \left( \frac{1}{x+k} + \frac{1}{\beta-x+k} \right) = \\
\pi^{-3} \sin \pi \beta \sin \pi \chi \sin \pi \left( \beta - \chi \right) \Gamma(\chi) \Gamma(\beta - \chi) \Gamma(1 - \beta),
\]
\[
\text{Re } \beta < 1, \quad -\chi \notin \mathbb{N}_0, \quad \chi - \beta \notin \mathbb{N}_0.
\]

\[
(3.20) \sum_{k=0}^{\infty} \left( -\beta \right)^{2k} \left( \frac{1}{m+k} + \frac{1}{\beta-m-1+k} \right) = 0,
\]
\[
\text{Re } \beta < 1, \quad m+1-\beta \notin \mathbb{N}_0.
\]

\[
(3.21) \sum_{k=0}^{\infty} \left( -\beta \right)^{2k} \frac{1}{\beta+2k} = \pi^{-3} \sin \pi \beta \sin \frac{\pi}{2} \pi \beta \Gamma \left( \frac{1}{2} \beta \right) \Gamma \left( 1 - \beta \right),
\]
\[
\text{Re } \beta < 1, \quad -\beta/2 \notin \mathbb{N}_0. \text{ From (3.19) with } \chi = \beta/2.
\]

\[
(3.22) \sum_{k=0}^{n} \left( \frac{n}{k} \right)^{2k} x^{k}. \text{ See (3.89), (3.100), (3.138), (3.178)}
\]

\[
(3.23) \sum_{k=0}^{n} \left( \frac{n}{k} \right)^{x} \left( 1+x \right)^{n-k}. \text{ See (3.89), (3.100), (3.138)}
\]

\[
(3.24) \left( \frac{x-1}{n} \right)^{n} \sum_{k=0}^{n} \left( \frac{n}{k} \right) \left( \frac{x+1}{x-1} \right)^{k} = P_{n}(x),
\]
\[
\text{Legendre polynomial, see S (3).}
\]

\[
(3.25) \sum_{k=1}^{n} \left( \frac{n}{k} \right) H_{k} = \binom{2n}{n} \left( 2H_{n} - H_{2n} \right), \quad n \geq 1
\]
\[
H_{m} = \sum_{j=1}^{m} j^{-1}.
\]
(3.26) \[ \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{z}{k+1} 2^{k+1} \]

Mittag-Leffler polynomial. Rainville (1960), p. 296. Roman (1984), Ch. 4.1.6, mind the factor \( n! \). In the second formula on p. 76 \( \binom{n}{k} \) should be \( n! / k! \). See (3.74).

(3.27) \[ \sum_{k=0}^{n} \binom{n}{k} (-1)^k = \prod_{j=1}^{n} (1 - j^{-2} \beta^2) = \binom{n+\beta}{n} \binom{n-\beta}{n}, \quad (\text{with empty product} = 1). \]

(3.28) \[ \sum_{k=0}^{n} \binom{n}{k} (-1/2)^k \binom{1/2}{k} = (2n+1) \binom{-1/2}{n}. \]

(3.29) \[ \sum_{k=0}^{n} \binom{n}{k} (-1/2)^k 2^k : (3.491) \text{ and } B(13). \]

(3.30) \[ \sum_{k=0}^{n} \binom{n}{k} (-n)^k (x+k)^{-1}. \]

With \( D(24) \) for second factor and then (3.202).
\[(3.32) \quad \sum_{k=0}^{\infty} \binom{n}{k} (\frac{x}{k}) = \left(\frac{x+n}{\eta + \zeta}\right), \eta = 0, \ldots, n.\]

\[(3.33) \quad \sum_{k=0}^{\infty} \binom{n}{k} (\frac{x}{k+\zeta}) = \left(\frac{x+n}{\zeta + \eta}\right).\]

\[(3.34) \quad \sum_{k=n}^{\infty} \frac{(x)^{x}}{(k-n)^{k-n}} = \sum_{h=0}^{\infty} \frac{(x)^{h}}{(n+h)^{h}} =
\]

\[2^{x+n} \sum_{k=n}^{\infty} \frac{(x)^{x}}{(2k-n)^{k-n}} y^{-k} =
\]

\[2^{x+n} \sum_{k=n}^{\infty} \frac{(x)^{x}}{(k)^{k-n}} y^{-k}, \text{Re} \ x > -\frac{1}{2}.\]

\[(3.35) \quad \sum_{k=0}^{n} \binom{x}{k+y} (\frac{1}{k-n}) = \sum_{h=0}^{n+y} \binom{x}{h} (\frac{1}{n+h}).\]

Now maybe (3.39) or (3.42) – (3.45).

\[(3.36) \quad \sum_{k=0}^{m+\zeta} \frac{2k}{(m+k)^{2m}} (\frac{2m}{m+k}) = \frac{m+\zeta}{m+\zeta} \binom{2m}{m} (\frac{2m}{\zeta}), \quad m+\zeta \geq 1.
\]

\[(3.37) \quad \sum_{k=0}^{\infty} \frac{(-x)^{x}}{(k)^{x+k}} = \frac{x}{x-\zeta} \binom{x-1}{n} (\frac{x-1}{n+\zeta}), \quad x \neq \zeta.
\]

\[(3.38) \quad \sum_{k=0}^{a \wedge b} \frac{(-x)^{x}}{(a-k)(b-k)} = \frac{(-1)^{a+b}}{x+a-b} \binom{a+x}{a} (\frac{b-x}{b}).
\]

\[= \binom{a+x}{b} (\frac{b-x}{a}), \quad a, b \in N, \quad x \neq b-a \text{ for first equality} \]
\begin{align*}
(3.40) \quad & \sum_{k=0}^{\infty} \binom{x}{k} \binom{x}{2m-k} = \binom{2x}{2m} + \binom{x}{m} \\
(3.41) \quad & \sum_{k=0}^{\infty} \binom{x}{k} \binom{x}{2m+1-k} = \binom{2x}{2m+1} \\
(3.42) \quad & \sum_{k=0}^{n-x} \binom{x}{k} \binom{n-x}{n-k} = \frac{n-x}{n} \binom{n-1}{x-1} \binom{n-x-1}{n-x} \\
& \quad, \quad \varepsilon = 0, \ldots, n, \quad n \geq 1 \\
(3.43) \quad & \sum_{k=0}^{n-x} \binom{x}{k} \frac{(1-x)(n-x)}{(n-1)(n-x)} = \frac{(n-1)(1-x)-x}{n(n-1)} \binom{x-1}{x-1} \binom{n-x-1}{n-x-1} \\
& \quad, \quad \varepsilon = 0, \ldots, n-1, \quad n \geq 2 \\
(3.44) \quad & \sum_{k=m}^{n} \binom{n}{k} \binom{a+b-n}{a-k} = \sum_{j=m}^{n} \frac{(j-1)(a+b-j)}{(a-k)(a-k)} \\
& \quad, \quad a \in \mathbb{N}_{1}, \quad b \in \mathbb{N}_{0}, \quad 1 \leq m \leq a, \quad m \leq n \leq a+b \\
(3.45) \quad & \binom{a+b}{n}^{-1} \sum_{k=m}^{n} \binom{a-b}{k} \binom{a-b}{n-k} = \\
& \quad, \quad a \in \mathbb{N}_{1}, \quad b \in \mathbb{N}_{0}, \quad 1 \leq m \leq a, \quad m \leq n \leq a+b \\
(3.46) \quad & \sum_{k \leq n} \binom{u}{k} \binom{v}{n-2k}. \quad \text{See (3.81), (3.45)}
\end{align*}
\[(3.47) \quad \sum_{2j \leq n} \binom{x}{2j} \binom{y}{n-2j} = \binom{x+y}{n} + \sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{y}{n-k}.\]

\[(3.48) \quad \sum_{2j+1 \leq n} \binom{x}{2j+1} \binom{y}{n-2j-1} = \binom{x+y}{n} - \sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{y}{n-k}.\]

\[(3.49) \quad \sum_{j=0}^{m} \binom{x}{2j} \binom{y}{2m-2j} = \binom{x+y}{2m} + \sum_{j=0}^{m} (-1)^{m-j} \binom{x-y}{2j} \binom{y}{m-j}.\]

\[(3.50) \quad \sum_{j=0}^{m-1} \binom{x}{2j+1} \binom{y}{2m-2j-1} = \binom{x+y}{2m} - \sum_{j=0}^{m} (-1)^{m-j} \binom{x-y}{2j+1} \binom{y}{m-j}, \quad m \geq 1.\]

\[(3.51) \quad \sum_{j=0}^{m} \binom{x}{2j} \binom{y}{2m+1-2j} = \binom{x+y}{2m+1} - \sum_{j=0}^{m} (-1)^{m-j} \binom{x-y}{2j} \binom{y}{m-j}.\]

\[(3.52) \quad \sum_{j=0}^{m} \binom{x}{2j+1} \binom{y}{2m-2j} = \binom{x+y}{2m+1} + \sum_{j=0}^{m} (-1)^{m-j} \binom{x-y}{2j+1} \binom{y}{m-j}.\]
\( (3.54) \) \[2 \sum_{j=0}^{n-1} \binom{x}{2j} \binom{n-x}{2m+1-2j} = \binom{2n}{2m+1}.\]

\( (3.55) \) \[2 \sum_{j=0}^{n-1} \binom{y}{2j+1} \binom{y}{2m-x-j-1} = \binom{2y}{2m} + (-1)^{m+1} \binom{y}{m}, \quad m \geq 1.\]

\( (3.56) \) \[2 \sum_{j=0}^{m} \binom{y}{2j+1} \binom{y}{2m-2j} = \binom{2y}{2m+1}.\]

\( (3.57) \) \[2 \sum_{j=0}^{m} \binom{y+1}{2j} \binom{y}{2m-2j} = \binom{2y+1}{2m} + (-1)^m \binom{y}{m}.\]

Further relations of this type may be derived from
\( (3.49) - (3.52).\)

\( (3.58) \) \[2 \sum_{2j \leq n} \binom{x}{2j} \binom{2n-x}{n-2j} = \binom{2n}{n} + (-4)^n \binom{1}{x-\frac{1}{2}}.\]

\( (3.59) \) \[2 \sum_{2j+1 \leq n} \binom{x}{2j+1} \binom{2n-x}{n-2j-1} = \binom{2n}{n} - (-4)^n \binom{1}{x-\frac{1}{2}}.\]

\( (3.60) \) \[\sum_{3j \leq n} \binom{x}{j} \binom{y-3x}{n-3j}.\] See (3.42).
(3.61) \[ \sum_{3h \leq n} \binom{x}{h} \binom{-x}{n-3h} = (-1)^n \sum_{k=0}^{n} \binom{x}{k} \binom{k}{n-k}. \]

(3.62) \[ \sum_{k=0}^{n} \binom{n}{k} \binom{u}{u-k} = \binom{v+n}{u}, \]
\[ u, v \in \mathbb{C} \text{ with notation and restriction } \mathcal{D} (16). \]

(3.63) \[ \sum_{2k \leq n} \left\{ \binom{n}{k} - \binom{n}{k-1} \right\}^{2v} = \frac{1}{n+1} \binom{2n}{n}. \]

(3.64) \[ \sum (qk)(pk), \ qk \leq m, \ pk \leq m, \]
\[ \text{ } p, q \text{ rel. prime. See Bragg (1999).} \]
\[(3.65) \sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{y}{n-k}. \text{ See (3.47), (3.48)}\]

\[(3.66) \sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{y}{n-k} = \sum_{2k \leq n} (-1)^k \binom{x}{k} \binom{y-x}{n-2k}.\]

\[(3.67) \sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{n}{n-k} = (-1)^m \binom{x}{m}, n = 2m.\]

\[(3.68) \sum_{k=0}^{n} (-1)^k \binom{y+1}{k} \binom{y}{2m-k} = (-1)^m \binom{y}{m}.\]

\[(3.69) \sum_{k=0}^{n} (-1)^k \binom{y+1}{k} \binom{y}{2m+1-k} = (-1)^{m+1} \binom{y}{m}.\]

\[(3.70) \sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{n-x}{n-k} = (-1)^m \binom{1/2 x - 1/2}{n}.\]

\[(3.71) \sum_{k=0}^{n} (-1)^k \binom{2x}{k} \binom{y-2x}{n-k} = \sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{y-2x}{n-k} y^k, \text{ see (3.427e).}\]

\[(3.72) \sum_{k=0}^{n} (-1)^k \binom{1/2}{k} \binom{-1/2}{n-k} = y^{-m} \binom{2m}{m}, n = 2m, -y^m \binom{2m}{m}, n = 2m+1.\]

\[(3.73) \sum_{i=0}^{n} (-1)^i \binom{x}{i} \binom{n-x}{n-i} = \sum_{i=0}^{n} (-1)^i \binom{x}{i}. \text{ See also (3.84a, b, c).}\]
\[(3.74) \sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{n}{n-k} = \sum_{k=0}^{n} \binom{-x}{k} \binom{n-1}{n-k} 2^k.\]

\[(3.75) m \sum_{k=0}^{m} (-1)^k \binom{x}{k} \binom{2m-k}{m-k} = (-1)^m \binom{x}{m} + \binom{m}{m} \binom{x}{2m}.\]

\[(3.76) \sum_{k=-m \ldots n} (-1)^k \binom{2m}{m-k} \binom{2n}{n-k} = \frac{(2m)! (2n)!}{m! n! (m+n)!}.\]

\[(3.77) \sum_{2k \leq n} (-1)^k \binom{x}{k} \binom{-x}{n-2k} = (-1)^n \binom{x}{n}.\]

Generating functions: \((1-z^x) (1+z)^{-x} = (1-z)^x.\)

\[(3.78) \sum_{2k \leq n} (-1)^k \binom{x}{k} \binom{2n-2x}{n-2k} = (-4)^n \binom{\frac{1}{2} x - \frac{1}{2}}{n}.\]

From (3.66) with \(y = 2n-x\) and (3.70).

\[(3.79) \sum_{j=0}^{m} (-1)^{m-j} \binom{x-y}{2j} \binom{y}{m-j}. \text{ See (3.49), (3.50).}\]

\[(3.80) \sum_{j=0}^{m} (-1)^{m-j} \binom{x-y}{2j+1} \binom{y}{m-j}. \text{ See (3.51), (3.52).}\]

\[(3.81) \sum_{2k \leq n} (-1)^k \binom{x}{k} \binom{-x}{n-4k} = (-1)^n \sum_{2k \leq n} \binom{x}{k} \binom{x}{n-2k}.\]
\[(3.82) \quad \sum_{h=0}^{m} (-1)^{m-h} \binom{-x}{2h} \binom{x}{m-h} = \binom{x}{2m}.
\]
\[(3.83) \quad \sum_{h=0}^{m} (-1)^{m-h} \binom{-x}{2h+1} \binom{x}{m-h} = -\binom{x}{2m+1}.
\]
\[(3.84a) \quad \sum_{k=0}^{n} (-1)^{k} \binom{x}{k} \binom{n-1-x}{n-k} = (-2)^{n} \binom{x}{n}.
\]
\[(3.84b) \quad \sum_{2j \leq n} \binom{x}{2j} \binom{n-1-x}{n-2j} = (-2)^{n} \binom{x}{n}, n \geq 1.
\]
\[(3.84c) \quad \sum_{2j+1 \leq n} \binom{x}{2j+1} \binom{n-1-x}{n-2j-1} = (-2)^{n-1} \binom{x}{n}, n \geq 1.
\]
\[(3.85) \quad \sum_{k=0}^{n} (-1)^{n-k} \binom{2^{u}}{2k+1} \binom{n-u}{n-k} = 2^{2n+1} \binom{n+u}{2n+1}.
\]
\[(3.86) \quad \sum_{k=0}^{n} (-1)^{n-k} \binom{2^{u}}{2k+1} \binom{n-1-u}{n-k} = \frac{2^{u} + n}{(u+n)(2n+1)} 4^{n} \binom{n+u}{2n}.
\]
\[(3.87) \quad \sum_{k=0}^{n} (-1)^{n-k} \binom{2^{u}}{2k} \binom{n-u}{n-k} = \sum_{\ell=0}^{2n} (-\lambda)^{\ell} \binom{n+\ell}{\lambda},
\]
\[(3.88) \quad \sum_{k=0}^{n} (-1)^{n-k} \binom{2^{u}}{2k} \binom{n-1-u}{n-k} = \frac{u}{u+n} 4^{n} \binom{n+u}{2n}, u \neq n.
\]
\[ 
\sum_{j=0}^{n} \binom{n}{j} \binom{n-j}{m-j} (z-1)^j, \quad a, b \in \mathbb{C}.
\]

\[ 
(3.90) \quad \sum_{k=0}^{n} \binom{n}{k} \binom{x^k}{k} z^k = \\
\sum_{j=0}^{n} \binom{n}{j} (x+n-j) (z-1)^j.
\]

\[ 
(3.91) \quad \sum_{k=0}^{n} \binom{n}{k} \binom{b}{k} x^{n-k} y^k = \\
\sum_{i=0}^{n} \binom{n}{i} \binom{b+i}{i} (x-y)^{n-i} y^i.
\]

\[ 
(3.92) \quad \sum_{k=0}^{n \wedge m} \binom{n}{k} \binom{x}{m-k} z^k (z-1)^{n-k} = \\
\sum_{h=0}^{n \wedge m} \binom{n}{h} (x+n-h) (z-1)^{n-h} = \\
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{x+k}{m} z^k,
\]

\[ 
(3.93) \quad \sum_{k=0}^{n} \binom{n}{k} \binom{x^k}{k} x^k = \sum_{i=0}^{n} \binom{n}{i} \binom{x+i}{n} = \\
\sum_{i=0}^{n} \binom{x+i}{i} \binom{x}{n-i}.
\]
\[(3.94) \quad \sum_{k=0}^{n} \binom{x}{k} \binom{n-k}{u-k} u^k v^{n-k} =
\]
\[\sum_{j \leq n} \binom{n-j}{j} \binom{x}{n-j} (u+v)^j (u+v)^{n-j} =
\]
\[\sum_{j \leq n} \binom{x}{j} \binom{x-j}{n-2j} (u+v)^j (u+v)^{n-2j} =
\]
\[
2^{-n} \sum_{2k \leq n} (-1)^k \binom{2x}{k} \binom{2x-2k}{n-2k} (u-x)^{2k} (u+v)^{n-2k}.
\]

\[(3.95) \quad \sum_{k=0}^{n} \binom{2x}{k} \binom{y}{n-k} 2^k \quad \text{See (3.89),}
\]

\[(3.96) \quad \sum_{k=0}^{n} \binom{x}{k} \binom{n-1-y}{n-k} t^k =
\]
\[\sum_{j=0}^{n} (-1)^{n-j} \binom{n-j}{j} \binom{y-x}{n-j} (t-1)^j.
\]

\[(3.97) \quad \sum_{k=0}^{n} \binom{x}{k} \binom{n-1-y}{n-k} 2^k =
\]
\[\sum_{2k \leq n} (-1)^{n+k} \binom{x}{k} \binom{y-2x}{n-2k} =
\]
\[\sum_{2k \leq n} E(k)^k \binom{y-x}{k} \binom{2x-y}{n-2k}.
\]

\(\text{See also } (3.84)\).
\[
(3.99) \quad \sum_{j=0}^{v} (-1)^j \binom{u}{j} (v-j)^{u-j} (1-z)^{v-u-j} = \\
\frac{1}{\xi} \frac{d^v}{dz^v} z^u (1-z)^v.
\]

\[
(3.100) \quad \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} (-\lambda)^{n-k} = \\
\sum_{k=0}^{n} (-x-1)^{k} \binom{n-k}{k} \lambda^k (\lambda+1)^{n-k}.
\]

\[
(3.101) \quad \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k+1} (-\lambda)^{n-k} = \\
\sum_{k=0}^{n+1} (-x)^{k} \binom{n+1-k}{k} \lambda^k (\lambda+1)^{n+1-k}.
\]

\[
(3.102) \quad \sum_{k=0}^{n} \binom{n+\frac{1}{2}}{k} \binom{n+\frac{1}{2}}{n-k} \frac{1}{5^k} = \frac{(2n)}{(\frac{n}{2})} F_{\frac{n}{2}}.
\]

\[
(3.103) \quad \sum_{k=0}^{n} \binom{n+\frac{1}{2}}{k} \binom{n+\frac{1}{2}}{n-k} \frac{1}{5^k} = \frac{1}{2} \frac{(2n+2) F_{\frac{n+1}{2}}}{(n+1)}.
\]
\[ (3.104) \sum_{k=0}^{n} \binom{n-k/2}{k} \binom{n+k/2}{n-k} (1+4x)^k = \binom{2n}{n} \Phi_{2n} (x). \]

\[ (3.105) \sum_{k=0}^{n} \binom{n+k/2}{k} \binom{n+k/2}{n-k} (1+4x)^k = \frac{1}{2} \binom{2n+2}{n+1} \Phi_{2n+1} (x). \]

\[ (\Phi_m (x): \text{Fibonacci-like polynomial, see \textit{Chapter F}}) \]

\[ (3.106) \sum_{k=0}^{n} \binom{n-k/2}{k} \binom{n-k/2}{n-k} 5^k = \frac{1}{2} \binom{2n}{n} L_{2n}, \]

\[ (3.107) \sum_{k=0}^{n} \binom{n+k/2}{k} \binom{n-k/2}{n-k} 5^k = \binom{2n}{n} L_{2n+1}, \]

\[ (L_m: \text{Lucas number, see \textit{Chapter F}}) \]

\[ (3.108) \sum_{k=0}^{n} \binom{n-k/2}{k} \binom{n-k/2}{n-k} (1+4x)^k = \frac{1}{2} \binom{2n}{n} \Lambda_{2n} (x). \]

\[ (3.109) \sum_{k=0}^{n} \binom{n+k/2}{k} \binom{n-k/2}{n-k} (1+4x)^k = \binom{2n}{n} \Lambda_{2n+1} (x). \]

\[ (\Lambda_m (x): \text{Lucas-like polynomial, see \textit{Chapter F}}) \]

\[ (3.110) \sum_{k=0}^{n} \binom{n+k}{k} \binom{n+k}{n-k} \left( \frac{1}{2} x + \frac{1}{2} \right)^k \left( \frac{1}{2} x - \frac{1}{2} \right)^{n-k} = P_n (a, b) (x): \text{Jacobi polynomial, See Rainville} \]

\[ (n! \Gamma \left( 1, \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2} \right) P_n (a, b) (x) \]
\[
(3.111) \sum_{k=0}^{n} \binom{n}{k} \binom{a}{k}(b)^{n-k} = (a)_n \left( \frac{a+b-r}{n} \right) = \\
\frac{(a)_n (n)_r}{(a+b)_n} \left( \frac{a+b}{n} \right), \quad a, b \in \mathbb{C}, \quad r = 0, \ldots, n,
\]
\[a+b \notin 0, \ldots, n-1.\]
\[
(3.112) \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} \left( k - \frac{n}{a+b} \right)^2 = \\
\frac{na^2b(a+b-n)}{(a+b)^2(a+b-1)} \left( a+b \right)_n, \quad a, b \in \mathbb{C}, \quad a+b \neq 0,
\]
\[a+b \neq 1.\]
\[
(3.113) \sum_{2k \leq n} \frac{(a)_k (b)}{k(n-k)} \left( k - \frac{1}{2} n \right)^2 = \\
\frac{1}{8} \frac{n(2a-n)}{2a-1} \binom{2a}{n}, \quad a \in \mathbb{C}, \quad a \neq \frac{1}{2}.
\]
\[
(3.114) \sum_{k=0}^{n} \binom{a}{k} \binom{n-a}{n-k} k^m = a^m, \quad m = 0, \ldots, n.
\]

From \( G (3.5) \), with \( q(x) = x^m \).
\[
(3.115) \sum_{k=0}^{j} \frac{na-k(a+b)}{k} \binom{a}{k} \binom{b}{n-k} = \\
(j+1)(n-j) \binom{a}{j+1} \binom{b}{n-j}, \quad j = 0, \ldots, n.
\]
\[
(3.116) \quad \sum_{k=0}^{n} k - \frac{na}{a+b} \binom{a}{k} \binom{b}{n-k} = \frac{2}{a+b} \binom{j+1}{n-j} \binom{j+1}{n-j}, \quad j = \left\lfloor \frac{na}{a+b} \right\rfloor, \\
a \in \mathbb{N}_0, \quad b \in \mathbb{N}_0, \quad 1 \leq n \leq a+b.
\]

\[
(3.117) \quad \sum_{k=0}^{n} \frac{1}{k+1} \binom{a}{k} \binom{b}{n-k} z^k = \frac{1}{a+1} \sum_{h=1}^{n+1} \binom{a+1}{h} \binom{b}{n+1-h} z^{h-1}, \quad a \neq -1.
\]

From (3.10) and \(k+1 = h\).

\[
(3.118) = C (3.99) \quad \sum_{k=0}^{n} \binom{x}{k} \frac{y+k\nu}{y+n\nu} \binom{y+n\nu}{n-k} = (x+y)(x+y+n\nu)^{-1} \binom{x+y+n\nu}{n}.
\]

\[
(3.119) \quad \sum_{k=0}^{n} \left( \frac{x-1}{k} \right) \left( \frac{-x}{n-k} \right) \frac{m}{m+k} = \binom{m+n}{n} \binom{-x-m}{n}, \quad m \geq 1.
\]

\[
(3.120) \quad \sum_{k=0}^{m} \binom{k}{m-k} \binom{y}{m+k}, \quad \nu \geq 0.
\]

\[
(3.121) \quad \sum_{k=0}^{m} \left( \frac{k+1}{2} \right) \binom{y}{m-k} \binom{y}{m+k+1}, \quad \nu \geq 0.
\]
\[(3.130) \sum_{k=0}^{n} \binom{n}{k} \binom{k}{\frac{1}{\varepsilon}} = \binom{n}{\frac{1}{\varepsilon}} 2^{n-\frac{1}{\varepsilon}}.\]

\[(3.131) \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{\frac{1}{\varepsilon}} = \sum_{h=0}^{n} \binom{n}{h} \binom{h}{\frac{1}{\varepsilon}}, \quad (k = n-h).\]

\[(3.132) = \Phi(j4) \sum_{2k \leq n} \binom{n}{2k} \binom{k}{\frac{1}{\varepsilon}} = \frac{n}{n-\varepsilon} \binom{n-\varepsilon}{\frac{1}{\varepsilon}} 2^{n-2\varepsilon-1}, \quad 2\varepsilon < n. \quad \text{For } 2\varepsilon > n \text{ the l.h.s. is zero.}\]

\[(3.133) = \Phi(j4) \sum_{2k \leq n} \binom{n+1}{2k+1} \binom{k+1}{\frac{1}{\varepsilon}} = \binom{n-\varepsilon}{\frac{1}{\varepsilon}} 2^{n-2\varepsilon}, \quad \text{cf. Jolley (1961)}\]

\[(3.134) \sum_{k=0}^{n} \binom{n}{k} \binom{k}{\frac{1}{\varepsilon}} = 2^{n-2\varepsilon} \frac{n}{n-\varepsilon} \binom{n-\varepsilon}{\frac{1}{\varepsilon}}, \quad n \neq \varepsilon, \quad (-1)^{\frac{1}{\varepsilon}} = 2^{-\varepsilon}, \quad n \varepsilon \geq 1 \implies -1, \quad n = \varepsilon = 0.\]

\[(3.135) \sum_{k=0}^{\infty} \binom{y}{k} \binom{k}{\frac{1}{\varepsilon}} = 2^{y-2\varepsilon} \frac{y}{y-\varepsilon} \frac{y-\varepsilon}{\varepsilon}, \quad \text{Re} \ y > \varepsilon, \quad y \notin \mathbb{N}.\]

\[(3.136) \sum_{2k \leq n} \binom{n+1}{2k+1} \binom{x+k}{n} = \binom{2x}{n}.\]

\[(3.137) \sum_{2k \leq n+1} \binom{n+1}{2k} \binom{x+k}{n} = \binom{2x+1}{n}.\]
\[(3.138a) \sum_{2k \leq n} \binom{n}{2k} \left( \frac{k}{\kappa} \right) \left( \frac{k-\frac{1}{2}}{\kappa} \right) = \left( \frac{n-1}{\kappa} \right)^2 n^{-2\kappa-1}, \ \kappa < n, \]

\[(3.138b) \sum_{2k \leq n} \binom{n}{2k} \left( \frac{k}{\kappa} \right) \left( \frac{k-\frac{1}{2}}{\kappa} \right) = \left( \frac{n-1}{\kappa} \right)^2 2^{-n-2\kappa}, \ \kappa \geq n. \]

\[(3.138c) \sum_{2k+1 \leq n} \binom{n}{2k+1} \left( \frac{k+\frac{1}{2}}{\kappa} \right) = -\frac{n}{n-\kappa} \left( \frac{n-\kappa}{\kappa} \right)^2 2^{-n-2\kappa}, \ \kappa < n, \]

\[(3.138d) \sum_{2k+1 \leq n} \binom{n}{2k+1} \left( \frac{k+\frac{1}{2}}{\kappa} \right) = (-1)^{\kappa-1} 2^{-\kappa}, \ \kappa = n \geq 1, \]

\[(3.139 \epsilon) \sum_{2k+1 \leq n} \binom{n}{2k+1} \left( \frac{k+\frac{1}{2}}{\kappa} \right) = (-1)^{\kappa+1} \frac{n}{2\kappa-n} \left( \frac{2\kappa-n}{\kappa} \right)^2 2^{-n-2\kappa}, \]

\[(3.139 \delta) \sum_{2k+1 \leq n} \binom{n}{2k+1} \left( \frac{k+\frac{1}{2}}{\kappa} \right) = 0, \ h = 0, \]

\[(3.140) \sum_{2k \leq n} \binom{n}{2k} \left( \frac{a+k}{\kappa} \right) = (-1)^n \sum_{j=0}^{n+2a=2\kappa} \binom{\kappa+j}{j} \left( \frac{a+1}{n+2a=2\kappa-j} \right) (-2)^j, \]

\[a = 0, \ldots, \kappa, \ n \geq 2\kappa - 2a. \]
(3.141) \[ \sum_{2k \leq n} \binom{n+1}{2k+1} \binom{a+k}{\varepsilon} = \]

\[ (-1)^n \sum_{j=0}^{n+2a-2\varepsilon} \binom{\varepsilon+j}{j} \binom{2a}{n+2a-2\varepsilon-j} (-2)^j , \]

\[ \alpha = 0, \ldots, \varepsilon, \quad n \geq 2\varepsilon - 2\alpha . \]

(3.142) \[ \sum_{j=0}^{n} \binom{n}{j} \binom{j+1}{\varepsilon} = \frac{n+\varepsilon+1}{n+1} \binom{n+1}{\varepsilon} 2^{n-\varepsilon} . \]

(3.143) \[ \sum_{k=0}^{n} \binom{n}{k} \binom{x+\frac{1}{2}k}{m} \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k} \binom{x-\frac{1}{2}k}{m} . \]

(3.144) \[ \sum_{i=0}^{n} \binom{n}{i} (x+i) . \text{See (3.93), (3.154)} \]
\[(3.146) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left( \frac{a-x-k}{m} \right) = \Delta^n f(x),\]
\[ (3.151) \sum_{k=0}^{n-r} (-1)^k \binom{x}{k} \binom{n-k}{r} = \binom{n-x}{n-r}, \quad r \leq n. \]

\[ (3.152) \sum_{2k \leq n} (-1)^k \binom{n}{2k} \binom{a+k}{r} = \]
\[ n+r-a \geq n+2a-2r \left( r+1 \right) / \left( 2a+r+1 \right) \]

\[ a=0, \ldots, r, \quad n \geq 2r-2a. \]
\[(3.155) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2k}{m} = \Delta^n h(0),\]

\[h(x) = \binom{x}{m}, \quad \Delta^n h(0) = 0, \quad n > m,\]

\[\Delta^n h(0) = \left(\frac{n}{m-n}\right)^2 \frac{2^{2n-m}}{n}, \quad n \leq m.\]

\[(3.156) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{\nu-2k}{m} =\]

\[\sum_{i=0}^{n} \binom{n}{i} \binom{\nu-n-i}{m-n} =\]

\[\sum_{h=0}^{m-n} \binom{\nu-2n}{h} \binom{n}{m-n-h} 2^{2n-m+h} =\]

\[(-1)^{m+n} \sum_{h=0}^{m-n} \binom{m-\nu-1}{h} \binom{n}{m-n-h} 2^{2n-m+h}, \quad n \leq m,\]

(And zero for \(n > m\)). See (3.172).

\[(3.157) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2k+1}{m} = \frac{m+1}{n+1} \binom{n+1}{m-n} 2^{2n-m}, \quad n \leq m,\]

(And zero for \(n > m\)).

\[(3.158) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{3n-2k}{2n} = \sum_{i=0}^{n} \binom{n}{i} \binom{2n-i}{n} =\]

\[\sum_{h=0}^{n} \binom{n}{h} 2^h = \sum_{h=0}^{n} (-1)^{n-h} \binom{n}{h} \binom{n+h}{h} 2^h =\]

\[\Delta^n \left(\frac{n+x}{x}\right) \bigg|_{x=0}.\]
\[ (3.162) \sum_{m_i \leq N-s} (-1)^i \left( \begin{array}{c} s \\ i \end{array} \right) \left( \begin{array}{c} N-mi-1 \\ s-1 \end{array} \right) ; \]

Number of ordered partitions of \( N \) into \( s \) parts, \( \forall i \leq m \). \cite{David and Barton (1962), pp. 227, 228}.

\[ (3.163) \sum_{k \leq j} (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} n-1+j-k \zeta \\ n-1 \end{array} \right) ; \]

\[ \sum_{k \leq j} (-1)^k \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} n+j-k \zeta \\ n \end{array} \right), \ \zeta \geq 2, \ n \geq 1. \]

\[ (3.164) \sum_{h=0}^{n-m} (-1)^{n-m-h} \left( \begin{array}{c} n-m \\ h \end{array} \right) \left( \begin{array}{c} h \zeta \\ M \end{array} \right). \]
(3.169) \[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2x+k}{m} = \Delta^n \binom{2x}{m} = \sum_{k=0}^{m-n} \binom{n}{k} \binom{2x}{m-n-k} 2^{n-k}, \quad n \leq m.
\]
(3.172) \[ \sum_{2k \leq n} (-1)^k \binom{m}{k} \binom{m+n-2k-1}{m-1} = \binom{m}{n}, \quad m \geq 1. \]

(3.174) = \sigma(93) \leq \sum_{j=0}^{\infty} (-1)^j \binom{n}{m-j} =
\begin{pmatrix} n \vspace{-1cm} \\ m \end{pmatrix} \binom{n-x}{m}, \quad m \leq n, \quad \text{Cf. Nieuw Arch. (3) 13, 1965, p. 12d, (i).}

(3.175) \[ \sum_{j=0}^{n} (-1)^j \binom{2n}{n-j} \binom{a j^2 + b}{p} = \frac{1}{2} \binom{2n}{p}, \]

\[ p \quad \text{and} \quad j. \]
(3.177) \( \sum_{j=0}^{n} \binom{j}{j} (\cdots \binom{1}{n}) (z-1)^j \), see (3.90).

(3.179) \( \sum_{j=0}^{n} \binom{n}{j} (2n-j) (-x)^j = (-1)^{n/2} \frac{1+(-1)^{n}}{2} \binom{n}{n/2} \).

\[ 2^{-n} \sum_{2j \leq n} (-1)^j \binom{n}{j} \binom{2n-2j}{n} (y-1)^{x-j} (y+1)^{n-x} \]

(3.183) \( \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{x+k}{m} z^k \); see (3.92).
(3.184) \[ \sum_{h=0}^{n} (-1)^{h-r} \binom{n+1}{h+1} x^h. \]

See (1.47) with \( a = 0 \).

(3.185) \[ \sum_{h=0}^{n} (-1)^{h-r} \binom{n}{h-1} x^h, \quad r \geq 1. \]

See (1.47) with \( a = 1 \).

(3.186) \[ \sum (-1)^k \binom{n}{k} \frac{2n-2k}{n} x^{n-2k} = 2^n \mathcal{P}(x). \]

(3.189) \[ \sum_{2i \leq n+2i} (n+2i) \binom{n+i}{i} \left( \frac{2n+2i-2i}{n} \right)^i (4y-1)^i = \]

\[ 2^{n+2i} \sum_{2i \leq n} \binom{n+2i}{i} \left( \frac{n-i}{i} \right)^{r+i} \]

\[ 2^{n+2i} \sum_{2i \leq n} \binom{2i+2i}{i} \left( \frac{n+i}{2i+2i} \right)^{r+i} \]

(3.190) \[ \sum_{k=0}^{\infty} (-1)^k \binom{u}{k} \left( \frac{1}{e} \right)^{u+k} z^{u+k-\epsilon} = \]

\[ \frac{1}{e^z} \frac{dz}{dz} (1-z)^\eta, \quad 0 < z < 1. \]
\[
\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (2n-k) = (-1)^{n} \binom{2n-2}{n-1}.
\]
\[ (3.197) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{a+x+k}{m} (b_x + b_k + c) = \]
\[ \Delta^n \left( \binom{a+x}{m} (b_x + c) \right) = 0, \quad n \geq m+1; \quad = n b, \quad n = m+1; \]
\[ = (b_x + c) \binom{a+x}{m-n} + n b \binom{a+x+1}{m-n+1}, \quad n \leq m. \]

\[ (3.198) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{a-x-k}{m} (b_x + b_k + c) = \]
\[ \Delta^n \left( \binom{a-x}{m} (b_x + c) \right) = 0, \quad n \geq m+1; \quad = n b (-1)^{n-1}, \quad n = m+1; \]
\[ = (-1)^n (b_x + c) \binom{a-x-n}{m-n} + (-1)^{n-1} n b \binom{a-x-n}{m-n+1}, \quad n \leq m.\]
(3.202) \[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{b - ak}{m} \right) (x+k)^{-1} = \sum_{m=1}^{n} \frac{1}{(x+n)^{-1}(ax+b^{-1})}, \quad m \leq n, \quad -x \notin \mathbb{N} \]

(3.206) \[ \sum_{k=0}^{c} (-1)^{c-k} \binom{c}{k} \frac{x+k}{x+k+ma} \binom{x+k+ma}{m} = \]

\[ \Delta_{\varepsilon}^{c} \frac{x}{x+ma} \left( \frac{x+ma}{m} \right) = \frac{x+\varepsilon a}{x+ma} \left( \frac{x+ma}{m-\varepsilon} \right), \quad \varepsilon \leq m, \]

and zero for \( \varepsilon > m \).
\[ (3.207) \quad \sum_{h=0}^{\epsilon} (-1)^h \binom{\epsilon}{h} \frac{y-h}{y-h+ma} \left( \frac{y-h+ma}{m} \right) = \frac{1}{123} \]

\[ \frac{y-\epsilon+ma}{y-\epsilon+ma} \binom{y-\epsilon+ma}{m}, \quad \epsilon \leq m, \]

and zero for \( \epsilon > m \).

\[ (3.208) \quad \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{x+2k}{x+2k+ma} \left( \frac{x+2k+ma}{m} \right) = \sum_{j=0}^{n} \binom{n}{j} \frac{x+j+ma}{x+j+ma} \left( \frac{x+j+ma}{m-n} \right), \quad n \leq m, \]

(3.209) \quad \sum_{j=m-n}^{m} (-1)^{m-j} \binom{m-j}{j} \binom{2m-j}{m} \binom{x}{j-n} 2^{j-2m}, 

see (3.170)

(3.210) \quad \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{x+k}{x+k+ma} \left( \frac{x+k+ma}{m} \right) \nu^k = \sum_{k=0}^{m} \binom{n}{k} \frac{x+ka}{x+ma} \left( \frac{x+ma}{m-k} \right) \nu^k (\nu-1)^{n-k} = \sum_{h=0}^{m} \binom{n}{h} \frac{x+n-h+ha}{x+n-h+ma} \left( \frac{x+n-h+ma}{m-h} \right) (\nu-1)^{n-h}.

(3.211) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{1}{k+1} = \delta_{nk}.
\[(3.212) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{a + ak + \delta}{m} \frac{1}{y + x + k} \]

\[= (-1)^{n} \left( \frac{b - a f}{m} \right) \left( \frac{x + y + n}{n} \right)^{-1}, \quad m \leq n. \]

And see (3.164), p. 1192, and (3.208)

\[(3.213) \sum_{k=0}^{n} (x) \binom{2n-k}{n-k} \frac{k}{n-2k} (-a)^{k} \]

\[= (3.1173).\]
\[
(3.218) \sum_{k=\ell}^{n-s} \binom{k}{\ell} \binom{n-k}{s} = \sum_{k=0}^{n} \binom{k}{\ell} \binom{n-k}{s} = (n+1)^{\ell} \text{ first equality for } n \geq \ell + s.
\]

\[= 0, \quad n \text{ odd.} \]

\[= n \ell \cdots \ell \quad n \text{ even.} \]

\[
(3.221) \sum_{k=\ell}^{n} (-1)^{k} \binom{a+k+p}{\ell+b} \binom{b+n-k+q}{s+q} =
\]

\[= 0, \quad \text{elsewhere. For:} \]

\[s+q = \ell+p, \quad \ell \in \{0, \ldots, a\}, \quad s \in \{0, \ldots, b\}, \]

\[p \in \mathbb{N}_{0}, \quad q \in \mathbb{N}_{0}. \]

\[
(3.222) \sum_{j=m}^{n} (j-1) \binom{a+b-j}{a-m}, \text{ see } (3.44), (3.45).
\]
\[
\begin{align*}
(3.22)& \quad \sum_{k=0}^{n} k \binom{x+k}{m} \binom{x-k}{m} = \\
& \quad \left( x+1 \right) \left( x \right) \ldots \left( x+n+1 \right) \left( x-n \right) \\
(3.222) & \quad \sum_{k=0}^{\infty} j \left( j \right) = \\
& \quad k=0 \left( k \right)^{s-1} \sum_{k=0}^{t+s} \binom{s}{k} \binom{s}{s-k} z^{t+s-k} \\
(3.237) & \quad \sum_{k=0}^{m} \binom{a+2k}{a-1} \binom{a+2m-2k}{a-1} = \\
& \quad \left( \frac{2a+2m+1}{2a-1} \right) - \left( \frac{a+m}{a-1} \right), \quad a \in \mathbb{N}
\end{align*}
\]
\( T \leq \frac{n \left( 2x \right) \left( y + 2x - k \right)}{4} \)
(3.340) \[ \sum_{j=0}^{n} \binom{2x}{n-2j} \binom{x-\frac{1}{2}n+j}{j} = \frac{2x}{2x+n} \binom{x+\frac{1}{2}n}{n} 2^n. \]

(3.341) \[ \sum_{k=0}^{m} \binom{2x}{2k} \binom{x-k}{m-k} = \frac{x}{x+m} \binom{x+m}{2m} y^m. \]

From (3.340) with \( n = 2m+1 \), put \( m - j = k \).

(3.343) \[ \sum_{k=0}^{m} \binom{2y+1}{2k} \binom{y+\frac{1}{2}-k}{m-k} = \frac{2y+1}{2y+2m+1} \binom{y+m+\frac{1}{2}}{2m} y^m. \]

From (3.341) with \( x = y+\frac{1}{2} \).

(3.344) \[ \sum_{k=0}^{m} \binom{2y+1}{2k+1} \binom{y-k}{m-k} = \frac{2y+1}{2m+1} \binom{y+m}{2m} y^m. \]

From (3.342) with \( x = y+\frac{1}{2} \) and \( D(1y) \) in e.h.s.
(3.348) \sum_{k=0}^{m} \binom{2x+1}{2k+1} \binom{x-k-\frac{1}{2}}{m-k} = \left(\frac{x+m+\frac{1}{2}}{2m+1}\right) x^{2m+1}.

From (3.346) with \( n = 2m+1 \). Put \( m-j = k \).

(3.349) \sum_{k=0}^{m} \binom{2y}{2k} \binom{y-k-\frac{1}{2}}{m-k} = \left(\frac{y-\frac{1}{2}+m}{2m}\right) y^{m}.

With \( x = y-\frac{1}{2} \) in (3.347).

(3.350) \sum_{k=0}^{m} \binom{2y}{2k+1} \binom{y-k-\frac{1}{2}}{m-k} = \left(\frac{y+m}{2m+1}\right) x^{2m+1}.

From (3.348) with \( x = y-\frac{1}{2} \).

(3.351) \sum_{k=0}^{m} \binom{2y}{2k+1} \binom{y-k-\frac{1}{2}}{m-k} = \frac{2y}{2m+1} \left(\frac{y+m-\frac{1}{2}}{2m}\right) y^{m}.

(3.352) \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} = 4^{-n} \sum_{h=0}^{n} \binom{2h}{h} \binom{2n-2h}{n-h} 5^{h}.

(3.353) \sum_{2k \leq n} \binom{n}{k} \binom{2k}{k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{2k}{k} \binom{2n-2k}{n-k} 3^{k}.

See (3.370)

(3.354) \sum_{2j \leq n-1} \binom{n-1-2j}{j} \binom{x-\frac{j}{2}n+j}{j} = \frac{n}{x+\frac{1}{2}n} \binom{x+\frac{1}{2}n}{n} x^{n-1},

\( n \geq 1 \).
\[(3.356) \sum_{h=0}^{m-\varepsilon} \binom{2m+1}{2h+1}\binom{\varepsilon+h}{h} = (2m-\varepsilon)_{2m-2\varepsilon}\]

\[(3.358) \sum_{1}^{m-\varepsilon} \binom{2m}{2h} = m \binom{2m-\varepsilon}{2m-2\varepsilon}\]

\[(3.360) \sum_{k=0}^{x+k-1} \binom{x+k-1}{k} \binom{x}{n-2k} = \binom{x+n-1}{n}\]

\[(3.361) \sum_{x \leq n} \binom{2x+1}{2j} \binom{x-2j}{j} = 4 \binom{n-2}{2n}\]
\begin{align*}
(3.365) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (y_{n-k})^2 &= (y_n - x) \\
(3.366) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^2 &= (n^2) \\
(3.367) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+n)^2 &= (-1)^n \binom{n}{n} \\
(3.368) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (N+m-k)(s+1-1) &= \frac{N-1}{m-k}s \\
(3.369) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (N+m-k)(N-1) &= \frac{N-1}{m-k}s 
\end{align*}
\[ y^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{2k}{n} \right)^k = \sum_{k=0}^{n} (-1)^k \left( \frac{2n-2k}{n} \right)^k \left( \frac{2k}{n} \right)^k \frac{1}{n} \] 

See (3.353)
\[(3.377) \sum_{k=0}^{n} (-1)^{k} \binom{x}{n-k} \binom{k}{y+k} = \binom{x-y-1}{n} \]

\[(3.378) \sum_{k=0}^{n} (-1)^{k} \binom{x}{n-k} \binom{y+k}{k+1} = \binom{x}{n+1} - \binom{x-y}{n+1} \]

\[(3.379) \sum_{h=0}^{n} (-1)^{h} \binom{x}{n-h} \binom{n-h}{h} \quad \text{See (3.61)} \]

\[(3.380) \sum_{j=0}^{r} (-1)^{j} \binom{y}{j} \binom{x-j}{k-j} = \delta_{k} \]

\[(3.381) \sum_{0 \leq j \leq n} \binom{2x}{n-j} \binom{x-\frac{n}{2} n+j}{j} \quad \text{See (3.340)} \]

\[(3.382) \sum_{2k \leq m} \frac{2m}{(m-2k)(\frac{1}{2} m+k)} = \frac{1}{3} \binom{3/2 m}{m} 2^{m+1}, \quad m \geq 1. \]

\[(3.383) \sum_{2k \leq m} \frac{2m+1}{(m-2k)(\frac{1}{2} m+k)} = \binom{3/2 m}{m} 2^{m}, \quad m \geq 1. \]

\[(3.384) \sum_{2k \leq m-1} \frac{2m}{(m-2k-1)(\frac{1}{2} m+k)} = \binom{3/2 m-1}{m-1} 2^{m-1}, \quad m \geq 1. \]

\[(3.385) \sum_{k=0}^{n} (-1)^{n-k} \binom{x}{n-k} \binom{y+2k}{2k} \quad \text{special cases}. \]
(3.386) \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n-k}{k} \left(\frac{y+2k+1}{2k+1}\right). 

Special cases.
\[ (3.391) \sum_{j=0}^{n} \binom{a+j}{j}(a+b-j)(z-1)^j. \text{ See (3.89).} \]

\[ (3.392) \sum_{k=0}^{n-k} \binom{k}{k}(n-k)! (y-k)! = \binom{n-k}{n-k} y \ binom{y}{n} \text{, } n \text{ odd.} \]

\[ (3.396) \sum_{k=0}^{n} \binom{n}{k} \binom{y+k}{n-k} y^k = \binom{y+1}{n} + \binom{y}{n-1}, \text{ } n \geq 1. \]

\[ (3.397) \sum_{j=0}^{n} \binom{j}{j}(n-j)^2 \binom{2j-n}{n} = \binom{2y}{n}. \]

\[ (3.398) \sum_{k=0}^{n} \binom{x}{k}(n-k)! (-2x+n+k-1)! \binom{n}{n-k} x^k = \]
\[ \sum_{2k \leq n} \binom{x}{n-k} \binom{n-k}{k} (\lambda - 2)^{n-2k} = \sum_{k=0}^{n} \binom{x}{k} \left( -2 + \left\lfloor \frac{n}{k} \right\rfloor \right)^{n-k} = (2x)^n. \]

\[ (3.402) \sum_{k=0}^{n} \binom{x}{k} \left( -2x + n + k - 1 \right)^{n-k} y^k = (2x)^n. \]

\[ (3.7.97) \sum_{k=0}^{n} \binom{x}{k} \binom{n-k}{k} n^k = (x - 1)^m, \quad n = 3m, \quad = 0, \quad n = 3m + 1 \text{ or } n = 3m + 2. \]
\[ (3.405) \sum_{i=0}^{n} \binom{n}{i} (b+i) y^i (x-y)^{n-i} \quad \text{See (3.91).} \]

\[ (3.406) \sum_{h=0}^{n} \binom{n}{h} (n+h) (z-1)^{n-h}. \quad \text{See (3.178).} \]

As Legendre polynomial: See S (5).

\[ (3.407) \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n+k) x^k = \frac{1}{n} D^n x^n (1-x)^n \quad \text{Equiv. with (3.192), (3.542)} \]

\[ (3.408) \sum_{2i \leq n} \binom{n+i}{i} \binom{n-i}{i} y^i \quad \text{See (3.189).} \]

\[ (3.409) \sum_{k=n}^{\infty} \binom{x}{k} \binom{x-k}{k-n} y^{-k} \quad \text{See (3.34).} \]

\[ (3.410) \sum_{h=0}^{n+m} \binom{n}{h} (x+n-h) (z-1)^{n-h} \quad \text{See (3.92).} \]

\[ (3.411) \sum_{2k \leq n} \binom{n-k}{k} \binom{n-k}{k} z^{n-2k} = \binom{2y}{n} \quad \text{See (3.397).} \]

\[ (3.412) \sum_{2j \leq n} \binom{n-j}{j} \binom{n-j}{j} (uv)^j (u+v)^{n-2j}. \quad \text{See (3.94).} \]

\[ (3.413) \sum_{h=r}^{n} (-1)^h r (h-a) (n-a+1) x^h \quad \text{See (1.47)} \]

\[ \sum_{k=0}^{n-r} (-1)^k \binom{n}{k} (n+k+r) 2^{n-r-k} = \varepsilon \leq n, \quad 0, \ n-r \ odd; \ (-1)^m \binom{n}{m}, \ n-r = 2m, \]
\[ (3.415) \sum_{k=0}^{n} \binom{x}{n-k} \binom{2x+k}{k} (-2)^{n-k} \text{, See C(188).} \]

\[ \binom{n}{i} = \sum_{j=0}^{n} \left( -\frac{1}{2} \right)^{j} \binom{n-j}{j} (n+\lambda^2)^j (2\lambda)^{-2j} \]

\[ (3.417) \sum_{i=0}^{n} \binom{h+2-i}{i} \binom{\lambda^2}{h+i} (\lambda-y)(-2\lambda) \text{, See C(211)} \]

\[ \sum_{k=0}^{n} (-1)^{n-k} \binom{\beta+n}{n-k} \binom{\alpha+\beta+n+k}{k} (\frac{1}{2}x+\frac{1}{2})^n = \mathcal{P}_{\beta}^{\alpha}(x). \]

\[ (3.717) \sum_{k=0}^{s} \binom{k}{k} (s-k)^{-k-1} \text{, See (3.226).} \]
many special cases from the same general generating function.

\[
(3.429) \sum_{h=0}^{m} \binom{m}{m+h+1} \binom{2h+1}{h+1} 2^{2h+1} = \binom{2x}{2m+1}.
\]

\[
(3.431) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} 2^{-k} = \binom{2m}{m} 4^{-m}, \quad n=2m, \quad n \text{ odd}.
\]

\[
(3.432) \sum_{k=\tau}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} 4^{-k} = (-1)^\tau 4^{-n} \binom{2n}{n+\tau}, \quad \tau \leq n.
\]
\[
(3.433) \sum_{k=\varepsilon}^{n} (-1)^{k} \binom{n}{k} \left(\frac{2k}{k-\varepsilon}\right)^{2-k} x^{-k} = 0, \; n-\varepsilon \text{ odd},
\]
\[
= (-1)^{\varepsilon} \binom{n}{\varepsilon} x^{-n}, \; n-\varepsilon = 2j, \; (\varepsilon \leq n).
\]

\[
\sum_{k=\varepsilon}^{n} (-1)^{k} \binom{n+u}{n-k} \left(\frac{2k}{k-\varepsilon}\right)^{t} x^{t},
\]

(3.435) \sum_{k=\varepsilon}^{n} (-1)^{k} \binom{n-k}{k} x^{k}.

See (14) - (17) in (3.434).

See (14) in (3.434).
\[(3.439) \sum_{2k \leq n} \binom{n}{n-2k} \binom{y}{k}^2 \binom{n-2k}{2} = \binom{2y}{n} \]

\[(3.441) \sum_{s \leq 2k \leq m+s} (2k-s) \binom{m+s}{k}^2 = \binom{m+s}{n} \]

\[(3.442) \sum_{2k \leq n} \binom{n}{2k} \binom{2k}{k} y^{-k} x^{n-2k} (x-1)^k = P_n(x) \]

\[(\cdot 77^1) \sum_{k=0}^{n} \binom{n}{k} x^k (x+1)^{n-k} \quad \text{cf. (3.538) - (3.540) in (3.434)} \]

\[(3.444) \sum_{2k \leq n} \binom{n}{2k} \binom{2k}{k} y^{-k} x^{n-2k} = \binom{2n}{n} \]

\[\therefore \quad n \Rightarrow n \text{ or } n-k/n \Rightarrow k \]
\[
(3.449) \sum_{k=0}^{n} \binom{2k+1}{n-k} (2k-1) x^{2k+1} = \binom{2m+1}{2n+1}.
\]

\[
(3.451) \sum_{k=n}^{\infty} \binom{x}{2k-n} (2k-n) y^{-k}. \text{ See } (3.34).
\]

\[
\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} (y)^{k}.
\]
(3.456) \[ \sum_{k=0}^{n} (-1)^k \binom{n+u}{k} \binom{2k}{n} k \times k^{-1} = \]
\[ (-1)^n \sum_{k=0}^{n} \binom{2k}{k} \left( -\frac{u}{k} \right)^k \binom{k}{n-k} \frac{1}{k+1} \] 

(3.457) \[ \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \binom{2k}{k} k \times k^{-1} = \]
\[ (-1)^n \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} k \left( 1-2x \right)^{k-1} \] 

(3.458) \[ \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \binom{2k}{k} k y^{n-k} = 2 \binom{2n-2}{n-1}, \quad n \geq 1. \]

Let \( x \to y \) in (3.457): 

(3.459) \[ C(197) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k+1} \binom{2k}{k} y^{n-k} = \]
\[ \binom{2n+1}{n} \] 

(3.460) \[ C(200) \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \binom{2k}{k} (-2)^{n-k} = \]
\[ (-1)^{n} 2^{2n+1} \left( \frac{1}{2n+1}, n = 2 \varepsilon + 1, \right) = (-1)^{n} 2^{2n+1} \left( \frac{1}{2n+1}, n = 2 \varepsilon + 1. \right) \] 

(3.461) \[ C(212) \sum_{2k \leq n} \binom{2k}{1} \frac{1}{k+1} \binom{2k}{k} 2^{-2k} = \frac{1}{n+2} \binom{2n+2}{n+1}. \]
(3.462) = C(218) + C(219)

\[ \sum_{h} (-1)^{h} \left( \frac{n}{2h} \right) \left( \frac{1}{2h+1} \right) y^{h} = \]

\[ \sum_{j=0}^{n} (j+1) \left( \frac{n}{j} \right) (-2)^{n-j}. \]

(3.463) \[ \sum_{k=0}^{n} \binom{\frac{n}{k}}{\frac{y+n}{y+k}} \left( \frac{y+k}{n-k} \right) y^{k} = \]

\[ \sum_{k=0}^{n} \binom{2x+1}{k} \left( \frac{y+2x-k}{n-k} \right). \]

(3.465) \[ \sum_{j=0}^{n} \frac{(-1)^{j}}{j+1} \binom{x}{j} \left( \frac{y-j}{n-j} \right) = \frac{1}{x+1} \left\{ \left( \binom{y+1}{n+1} + (-1)^{n} \binom{n+x-y}{n+1} \right) \right\}, \]

\[ x \neq -1. \]
\[(3.466) \sum_{h=n}^{m} \frac{n}{2h-n} (2h-n)(-1)^{h-n} n^{-xh} (-m-h), \]

\[1 \leq n \leq m. \text{ See (3.168)}, \]

\[(3.467) \sum_{k=1}^{n} (-1)^{k-1} k^{-1} \left( \frac{n}{k} \right) \left( \frac{n+k}{k} \right) = 2 \sum_{k=1}^{n} k^{-1}, n \geq 1. \]

\[(3.468) \sum_{h=0}^{m} \frac{x}{x+h+\frac{1}{2}} \left( \frac{x+h+\frac{1}{2}}{2h+1} \right) x^{2h+1} \left( \frac{1}{2m} \right)^{m-h} =\]

\[(2x+n)(2m+1)^{-1} (x+m)^{-1} \left( \frac{x+m}{2m} \right)^{m}. \]

\[(3.469) \sum_{h=0}^{m} \left( \frac{x}{m+h} \right) \frac{2m+1}{m+h+1} \left( \frac{m+h+1}{2h+1} \right)^{m} = \]

\[\sum_{h=0}^{m} \left( \frac{x}{m+h} \right) \frac{2m+1}{2h+1} \left( \frac{m+h}{2h} \right)^{m} = \left( \frac{2x+1}{2m} \right)^{m}. \]

\[(3.470) \sum_{h=0}^{m+1} \left( \frac{x}{m+h} \right) \frac{m+1}{m+h+1} \left( \frac{m+h+1}{2h} \right)^{m} = \left( \frac{2x+1}{2m+1} \right)^{m+1}. \]

\[(3.471) \sum_{h=0}^{m-n} (-1)^{h} \left( \frac{m-h}{m+n} \right) \left( \frac{x}{m-n} \right)^{m-h} = \]

\[\Delta^{n} \left( \frac{x}{m} \right), \quad n \leq m, \quad m \geq 1. \text{ Put } m-j = h \text{ in (3.170)}. \]

\[\text{Cf. (3.168)}, \]

\[(3.472) \sum_{k=r}^{n} (-1)^{k} \left( \begin{array}{c} a-r \end{array} \right) \left( \begin{array}{c} a+b-k \end{array} \right) \frac{1}{k}. \quad \text{pp. M21-22}. \]

\[(3.473) \sum_{k=0}^{n} \left( \begin{array}{c} \frac{2n-k}{n-k} \end{array} \right) \frac{k}{2n-k} (-a)^{k}. \text{ Special cases.} \]
\[(3.475) \sum_{2k \leq n} (-1)^k \binom{n}{k} \binom{n-k}{k} k! \left(\frac{1}{2x}\right)^{n-2k} = \]

\[\sum_{2k \leq n} (-1)^k \binom{n}{2k} \binom{2n}{k} k! \left(\frac{1}{2x}\right)^{n-2k} = H_n(x), \]

Hermite polynomial, see \ref{S10}-\ref{S12}.

\[(3.476) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} x^k / k! = Z_n(x). \]


\[(3.477) \sum_{k=1}^{n} (-1)^k \binom{n}{k} \binom{n+k-1}{k} \sum_{j=1}^{k} j^{-i} = \frac{1}{n} (-1)^n, \]

\[n \geq 1.\]

\[(3.478) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k-1}{k} \sum_{j=1}^{k+1} j^{-i} = \frac{1}{n} (-1)^n, \]

\[n \geq 2.\]

\[(3.479) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k-1}{k} \sum_{j=1}^{n+k-1} j^{-i} = \frac{1}{n} (-1)^n, \]

\[n \geq 2.\]
\[(3.485) \sum_{k=0}^{n-m} \binom{n-k}{m} \binom{x+k-1}{k} = \binom{x+n}{n-m}, \quad m \leq n.\]

\[(3.486) \sum_{k=\epsilon}^{n} \binom{k}{\epsilon} \binom{y-k}{n-k} = \binom{y+1}{n-\epsilon}, \quad \epsilon \leq n.\]

\[(3.487) \sum_{k=\epsilon}^{n} \binom{k}{\epsilon} \binom{x+k}{k} = \sum_{k=\epsilon}^{n} \binom{x+k}{\epsilon} \binom{x+n+1}{n-k}.\]

\[(3.488) S_{mn} = \sum_{k=0}^{n} \binom{k+u}{k} \binom{k+u+m+1}{m},\]

\[S_{mn} = S_{nm} = \binom{u+m}{m} \sum_{j=0}^{m+n} \binom{m}{j} \binom{u+n+1}{u+j+1}.\]

\[(3.489) \sum_{k=0}^{n} \binom{x-k}{n-k} \binom{y+k+1}{m} = \sum_{k=0}^{m} (y-k) \binom{x+1}{n-k} = \sum_{h=0}^{m+n} (y+1) \binom{x+1}{n-h}.\]

\[(3.490) \sum_{h=0}^{\infty} \binom{2h+m}{h} \binom{2h+2m}{m} x^h = \left(\frac{2m}{h+m}\right) \binom{h+m}{m} x^h = \left(3.546\right)\]

\[\frac{1}{m!} D^m (1-yx)^{-\frac{1}{2}} = \left(\frac{2m}{m}\right) (1-yx)^{-m-\frac{1}{2}}, \quad |x| < 1/y.\]
(3.491) \( \sum_{k=0}^{n} \binom{s+k}{s} (a-s-1+b-k) = \sum_{h=0}^{s} (\alpha+h) (b-s-1+a-h) = (a+b), \)

\( s+1 \leq a \in \mathbb{N}, \quad \varepsilon + 1 \leq b \in \mathbb{N} \)

(3.492) \( \sum_{j=0}^{n} \binom{n+\varepsilon-j}{j} \binom{n+\varepsilon-2j}{\varepsilon-j} x^j = F^x_n(x), \)


(3.493) \( \sum_{j=0}^{n} \binom{2j}{j} \binom{j}{\varepsilon-j} y^j = \left( \frac{2\varepsilon}{\varepsilon} \right)^{n-\varepsilon} \sum_{k=0}^{n-\varepsilon} \binom{n-\varepsilon-k}{k} (-4y)^k, \quad n \geq \varepsilon. \)

(3.494) \( \sum_{k=m}^{N} \binom{k}{m} \binom{N+t+\lambda k}{N-k} u^k, \quad m \leq N. \)

Some identities and special cases, e.g.

\( \sum_{k=m}^{N} \binom{k}{m} \binom{N+k}{2k} (-4)^k = (-1)^m \binom{N+m}{2m+1} \frac{4^m}{2m+1}, \)

and related relations, e.g. with \((N+k)^{-1}\)

and with 2k replaced by \(2k+1\).

(3.495) \( \sum_{2j \leq N-m} (-1)^j \binom{N-2j}{m} \binom{N-j}{j} 2^{N-2j} = \binom{N+m-1}{2m+1} 2^m, \)

see (12) in (3.494).
\[(3.77) \sum_{k=\ell}^{n-1} (k) \frac{n!}{(n-k)!} (\ell) \frac{n!}{(n-k)!} (\ell) = 2^{n-\ell} \binom{n}{\ell}, \ \ell \leq n, \ \text{See (1) in proof of (3.132), (3.133)}.\]

\[(3.497) \sum_{k=\ell}^{n-1} (k) \frac{n!}{(n-k)!} (\ell) \frac{n!}{(n-k)!} (\ell) = 2^{n-\ell} \binom{n}{\ell}, \ \ell \leq n-1, \ n \geq 1.\]

\[(3.498) \sum_{k=\ell}^{n} (k) \frac{n!}{(n-k)!} (\ell) \frac{n!}{(n-k)!} (\ell) \frac{2n}{2n-k} (2n-k).\]
(3.507) \[ \sum_{k=0}^{n} (-1)^k \binom{z+k}{k} \binom{z+n-k}{n-k} = 0, \quad n \text{ odd,} \]

\[ = \binom{z+m}{m}, \quad n = 2m. \]

(3.508) \[ \sum_{k=0}^{n} \binom{x-k}{k} \binom{y+n+k}{n-k} = \]

\[ \sum_{k=0}^{n} (-1)^k \binom{2k-1}{k} \binom{x+y-n-k}{n-k} = \]

\[ \sum_{k=0}^{n} \frac{x+y+2k}{x+y+2k-k} \binom{x+y+2k-k}{k} (-1)^n-k. \]

(3.509) \[ \sum_{k=0}^{n} \binom{x+2k}{k} \binom{y+2n-2k}{n-k} = \]

\[ = \sum_{k=0}^{n} \binom{2k}{k} \binom{x+y+2n-2k}{n-k} = \]

\[ \sum_{k=0}^{n} \frac{x+y}{x+y+2k} \binom{x+y+2k}{k} 4^{-n-k}. \]

(3.510) \[ \sum_{k=0}^{m} \binom{x-k}{k} \binom{m-k}{k} = \]

\[ \sum_{k=0}^{m} \binom{x+1}{k} (-2)^{m-k} = \sum_{j=0}^{m} (-x)^j \binom{m-j}{m-j}. \]

(3.511) \[ \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} = 4^n. \]
\[(3.5.12) \sum_{k=0}^{n} (-1)^k \binom{2k}{k} \binom{2n-2k}{n-k} = 0, \quad n \text{ odd}, \]
\[= \left(\frac{2m}{m}\right)^m, \quad n = 2m. \]

\[(3.5.13) \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k-1}{n-k} = \frac{1}{2^n} + \frac{1}{2} \binom{2n}{n} + \frac{1}{4} \delta_{0n}, \]
\[(3.5.14) \sum_{k=0}^{n} (-1)^k \binom{2k-1}{k} \binom{2n-2k-1}{n-k} = 0, \quad n \text{ odd}, \]
\[= \left(\frac{2m}{m}\right)^{m-1} + \frac{1}{2} \left(\frac{4m}{2m}\right) + \frac{1}{4} \delta_{0m}, \quad n = 2m. \]

\[(3.5.15) 2 \sum_{2h \leq n} \binom{y}{2h} \binom{2n-y}{n-2h} = \frac{1}{2^n}, \quad n \text{ odd}, \]
\[= \frac{1}{2^{2m}} + \left(\frac{2m}{m}\right)^m, \quad n = 2m. \]

\[(3.5.16) 2 \sum_{2h+1 \leq n} \binom{y+h+2}{2h+1} \binom{2h-y-2}{n-2h-1} = \frac{1}{2^n}, \quad n \text{ odd}, \]
\[= \frac{1}{2^{2m}} + \left(\frac{2m}{m}\right)^m, \quad n = 2m. \]

\[(3.5.17) \sum_{2i+1 \leq M} \binom{M-i-1}{i} \binom{y+i}{2i+1} + \]
\[\sum_{2j \leq M} \binom{M-j}{j} \binom{y+j}{2j} = \binom{M+y}{M}. \]
(3.518) \[ \sum_{i=0}^{N-1} \left( -\frac{x-\ell-i}{2i+1} \right) \left( \frac{N+i}{2i+1} \right) \]

\[ + \sum_{j=0}^{N} \left( \frac{x-j}{j} \right) \left( \frac{N+j}{2j} \right) = \left( \frac{N+x}{N} \right), \quad N \geq 1, \]

for \( x \geq 0 \) with the first sum zero.

(3.519) \[ \sum_{\frac{1}{2} M \leq i \leq N-1} \left( -\frac{M-i-1}{2i+1} \right) \left( \frac{N+i}{2i+1} \right) + \]

\[ \sum_{\frac{1}{2} (M+1) \leq j \leq N} \left( -\frac{M-j}{j} \right) \left( \frac{N+j}{2j} \right) = 0, \quad M \leq 2N-2. \]

(3.520) \[ \sum_{2i+1 \leq M} \left( -\frac{M-i}{i+1} \right) \left( \frac{N+i}{2i+1} \right) + \]

\[ \sum_{2j \leq M} \left( -\frac{M-j}{j} \right) \left( \frac{N+j-1}{2j} \right) = \left( \frac{N+M}{M} \right), \]

(3.521) \[ \sum_{i=0}^{N-1} \left( -\frac{x-\ell-i}{2i+1} \right) \left( \frac{N+i}{2i+1} \right) + \]

\[ \sum_{j=0}^{N-1} \left( -\frac{x-j}{j} \right) \left( \frac{N+j-1}{2j} \right) = \left( \frac{N+x}{N} \right), \quad N \geq 1. \]

(3.522) \[ \sum_{\frac{1}{2} M \leq i \leq N-1} \left( -\frac{M-i}{i+1} \right) \left( \frac{N+i}{2i+1} \right) + \]

\[ \sum_{\frac{1}{2} (M+1) \leq j \leq N-1} \left( -\frac{M-j}{j} \right) \left( \frac{N+j-1}{2j} \right) = 0, \quad M \leq 2N-3. \]
(3.523) \[ \sum_{2 \leq j \leq n} \binom{n-j}{j} \frac{(x+n-j-1)}{n-j} = \]

\[ \sum_{2 \leq j \leq n} \frac{(x+j-1)}{j} \cdot \frac{(x+n-j-1)}{n-j} \text{; Fibonacci convolution, see pp. C52-55, F27-32.} \]

(3.524) \[ \sum_{j=0}^{n} \binom{\lfloor j/2 \rfloor}{\lfloor j/2 \rfloor} \frac{x-n+j+\lfloor j/2 \rfloor}{j} = \]

\[ \sum_{k=0}^{n} \binom{n-k}{k} (x-k). \]

(3.525) \[ \sum_{j=\varepsilon}^{n} \binom{\lfloor j/2 \rfloor}{\lfloor j/2 \rfloor} \frac{-(n+j+\lfloor j/2 \rfloor)}{j-\varepsilon} = \binom{n-\varepsilon}{\varepsilon}, \varepsilon \leq n. \]

(3.526) \[ \sum_{k=0}^{n} (-1)^k \binom{n-k}{k} \frac{x-\frac{k}{2}}{n-k} = 1 \quad (= 3.531) \]

(3.527) \[ S_{\varepsilon} (x, y) = \sum_{j=0}^{\varepsilon} \binom{x+j}{j} \binom{y+(\varepsilon-j)a}{\varepsilon-j} = \]

\[ \binom{x+y+\varepsilon a}{\varepsilon} + a S_{\varepsilon-1} (x+y, y-u+a-1), \varepsilon \geq 1, u \in \mathbb{C}. \]

(3.528) \[ \sum_{j} \binom{n-j}{j} \frac{(m-\varepsilon+j)}{n-j} + \sum_{i} \binom{n-1-i}{i} \frac{(m-\varepsilon+i)}{(\varepsilon-1-i)} = \]

\[ \binom{n+m-\varepsilon}{\varepsilon}, \quad 1 \leq \varepsilon \leq \lfloor (n+m)/\varepsilon \rfloor, \text{ with summations} \]

\[ 0 \leq (\varepsilon - \lfloor m/\varepsilon \rfloor) \leq j \leq \varepsilon \land \lfloor n/\varepsilon \rfloor \quad \text{and} \]
\[ 0 \leq i \leq (n-1) \land \left\lfloor \frac{n-1}{2} \right\rfloor, \]

Empty sums being zero, cf. (3.621) - (3.629).

\[
\begin{align*}
(3.529) \quad & \sum_{k=0}^{n} \binom{a+k}{k+1} \binom{b+n-k}{n-k+1} = \\
& \binom{a+b+n+1}{n+2} - \binom{a+n+1}{n+2} - \binom{b+n+1}{n+2}.
\end{align*}
\]

\[
(3.530) \quad \sum_{k=0}^{n} (-1)^k \binom{a+k}{k+1} \binom{b+n-k}{n-k+1} = 0, \text{ } n \text{ odd,}
\]

\[
= 2 \binom{a+2m+1}{2m+2} - \binom{a+m}{m+1}, \quad n=2m.
\]

\[
(3.531) \quad \sum_{i=0}^{n} (-1)^i \binom{x-i}{i} \binom{x+2i}{x-i} = 1 \quad (\text{cf. (3.526)}).
\]

\[
(3.532) \quad \sum_{i=0}^{[n/2]} (-1)^i \binom{n-i}{i} \binom{n-2i}{n-i} = 1, \quad \delta \leq n.
\]
\[(3.538) \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} u^k v^{n-k} = \]

\[\sum_{2j \leq n} \binom{n-j}{j} \binom{2n-2j}{n-j} (-4uv)^j (u+v)^{n-2j} = \]

\[\sum_{2k \leq n} \binom{2k}{k} \binom{n}{2k} 2^{n-2k} (u-v)^{2k} (u+v)^{n-2k} = \]

\[(3.539) = (24) \text{ in (3.434)} \]

\[\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} (1-4x)^k = 4^n \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} (-x)^k, \]

\[(3.540) \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} x^k = (4x)^n \frac{p(\frac{1}{2}x+\frac{1}{2}x^2)}{n!} \]

\[= \sum_{k=0}^{n} \binom{n}{k} (x+1)^{2k} (x-1)^{2n-2k} \quad (\text{Legendre pol., Ch. 5}) \]

\[(3.541) \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} (-1)^k 3^{n-k} = \]

\[2^n \sum_{2j \leq n} \binom{n-j}{j} \binom{2n-2j}{n-j} 3^j = \]

\[4^n \sum_{2k \leq n} \binom{2k}{k} \binom{n}{2k} \quad \text{Sec (3.192), (3.407)} \]

\[(3.542) \sum_{k=0}^{n} \binom{n+k}{2k} \binom{n}{k} x^k = (\text{Legendre pol., Ch. 5}) \]

\[= \sum_{k=0}^{n} \binom{n}{k} x^k (x+1)^{n-k} = \frac{p_n(2x+1)}{n!}, \]
\[(3.543) \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k} \binom{2k}{k} x^{n-k} = 0, \quad n \text{ odd,} \]

\[= (-1)^m \binom{2m}{m}, \quad n = 2m. \]

\[(3.544) \sum_{k=0}^{n} \binom{2k}{k} \binom{x-k}{n-k} y^{-k} = \binom{x+1/2}{n} \binom{n-2j}{2j}. \]

\[(3.545) \sum_{2j \leq n} \binom{2j}{j} \binom{x-2j}{n-2j} y^n = \binom{2x+1-n}{n}. \]

This is (14) in (3.494), p. 293.

\[(3.546) \sum_{h=0}^{\infty} \binom{m+h}{h} \left(\frac{2m+2h}{m+h}\right) x^h = \left(\frac{3.490}{m!} \sum_{h=0}^{\infty} \binom{m+h}{h} \left(\frac{2m+2h}{m+h}\right) x^h \right)

\[= \frac{1}{m!} \sum_{h=0}^{\infty} \binom{2m}{m} \left(1-x\right)^{-m} x^h. \]

\[(3.547) \sum_{k=0}^{n} \binom{m+k}{k} \binom{2m+2k}{m+k} y^{-k} = \frac{m+1}{4m+2} \binom{m+n+1}{n} \binom{2m+2n+2}{m+n+1} = \binom{2m}{m} \binom{m+n+1}{n}. \]

\[(3.548) \sum_{k=0}^{n} (-1)^k \binom{x-k}{n-k} \binom{y-k}{k} y^{-k} = \binom{x-1/2}{y} \binom{y+1}{n-k} \frac{y+1-n+k}{n-k}.

\[\sum_{k=0}^{n} y^k \binom{x-1/2}{k} \frac{y+1-n+k}{n-k} = \sum_{k=0}^{n} y^k \binom{x-1/2}{k} \binom{y-n+k}{n-k}, \text{and special cases.} \]
(3.549) \[ \sum_{k=0}^{n} (-1)^k \binom{2k}{k} \binom{2x+n-k}{n-k} = \sum_{h=0}^{n} \frac{2x+3-h}{2x+3-h} \binom{2x+3-h}{h} (-4)^{n-h}. \]

(3.550) \[ \sum_{x_k \leq n} (-1)^k \binom{x+k}{k} \binom{n-k}{k} 2^{n-x_k} = \sum_{x_j \leq n} (-1)^j \binom{x}{j} \binom{n+1}{j+1} = \sum_{k=0}^{n} \binom{2x+n+k}{k} \binom{x}{n-k} (-2)^{n-k}. \]
\[
(3.555) \sum_{k=0}^{n} k \binom{2k}{k} \binom{2n-2k}{n-k} = n \cdot 2^{2n-1} \\
\]

\[
(3.556) \sum_{h=0}^{n} (2h+1) \binom{2h}{h} \binom{2n-2h}{n-h} (1-4x)^{n-h} = \\
4^n \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \binom{2k}{k} x^k \\
\]

\[
(3.557) \sum_{k=0}^{n} \frac{1}{2k-1} \binom{2k}{k} \binom{2n-2k}{n-k} = -\delta_{on} \\
\]

\[
(3.558) \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{x+k} = \\
x^{-1} \left( x+n \right)^{-1} \left( x+n-\frac{1}{2} \right)^{-1} x^n, \quad -x \notin \mathbb{N}_0 \\
\]

\[
(3.559) \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{2n-2k-1} \binom{2n-2k}{n-k} \frac{1}{x+k} = \\
4^n x^{-1} (2x-1)^{-1} \left( x+n \right)^{-1} \left( n+x-\frac{3}{2} \right), \quad x \neq \frac{1}{2}, -x \notin \mathbb{N}_0 \\
\]

\[
(3.560) \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{2k+1} \binom{2n-2k}{n-k} \frac{1}{2n-2k-1} = \\
\frac{1}{2} \binom{2n+1}{n+1} \binom{2n}{n+1} \\
\]

\[
(3.561) \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{2k+1} = 4^n \frac{1}{n} \binom{n+1/2}{n}^{-1} = \\
2^{4n+1} \binom{n+1}{n}^{-1} \left( 2n+1 \right)^{-1} \left( 2n \right)^{-1} \\
\]
\[ (3.563) = \sum_{k=0}^{n} \frac{1}{k+1} \left( \frac{2^k}{k} \right) \left( \frac{2n-2^k}{n-k} \right) = \frac{1}{(2n+1)} \binom{2n+1}{n} \]

\[ (3.564) \sum_{k=0}^{n} \frac{1}{k+1} \left( \frac{2^k}{k} \right) \left( \frac{2n-2^k}{n-k} \right) = \frac{n-\varepsilon+1}{2n+2} \binom{2\varepsilon}{\varepsilon} \binom{2n+2-2\varepsilon}{n+1-\varepsilon}, \quad \varepsilon = 0, \ldots, n. \]

Because of identities C(112)-(116), e.g.

\[ \frac{m}{2m+1} \binom{2m+1}{m} = \binom{2m}{m+1} = \frac{m}{m+2} \binom{2m+1}{m+2}, \]

there are relations in various forms, e.g.

\[ C(117) = C(127), \text{ that look similar to } (3.560) \text{ and } (3.561). \text{ They are special cases of the identities for the Gould polynomials } x(x+n\alpha)^{-1}(x+n\alpha) \text{ with } \alpha = \varepsilon. \]

See Chapter C, p. 36-49.

Example:

\[ C(121) \sum_{k=0}^{n} \binom{2^k}{k+2} \frac{1}{n-k+1} \left( \frac{2n-2^k+2}{n-k} \right) = \binom{2n+3}{n+4}. \]
\[
(3.965) \sum_{k=0}^{n} (-1)^k \binom{n-k}{k} \binom{2n-2k}{n-k} (1+2x)^{n-2k} =
\]
\[
2^n \sum_{h=0}^{n} \binom{n}{h} \binom{n+h}{n} x^h.
\]
\[
(3.966) \sum_{k=0}^{n} \frac{(-2)^k (2k)_{2k}}{(2k-1)!} = -(-2n+1) \frac{(-2n)_n}{n!}
\]
From \(B(13), B(15)\) and \(3.18\).
\[
(3.967) \sum_{k=0}^{\infty} \frac{(-2)^k (2k)_{2k}}{(2k-1)!} \frac{1}{2k-1} = \frac{\pi}{2}.
\]
From \(B(13), B(15)\) and \(3.15\).
\[
(3.968) \sum_{k=0}^{n} \frac{2k+1}{1-2n+2k} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{m}{m+k} =
\]
\[
\frac{2m-1}{2m+2n-1} \binom{2m+2n}{m+n} \binom{2m}{m}, \quad m \geq 1.
\]
From \(3.119\) with \(x = -\frac{1}{2}\) and \(B(14), B(15)\).
\[
(3.969) \sum_{k=0}^{n} (1-2n+2k)^{-1} \binom{2n-2k}{n-k} \binom{2m+2k}{m+k} =
\]
\[
\frac{m}{m+n} \binom{2n}{n} \binom{2m}{m}, \quad n+m \geq 1.
\]
\[
(3.970) \sum_{k=0}^{n} \binom{2n-2k}{n-k} \binom{2m+2k}{m+k} (1-2m-2k)^{-1} =
\]
\[
\frac{m}{m+n} \binom{2n+1}{n} \binom{2m}{m}, \quad n+m \geq 1.
\]
\[(3.571) \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k} \binom{2k}{k} (x+k)^{-l} = \]
\[(-1)^n x^{-l} \binom{x-1}{n} \binom{n+x}{n}^{-l}, \quad -x \notin \mathbb{N}_0.\]

\[(3.572) \sum_{k=0}^{n} (-1)^k \frac{n}{n+k} \binom{n+k}{2k} \binom{2k}{k} (x+k)^{-l} = \]
\[(-1)^{n-1} x^{-1} \binom{x-1}{n-1} \binom{x+n}{n}^{-l}, \quad n \geq 1, \quad -x \notin \mathbb{N}_0.\]

\[(3.573) \sum_{k=0}^{n} (-1)^k \binom{2k+y}{k} \binom{n+k+y}{n-k} (x+k)^{-l} = \]
\[(-1)^n x^{-1} \binom{x+n}{n}^{-l} (x+y-1), \quad -x \notin \mathbb{N}_0.\]

\[(3.574) \sum_{h=0}^{m} \frac{1}{2h+1} \binom{2h}{h} (2m-2h+1) \binom{2m-2h}{m-h} = \]
\[-\frac{1}{4} + \frac{m+1}{4} \binom{2m+2}{m+1}^{-l}.\]

\[(3.575) \sum_{k=0}^{n} \frac{1}{2k+1} \binom{2k}{k} \binom{n-2k-1}{n-k} = \]
\[\frac{1}{4} \binom{2n}{n} \{ (2n+1)(2n+1) \}^{-l}, \quad n \geq 1.\]
\[(3.577) \sum_{k=0}^{m} \frac{x}{x-k} \binom{x-k}{k} \binom{k}{m-k} = \binom{x}{m}, \quad x \notin \{0, 1, \ldots, m\}.
\]
\[(3.578) \sum_{2k \leq m} \frac{m}{m-k} \binom{m-k}{k} \binom{k}{m-k} = \delta_{m}, \quad m \geq 1.
\]
\[(3.579) \sum_{k=0}^{n} (-1)^{k} \frac{x}{x-k} \binom{x-k}{k} \binom{x-2k}{n-k} = \delta_{on}.
\]
\[(3.580) \sum_{k=0}^{m} \frac{(2k+1)}{m+k+1} \frac{x}{x-k} \binom{x-1-k}{m-k} \binom{x+k}{m+k}, \quad n \geq 0.
\]
\[(3.581) S_{nm} = \sum_{k=0}^{\infty} \binom{n+ek}{m+ek+k} \binom{ek+k}{k} (-1)^{k} \frac{1}{ek+k+1} = \binom{n-1}{m-1}, \quad n \geq 1, m \geq 1, \quad S_{no} = \delta_{on}, \quad S_{om} = \delta_{om}, \quad \epsilon \in \mathbb{N}.
\]
\[(3.582) S_{nm} = \sum_{k=0}^{\infty} \binom{n+ek}{m+ek+k} \binom{ek+k}{k} (-1)^{k} \frac{1}{ek+k+1} \binom{n-1}{m-1} + \frac{\epsilon}{2\epsilon+4} \binom{n-2\epsilon}{m-1}, \quad n \geq 2\epsilon, m \geq 1,
\]
\[S_{om} = \frac{1}{2} \delta_{om}, \quad S_{o0} = \frac{\epsilon}{2\epsilon+4}, \quad S_{no} = 0, \quad n \geq 2\epsilon, \quad \epsilon \in \mathbb{N}.
\]
\[(3.583) \sum_{k=\lfloor \frac{m}{2} \rfloor}^{n-2k} \frac{(-1)^{k} n}{n-2k} \binom{n-2k}{k} \binom{n-3k}{m-2k} = \frac{n}{n-m} \binom{n-m}{m}, \quad m < n; \quad = 1, \quad m = n \geq 1.
\]
\[(3.584) \sum_{k \leq m \wedge \frac{m}{n} \leq 3} (-1)^k \frac{n}{n-2k} \binom{n-2k}{k} \binom{n-3k}{m-k} = \frac{n}{m} \binom{n}{m}, \quad 1 \leq m \leq n.\]

\[(3.585) \sum_{[n-\epsilon] \leq 3k \leq n} \frac{n}{n-2k} \binom{n-2k}{k} \left( \frac{n-2k+n(\theta-1)-1}{k+3k-n} \right) = \frac{n^{\theta+\epsilon-1}}{\epsilon} + \sum_{2k \leq n} (-1)^{n-k} \frac{n}{n-k} \binom{n-k}{k} \binom{k+n(\theta-1)+\epsilon-1}{k}.\]

\[(3.586) \sum_{m \leq k \leq n/2} k^{\epsilon} \binom{k}{n-2k} \binom{n-3k}{k-m} = \frac{1}{n-m} \binom{n-m}{m}, \quad 1 \leq m \leq n/2.\]
The identities (3.591) - (3.645) are derived in the theory of the Gould polynomials, a special case of the theory of polynomials of convolution type (Chapter C), or have a connection with Gould polynomials.

\[(3.591) = C(85) \frac{(x+y)(x+y+n\alpha)^{-1}(x+y+n\alpha)}{\sum_{k=0}^{n} \frac{x}{x+k\alpha} \binom{x+k\alpha}{k} \frac{y}{y+(n-k)\alpha} \binom{y+(n-k)\alpha}{n-k}}.\]

\[(3.592) = C(90) \frac{(x+y)(x+y+n\beta)^{-1}(x+y+n\beta)}{\sum_{k=0}^{n} \frac{x}{x+k\alpha} \binom{x+k\alpha}{k} \frac{y+k\beta-k\alpha}{y+n\beta-k\alpha} \binom{y+n\beta-k\alpha}{n-k}}.\]

\[(3.593) = C(91) \sum_{k=0}^{n} \frac{x}{x+k\alpha} \binom{x+k\alpha}{k} \frac{y-k\alpha}{y-n-k\alpha} \binom{y-n-k\alpha}{n-k}.\]


\[(3.594) = C(93) \frac{n(x+y+n\alpha)^{-1}(x+y+n\alpha)}{\sum_{k=0}^{n} \frac{k}{x+k\alpha} \binom{x+k\alpha}{k} \frac{y}{y+n\alpha-k\alpha} \binom{y+n\alpha-k\alpha}{n-k}}.\]
(3.595) = C(q^a) \frac{x+y+n b}{x+y+n b+n a} \left( \frac{x+y+n b+n a}{n} \right) = 

\sum_{k=0}^{n} x+k b \binom{x+k b+n a}{k} y \binom{y+(n-k)(b+a)}{n-k}.

(3.596) = C(q^a) \frac{x+y+2}{x+y+2+n a-n} \left( \frac{x+y+n a+1}{n} \right) = 

\sum_{k=0}^{n} x+1 \binom{x+k a}{k} y+1 \binom{y+n a-k a}{n-k}.

(3.597) = C(q^a) \frac{x+y+2}{x+y+2+n b-n} \left( \frac{x+y+n b+1}{n} \right) = 

\sum_{k=0}^{n} x+1 \binom{x+k a}{k} y+1+k(b-a) \binom{y+n b-k a}{n-k}.

(3.598) = C(q^a) \frac{n(x+y+2+n a-n)}{x+y+2+n a-n} \left( \frac{x+y+n a+1}{n} \right) = 

\sum_{k=0}^{n} k \binom{x+k a}{k} y+1 \binom{y+n a-k a}{n-k}.

(3.599) = C(q^a) \frac{x+y+z+n(b-1)}{x+y+z+n(b+a-1)} \left( \frac{x+y+1+n(b+a)}{n} \right) = 

\sum_{k=0}^{n} x+1+k(b-1) \binom{x+k(b+a)}{k} y+1 \binom{y+(n-k)(b+a)}{n-k}.

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\[(3.600) = C_1(98) \frac{1}{(x+y) ![\(n\)]} (x+y+nu)^{-1} \frac{1}{n} (x+y+nu) = \]
\[
\sum_{k=0}^{n} \binom{k}{x+k} \frac{u+k(u+v-1)}{y+k(u+v-1)+nu} \left( \frac{y+n\nu}{n-k} \right).
\]

\[(3.601) = C_1(99) \frac{1}{(x+y) ![\(n\)]} (x+y+nu)^{-1} \frac{1}{n} (x+y+nu) = \]
\[
\sum_{k=0}^{n} \left( \binom{k}{x} \frac{u+k\nu}{y+nu} \left( \frac{y+nu}{n-k} \right) \right).
\]

\[(3.602) = C_1(101) \frac{1}{(x+y) ![\(n\)]} (x+y+nu)^{-1} \frac{1}{n} (x+y+nu) = \]
\[
\sum_{k=0}^{n} \binom{k}{x+k} \frac{u+kt}{y+kt} \left( \frac{w+kt}{n-k} \right).
\]

\[(3.603) = C_1(102) \frac{1}{(x+y) ![\(n\)]} (x+y+nu)^{-1} \frac{1}{n} (x+y+nu) = \]
\[
\sum_{k=0}^{n} \binom{k}{x+ka+k\beta} \frac{u+(n-k)a}{y+(n-k)(a+\beta)} \left( \frac{y+(n-k)(a+\beta)}{n-k} \right).
\]

This depends only on \(x+y\).

\[(3.604) = C_1(127) \frac{1}{(x+y) ![\(n\)]} (x+y+nu)^{-1} \frac{1}{n} (x+y+nu) = \]
\[
\sum_{k=0}^{n} \binom{2k+x}{k} \frac{1}{n+1-k} \left( \frac{2n-2k+2}{n-k} \right) = \left( \frac{2n+x+2}{n} \right).
\]

From \((3.595)\) with \(a=0, \quad b=2, \quad y=2\).
\[
(3.605) = C(97), \quad n \geq 1, \quad ab \neq 0, \\
\sum_{k=1}^{n} \binom{kb+ka}{k} \frac{1}{k!(b+a)-na} \left( \frac{na-ka-kb}{n-k} \right) = \frac{b+a}{nab} \binom{na}{n}.
\]

\[
(3.606) = \sum_{k=1}^{n} k^{-1} \binom{kb}{k} (-kb)^{n-k} = (-1)^{n-1} \frac{b}{n}, \quad n \geq 1.
\]

\[
(3.607) = \sum_{j=0}^{m} \binom{j}{j} \frac{(-1)^j (j+b+j)!}{1+j \cdot b} \left( \frac{x+jb+m}{m-j} \right) = \binom{x+m-1}{m},
\]

\[
(3.608) = C(173) \sum_{k=1}^{n} \frac{1}{ka} \binom{ka}{k} \frac{y}{y+na-ka} \binom{y+na-ka}{n-k} = \\
\frac{d}{dy} \binom{y+na}{n} = \\
\left( \frac{y+na}{n} \right) \left\{ \frac{na}{(y+na)^2} + \frac{y}{y+na} \sum_{i=0}^{n-1} \frac{1}{y+na-i} \right\}, \quad n \geq 1.
\]

\[
(3.609) = C(174) \sum_{k=1}^{n} \frac{1}{k} \binom{ka}{k} \frac{1}{n-k} \binom{na-ka}{n-k} = \\
2an^{-1} \binom{na}{n} \sum_{i=1}^{n-1} \frac{1}{na-i}, \quad n \geq 2.
\]

\[
(3.610) = C(175) \sum_{k=1}^{n} \frac{1}{ka} \binom{ka}{k} \binom{y+na-ka}{n-k} = \\
\binom{y+na}{n} \sum_{i=0}^{n-1} \frac{1}{y+na-i}, \quad n \geq 1.
\]
\[(3.6.11) = C(238)\]

\[
\sum_{k=1}^{n} \frac{1}{ka} \left( \frac{ka}{k} \right) \frac{y + kb - ka}{y + nb - ka} \left( \frac{y + nb - ka}{n - k} \right) =
\]

\[(y + nb)^{-1} \left( \frac{y + nb}{n} \right) \left\{ 1 + y \sum_{i=1}^{n-1} \left( \frac{y + nb - i}{n} \right)^{-1} \right\}, \quad n \geq 2.
\]

\[(3.6.12) \sum_{k=0}^{n} \left( \frac{x + k z}{k} \right) \left( \frac{y - k z}{n - k} \right) =
\]

\[
\sum_{k=0}^{n} \left( \frac{x + y - k}{n - k} \right) z^k = \sum_{k=0}^{n} \left( \frac{x + y + 1}{n - k} \right) (z - 1)^k.
\]

\[(3.6.13) \sum_{k=0}^{n} \left( \frac{x + k z}{k} \right) \left( \frac{\varepsilon - x - k z}{n - k} \right) =
\]

\[
z^\varepsilon + 1 \left( \frac{z - 1}{n - \varepsilon - 1} \right), \quad \varepsilon = 0, \ldots, n - 1,
\]

\[
= (z - 1)^{-1} \left( \frac{z^{n+1} - 1}{z - 1} \right), \quad \varepsilon = n.
\]

\[(3.6.14) \sum_{k=0}^{n-1} \frac{1}{k + nb + k} \left( \frac{h - k b + k}{k} \right) \left( \frac{(n - k)(b + 1)}{n - 1 - k} \right) =
\]

\[
\left( \frac{n b + n - 1}{n - 1} \right), \quad n \geq 1.
\]
The identities (3.620)–(3.645) contain
Gould polynomials \( x(x+na)^{-1}(x+na)^{n-1} \) with
\( a = -1 \), \( a = 2 \) and \( a = \frac{1}{2} \), or related quantities:

\[
(3.620) \sum_{2j \leq n} (-1)^j \binom{n-j}{j} \binom{2n-2j}{n-j} (1+\lambda)^j \lambda^{-2j},
\]

see \( C(190) \), \( (3.528) \), \( (3.565) \).

\[
(3.621) \sum_{k=0}^{\varepsilon} \binom{x-k}{k} \binom{y+k}{y-k} + \binom{y}{\varepsilon} (= \Phi(78)),
\]

\[
\sum_{h=0}^{\varepsilon-1} \binom{x-1-h}{h} \binom{y+1-h}{y+h} = \binom{x+y}{\varepsilon}, \quad \varepsilon \geq 1.
\]

This relation, with \( x = m \) and \( y = m - \varepsilon \), may contain terms that are not present in
the next identity:

\[
(3.622) = \Phi(77), \quad \sum' \binom{n-j}{j} \binom{m-\varepsilon+j}{m-j} +
\]

\[
\sum'' \binom{n-1-j}{j} \binom{m-\varepsilon+j}{m-j} = \binom{n+m-\varepsilon}{\varepsilon}, \quad 1 \leq \varepsilon \leq \lfloor (n+m)/2 \rfloor.
\]

Here \( \sum' \) sums over \( j \) with \( 0 \leq \varepsilon - \lfloor m/2 \rfloor \leq j \leq \varepsilon \) \& \( \lfloor n/2 \rfloor \) and \( \sum'' \)
over \( j \) with \( 0 \leq \varepsilon - \lfloor (m+1)/2 \rfloor \leq j \leq (\varepsilon-1) \Delta [\lfloor n/2 \rfloor]. \)

So the extra terms in (3.621) should sum to zero.
\[(\text{3.623}) \quad \sum_{k=0}^{n} \binom{x-k}{n-k} \binom{y-n+k}{x-k} = \sum_{k=0}^{n} \frac{x+y+k}{x+y+k-n} \binom{x+y+n-k}{k} (-y)^{n-k}.
\]

See also (3.612), (3.613), (3.508).

\[(\text{3.624}) = \Phi(80) \quad \sum_{j=0}^{r} \frac{x-j}{x-j} \binom{x-j}{j} \binom{y-x+j}{x-j} + \sum_{j=0}^{r-1} \frac{x-1-j}{x-1-j} \binom{x-1-j}{j} \binom{y-x+j}{x-1-j} = \frac{x+y+y}{x+y-x}, x \geq 1.
\]

From \(C(91)\), these sums may contain terms when \(x=n, y=m\), that are not present in the next identity:

\[(\text{3.625}) = \Phi(79) \quad \sum' \frac{n}{n-j} \binom{n-j}{j} \binom{m-x+j}{x-j} + \sum'' \frac{n-1}{n-1-j} \binom{n-1-j}{j} \binom{m-x+j}{x-1-j} = \frac{n+m}{n+m-x}, 1 \leq \xi \leq \left\lfloor \frac{n+m}{2} \right\rfloor, m \geq 1, n \geq 2.
\]

Here \(\sum'\) sums over \(j\) with \(0 \leq j \leq \left\lfloor \frac{m}{x} \right\rfloor\) and \(\sum''\) sums over \(j\) with \(0 \leq j \leq \left\lfloor \frac{m+1}{x} \right\rfloor\).

\[(\text{3.626}) \quad \sum_{j=0}^{x} \frac{x-j}{x-j} \binom{x-j}{j} \binom{y-x+j}{y-x+j} = (x+y)(x+y+y-\xi)^{-1} \binom{x+y-y}{y} = \Phi(82), \text{see } C(85)).
\]
This relation is different from

\[
(3.627) = \Phi(81) \sum_{n} \frac{n}{n-j} \left( \begin{array}{c} m+n-\varepsilon \\ m-\varepsilon \end{array} \right) \left( \begin{array}{c} n-\varepsilon \\ n-j \end{array} \right) = \\
\sum_{m+n-\varepsilon} (-1)^m \frac{n-m}{n-\varepsilon} \left( \begin{array}{c} n-\varepsilon \\ n-\varepsilon \end{array} \right),
\]

\[1 \leq m \leq n, \ m \leq \varepsilon \leq \left\lfloor \frac{(n+m)/2} \right\rfloor, \text{ Here the sum is over } j \text{ with } \text{ov}(\varepsilon - [m/2]) \leq j \leq \varepsilon \wedge \left\lceil \frac{n/2} \right\rceil.\]

\[
(3.628) = \Phi(84) \geq_{j=0}^{\varepsilon} \left( \begin{array}{c} x-j \\ y-j \end{array} \right) \frac{Y_{-\varepsilon+j}}{Y_{-\varepsilon-j}} = \left( \begin{array}{c} x+y-\varepsilon \\ x \end{array} \right).
\]

From C(91) with \(a = -1\). This relation is different from

\[
(3.629) = \Phi(83) \sum_{j} \left( \begin{array}{c} n-j \\ m+\varepsilon+j \end{array} \right) = \\
\left( \begin{array}{c} n+m-\varepsilon \\ \varepsilon \end{array} \right) + (-1)^m \left( \begin{array}{c} n-\varepsilon \\ \varepsilon -m \end{array} \right), \ 1 \leq m \leq n, \ m \leq \varepsilon \leq \left\lfloor \frac{(n+m)/2} \right\rfloor.
\]

The sum is over \(j\) with \(\text{ov}(\varepsilon - [m/2]) \leq j \leq \varepsilon \wedge \left\lceil \frac{n/2} \right\rceil.

For a slightly more general version of (3.627) and of (3.629), with a different proof, see \(\Phi(335)\) and \(\Phi(333)\).
(3.632) \sum_{k=0}^{n-s} \frac{\binom{e}{2k-e} \binom{2n-k}{k} \frac{s}{2(n-k)-s} \binom{2(n-k)-s}{n-k}}{(e+s)(2n-3-s)} = \frac{\binom{2n-s}{n}}{n+1}, \quad e+s \leq n.

(3.633) \sum_{k=0}^{n} \binom{x+y+2k}{k} \binom{y+2n-2k}{n-k} = \sum_{k=0}^{n} \frac{x+y}{x+y+2k} \binom{x+y+2k}{k} y^{n-k}.

See also (3.612), (3.613), (3.623), (3.645).

(3.634a) \sum_{j=0}^{n} (-1)^{j} \binom{e-j}{j} \frac{1}{n-j+1} \binom{2n-2j}{n-j} = \binom{2n-s}{n}.

(3.634b) \sum_{j=0}^{\left[\frac{e}{2}\right]} (-1)^{j} \binom{e-j}{j} \frac{1}{n-j+1} \binom{2n-2j}{n-j} = \frac{e+1}{n+1} \binom{2n-s}{n}, \quad n \geq e \in \mathbb{N}.

(3.635) \sum_{k=0}^{n} \frac{1}{k+1} \binom{2k}{k} \binom{y-2k}{n-k} = \binom{y+1}{n}.

(3.636) \sum_{k=0}^{\infty} \frac{1}{2k+1} \binom{2k}{k} \binom{y+k+2}{x+k+1} y^{x-k}.

Because of the identities \( C(106), C(112) \), e.g.,

\[
\frac{m}{2m+1} \binom{2m+1}{m} = \binom{2m}{m+1}, \quad \frac{m}{m+2} \binom{2m+1}{m} = \binom{2m+1}{m+2},
\]

some identities for the Gould polynomials with \( a = 2 \) may be written in different forms, especially for \( x = \pm 1, y = \pm 1 \). We refer to \( C(117) = (127) \) and only give some examples:

\[
(3.638) = C(117), \quad \text{convolution of Catalan numbers:} \quad \sum_{k=0}^{n} \binom{k+1}{k}^{-1} \binom{2k}{k}^{-1} \binom{n-k+1}{n-k} = \binom{n+2}{n+1}^{-1} \binom{2n+2}{n+1}^{-1}.
\]

\[
(3.639) = C(122) \quad \sum_{h=0}^{m} \binom{2h+3}{h} (m+1-h)^{-1} \binom{2m-2h+2}{m-h} = \binom{2m+5}{m}.
\]
\[ (3.645) \sum_{k=0}^{n} \binom{x+\frac{1}{2}k}{k} \binom{y+\frac{1}{2}n-\frac{1}{2}k}{n-k} = \] 

\[ \sum_{2h \leq n} \left( \begin{array}{c} n \cr h \end{array} \right) \frac{x+y+1}{x+y+1+\frac{1}{2}n-h} \left( \frac{x+y+1+\frac{1}{2}n-h}{n-2h} \right). \]

See also (3.612), (3.613), (3.623), (3.633).

(3.646) \[ \sum_{2k \leq n} \frac{1}{2k+1} \binom{x+k}{k} \binom{x-k}{n-2k} = \frac{1}{2x+1} \binom{2x+1}{n+1}. \]

(3.647) \[ \sum_{2h \leq m} \frac{1}{2h+2} \binom{x+1+h}{2h+1} \binom{x-h}{m-2h} = \frac{1}{m+2} \left\{ \binom{2x+1}{m+1} - \binom{x}{m+1} \right\}. \]

(3.648) \[ \sum_{2j \leq n} \frac{2x}{x+j} \binom{x+j}{2j} \binom{x-j}{n-2j} = \binom{2x}{n} + \delta_{on}. \]
\[ x \not\in \{0, 1, \ldots, n-1\} \]. For \( x = -1 \) see (1.2).

\[ \sum_{k=0}^{n} \binom{1/2}{n-k} \left( -\frac{3}{2} \right)^{-k} = \frac{(-4)^{-n}}{2n+1} \binom{2n}{n} \]
\[
\sum_{k=1}^{\infty} \binom{n}{k} (k^{-1})^{-(k-1)} = \frac{1}{n-1} \cdots \cdots \cdots \cdot (-1)^{n-1} \cdot \cdots \cdot (-1)^1 \cdots \cdot (-1)^1
\]
(7.11) \sum_{k=0}^{\infty} (-1)^{k} \binom{k}{k} \binom{k+x}{k} = \\
\frac{(2m)!}{m! (2m+2x)!} \quad n = 2m ,

(4.12) \sum_{k=0}^{n} \frac{(-1)^{k} \binom{n}{k} (2n-x-k)}{2^{k+1}} = \\
\frac{x+1}{2n-x} \sum_{h=0}^{n} \binom{n}{h} (2n-x-h)^{-1} 2^{h} = \\
\frac{x+1}{x+1-n} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{x-n}{k+1} , \quad x \notin \{0, 1, \ldots, 2n\} .

\therefore \quad \sum_{k=0}^{n} \frac{(-1)^{k} \binom{n}{k} (x-n)}{k+1} \binom{x}{k} \binom{x-n}{k+1} =
\[(4.14) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{2n}{2k} \right)^{-1} = \frac{2n+1}{2n+2} \left( 1 + (-1)^n \right) \]

\[(4.15) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{2n+1}{2k+1} \right)^{-1} = \frac{n+1}{n+2}, \quad n \text{ odd}, \]

\[= 1, \quad n \text{ even} \]

\[(4.17) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{2n+2}{2k+1} \right)^{-1} = \frac{2n+3}{y(n+1)(n+3)} \left( 1 + (-1)^n \right) \]

\[(4.18) \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \left( \frac{2n}{2k+1} \right)^{-1} =
\]

\[(2n+1) \int_{-1}^{1} H \quad \text{if} \quad \text{some condition on } h(n) \quad \text{is met} \]
(4.22) $A_m = \sum_{k=0}^{m} (-1)^k \binom{2m}{2k} \binom{m}{k} = (2^{-2m})(1+(-1)^m)$, $m \neq 1$, $A_1 = 0$.

(4.23) $B_m = \sum_{k=0}^{m} (-1)^k \binom{2m+1}{2k+1} \binom{m}{k} = 1+(2m)\binom{-1}{m}$, $m \geq 1$. 

$x+1 \notin \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor \}$. 
\[ \mathcal{D}_m = \sum_{k=0}^{m} (-1)^k \binom{2m+1}{2k} \binom{m}{k} = (-1)^m B_m. \]

\[ \int_{-1/2}^{1/2} \left( 1 - \sum_{n=1}^{\infty} \frac{\left( \frac{1}{n+1} \right)^{-1}}{n} \right) \left( \frac{1}{n} \right)^{-1} \]

\[ \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \binom{n+1-k}{k} \]

\[ \sum_{k=0}^{n} \binom{n+2}{k+1} \binom{n+2-k}{k} \quad 3k \leq n \]

\[ (-1)^n \binom{n+2}{n+1} + \sum \binom{n+2}{k+1} \binom{n+2-k}{k} \]

\[ \cdots \]
\begin{align*}
&\sum_{k=0}^{n} k \binom{n}{k} / x^{k-1} \\
&(4.32) \sum_{h=0}^{n} \binom{x-n+h}{h} a^{n-h}, \\
&x \notin \{0,1,\ldots,n-1\}. \text{ And some special cases} \\
&(4.34) \sum_{h=0}^{n} \binom{n}{h} \binom{2n-x-1}{h+1} 2^h. \text{ See (4.12),} \\
&(4.35) \sum_{h=0}^{n} \binom{n}{h} \binom{2n-x-2}{h} 2^h. \text{ See (4.13), (4.33),} \\
&(4.36) \sum_{k=0}^{n} 2^k \binom{n}{k} \binom{2n}{k} = 4^n / (2n). 
\end{align*}
\[ m + n \geq 1, \quad x \notin \{0, 1, \ldots, m-1\} \text{ when } m \geq 1. \]

\[
\sum_{k=0}^{\infty} (-1)^k \binom{x}{k} \left( \frac{z+k+\varepsilon}{\varepsilon+1} \right)^{-1}
= (\varepsilon+1) \frac{\Gamma(z) \Gamma(x+\varepsilon+1)}{\Gamma(z+x+\varepsilon+1)}. \]

\[
\sum_{k=0}^{\infty} \binom{n}{k} (k)(n+1) \quad \text{if} \quad x \notin \{0, 1, \ldots, mn\}. \]
\[ -\lambda \notin \mathbb{N}_1. \]
\[(4.56) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{k}^{-1} = \frac{x}{x+n}, \quad -x \notin \mathbb{N}^+.
\]

\[
(4.58) \quad \sum_{j=0}^{n} (-1)^j \binom{y+j}{n-j} \binom{a+j}{j}^{-1} = \sum_{k=0}^{n} \binom{y-k}{n-k} \frac{a}{a+k}.
\]
\[
(\gamma_n^{(l)}) \leq \sum_{k=0}^{\infty} \left( \begin{array}{c}
-7 \\
4 + k
\end{array} \right) = \\
(1 - 2z)^{-\frac{1}{2}} \left\{ 1 - \left( \frac{z}{n+1} \right) \left( \frac{\sqrt{2}}{n+1} \right)^{-1} \right\}, \quad z \neq \frac{1}{2}.
\]

From B(13) and (4.2),

\[
\sum_{i=0}^{\infty} (-1)^i \left( \begin{array}{c}
n \\
i
\end{array} \right) \left( \begin{array}{c}n \\\n-i
\end{array} \right) \left( \begin{array}{c}n \\\n
\right)^{-1} = \\
(-1)^i i! \left( \begin{array}{c}
y+n-i-1 \\
\end{array} \right) \left( \begin{array}{c}y+n \\
n-i
\end{array} \right)^{-1}, \quad i \leq n.
\]
(4.71) \[ \sum_{k=1}^{n'} (-1)^{k-1} \frac{1}{k} \binom{n}{k} \binom{x}{k}^{-1} y^k = \]

(4.72) \[ \sum_{k=1}^{n-1} (2k+1)^{-1}, \quad n>1. \]

\[ \frac{1}{2} f(0) + \frac{1}{2} \binom{-1}{n} (-1) = f(-1). \]

(4.73) \[ (\forall x) \quad \text{when } \quad f(x) = -f(-x-1). \]

(4.74) \[ \sum_{k=0}^{\infty} (-1)^{k+1} \binom{\varepsilon}{k} \binom{n-1}{k} k^{\varepsilon} = 0, \quad \varepsilon = 5, \ldots, n-1; \]

\[ = \frac{1}{2}, \quad \varepsilon = 0, \quad n \geq 1; \quad = \frac{1}{2} (-1)^n (n!)^2, \quad \varepsilon = n \geq 1. \]
\[ \sum_{k=1}^{n} (n)(n+k) \leq n^2(n+1) - 1\ . \]
(4.82) \[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{a+k}{k}\right)^{-1} \times k! = \frac{1}{(a+n)^{-1}(n)} \] generalized Laguerre

(4.84) \[ \sum_{k=0}^{n} \binom{n}{k} \frac{1}{\zeta_{b-x-k}} \left(\frac{b-x}{k}\right)^{m} \Delta_{k}^{m}(x+k)'' = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} \left(\frac{b-x}{k+1}\right)^{-1} \Delta_{n-k}^{m}(x+k)^{m} = \] 

\[ \lambda_{m} \left(\lambda_{b-x-n}\right)^{-1} \left(\lambda_{b-x}\right)^{-1}, m \leq n, b-x \notin \{0, \ldots, n\}. \]

(4.86) \[ \text{IR(64)} \sum_{k=0}^{n} \binom{n}{k} \frac{a+kb-k}{a+nb-k} \left(\frac{a+nb}{k}\right)^{-1} b^{k} = 1 \]

(4.87) \[ \sum_{k=0}^{n} \frac{n! (a+k+k b)}{(n-k)! (n+a)(n+a+b) \ldots (n+a+kb)} = 1. \]
\[ n + a + 1 \leq \sum_{h=0}^{n+(n+2-(h+1)^2)} \cdot 2^h \cdot (y-k)^{a+k} \]
(4.97) \( \sum_{\nu=1}^{\infty} \frac{r}{(\beta+j-1)(\alpha+\beta+\gamma+j-1)} = \)

(4.98) \( \sum_{k=1}^{n} \frac{(2n-2k)}{n-k} y^{2k} \left\{ \frac{1}{2k(2k+1)(2k)} \right\}^{-1} = \)

\[ 2n (2n+1)^{-1} \binom{2n}{n}, \quad n \geq 1. \]
\[ \sum_{2k \leq n} (-1)^k \binom{x+1}{n-2k} \binom{x}{n-k}^{-1} \frac{n+1}{n+1-k}, \quad k \neq 0, \ldots, n-1 \]

\[ (5.2) \sum_{k=0}^{\lfloor n/2 \rfloor} (n-k)^{-2} = \frac{3}{2n+3} \left( \binom{2n+2}{n+1} \right)^{-1} \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{1}{k} \binom{2k}{k} = \]

\[ \sum_{2k \leq n} (-1)^k \frac{n+1}{n+1-k} \binom{n+1}{2k+1} \binom{n}{k}^{-1} \]
(1.3) Induction on \( n \) with \( D(18) \). See Feller I (1957), Ch. II. 12, (12.7); Egorychev

\[
2^{2m} = \sum_{k=0}^{m} \binom{2m}{k} + \sum_{k=m+1}^{2m} \binom{2m}{k} - \binom{2m}{m} =
\]

\[
2 \sum_{k=0}^{m} \binom{2m+1}{k}.
\]
(1.9) In the notation of Feller I (1957), Ch. IV and XIII. 3 the identity (1.9)
for \( x = \frac{1}{h} \) states that
\[ u_{0,n} = \sum f_{2k} . \]

(1.11) - (1.14) The first equalities follow from
Eq. (2.27). With these equalities we find
(1.11) - (1.14) from (1.5), (1.6), and (1.7), (1.8)
or (1.3) by putting \( m-k = h \) or \( m-1-k = h \).
Or (1.12) from (1.71) with \( k+1 = h \) and
(1.14) from (1.13) with \( k+1 = h \).

\[ \sum_{2k \leq m} \binom{2m}{2k} = \sum_{2k \leq m} \binom{2m}{2k} \]
\[ + \sum_{2k \leq m} \binom{2m}{2m-2k} = \]
\[
\sum_{2k \leq m-1} (-1)^k \binom{2m}{2k+1} + \sum_{m-1 \leq 2h \leq 2m-2} (-1)^h \binom{2m}{2h+1} = \]
\[
m-1, \quad k \in \{m, \ldots, m\} \quad \text{and} \quad m-1, \ldots, 1, 2, \ldots, m-1.
\]
\[ (12m - 19m) = 12m \]

\[ |m-j| \text { even. It is equal to } 2^{m-j}, \text { by (1.15) when } m \text { is even and by (1.16) when } m \text { is odd.} \]

\( (1.27) \) This is (13), trisection of the binomial formula D (19), for \((1+z)^n\) with \(z=1\). We have \( s^3 = \exp(2\pi i/3) \), \( s^2 = \exp(-2\pi i/3) \), \( 1+s = \exp(\pi i/3) \).

\( (1.28) \) This is (12), \( 1 \leq k \leq \ldots \).

\( (1.30), (1.31) \). For \( A_n (0) \) see Riordan (1968), Ch. 2, Exercise 18 &.
We have with (1.5) and (1.6)

\[(1) \sum_{j=0}^{2} A_n(j) = \sum_{h=0}^{n} \binom{2n}{n-h} = 2^{2n-1} + \frac{1}{2} \binom{2n}{n},\]

\[(2) \sum_{j=0}^{2} B_n(j) = \sum_{h=0}^{n} \binom{2n+1}{n-h} = 2^{2n}.\]

With (18),

\[\sum_{n=1}^{\infty} \frac{1}{2n-1} \leq \frac{1}{2n-1}.\]

\[B_n(0) = A_n(0) + A_n(1),\]

\[D(0) = D(1) \cup A(1) \cup A(0).\]
(7) and (1) we have
\[ A_{n+1}(1) = B_n(0) + B_n(1) = A_n(1) + 2^{2n-1} + \frac{1}{2} \binom{2n}{n}, \]
and this gives (1.30) for \( A_n(1) \) since \( A_0(1) = 0 \).
From (5), (7), (8) and (1)
\[ A_{n+1}(2) = B_n(1) + B_n(2) = A_n(2) + 2^{2n-1} - \frac{1}{2} \binom{2n}{n}, \]
and this gives (1.30) for \( A_n(2) \) since \( A_0(2) = 0 \).
One also could apply (1) and (1.30) for \( A_n(0), A_n(1) \).
The relations (1.31) now follow with (6), (7), (8), with (2) as a control.

(1.32), (1.33) The proof is similar to the proof of (1.30), (1.31). From (1.7) and (1.8)
(1) \[ \sum_{j=0}^{2} U_n(j) = \sum_{h=0}^{n} \binom{2n}{n-h} (-1)^h = \binom{2n-1}{n}, \]
(2) \[ \sum_{j=0}^{2} V_n(j) = \sum_{h=0}^{n} \binom{2n+1}{n-h} (-1)^h = \binom{2n}{n}. \]
With $D(18)$,

$$(3) \mathcal{U}_n(0) = \sum_{3k \leq n} \left(\frac{2^{n-1}}{n-3k}\right)(-1)^k + \sum_{3k \leq n-1} \left(\frac{2^{n-1}}{n-1-3k}\right)(-1)^k =$$

$$= \sum_{3k \leq n} \left(\frac{2(n-1)+1}{n-1-3(k-1)-2}\right)(-1)^k + \sum_{3k \leq n-1} \left(\frac{2(n-1)+1}{n-1-3k}\right)(-1)^k =$$

$$= \sum_{-2 \leq 2i \leq n-2} \left(\frac{2(n-1)+1}{n-1-3i-2}\right)(-1)^{i+1} + \mathcal{V}_{n-1}(0) =$$

$$(\frac{2^{n-1}}{n}) - \mathcal{V}_{n-1}(2) + \mathcal{V}_{n-1}(0), \quad n \geq 1.$$  

In the same way

$$(4) \mathcal{U}_n(1) = - \mathcal{V}_{n-1}(0) + \mathcal{V}_{n-1}(1), \quad n \geq 1,$$

$$(5) \mathcal{U}_n(2) = - \mathcal{V}_{n-1}(1) + \mathcal{V}_{n-1}(2), \quad n \geq 1,$$

$$(6) \mathcal{V}_n(0) = \mathcal{U}_n(0) - \mathcal{U}_n(1),$$

$$(7) \mathcal{V}_n(1) = \mathcal{U}_n(1) - \mathcal{U}_n(2),$$

$$(8) \mathcal{V}_n(2) = \mathcal{U}_n(2) - \mathcal{U}_n(0) + \left(\frac{2n}{n}\right).$$

From (3), (8), (6) and (1), noting that $\left(\frac{2n}{n}\right) = 2\left(\frac{2^{n-1}}{n}\right)$ for $n \geq 1.$
From (4), (6), (7) and (8)

\[ U_{n+1}(1) = 3 \ U_n(1) - \binom{2n-1}{n}, \quad n \geq 0, \]

\[ n \geq 1, \]

\[ U_{n+1}(2) = 3 \ U_n(2) - \binom{2n-1}{n}, \quad n \geq 1, \]

\[ \ell_n(\ell) - \sum_{k=1}^{\ell} \binom{\ell}{k} = -3^{n-1} + \sum_{k=0}^{n-1} 3^{n-1-k} \binom{2k-1}{k}, \quad n \geq 2. \]

This also holds for \( n = 1. \)

The identity (1.33) then follows from (6)-(8).
(1.42) We apply Newton’s interpolation formula \( G(44) \) for \( y=0 \) to the right-hand side.

\[
\begin{align*}
\sum_{k=r}^{n-r} \binom{n-r}{k-r} (-1)^{k-r} (1+z)^{-n} &= \\
&= \sum_{k=0}^{n-r} \binom{n-r}{k} (-1)^{k} \frac{z^{k}}{(1+z)^{n-r-k}} 
\end{align*}
\]

D (24).

(1.44) From (1.42) with \( x = n+a, z = u/v \), Cf. Graham, Knuth and Patashnik (1988), (5.19). Sequence of Bernoulli experiments with probability of success \( v \) and probability of failure \( u \). Probability of at most \( n \) failures in the first \( n+a \) experiments is equal to the probability of at most \( n \) failures before the \( a \)-th success.

(1.45) From (1.42) with \( z \) replaced by \( z^{-1} \) and \( k = n-h \) in right-hand side. Or \( z \) as (1.42).

(1.46) From (1.44) with \( a = \varepsilon, n = m + \varepsilon \), on the left \( h = m - k \), on the right D (24) and \( i = k + \varepsilon \). Experiments as in (1.44). The probability of at least \( \varepsilon \) successes in the first \( m \) experiments is equal to the probability that the
With \( \lbrack 1.4.1 \rbrack \), rearrangement of factorials and \( (1.3.17) \) the first member is equal to

\[
\sum_{h=e}^{n} j (n-h) \sum_{i=e}^{h} (-1)^{h-j} (h-a+1) =
\]

With \( \lbrack 1.4.1 \rbrack \), rearrangement of factorials and \( (1.3.17) \) the second member of \( (1.4.7) \) is equal to

\[
\sum_{j=e}^{n} \frac{(j-a)}{(j-e)} \sum_{i=0}^{j-e} \frac{(-1)^{i} x_{e+1}^i}{(j-i)} =
\]

\[
\sum_{j=e}^{n} \frac{(j-a)}{(j-e)} \sum_{h=e}^{j-e} \frac{(-1)^{h-e} x_{h-e}}{(h-e)} =
\]

\[
\sum_{h=e}^{n} \frac{(-1)^{h-e} x_{h-e}(h-a)}{(h-e)} \sum_{k=0}^{n-h+1} \binom{h-a+k}{k} =
\]
\[
\sum_{h=r}^{n} (-1)^{h-r} \binom{n-a}{h-r} \binom{n-a+1}{n-h} x^h
\]

For \(a = 1\): W. C. Guenther (1967).

(1.48), (1.49) (Oral communication by W. Schaarasma).

Consider a sequence of Bernoulli experiments with probability \(p\) of success and \(q = 1-p\) of failure. Let \(S_n\) be the number of successes and \(T_n\) be the number of failures in the first \(n\) experiments. We have

\[
1 (2n \geq n+1) = 1 (2n - 1 \geq n+1) + p \cdot 1 (2n - 1 = n)
\]

\[
= P(S_{2n-1} \geq n) - q P(S_{2n-1} = n), \quad n \geq 1.
\]

Since \(P(S_{2n-1} = n-1) = q P(S_{2n-1} = n)\),
\[
P(S_{2n-1} < n-1) - P(S_{2n-1} > n+1) =
\]
\[
\sum_{\substack{k=0 \leq m-\varepsilon \leq m\ \text{and} \ 2k \leq m-\varepsilon \leq m}} \binom{m-\varepsilon}{2k} = (-1)^{m-\varepsilon} \sum_{2k \leq m-\varepsilon} \binom{m-\varepsilon}{2k+1} = (-2)^{m-\varepsilon}.
\]

With Newton's interpolation formula (1.56), with \( y = 0 \). From G.(38), D.(41) and (1.16)

\[
\Delta^\varepsilon \sum_{2k \leq m \leq m-\varepsilon} \binom{x-2-2k}{m-2k} = \sum_{2k \leq m-\varepsilon} \binom{-2-2k}{m-\varepsilon-2k}
\]

Taking \( p = \frac{x}{x+y}, q = \frac{y}{x+y} \), one finds (1.48) for \( x > 0, y > 0 \), and then for \( x, y \in \mathbb{C} \), since both sides are polynomials.

In (1) put

\[
P(S_{2n} \geq n+1) = 1 - P(S_{2n} \leq n-1) - P(S_{2n} = n)
\]

and

\[
P(S_{2n-1} \geq n) = 1 - P(S_{2n-1} \leq n-1)
\]

to give

\[
2P(S_{2n} \leq n-1) + P(S_{2n} = n) = 2P(S_{2n-1} \leq n-1), \quad n \geq 1,
\]

and (1.49) follows similarly to (1.48).

Let \( 2n \) people vote independently with probability \( p \) for and with probability \( q \) against a proposal. When votes are evenly divided, the proposal is carried by lot.
$$\frac{n}{2h \leq n} (\sin \frac{\pi}{2} r) \leq - \frac{(\pi r)}{2} (n/2).$$

(1.60) \((2i)^n (\sin x)^n = e^{ix} \left( 1 - e^{-2ix} \right)^n =
\sum \binom{n}{k} (-1)^k e^{ix(n-2k)} + \sum \binom{n}{k} (-1)^k e^{ix(n-2k)}
\sum_{2h \leq n} \binom{n}{h} (-1)^{n-h} e^{ix(2h-n)} - \frac{1+(-1)^n}{2} (-1)^{n/2} \binom{n}{n/2}.$$

(1.61) From (1.50a)
$$2 \sum_{2k \leq n} (2k) e^{ikx} = (1+e^{ix})^n + (1-e^{ix})^n \equiv$$
\sin x (n-1) \sin x (n-2) \sin x (n-3) \cdots \sin x.
\[(1.63)\]
\[2 \sum_{k=0}^{m} \binom{2m}{m-k} (u^k + u^{-k}) =
\sum_{i=0}^{m} (2m-i) (u^i + u^{-i}) + \sum_{i=0}^{m} (2m-i) (u^{i+1} + u^{-i})
\]

\[
\sum_{k=0}^{m} (-1)^{m-k} \binom{2m+1}{m-k} (u^{k+1} + u^{-k}) =
\]

\[
\sum_{k=0}^{m} (-1)^{m-k} \binom{2m+1}{m+k+1} (u^{k+1} + u^{-k}) =
\]

\[
\sum_{h=0}^{2m+1} (-1)^h \binom{2m+1}{h} u^{h-m} + \sum_{h=0}^{2m+1} (-1)^h \binom{2m+1}{h} u^{m+1-h}.
\]
\[
\frac{n!}{(n-c)!} \sum_{h=0}^{n-c} \binom{n-c}{h} = \frac{n!}{(n-c)!} 2^{n-c}, \quad c \leq n.
\]

\[
\sum_{h=0}^{c} (-1)^h \binom{x}{k} \frac{x!}{(x-k)!(k-h)!} = \sum_{k=c}^{\infty} (-1)^k \frac{x!}{(x-k)!(k-c)!}.
\]

\[\text{We, equivalently, induction on } n.\]
(1.73), (1.74). Differentiate (1.50) with respect to $u$ and put $u = v = 1$. For (1.74) subtract (1.50) with $u = v = 1$.

(1.75), (1.76) Differentiate (1.50) with respect to $u$ and put $u = i, v = 1$. For (1.76) subtract (1.18).

(1.77) - (1.84). These relations are special cases of $\Phi (125) - (128)$, obtained by substituting the values of $\Phi_i (x)$ and $\Lambda_i (x)$ for special $x$.

From $\Phi_i (125), (126)$ we obtain (1.77) and (1.78) for $x = -1/4$ with $\Phi (128)$. From $\Phi (127), (128)$ and $\Phi (128)$ we obtain (1.5) and (1.6).
\[ i^{-j} \sum_{j=0}^{m} (m+j) = \binom{m}{j} + \binom{m}{j} \quad \text{for} \quad i^{m} \leq i^{k} \]

where the last sum is over \( k \) with.

a linear combination of the relations in \((1.31)\). See also Riordan (1968), Ch. 2, Exercise 18.6.

From \( \Phi(125), (126) \), we obtain \((1.83)\) and from \( \Phi, (126), (127) \), we derive \((1.84)\). Taking \( \Phi(98) \) in \( \Phi(127) \) leads to \((1.63)\) with \( u = -2 \), and \( \Phi(98) \) in \( \Phi(128) \) gives \((1.64)\).

with \( x = -1 \).

\((1.87)\) Differentiation \((i\text{ times})\) w.r. to \( z \) of \((1.41)\)
(1.88) Since \( ax - k = (a - k)x - k(1-x) \), the right-hand side of (1.88) equals

\[
\sum_{k=0}^{j} \frac{a!}{k!(a-k-1)!} x^{k+1}(1-x)^{-k} - \sum_{j} a! \frac{1}{x^j(1-x)^{j-k}} = (j \geq 1)
\]

of the binomial distribution \( B(n, p) \). When the integer-valued random variable \( X \) has expectation \( EX = \mu \), we have

\[
\sum_{k} kP(X=k) - \mu = 0. \quad \text{So}
\]

\[
-1 \leq \sum_{k} kP(X=k) - \mu \leq 1
\]

For the binomial distribution \( B(n, p) \) this gives, with (1.88) and \( j = [\mu] = [np] \)
\[ 2(j+1)(j+1) \binom{j}{i} (1-\mu)^j \nu \]

For the limiting Poisson distribution we have

\[ a \in j, j \to \infty \]

from (*) with the relation

\[ \binom{j}{i} \to \frac{1}{i!} \]

For the hypergeometric and negative binomial distributions see (3.11b) and (1.361).

References: Frame (1945), Johnson (1959), Ramasubban (1958), Crow (1958)

(1.93) Each of my independent balls is put with probability \( \frac{1}{n} \) into one of \( n \) cells.
(1.102) Induction on \( n \). We have \( S_k = \varepsilon^k \). With \( D(18) \) and with (1.2) and (1.3)
\[
\alpha \leq n, \quad k \leq \alpha + (n-1) \quad \varepsilon_n, \quad k \leq \alpha + (n-1) \quad \varepsilon_n
\]


(1.103) From \( D(14) \), and from (1.3) we see that the left-hand side equals

\[
\frac{1}{k} + \frac{1}{n} \sum_{k=1}^{n-1} \frac{(2n)!}{(n-k)! (n+k)!} \frac{1}{k(n+k)} =
\]

\[
\frac{1}{n} \sum_{k=0}^{n-1} \frac{(2n)!}{(n-k)! (n+k)!} \left( \frac{1}{k} - \frac{1}{n+k} \right) =
\]

\[
\frac{1}{n} \sum_{k=0}^{n-1} \frac{(2n)!}{(n-k)! (n+k)!} \frac{2(2n-1)}{n} \frac{A_n - 2 \sum_{k=1}^{n-1} (2n-1)}{(n-1-k)} =
\]

\[
\frac{1}{n} \sum_{k=0}^{n-1} \frac{(2n)!}{(n-k)! (n+k)!} \frac{2(2n-1)}{n} \frac{A_n - 2 \sum_{k=1}^{n-1} (2n-1)}{(n-1-k)} - \frac{2}{n} 2^{2n-2}
\]
\[ \frac{2(2n-1)}{n} A_{n-1} + \frac{1}{n} \binom{2n-1}{n} . \]

So \( A_n \) satisfies the recurrence

\[ n A_n = (4n-2) A_{n-1} + \binom{2n-1}{n}, \quad n \geq 2, \]

cf. Riordan (1968), Ch. 2, Problem 17. The right-hand side of (1.104) satisfies the same recurrence, and has the same value for \( n=1 \).

With D (13) and with (1.6)

\[ B_n = \frac{2n+1}{n+1} \sum_{k=1}^{n} \frac{(2n)!}{(n-k)!(n+k)!} \left( \frac{1}{k} - \frac{1}{n+k+1} \right) = \]

\[ \frac{2n+1}{n+1} A_n - \frac{1}{n+1} \sum_{k=1}^{n} \binom{2n+1}{n-k} = \]

\[ \frac{1}{2} \binom{2n+1}{n} H_n - \frac{1}{n+1} H^n + \frac{1}{n+1} \binom{2n+1}{n} . \]

With D (18), for \( n \geq 2, \)

\[ B_n = \sum_{k=1}^{n} \binom{2n}{n-k} \frac{1}{k} + \sum_{k=1}^{n-1} \binom{2n}{n-1-k} \frac{1}{k} , \]

so \( C_n = B_n - A_n \) and (1.106) follows from (1.104) and (1.106).

Similarly, with (1.6),

\[ D_n = \frac{1}{n+1} + \frac{1}{n+1} \sum_{k=1}^{n} \frac{(2n+1)!}{(n-k)!(n+k)!} \left( \frac{1}{k} + \frac{1}{n+1-k} \right) = \]

\[ \frac{2n+1}{n+1} A_n + \frac{1}{n+1} \sum_{k=1}^{n+1} \binom{2n+1}{n+1-k} = \]
$$\frac{1}{2} \left(\begin{array}{c} 2n+1 \\ n \end{array}\right) H_n + \frac{1}{n+1} y^n.$$ 

(1.108)-(1.119) Put

$$V_n(a, u) = \sum_{k=0}^{n} \left(\begin{array}{c} 2n+1 \\ n-k \end{array}\right) \frac{u^k}{k+1}.$$ 

We have

$$U_n(a, u) = \frac{1}{n+1-a} \sum_{k=0}^{n} \frac{(2n+1)!}{(n-k)! (n+k)!} \left( \frac{u^k}{k+1} - \frac{u^k}{n+k+1} \right) =$$

$$\frac{2n+1}{n+1-a} V_n(a, u) - \frac{1}{n+1-a} \sum_{k=0}^{n} \left(\begin{array}{c} 2n+1 \\ n-k \end{array}\right) u^k$$

$$V_n(a, u) = (n+a)^{-1} u^n +$$

Once known, (1) and (2) can be derived by comparing the coefficients of $u^k$ on both sides for $n=1$ and $n=-1$ (as (1.5)-(1.8)).
\( (4) \ V_n(a, l) = \frac{2^n}{n+a} U_{n-1}(a, l) + \frac{2^{n-1}}{n+a} + \frac{1}{2n+2a} \left( \binom{2n}{n} \right), n \geq 1, \)

\( (5) \ V_n(u, l) = \frac{u}{n+a} \left( U_{n-1}(u, l) + (n+a) \binom{n}{n} \right), n \geq 1. \)

From (4) and (5)
which is (1.100) since \( u(\frac{1}{2}) = 1 \), following (1.107).

In the same way from (10)

\[
(2n+1) 4^{-2n} \binom{2n}{n} U_n \left( \frac{1}{2}, -1 \right) = (2n-1) 4^{-2n-1} \binom{2n}{n-1} U_{n-1} \left( \frac{1}{2}, -1 \right),
\]

\[
U_n \left( \frac{1}{2}, -1 \right) = \left( n + \frac{1}{2} \right)^{- \frac{3}{2}} 4^{2n} \binom{2n}{n}^{-1},
\]
since \( U_0 \left( \frac{1}{2}, -1 \right) = 2 \). This is (1.110). Then (1.111) follows with (5). Cf. Prob. 4519, Monthly, 61 (1954).

\[
\frac{1}{4} n + \left( n + \frac{1}{2} \right) \binom{2n}{n}, \quad n \geq 1,
\]

(12) \( (n-\frac{1}{2}) (n+\frac{3}{2}) U_n \left( -\frac{1}{2}, -1 \right) = 2n (2n+1) U_{n-1} \left( -\frac{1}{2}, -1 \right) \)

\[
+ \binom{2n}{n}, \quad n \geq 1.
\]
\[ \sum_{k=0}^{n} \frac{1}{2k-1} \binom{n}{k} H_k + \sum_{k=0}^{n} \frac{\binom{n}{k} (\sum_{j=1}^{k} \binom{j}{k} H_j)}{2k-1} \cdot \] 

With B (15), (1.3) and B (13) the first sum on the right is equal to

\[ -\sum_{k=1}^{n} \frac{1}{k} (\frac{1}{2})^k (-1)^k = (-1)^n (\frac{1}{2})^n = -\frac{1}{2} \binom{2n}{n} , \]

This is (1.114) and (1.115) follows with (5).
For \( a = 1 \) the relations (7) and (8) become

\[(13) \quad n(n+1) U_n(1,1) = 2n(2n+1) U_{n-1}(1,1)\]

\[-\frac{1}{2} \binom{2n}{n}, \quad n \geq 1.\]

Multiplying (13) with \( (n-1)! n!/(2n+1)! \) we find

\[(2n+1) \binom{n}{n} U_n(1,1) = \binom{2n-1}{n-1} U_{n-1}(1,1) + \frac{1}{2n} - \frac{(n-1)! n! 4^n}{2 (2n+1)!}, \quad n \geq 1\]

\[(2n+1) \binom{n}{n} U_n(1,1) = 1 + \sum_{k=1}^{n} \frac{1}{2k} - \sum_{k=1}^{n} \frac{4^k}{2k(2k+1)} (2k)^{-1}, \quad n \geq 1, \quad 1, 1\]
(1.124) With D. (14) and (1.3)
\[ \sum_{k=0}^{n} \frac{(n+x-1)}{n-k} \left( \frac{-1}{x+k} \right)^k = \frac{1}{n+x} \sum_{k=0}^{n} \frac{(-1)^k (n+x)}{n-k} = \]
\[ \frac{1}{n+x} \binom{n+x-1}{n}. \]

(1.125) Writing \( S_n \) for the l.h.s. of (1.125) we have with D. (18) and (1.124)
\[ S_n = \sum_{k=0}^{n} \frac{(n-1+x)}{n-k} \left( \frac{-1}{x+k} \right)^k + \sum_{k=0}^{n-1} \frac{(n-1+x)}{n-1-k} \left( \frac{-1}{x+k} \right)^k = \]

(1.126) For small \( z \), with D. (25),
\[ \sum_{n=0}^{\infty} z^n S_n (a, x) = \sum_{k=0}^{\infty} a_k \frac{z^k}{k+1} \sum_{m=0}^{\infty} \frac{1}{m+1} \binom{n+x}{n-k} z^m = \]
\[ \sum_{k=0}^{\infty} a_k \frac{z^k}{k+1} \sum_{m=0}^{\infty} \frac{1}{m+1} \binom{k+x}{m+1} z^m = \]
\[ \sum_{k=1}^{\infty} a_k \frac{z^k}{k+1} \frac{1}{1-z} = \frac{1}{1-z} \left[ \frac{1}{1-z} \right] = \]
\[ (1-z)^{-x} \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{(a+1)^{n+1} - 1}{a} z^k, \]
and (1.126) follows with D. (25) and the conv.
\[
\sum_{k=1}^{n} (-1)^{n-k} k^{-1} \binom{n}{n-k} = \sum_{k=1}^{n} k^{-1} \frac{(-k^{-1})^{-1}}{n-k} = \\
\left(-\frac{1}{2}\right) \sum_{k=1}^{n} k^{-1} \binom{n}{k} \left(-\frac{1}{2}\right)^{-1} = \\
\text{(1.130) Trivial for } n=0. \text{ For } n \geq 1, \text{ by (14) the left-hand side is equal to} \\
1 + \sum_{k=1}^{n} \binom{k}{k} - \binom{k}{k-1} = \binom{x}{n}.
\]
\[ \sum_{k=0}^{n} (k) \sum_{k=0}^{n} (k) z^k \]

Interchanging sum and integral is justified

series in (1.131) converges absolutely for \( \Re x > -1 \) and all \( z \in \mathbb{C}, -z \notin \mathbb{N} \). The equality in (1.131) in the latter case follows


(1.132), (1.134) Differentiation (once and twice) of (1.100) w.r.t. to \( x \), using \( B(39) \), Cf. G(60) with \( f(x) = 1 \). See Prob. 4757, Monthly 65, 1299, 1311, 525.
G, (47), (51). The absolute convergence follows since the kth term in the series is \(O(k^{-c})\), with \(c = Re x + \varepsilon + 2\) as \(k \to \infty\).

\[
\sum_{k=1}^{\infty} \frac{(x)^{(-1)^k}}{z+k} = \frac{\Gamma(z) \Gamma(x+1)}{\Gamma(x+z+1)} - z^{-1} = \left\{ \begin{array}{ll} 1 & \text{if } z \in \mathbb{N} \\ \Gamma(x) & \text{if } z \in \mathbb{C} \end{array} \right.
\]

\[(1.137), (1.138)\] From \(D(14)\), and \(D(20)\),

\[
\sum_{k=0}^{\infty} \binom{x}{k} \frac{z^k}{k+1} = (x+1)^{-1} \sum_{h=0}^{\infty} \binom{x+1}{h} z^h = (x+1)^{-1} \sum_{h=1}^{\infty} \binom{x+1}{h} z^{h-1} = (1+x)^{-1} z - \left\{ (1+x)^{x+1} - 1 \right\}.
\]

The proof of \((1.138)\) is similar, with the binomial formula \(D(19)\).

\[(1.139)\] From \((1.102)\) and \((1.138)\) since

\[
k^{-1}(k+1)^{-1} = k^{-1} - (k+1)^{-1}.
\]
From (1.102): For $1 \leq \varepsilon \leq j$
\[
\sum_{k=\varepsilon}^{j} (-1)^{k-\varepsilon} \binom{k-1}{\varepsilon-1} = \sum_{k=\varepsilon}^{j} \binom{k-1}{\varepsilon-1}.
\]
Summing both sides w.r.t. to $j$ we obtain
\[
\sum_{j=\varepsilon}^{n} \sum_{k=\varepsilon}^{j} k^{-1} \binom{k-1}{\varepsilon-1} = \sum_{k=\varepsilon}^{n} k^{-1} \binom{k-1}{\varepsilon-1} (n-k+1) = (n+1) \sum_{k=\varepsilon}^{n} k^{-1} \binom{k-1}{\varepsilon-1} - \sum_{h=\varepsilon-1}^{n-1} \binom{h}{\varepsilon-1} = (n+1) \sum_{k=\varepsilon}^{n} k^{-1} \binom{k-1}{\varepsilon-1} - \binom{n}{\varepsilon}.
\]
Of Netto (1927), Ch. 14, (35) (Brun).
\[
\sum_{k=0}^{n} \frac{(-1)^k\binom{n}{k}(n+k+1)^{-1}}{2} \int_0^1 t^n (1-t)^{-1/2} \, dt,
\]

by the use of generating functions and a complicated transformation of integrals. Then, with the beta integral $D(33)$ and $P(94)$. 
(1.150) Integrate (1.45) over \([0, y]\) w.r. to \(z\).

(1.151) From (1.45): 

\[
(-1)^n \sum_{j=0}^{n} \binom{n}{j} \Delta^j \frac{1}{a + x} E^j \Delta^{-j} c^x \bigg|_{x=0} = \sum_{j=0}^{n} (-1)^{n-j} \frac{n!}{(n-j)!} \frac{(a-1)!}{(a+j)!} \frac{(c-1)^{n-j}}{c^j} = 
\]

\[
\sum_{1 \leq k \leq \frac{1}{2} n} \binom{n-1}{2k-1} \frac{z^{2k}}{2k} = \frac{1}{n} \sum_{1 \leq k \leq \frac{1}{2} n} \binom{n}{2k} z^{2k}. 
\]
\[
\sum_{2k \leq n-1} \frac{(2k+1)}{2k+1} + \int \left( (1+z) - (1-z)^{n+1} \right) \sqrt{(2n+2)}, \quad n \geq 1,
\]
and (1.155) follows since \( \psi = z \).

Or: bisection of (1.157). See (11).

\[
T_n = \sum_{1 \leq k \leq n/2} \binom{n-1}{2k} \frac{z^{2k}}{2k} + \sum_{1 \leq k \leq n/2} \binom{n-1}{2k-1} \frac{z^{2k}}{2k} = 
\]

\[
\sum_{1 \leq k \leq n} \binom{n-1}{2k} \frac{z^{2k}}{2k} + \frac{1}{2n} \left\{ (1+z)^n + (1-z)^n \right\} - 2 \left( \frac{1}{2n} \right)^n = 
\]

(1.157) The r.h.s. of (1.157) is equal to
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{c^k}{a+k} = \]
\[ V_n = \sum_{2k \leq n} \frac{(2k+1)}{(2k+1)^2} Z_{2k}^\prime + \sum_{2k \leq n} \frac{(2k)}{(2k+1)^2} Z_k = \]

\[ \bar{V} \quad \text{and} \quad \bar{V} \quad \text{for} \quad n \geq 3. \]

(1.162) Denote by $W_n$ the l.h.s. of (1.162).

From (1.18), (1.13) and (1.156), for $n \geq 3$,

\[ W_{n-1} + \frac{1}{n} \sum_{j=2}^{n-1} \frac{1}{2^j} \{ (4z)^j + (1-z)^j - 2j \} \]

The relation (1.162) now follows since it holds.
(1.163), (1.164). By putting \( a+k = x+k + a-x \) and noting \( G(\alpha^2), (2\alpha^a) \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} (a+k)^{x} (x+k)^{-1} = \sum_{i=0}^{c} \left( \binom{c}{i} (a-x)^{r-i} (-1)^n \Delta^n x^i \right),
\]
and (1.163) follows with (1.100). For (1.164)
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} (a+k)^{x} (x+k)^{-2} = \sum_{i=0}^{c} \left( \binom{c}{i} (a-x)^{r-i} (-1)^n \Delta^n x^i \right) - 2.
\]
For \( 1 \leq r \leq n+1 \) only the terms with \( i=0 \) and \( i=1 \) remain, and (1.164) follows with (1.100) and (1.133). For \( r=0 \) see (1.133).
See also G. (58), (60).

\[
(a+k)^{c} = \sum_{j=0}^{c} \binom{c}{j} k^{j} a^{c-j}.
\]
with (1.102),
\[ x \alpha^{-1} \sum_{k=1}^{n} (-1)^k \binom{n}{k} = -x \alpha^{-1} \text{ with (1.2)}, \]
\( (x - 1) \alpha^{-1} \sum_{k=1}^{n} (-1)^k \binom{n}{k} k^{-1} = \)

\[ \int_{0}^{1} t^\alpha (1-t)^\beta \, dt = \sum_{k=1}^{\infty} (-1)^k \binom{\beta}{k} t^{\alpha+k} \, dt = \]

Formally, for Re\( \alpha > -1 \) and 0 \( \leq y < 1 \)
\[ \int_{0}^{1} t^\alpha (1-t)^\beta \, dt = \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} t^{\alpha+k} \, dt = \]

Interchanging sum and integral is allowed since
\[ \int_{0}^{1} \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} t^{\alpha+k} \, dt = \]

\[ \int_{0}^{1} t^{\alpha} (1-t)^{\beta} \, dt = \int_{0}^{1} \frac{\tau^{\alpha}}{\tau^{\beta+1}} (1-\tau)^{\beta} \, d\tau = \sum_{k=0}^{\infty} \frac{(-1)^{k}}{\beta+k+1} \binom{\alpha}{k} (1-\gamma)^{\beta+k+1} \].

So the two sums in (1.167) add to
\[ \int_{0}^{1} t^{\alpha} (1-t)^{\beta} \, dt = \alpha! \beta! / (\alpha+\beta+1)! \]

and similarly for the other series in (1.167). So the series in the l.h.s. converge.
(1.168), (1.169) By B(38) the L.H.S. of (1.168) is equal to \[ D \left( \frac{x}{m+1} \right) \left. \right|_{x=0}^{x=0} \]

We obtain (1.169) from (1.168) by putting \( x = -y \) and applying \( D(24) \). For a direct proof note that the r.h.s. of (1.169) is equal to

\[
\left( \frac{u-1+m+1}{m+1} \right) \sum_{j=1}^{m+1} \frac{1}{u-1+j} = \frac{d}{du} \left( \frac{u-1+m+1}{m+1} \right)
\]

by B(39). The sequence of polynomials \( \binom{x+n-1}{n} \) is defined. We have

\[ D \left( \frac{x-1+m+1}{m+1} \right) = \sum_{k=0}^{\infty} c_k \binom{x+k-1}{k} \]
\[ c_k = (I - E^{-1})^k D \left( \frac{x-1+m+1}{m+1} \right) \bigg|_{x=0} = \]

\[ \bigg( x-1+m-k+1 \bigg) \bigg| \]
These relations contain Abel polynomials, their Sheffer sequences and quantities resembling them. The Abel poly-

$X$ has $n$ elements, i.e. $T$ is 'drawn at random' from the set of all $n^n$ mappings $X \to X$. We say that $y$ is a successor of $x$ if $y = T^k x$ for some $k \in \mathbb{N}$. So $x$ is a suc-

cessor of $x$. When $y$ is a successor of $x$, then $x$ is a predecessor of $y$. Let $S$ and $Z$ be

$$P(S=k) = \binom{n-1}{k-1} k! n^{-k} = \binom{n}{k} k! k^{-k-1},$$

$$= \sum_{k=1}^{n} P(Z=k) \cdot 1!$$

We can replace $y$ by a random variable.
A different interpretation of (1.180): In a sequence of independent experiments, each with probability \( p \) of...
A graph with vertex set $X$ and an edge from $x$ to $y$ if $Tx = y$. Then $P(S=k)$ is found by choosing $k-1$ vertices of a path from $x$ to some $y$, ordering them, choosing the closing edge from $y$, and finally choosing $v$ for all $v$ outside the path.

$$P(V=j, S=r) = \left( \begin{array}{c} m \\ j \end{array} \right) \left( \begin{array}{c} n-m-1 \\ r-j \end{array} \right) \frac{r!}{n-r}$$

$0 \leq j \leq m$, $r \geq j+1$. Necessarily $m \leq n-1$. So

$$P(V=j) = \sum_{r=j+1}^{n} \left( \begin{array}{c} m \\ j \end{array} \right) \left( \begin{array}{c} n-m-1 \\ r-j \end{array} \right) \frac{r!}{n-r} =$$

$$= \sum_{j=0}^{m} \left( \begin{array}{c} m \\ j \end{array} \right) \sum_{h=0}^{n-m-1} \left( \begin{array}{c} n-m-1 \\ h \end{array} \right) (h+j+1)! \frac{n^{-h-j-1}}{}$$

$$= \sum_{k=0}^{n-1} (k+1)! n^{-k-1} \sum_{h+j=k} \left( \begin{array}{c} m \\ j \end{array} \right) \left( \begin{array}{c} n-m-1 \\ h \end{array} \right)$$

$$= \sum_{k=0}^{n-1} \left( \begin{array}{c} n-1 \\ k \end{array} \right) (k+1)! n^{-k-1}, \text{ which is (1.181).}$$
The second member of (1.182) are $EZ$ and $ES$. For the third member note that $P(x \text{ is successor of } x) = 1$ and that for $y \neq x$

$$P(y \text{ is successor of } x \text{ in } k \text{ steps}) = \frac{(n-2)(n-1)!}{n^k}.$$ 

(1.181) For (1.181) cf. G.K.P. exercise (5.65).

(1.186) For $n > 2$ the l.h.s. equals
The relations (1.188) and (1.189) are special cases of $C$, Theorem 12. The sequences of polynomials $(x+na)^n/n!$ and 

$$
Y_n(x) = x^n/n!
$$

with $g_n(x) = x^n/n!$, or directly from $C$. Definition $g_n$. So (1.188) is $C$ (53), with 

$$
S_n(x) = n(x+na)^n/n! \quad \text{and} \quad T_n(x) = (x+na)^n/n!,
$$

where $Y_n$ is a variant of Jensen's (or Cauchy's) formula (3.612). See Fréchet (1954), Prob. 3.8. We prove (1.190) by induction on $n$. It holds 

$$(x+y)^n + an \sum_{h=0}^{n-1} \binom{n-1}{h} (x+a+ha)^{n-1} (y-a-ha)^h =$$

$$
(\ldots x^n, \ldots + \ldots y^n, \ldots)
$$

$$
n! \sum_{h=0}^{n} (x+y)^h a^{n-k}/k!.
$$

\[ \sum_{h=0}^{n} (x+n)^h / h! = \sum_{h=0}^{n-1} (x+n)^h / h! = (x+n)^n / n! \]  
(Françon (1974), Prop. 3.3).

(1.192) Generalization of (1.178) = C(172). In C(169) take \( y = z - n \alpha \):

\[ \sum_{k=1}^{n} \frac{1}{k} \frac{\mathcal{Q}(ka)}{\mathcal{Q}_k(z-ka)} \mathcal{Q}_{n-k}(z-ka) = \mathcal{Q}_n(z). \]
Then take for \( \mathcal{Q} \) the Abel polynomial (see C(75))

\[ \mathcal{Q}_n(x) = x (x+n\alpha)^{-1} / n! \]

Substituting into the above solution gives (1.192).

Trying a similar generalization of (1.176) and (1.177) by substituting the above

\[ (1.196) \text{ The first equality follows from G(64) with } f'(x) = \theta(x)(1-\theta(x))^{-1} \text{ in the last term. But } \]

and...
\[ \sum_{n=0}^{\infty} \sum_{k=1}^{N} \binom{N}{k} (-1)^{k-1} \binom{N}{k} \theta^k (1 - \theta^k)^{N-k} = \sum_{k=1}^{N} (-1)^{k-1} \binom{N}{k} \theta^k (1 - \theta^k)^{N-k}. \]


The rencontres problem. An urn contains mn balls, m of which bear the number \(j\), \(j = 1, \ldots, n\). One draws \(N \leq n\) balls at random, one by one, without replacement. When the \(i\)th drawing gives a ball with number \(i\), this is called a match, \(i = 1, \ldots, N\). For the

\[ P(X=j) = \frac{1}{(mn)!} \sum_{k=j}^{N} \binom{N}{j} \binom{N}{k} (mn-k)! m^k = \]
\[
\frac{\Gamma(j)}{(mn)!} \leq \frac{(-1)^m}{h^j} \leq \frac{1}{h^j}
\]

From (c) it follows by convergence of binomial moments that the distribution of \(X\) converges to the Poisson distribution with parameter \(\lambda\) when \(n \to \infty, N/n \to \alpha, m \text{ bounded.}\)

Again with $G(28)$ and $G(49)$

$$n! \sum_{x=0}^{\infty} \frac{x^n}{x!} \left( \frac{1}{e-1} \right)^{x} e^{aj} =$$
\[
\frac{u!}{(v+n)!} \sum_{j=0}^{n} \binom{v}{j} \binom{v+n}{n-j} e^{aj} (e^{-1})^{n-j},
\]

\[
\sum_{h=1}^{n} \sum_{j=0}^{\infty} \binom{r+1}{j} h^j = \sum_{h=1}^{n} \{ (h+1)^{r+1} - h^{r+1} \}.
\]
(1.208), (1.209). Putting \( H_m = \sum_{i=1}^{n} i^{-1} \), we have with \( \mathcal{D}(25) \):

\[
\sum_{n=1}^{\infty} \sum_{h=1}^{n} (-2)^h \binom{n}{h} H_h = \\
\sum_{h=1}^{\infty} (-2)^h H_h \sum_{m=0}^{\infty} \binom{m+h}{m} z^{m+h} = \\
(-z)^{-1} \sum_{h=1}^{\infty} \left\{ -2z \left( -\frac{1}{z} \right)^{-1} \right\}^h H_h.
\]

Since \( \sum_{h=1}^{\infty} w^h H_h = - (1-\log(1-w)) \),

\[
\sum_{n=1}^{\infty} z^n \sum_{h=1}^{n} (-2)^h \binom{n}{h} H_h = \\
- (1+z)^{-1} \log \left( \frac{1+z}{1-z} \right) = \\
- z (1+z)^{-1} \sum_{\varepsilon=0}^{\infty} (2\varepsilon+1)^{-1} z^{2\varepsilon+1} = \\
2 \sum_{n=1}^{\infty} z^n (-1)^n \sum_{2\varepsilon+1 \leq n} (2\varepsilon+1)^{-1},
\]

and (1.208) follows. Then, for \( n \geq 1 \),

\[
\sum_{h=0}^{n} (-2)^h \binom{n}{h} H_{h+1} = \\
1 + \sum_{h=1}^{n} (-2)^h \binom{n}{h} H_h + \sum_{h=1}^{n} (-2)^h \binom{n}{h} \frac{1}{h+1} = \\
1 + z (-1)^n \sum_{2\varepsilon+1 \leq n} (2\varepsilon+1)^{-1} - \frac{1}{2n+2} \sum_{j=2}^{n+1} \left( \begin{array}{c} n+1 \\ j \end{array} \right) (-2)^j = 
\]
\[
1 + 2 (-1)^n \sum_{\chi \leq n} (\chi + 1)^{-1} - (2n+2)^{-1} \left\{ (-1)^{n+1} - 1 + 2(-n+1) \right\} = \\
2 (-1)^n \sum_{\chi \leq n} (\chi + 1)^{-1} + (2n+2)^{-1} (1 + (-1)^n),
\]
\[
\sum_{j=1}^{n} (-1)^{j-1} \frac{1}{j} \binom{n-1}{j-1} = n^{-1}.
\]

\[\gamma_0 = x_0 = 0, \quad \gamma_m = \sum_{j=1}^{m} j^{-1}, \quad x_m = (-1)^{m-1} m^{-1}, \quad m \geq 1.\]

For (1.211) see Gould (1961a), Riordan (1968), Ch.1.2, Egorichev (1984), Ch.2.25.

(1.214) With (1.210) and B (30), the l.h.s. is equal to

\[
\sum_{k=1}^{n} (-1)^{k} \binom{n}{k} \sum_{i=1}^{k} \frac{1}{2i-1} = - \binom{n-\frac{1}{2}}{n-1}^{-1} =
\]

\[-(2n-1)^{-1} H^{n-1} \binom{2n-2}{n-1}^{-1} = - \frac{1}{2n} H^n \binom{2n}{n}^{-1}.
\]

(1.215) With (1.3) and (1.124) the l.h.s. is equal to
\[
\sum_{j=0}^{n} \frac{1}{x+j} \sum_{h=0}^{n-j} (-1)^{n-h} \binom{n+x}{h} = \\
5^n \frac{1}{(n+x-1)!} = \frac{1}{(x+n-1)}
\]

\[
\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \left\{ \sum_{i=1}^{k} \frac{1}{2i} + \sum_{i=0}^{k-1} \frac{1}{2i+1} \right\} = \\
\frac{1}{2n} + \frac{1}{2n} 4^n \binom{2n}{n}^{-1}
\]


From (1.210) since
\[
\sum_{j=1}^{x+k} (x+j)^{-1} = \sum_{i=0}^{x-1} \sum_{h=1}^{k} (x+hx-i)^{-1} = \\
<_{x-1} 1 <_{k} (1, -1, \ldots, 1)^{-1}
\]


(1.218), With (1.102), D(13) and then (1.211) the
\[
\begin{align*}
-t \cdot i^{-1} & \leq t \cdot (k+1) \leq t \cdot i^{-2} \\
\sum_{j=1}^{\infty} (j+1)^{j-1} |j| & = \sum_{k=1}^{\infty} \\
\text{This is Problem 10490, Monthly 106, 1999, 588-590 with four solutions, one of them generalizing } H_j \text{ to } \sum_{i=1}^{j} i^{-1} (1-x)^i. \\
(1.219) \text{ With (1.157) the l.h.s. is equal to } \\
-\sum_{j} \sum_{i} (i^{-1})(j^{-1}) \\
nH_n & \leq \sum_{i=1}^{n} j^{-1} \sum_{j=1}^{n} (h_{ij}) - \sum_{j} j^{-1}.
\end{align*}
\]
Write (1.15) as
\[ \sum_{2k \leq n} \binom{n}{2k} x_{n-2k} = y_n, \quad n \in \mathbb{N}_0, \]
with \( x_1 = 1, \ y_1 = 1, \ y_i = 2^{i-1}, \ i \geq 1 \). The companion of this relation, in the sense

the discussion of IR, (113) - (121). The equation IR (113) now has a solution, so IR, (116) - (121) hold. The relations IR (116)
\[
\sum_{2k+1 \leq n} \binom{n}{2k+1} x_k = y_n, \quad n \in \mathbb{N}_0,
\]
with \(x_i = 1, \quad y_0 = 0, \quad y_j = 2^{j-1}, \quad j \geq 1\). We now apply the discussion of \(IR, (122) -(130)\). The equation \(IR,(122)\) now has a solution. So \(IR\), \(125) -(130)\) hold. The relations \(IR, (125), (126)\) now become special cases of the binomial formula, \(IR\), (129) with \(m-k = h\) gives \(126\) and \(IR\), (130) with \(m-k = h\) gives \(127\),... 

with \(x_i = (-1)^i\), \(y_i = 2^{i/2} \cos \frac{i}{2} \pi\). We now need the discussion of \(IR\), (113) -(121). The equation of \(IR\), (113) now has a solution, so \(IR\), (116) -(120) hold. The relations \(IR\), (116), (117) give special cases of the binomial formula, \(IR\), (120) with \(m-k = h\), gives \(127\) with \(n = 2m\) and \(IR\), (121) gives \(127\).
\[ \sum_{2k \leq n} (-1)^{k+1} \binom{n}{2k+1} x_{n-2k} = y_n, \quad n \in \mathbb{N}_0, \]

\[ (i+1)^{\frac{1}{2}} \ldots \ldots \ldots \ldots \]

The discussion of \( IR, (122) \) – \( (130) \). The equation \( IR \) \( (122) \) now has a solution. So \( IR, (125) \) – \( (130) \) hold. The relations \( IR, (125), (126) \) now give special cases of the binomial formula, \( IR \) \( (129) \) gives \( (1.273) \) and \( IR \) \( (130) \) gives \( (1.272) \) with \( n = 2m+1 \). Take \( k = m-1 \) in \( IR \) \( (130) \).

\[ \sum_{k=0}^{\infty} \frac{(n+1-k)!}{k!} x_{n-k} = n! \]

and \( (1.274) \) follows from the convolution property.
(1.281) With \( D(x) \): \( (x+k) \equiv (x+k+1) - (x+k) \).

(1.283) From (1.281) with \( x=0 \), or with the same

\[ \sum_{i=1}^{m}(i-1)! = m! \]

Also: Let \( M \) be the max in a random sample of size \( s \) without replacement from \( \{1, \ldots, m\} \). Then, \( P(M=i) = (i-1)!/m \), \( 1 \leq s \leq i \leq m \).
\[
\frac{(x+n+1)!}{(z+1)! (x+n-z)!} \cdot \frac{n(z+2)-(x+z-x)}{z+2} + (x+1) = \\
n \left( \frac{x+n+1}{z+1} \right) - \left( \frac{x+n+1}{z+2} \right) + (x+1).
\]

Combinatorial interpretation: see (1.286).

\[
(x+1) \sum_{k=0}^{n} \left( \frac{x-k}{z+1} \right) - (z+1) \sum_{k=0}^{n} \left( \frac{x+1-k}{z+1} \right) = \\
(x+1) \left\{ \left( \frac{x+1}{z+1} \right) - \left( \frac{x-n}{z+1} \right) \right\} - (z+1) \left\{ \left( \frac{x+2}{z+2} \right) - \left( \frac{x+1-n}{z+2} \right) \right\} = \\
\frac{(x+1)!}{(z+1)! (x-n)!} \left( x+1 - (z+1) \frac{x+2}{z+2} \right) + \\
\frac{(x-n)!}{(z+1)! (x-n-z-1)!} \left( (z+1) \frac{x+1-n}{z+2} - x-1 \right) = \\
\frac{(x+1)!}{(z+1)! (x-n)!} \frac{x-z}{z+2} + \frac{(x-n)!}{(z+1)! (x-n-z-1)!} \frac{n-x-1-n(z+2)}{z+2}.
\]
For a combinatorial interpretation and a different proof in a special case, by putting \( k = (k) \) and applying (3.218), see \( \text{Graham, Knuth, and Patashnik (1988), Ch 5.2, Problems 2 and 4. Also (1.286)} \)

\[(1.286) \text{ From (1.284) with } x = 0. \text{ More directly, with (1.283) and then with } D(18) \text{ and } D(14), \]

\[\sum_{k} \frac{k^x}{k!} = \sum_{k} \left( \frac{1}{k!} \right) = \sum_{k} \frac{1}{k} \]

Let \( U_1, \ldots, U_n \) be the increasing order statistics of a sample of size \( n \) without replacement from \( \{1, 2, \ldots, N\} \), \( 2 \leq n \leq N \).

So

\[\sum_{j=n-1}^{N-1} \sum_{i=j+1}^{N} \frac{(j-1)}{n-2} = \sum_{j=n-1}^{N-1} \frac{(N-j)}{n-2} \binom{N-1}{n-1} = \binom{N}{n},\]
\[(\star) \sum_{k=0}^{\infty} \left( a + k \right) z^k = \sum_{k=t-a}^{\infty} \left( a + k \right) z^k = -\infty, t \in (-\infty, t-1) \]

\[\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \left( a + 2j \right) = \sum_{j=0}^{\infty} \left( a + 2j \right) \sum_{n=j}^{\infty} z^n = \]

\[\left( 1 - z^2 \right)^{-1} \sum_{j=0}^{\infty} \left( a + 2j \right) z^j = \]

\[\frac{1}{2} \left( 1 - z^2 \right)^{-1} \left\{ z^{-a} \left( 1 - z \right)^{-z-1} + (-z)^{t-a} \left( 1 + z \right)^{-t-1} \right\} = \]

\[\frac{1}{2} \left( 1 + z \right)^{-1} z^{-a} \left( 1 - z \right)^{-z-1} + \frac{1}{2} \left( 1 - z \right)^{-1} (-z)^{t-a} \left( 1 + z \right)^{-t-1} = \]

\[\frac{1}{2} \left( 1 + z \right)^{-1} \sum_{i=0}^{\infty} \binom{i}{t} z^i + \frac{1}{2} \left( 1 - z \right)^{-1} \sum_{i=0}^{\infty} \binom{i}{t} (-z)^i \]

\[\frac{1}{2} \left( 1 + z \right)^{-1} \sum_{h=t-a}^{\infty} \left( h + a + 1 \right) z^h + \frac{1}{2} \left( 1 - z \right)^{-1} \sum_{h=t-a}^{\infty} \left( h + a - 1 \right) (-z)^h = \]
Again by (8), and the convolution property (Theorem M1)

\[ \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \binom{n}{j} = \]

and (1.107) follows. The proof of (1.100) is similar. With G (9) and D (25), from (*)

\[ \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \binom{a+2j+1}{j} \sum_{j=0}^{\infty} \binom{a+j+1}{j} \sum_{j=0}^{\infty} z^{2n+1} \]

\[ \frac{1}{2} (1+z)^{-a} (1-z)^{-a} - \frac{1}{2} (1-z)^{-a} (-z)^{1-a} (1+z)^{-a-1} = \]

\[ \frac{1}{2} (1+z)^{-1} \sum_{h=0}^{\infty} \binom{h+a+1}{h} \frac{z^h}{2} - \frac{1}{2} (1-z)^{-1} \sum_{h=0}^{\infty} \binom{h+a+1}{h} (-z)^h = \]

\[ \sum_{h=0}^{\infty} \frac{z^{2h+1}}{2^{2h+1}} \sum_{h=0}^{\infty} \binom{h+a+1}{h+1} (-1)^{2n+1} \]

\[ f(1,80) = 1 \]
\[(1.289) \text{ For small } z, \text{ with } D(20), \]
\[
\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{j=0}^{n} \binom{n}{j} (y+2j) = \sum_{n=0}^{\infty} \frac{(y+2j)z^n}{n!} = \sum_{n=0}^{\infty} \frac{(1+z)^{y+2n+2}}{n!} - \frac{(1+z)^y}{n!} = \]
\[
= \frac{1}{2} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \left\{ \binom{3n+2+y-k}{i+1} - \binom{y-k-1}{i+1} \right\} \left( -\frac{1}{2} \right)^k z^{k+i} = \]
(1.292) - (1.294) \sum_{r=0}^{\infty} z^r \sum_{k=0}^{n} \binom{Y+k}{r} t^k = \sum_{k=0}^{n} t^k (1+z)^{Y+k} = (1-t-tz)^{-1} \left( (1+z)^Y - t^{n+1} (1+z)^{Y+n+1} \right) =

(1-t)^{-1} \sum_{r=0}^{\infty} z^r \sum_{h=0}^{r} \left\{ \binom{Y}{h} - t^{n+1} \binom{Y+n+1}{h} \right\} \left( \frac{t}{1-t} \right)^{r-h},

---

If (1.292) follows the solution (1.292).

Polynomial in k. Taking \( y = a \) in (1.293), with \( a \in \{0, 1, \ldots, e\} \) leads to (1.294). See also (*) in the proof of (1.287), (1.288) and Prob. Ex. 276, Monthly, 93, 1986, 62-63 for \( a = 0 \) (in the proof of Chapter 1).

(1.295), (1.296) From (1.319), or from (1.43) with
(1.302). For $m=0$ and $m=n-1$ the relation holds trivially. We now apply induction w.r. to $n$. For the step $n-1 \rightarrow n$ with $1 \leq m \leq n-2$ we have with D. (18)

$$A(n, m) = \sum_{k=m}^{n-1} \binom{k}{m} \frac{1}{n-k} =$$

$$\sum_{k=m+1}^{n-1} \binom{k-1}{m} \frac{1}{n-k} + \sum_{k=m}^{n-1} \binom{k-1}{m-1} \frac{1}{n-k} =$$

$$\sum_{l=1}^{n-2} \binom{l}{n-1} \frac{1}{l} + \sum_{l=1}^{n-2} \binom{l}{n-2} \frac{1}{l} =$$

$$\binom{n-1}{m} \sum_{k=m+1}^{n-1} k^{-1} + \binom{n-1}{m-1} \sum_{k=m}^{n-1} k^{-1} =$$
\[(\star) \quad \frac{x^{a-2m}}{x^{a-2m-1}} (m') = (m') - (m_{m-1}), \quad x \neq 0, \quad m \geq 1.\]
$(1.310)-(1.320)$ take $x = 2n + y + 1, \ z = x(1-x)$ in $(1.42)$ and apply $D$, $(24)$. For $y=0$ and $y=-1$
one obtains $(1.319)$ and $(1.320)$ by $(1.6)$. 

\[
\sum_{\nu=0}^{\infty} (\nu + a) (\nu + b + 1) / (\nu + 1)
\]

With the beta integral $D$, this is equal to 
\[
(a+1) \sum_{k=0}^{n} \binom{n}{k} x^k (k+a)! (n-k)! / (n+a+1)!
\]

\[
= x! \ q = 0 \ \binom{n}{q} x! \ (n-q)!
\]
(1.3.3) \rightarrow (1.3.25) From (7) (8) \text{ (11)} \text{ and } (1.2.10)

From (1.3.23) with \( y = 0, \ z = 1, \ x = \frac{1}{2} \) and (1.6)
one finds (1.3.24) and with \( y = -1, \ z = 1, \ x = \frac{1}{2} \)
and (1.5) the relation (1.3.25) follows.

\[
\sum_{k=0}^{r} \binom{r}{k} (1-x)^k =
\]
\[ \sum_{k=0}^{n} \binom{n+s+1}{k} (1-x)^k \times (n+s+1-k) + \sum_{h=n+1}^{n+s+1} \binom{n+s+1}{h} \times (n+s+1-h) (1-x)^h = 1. \]

A probabilistic proof of (1.327) is based on a sequence of Bernoulli experiments with probabilities \( p \in (0,1) \) and \( q=1-p \) of success and failure. Let \( W \) and \( \mathcal{V} \) be the numbers of successes and failures in the first \( n \) experiments, \( \mathcal{V}_n \) be the number of the experiment with the \( n \)th success and \( \mathcal{Z}_n \) the number of the experiment with

\[ P(W_{n+s+1} \geq n+s+1) + P(\mathcal{V}_{n+s+1} \geq n+1) = \]

\[ \sum_{i=0}^{n} \binom{i+s}{i} p^{s+1} q^i + \sum_{i=0}^{s} \binom{i+n}{i} q^{n+1} p^i. \]

The following proofs in terms of a game defined on the above experiments, of (1.327) and (1.329), though not mentioned explicitly there, and (1.330) were found by the author in Hinderer and Stieglitz (1987). Player I wins at time \( \mathcal{V}_n \) when the \( n \)th success occurs.
Consider the first entrance into \( \{X = \epsilon \cup Y = m \} \)
of a random walk on \( \mathbb{N}_0^2 \) with probabilities \( p \) and \( q \) of steps \((0,1)\) and \((1,0)\), start-

\[
1 = \sum_{j=\epsilon}^{\ell} (j-1) p^j q^{\ell-j} + \sum_{j=m}^{\ell} (j-1) q^j p^{\ell-m} = \\
\sum_{i=0}^{\ell-1} (\ell+i-1) p^i q^{\ell-i} + \sum_{i=0}^{\ell-1} (m+i-1) q^m p^i ,
\]

which is (1) for \( s = \epsilon - 1 \), \( n = m - 1 \).

Let \( g(\epsilon, m, p) \) and \( v(\epsilon, m, p) \) be the probabilities that I wins and that II wins, \( \epsilon, m \geq 1 \), as functions of \( \epsilon, m \) and the success probability \( p \). From (2) and (3)

\[
(4) \quad g(\epsilon, m, p) = p^\epsilon \sum_{i=0}^{\ell-1} (\ell+i-1) q^i , \quad \epsilon, m \geq 1 ,
\]
In particular for $m = \epsilon$

Since $q(1, 1, \beta) = \beta$, 

\[
q(\epsilon, \epsilon, \beta) = \beta + (\beta - \frac{1}{2}) \sum_{j=1}^{\epsilon-1} \binom{\epsilon - 1}{j} \beta^j q^j, \; \epsilon \geq 2,
\]

which is (1.328) by (4).
From (8) with \( m = \varepsilon - 1 \), for \( \varepsilon \geq 2 \),

\[
g(\varepsilon + 1, \varepsilon, \beta) = g(\varepsilon, \varepsilon - 1, \beta) + \left( \frac{\varepsilon - 2}{\varepsilon - 1} \right) \beta^2 q^{\varepsilon - 1}
\]

which gives (1.329) by (4).

From (2), (3), (4)

\[
EL = \sum_{j=\varepsilon}^{\varepsilon + m - 1} j (j - 1) \beta^2 q^{j - 1} + \sum_{j=m}^{\varepsilon + m - 1} j (j - 1) q^m \beta^{-m}
\]

(10)

\[
EL = \varepsilon \sum_{j=0}^{\varepsilon - 1} \left( \begin{array}{c} 2 \varepsilon - 1 \varepsilon \end{array} \right) (\beta q)^j \frac{1}{j + 1},
\]

Apply \( D'(24) \) again to \( D(24) \). Or write

\[
\frac{1}{x(x+k)} = \frac{1}{(x-1+k)} - \frac{1}{(x+1+k)}
\]

\[
\sum_{k=0}^{n} \frac{x-k}{x+k} \left( \frac{x+k}{k} \right) = \sum_{k=0}^{n} \frac{x}{x+k} \left( \frac{x+k}{k} \right)
\]

and \((1.342)\) follows with \( \Phi \). \( \Phi(95^a)\).

\( \varphi_{2m} \left( x(x+k) + (x+k) \varphi_{2m-2} (x+k) \right) \)

from \((1.341)\) and \( \Phi \). \( \Phi(96^w) \).
(1.347), (1.348) We apply the theory of inverse pairs, see Chapter IR, and show that (1.347) and (1.348) are companions in the sense of IR7 of simpler relations. For (1.347) we consider the inverse pair IR. (83) with \( u = b = 2 \):

\[
\frac{(-1)^{n-k}}{k+1} \left( n-k \right) = (-1)^{n-k+1} \left( 2k+1 \right).
\]

We have with (1.5) and (1.7)

\[
\sum_{k=0}^{n} a_{nk} \left( 1 + (-1)^k \right) = \sum_{h=0}^{n} \left( 2h+2 \right) +
\]

\[
a_{nk} = \left( n-k \right), \quad a_{nk} = \frac{(-1)^{n-k}}{2k+1} \left( n-k \right) = (-1)^{n-k} \frac{2n+1}{2k+1} \left( n+k \right).
\]
We also may prove (1.347) from (1.349) with \( m = n+1 \) and \( x = -\frac{1}{4} \), applying \( \Phi \) (28), and (1.348) from (1.350) with \( x = -\frac{1}{4} \).

Verge absolutely for \( |z| |1+z| < 4 \) and in (1.352) -- (1.354) also for \( |z|^2 |1+z|^{-1} \).

In (1.353) we see this by writing

V.e. Similarly, we now may prove these relations for sufficiently small \( z \), since the left-hand sides are analytic for \( |z|^2 |1+z|^{-1} < 4 \) and continuous.

We have, formally,

\[
\sum_{n=0}^{\infty} z^n = \sum_{2k+1} \frac{1}{2k+1} \binom{x+k}{k} \binom{x-k}{n-2k} =
\]
\[
\sum_{n=0}^{\infty} \frac{1}{n+1} \leq \sum_{2k \leq n} \frac{1}{2k+1} \leq \sum_{2k \leq n} \frac{1}{2k+1} \left( \frac{2k}{n-2k} \right) = \sum_{2k \leq n} \frac{1}{2k} \leq \sum_{n=2k}^{\infty} \frac{1}{n-2k} \leq \sum_{n=2k}^{\infty} \frac{1}{n} = \]

Convergence of the double series, to justify interchanging summations, follows as above, when needed with (*) or a similar
\[
\sum_{m=0}^{\infty} \frac{1}{2x+2} \left( \frac{2x+2}{m+2} \right) z^m - \sum_{m=0}^{\infty} \frac{1}{x+1} \left( \frac{x+1}{m+2} \right) z^m =
\]
\[
= 2 \left( \frac{1}{x+1} \right) \left\{ \left( 1 + z \right)^{2x+2} - \left( 1 + z \right)^{x+1} + 1 \right\}.
\]

\[
\sum_{k=0}^{\infty} \left( \frac{x+k}{2k} \right) \left\{ \frac{z^k}{(1+z)^{x+k}} + \frac{x-k}{2k+1} \frac{z^{2k+1}}{z} \left( 1 + z \right)^{x-k-1} \right\}.
\]
\[
\sum_{k=0}^{\infty} \binom{x+k}{n} \left( \frac{x+k}{n+1} \right)^{-} + \sum_{n=0}^{\infty} \binom{n+1}{n} \left( \frac{1}{x+k} \right) \sum_{n=0}^{\infty} \left( \frac{1}{n+1} \right) \left( \frac{1}{x+k} \right)^{-} =
\]

\[
(l+z)^x + \sum_{k=1}^{\infty} \binom{x+k}{2k-1} z^{2k-1} \left( 1+z \right)^{x+k-1} +
\]

\[
(1.357), (1.358), The left-hand sides con-
\]
\[(3.517) \text{ gives } (1.357): \text{ By I (25),} \]
\[(l_z)^{-y'} = \sum_{m=1}^{\infty} z^m \sum_{l \geq m_1} \binom{m-k-1}{k} (y+k) \]

\[\sum_{k=0}^{\infty} \binom{2k+1}{2k+1} (l_z) + \sum_{k=0}^{\infty} \binom{2k}{2k} (l_z)^{l_z} \]

Similarly, from (3.518) for (1.358)

\[\sum_{k=0}^{\infty} \binom{x-k-1}{k} (l_z)^{k} (l_z-2k-2) + \sum_{k=0}^{\infty} \binom{x-k}{k} (l_z)^{k} (l_z-2k-1) \]
In (1.361) we have the absolute first central moment of the negative binomial distribution with parameters \( p > 0 \) and \( \beta \in (0, 1) \). In (x) in the proof of (1.89) we have \( \mu = p \beta^{-1} (1-\beta) \) and then

\[
E |X_\mu| = z \sum_{k=0}^{I} \left( \frac{1-k}{p} \right) \binom{s+k-1}{k} \beta^k (1-\beta)^k
\]

with \( I \in [\mu] \) and (1.361) follows from (1.360).
are in Mason and Horadam (1990a) (4.10) – (4.18), (1990b) (5.16) – (5.24).
By putting \( m-i = k \) in (1.392) and (1.393) their relative reducible form.

For \( m \geq 3 \) the l.h.s. of (1.394) equals

\[
\frac{1}{2} \sqrt{V} + \sum_{i=1}^{m-3} \frac{2i}{(m-i)} (m-1-3i)\
\]

\[
\frac{1}{2} V + \frac{1}{2} U_{m-2} = \frac{1}{2} \frac{F_{2m-1}}{2m-1} + \frac{1}{2} \frac{F_{2m-2}}{2m-2},
\]

with \( F(1) \). Similarly, for the l.h.s. of (1.395):

\[
\frac{1}{2} V + \sum_{i=1}^{m-3} \frac{2i+1}{(m-i)} (m-2-3i)\
\]
\[ U_m + \sum_{3i \leq m-1} \frac{m+i+i}{m-i} \left( \frac{m-i}{2i+1} \right)^2 z^{m-1-3i} = \]

The l.h.s. of (1.397) is, with (1.393) and (1.394)

\[ V + \sum_{m-1} \frac{m+i+i}{m-i} \left( \frac{m-i}{2i+1} \right)^2 z^{m-3i} = F_{2m-1} + \frac{F_{2m-2}}{2m-2} \]

\[ = F_{2m} + \frac{F_{2m-2}}{2m-2} + 1 = \frac{1}{2} z^{m+1} \]

for \( m \geq 1 \).

\[ \sum_{n=0}^{\infty} \sum_{\lambda, u} \zeta_n(\lambda, u) z^n = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(n+k+\lambda)}{n-k} z^n = \]
Interchanging the summations is justified by the absolute convergence of the double series. With \( B \) (50), and \( D \) (25),

\[
\sum_{k=0}^{\infty} \frac{|u|^{k+1}}{k+1} \sum_{n=k}^{\infty} |\binom{n+k+\lambda}{n-k}| |z|^n = 
\]

... and the product property Theorem1 of generating functions. That \( E_i(u) = u \xi_i^2(u^{-1}) \) follows from \(\Phi \) (18). The latter relation then leads to (1.399) with \(\Phi \) (27).

When \( \lambda = 1 \), there is only the term with \( k=0 \) in (1.399). This proves (1.400).

When \( \lambda = 0 \), there are only the terms with \( k=0 \) and \( k=1 \) in (1.399), if \( n \geq 1 \). This proves (1.401).
\[ n \geq \frac{(\lambda + n - \xi - \epsilon) / \ldots / \psi}{\lambda + n - \xi - \epsilon - \lambda} \]

which is (1.402).

We have, for \( u \neq 0, \ u \neq -4 \)

\[ g'_i(u) = g_i(u) (u^4 + u^2)^{-\frac{1}{2}} = g_i(u) / (g_i(u) - g_2(u)) , \]

How gives (1.403). With \( g'_i = g_i / (g_i - g_2) \)
and \( g'_i(u) = u \c_i^2(u^2) \), \( i = 1, 2 \), we obtain
(1.404) from (1.403), noting \( \Phi(2b)(21) \), for \( u \neq 0, \ u \neq -4 \) and then by continuity for all \( u \). Applying \( \Phi(2b) \) to (1.403) and (1.404) gives
(1.405).

\[ \sum_{k=0}^{n} \binom{n-k}{k} \frac{a_n}{n-k} \]
This formula also may be derived by showing that both sides have the same derivative (by (1.428)) and the same value for \( z = 0 \). See also Lehmer (1935).

Consider the random walk \( Y_n \), \( n \in \mathbb{N} \).

\( u_m = \sum_{n=0}^{\infty} p(Y_n = m) = \sum_{k=0}^{\infty} \binom{m+2k}{k} \beta^k \gamma^{m+2k} \)

for \( m \geq 0 \). We have \( Y_n = m \) if \( k \) steps are \( 1 \) and \( n-k \) steps are \( 0 \), with \( n-2k = m \), i.e., \( n - m = 2k \), so

\[ p(Y_n = m) = \binom{n-k}{k} \beta^k \gamma^{n-2k} \]

Taking \( k \), with \( n = m + 2k \), as summation variable, gives the second equality in (\( \times \)).

Cf. Prob. 329, Stat. Neerland. 57(3), 1997, 379-381. Note that \( 1 - \beta p q = (\beta - q) \) when \( \beta + q = 1 \).
(1,431) With $B_{14}$, (1.3), $D_{13}$, and $B_{12}$,
\[\sum_{k=0}^{n} (2k+1) \binom{2k}{k} y^{-k} = \sum_{k=0}^{n} (-1)^k \left( \frac{-3/2}{k} \right) y^{-k}\]

\[y^{-n} \frac{(2n+2)!}{6n! (n+1)!} = \frac{1}{3} (2n+1) (2n+3) \binom{2n}{n} y^{-n} \cdot\]

After (1.3), one might apply $D_{13}$ and $B_{14}$. Induction on $n$ is possible, more direct but not shorter.

(1,432) With $B_{13}$, (1.87) and again $B_{13}$.

(1,433) With $B_{13}$, (1.67) and again $B_{13}$. cf.

---

the product property, Th. M1, of generating functions and the relation $(1+z)^{-2} = (1+z)^{1/2} (1+z)^{-1}$ we obtain.
\[
\binom{n}{\frac{1}{2}} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k},
\]

and (1.436) follows with B (13) and B (15). Cf. Riordan (1968), Example 2, § 4.2. One may also prove (1.436) by induction.

(1.437) From (1.436) and (1.435) with \( z = \frac{1}{2} \). Let \( T \) be the time of first entrance at 0 after \( t = 0 \) of a simple symmetric random walk starting at 0. Feller (1957) showed in Ch. III, (4.1) and (4.5), that \( P(T > 2n) = \binom{2n}{n} 4^{-n} \) and in Ch. XII, 3, (3.17) that 
\[
P(T = 2n) = \frac{1}{(2n+1)!} \quad k \geq 1.
\]
This proves

(1.438) Both sides of (1.438) have the same derivative. By (1.428) and are equal for \( z = 0 \). Cf. Lehmer (1985) and Problem 6638, Monthly 99(12), 1992, 172.

\[
\sum_{k=0}^{\infty} \binom{k}{4} = \sum_{k=0}^{\infty} (-1-k) = \sum_{k=0}^{\infty} (-1-(1-c)).
\]

For \( y > 0 \) we may integrate term by term over \( (0, 1) \), all terms being positive. With
the beta integral $D(y)$ and $D_1(y)$ one finds

$$
\sum_{k=0}^{\infty} \binom{2k}{k} \frac{y^{-k}}{k+1} = \Gamma(y) \Gamma\left(\frac{1}{2}\right) \sqrt{\pi} \Gamma\left(y + \frac{1}{2}\right)
$$

The left-hand side here is analytic for $-y \notin \mathbb{N}_0$ and the right-hand side for $-y \notin \mathbb{N}_0$, $-(y + \frac{1}{2}) \notin \mathbb{N}_0$. So the first equality in

\begin{equation}
(1.44.3) \quad \text{From (1.33a) with } \kappa = n+1 \text{ and (1.319)}
\end{equation}

\begin{equation}
\sum_{k=0}^{n} (-1)^k \binom{2k}{k} \frac{y^{-k}}{k+1} = \frac{\Gamma(y) \Gamma\left(\frac{1}{2}\right) \sqrt{\pi} \Gamma\left(y + \frac{1}{2}\right)}{y^n}
\end{equation}

\begin{equation}
\sum_{k=n+1}^{\infty} (-1)^k \binom{2k}{k} \frac{y^{-k}}{k+1} = \frac{\Gamma(y) \Gamma\left(\frac{1}{2}\right) \sqrt{\pi} \Gamma\left(y + \frac{1}{2}\right)}{y^n}
\end{equation}

\begin{equation}
(1.44.5) \quad \text{The limit for } x \to -\frac{1}{2} \text{ of the l.h.s. of}
\end{equation}
\[ \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=0}^{1} \sum_{i=1}^{n} \] 

With (1.129) and B (29)

\[ \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{j=0}^{1} \sum_{i=1}^{n} \]

From (1.470) follows from (1.470).
\[(1.451) \text{ With } (1.450) = (1.56) \text{ the l.h.s. is equal to}
\]
\[
\sum_{k=0}^{m} (-2)^{m-k} \binom{y+z}{k} + 2 \sum_{k=0}^{m-1} (-2)^{m-1-k} \binom{y+z}{k} =
\]
\[
\sum_{k=0}^{m} \binom{y+z}{k} - \sum_{k=0}^{m-1} \binom{y+z}{k}
\]

\[
\sum_{n=0}^{\infty} \sum_{0 \leq j \leq (n-b)(c-a)} (-1)^{n+j} (b+je)^{-n} =
\]
\[
\sum_{j=0}^{\infty} \sum_{m=0}^{\infty} z^{m} \binom{b+j(c-a)}{c-a} \sum_{m=0}^{\infty} z^{m} \binom{m+b+je}{m} =
\]
\[
\sum_{j=0}^{\infty} \frac{b+j(c-a)}{c-a} \sum_{m=0}^{\infty} z^{m} \binom{m+b+je}{m} =
\]
\[
\sum_{j=0}^{\infty} \frac{b+j(c-a)}{c-a} (l-z)^{c-b-1} c \binom{c}{c-a} (l-z)^{c-a} =
\]
\[
\frac{z^{b}(l-z)^{c-b-1} c \binom{c}{c-a} (l-z)^{c-a}}{z^{c-a} l^{c-a}} ,
\]
from which the sum may be found in-
(2.1) Denote the l.h.s. of (2.1) by $S_n$. From B (32)

\[ S_{n+1} = 1 + \sum_{k=1}^{n+1} \frac{(n)}{(k)} - \sum_{k=1}^{n+1} \frac{n-k+1}{k} (n+1)^{-1} \]

Solving this recurrence, with $S_0 = 1$, one finds (2.1). Or apply induction with the above recurrence.

The relation (2.1) is a special case of (2.4). Cf. Staver (1947), Rockett (1981).

(2.4) From B (32)

\[ S_{n+1}(x) = 1 + \sum_{k=1}^{n+1} x^k (n+1)^{-1} - \sum_{k=1}^{n+1} x^{n-k+1} (n+1)^{-1} \]

B. (47). The absolute convergence for $\text{Re} x < -2$ also follows from B (47).
\( n \rightarrow \frac{a^n h^m}{(n-h)l} \)

without reference to the polynomials \( U_{n+1} \) by expanding \((1 - atz + btz^2)\) in the relation following (1) into powers of \( z \). Com-

\[(2.7]\text{ The l.h.s. converges absolutely for } |x| < 1 \text{ by B(47) and the r.h.s. for } |x|/(1+x) < 1, \text{ i.e. } Re x > -\frac{1}{2}. \text{ We prove the identity for } |x| < \frac{1}{2} \text{ and then invoke analyticity. By D(25), G(23) and G(48) the r.h.s. is equal to}\]
\[ \sum_{n=0}^{\infty} x^n (y+1-n)^{-1} \left( \begin{array}{c} y+1 \\ n \end{array} \right)^{-1} = \sum_{n=0}^{\infty} x^{n+1} \left( \begin{array}{c} y \\ n \end{array} \right)^{-1}. \]

Here \( \Delta \) operates on \( t \). Changing the summation over \( k \) and \( j \) into the summation over \( n \) and \( j \) is justified by absolute

\[ \sum_{k=0}^{n+1} (-1)^{n-k} \left( \begin{array}{c} x+1 \\ k+1 \end{array} \right) = \sum_{k=0}^{n+1} (-1)^{n-k} \left( \begin{array}{c} x+1 \\ k \end{array} \right)^{-1}, \]

and (2.8) follows from (2.2).

(2.9) From (2.8). It follows from B (47) that the series converges absolutely and that the limit is \( - (x+1)(x+2)^{-1}(x+3)^{-1} \) for \( \Re x \leq -3 \).
(2.10), (2.11). The relation (2.10) follows with induction on \( r \). Then (2.11) follows for 
\( r = n \). An immediate proof of (2.11) is by

\[ A_n - A_{n-1} = (-1)^n(n+1)^{-2} - \sum_{k=0}^{n-1} (-1)^k k! (n-k-1)! / (n+1)! \]

\[ = (-1)^n(n+1)^{-2} - n^{-1}(n+1)^{-1} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \]

\[ = (-1)^n(n+1)^{-2} - n^{-1}(n+1)^{-2}(1+(-1)^{n+1}) = (-1 + 2(-1)^n)(n+1)^{-2}, \]

and (2.14) follows since \( A_0 = 1 \).

The form (2.15) of (2.14) is given in Gould (1973c) as (219) and attributed to Ljunggren.
\[ \sum_{k=0}^{\infty} (-1 + 2(-1)^k)(k+1) = \sum_{k=1}^{\infty} (k+m), \quad m \geq 1, \]

which may be proved by induction on \( m \).

\[ \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{(-1)^n \frac{x+1}{n+1}}{x+1} \sum_{j=1}^{\infty} \frac{x+1}{n+1} = \]

The relations (2.18) and (2.19) follow from (2.17) with \( n = x = 2m \) and with \( n = x = 2m - 1 \).

\[ \cap (\varepsilon - 1) \left\{ 1 - \left( \frac{n}{\varepsilon - 1} \right)^{-1} \right\} . \]

The relation (2.24) follows from (2.23) by letting \( n \to \infty \). The terms are nonnegative.

Note that \( \binom{m}{j} = \frac{m(m-1)\ldots(m-j+1)}{j!} \).

The sum and integral may be interchanged since the integrands are nonnegative. The series...
(2.37) With the beta integral \( I (34) \) the l.h.s. is
\[
\sum_{k=1}^{\infty} z^k (k-1)! x! / (k+x)! = .
\]

\[
\int_0^1 (1-t)^{(1-x)} (1-t^{x+1}) dt < \infty,
\]
which also proves the absolute convergence in (2.37).

(2.38) With \( I (24) \) and (2.7) the l.h.s. is equal to
\[
(x+1)^{-1} \sum_{k=1}^{\infty} (-z)^k \frac{x+k}{k-1} = (x+1)^{-1} \sum_{h=0}^{\infty} (-z)^{h+1} \frac{x+1+h}{h} =
\]
\[
-(x+1)^{-1} \sum_{h=0}^{\infty} \frac{h+1}{h} \left( \frac{-x-2}{h} \right)^h
\]
\[
- (x+1)^{-1} \sum_{h=1}^{\infty} \frac{-x-1}{k} \left( \frac{z}{x} \right)^k
\]
for $z$ satisfying $|z| < 1$ and $\text{Re} z > -\frac{1}{2}$.

The r.h.s. converges conditionally for $z = -\frac{1}{2}$, or $z/(1+z) = -1$, when $a > 0$. When $z$ goes from 0 to $-\frac{1}{2}$ along $\mathbb{R}$, $z/(1+z)$ descends from 0 to $-1$. So by Abel's continuity theorem the r.h.s. then tends to $-\infty$.

The l.h.s. of (x) converges for $z = 1$ when $a > 0$, by the alternating series theorem:

$(a_n)^{-1}$ decreases to zero as $n \to \infty$. Consi-

Setting $z \to 1$ in (x) gives (2.41) by noting continuity theorem.

We have, for $k \in \mathbb{N}$, and for $z = 1$ when
interchanging sum and integral (allowed at least for $Rea > 1$), putting $1-t = u$

(2.43) With the beta integral $D(D)$ the l.h.s. for $Rea > 0$ is equal to

$$a (1+x)^{-1} \sum_{n=0}^{\infty} \left( \frac{x}{1+x} \right)^n \int_0^1 t^{n+m} (1-t)^{a-1} \, dt =$$

$$a \int_0^1 t^m (1+x-tx)^{-1} (1-t)^{a-1} \, dt,$$

where interchanging sum and integral is justified by absolute finiteness for small $x$. The l.h.s. is then equal to

$$\left( \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! (a+k-1)!} \right)$$

$$\left( \sum_{k=0}^{\infty} (-1)^k \frac{1}{k! (a+k-1)!} \right).$$
again by the beta integral. Interchanging sum and integral is justified for small $x$ by absolute finiteness.

\[
\begin{equation}
(2.44) \quad \text{Denoting the l.h.s. by } S_n, \text{ we have for } n \geq 1, \text{ applying (2.34)}
\end{equation}
\]

\[
S_n - S_{n-1} = - \sum_{k=1}^{\infty} \frac{(k-1)! (n-1)!}{(k+n)!} = \ldots
\]

\[
S_{(\ell, n)} - S_{(\ell, n-1)} = \sum_{k=1}^{\infty} \frac{(k-1)! (n-1)!}{(k+n)!} = \ldots
\]

\[
\ldots - 1, \ldots, \frac{1}{n} \leq \frac{\ell - \frac{1}{n+1}}{\ell - \frac{1}{n+1}} - 2, \quad (n+\epsilon)^{\ell}
\]

\[
\ldots - 1, \ldots, \frac{1}{n} \leq \frac{\ell - \frac{1}{n+1}}{\ell - \frac{1}{n+1}} - 2, \quad (n+\epsilon)^{\ell}
\]
(2.46) From G (63), for \( f(x) = \frac{x!}{(v+x)!} \) with \( x=0 \), From G (49), D (11) and D (24) we have
\[ \sum_{k=1}^{\infty} (2x)^{2k} \int_{0}^{1} t^{k}(1-t)^{k-1} \, dt = \]
\[ \frac{1}{2} \log \left( x^2 + y^2 \right) \left| \frac{\arctan \frac{x}{\sqrt{x^2 - 1}}}{x} \right| = \]
\[ \frac{2x}{\sqrt{1-x^2}} \arctan \frac{x}{\sqrt{1-x^2}} = 2x \left( 1-x^2 \right)^{-\frac{3}{2}} \arcsin x. \]


(2.54) From (2.52) by replacing \( x \) by \( ix \). Or directly, first for \( 0 \leq x < 1 \) and then by analyticity for \( |x| \leq 1 \): With the Beta integral $D(34)$, putting \( x = \sqrt{1+x^{-2}} \), the l.h.s. is equal to

\[ \sum_{k=1}^{\infty} (-1)^k (2x)^{2k} \int_{0}^{1} t^{k}(1-t)^{k-1} \, dt = \]
\[ \int_{0}^{1} \left( t^2 - t - \frac{1}{4}x^2 \right)^{k-1} t \, dt = \]
\[
\frac{\alpha-1}{2\alpha} \log \left( \frac{t-\frac{1}{2}+\frac{1}{2}\alpha}{\frac{1}{2}+\frac{1}{2}\alpha-t} \right) = P(12)
\]

(2.56) We prove (2.56) for \( p \leq x \leq 1 \). The series converges absolutely for \( \|x\| < 1 \) and the identity then follows by analyticality. From the beta integral \( D(34) \), and since we may interchange sum and integral (everything is positive), we see that the \( l.h.s. \) is equal to

\[
x^{-1}(1-x^2) \arctan x (1-x^2) = x(1-x^2) \arcsin x.
\]

see Remark to (2.57)

(2.57) We prove (2.57) for \( 0 \leq x \leq 1 \). The series converges absolutely for \( \|x\| < 1 \) and (2.57) then follows by analyticality. With the beta integral \( D(34) \), the \( l.h.s. \) is equal to
\[
\sum_{k=0}^{\infty} (-1)^k (2x)^{2k} \int_0^1 t^k (1-t)^{k} \, dt = \\
\int_0^1 \left( -y^2 x^2 t^2 + y^2 x^2 t + 1 \right)^{-1} \, dt = \\
- \frac{1}{y^2 x^2} \log (1 + \frac{y}{2} x - t) \bigg|_0^1 + \frac{1}{y^2 x^2} \log (t + \frac{y}{2} x - \frac{1}{2}) \bigg|_0^1,
\]
where \( \alpha = (1 + x^{-2})^{1/2} \). Interchanging sum and integral is allowed by absolute finiteness. Substitution of \( 1 \) and \( 0 \) into the primitive gives (2.57).

One also may replace \( x \) by \( ix \) in (2.56).

Remark. Multiplying (2.56) and (2.57) by \( x \) and then differentiating we obtain (2.58) and (2.59). This leads to

\[
\sum_{k=1}^{\infty} k \binom{2k-1}{k} 2^k x^{2k-1} = \frac{d}{dx} \left( \text{arc} \sin x \right)^2,
\]
and integrate over \([0, y]\). The series converges absolutely for \( |y| \leq 1 \). Lehmer (1985), Koecher (1980), Borwein (1987), p. 139.

\[
- \frac{d}{dx} \left\{ \log (x + (1+x^{1/2})) \right\}^z,
\]
and integrate over \([0, y]\). The series con-
(2.63) Integrate (2.54) over \([0, y]\), noting that

\[\sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^{2k}}{2k+1} = (2z)^{-1} \arcsin \sqrt{z} \cdot\]
But from (3.575)
\[ \sum_{k=0}^{n} \frac{1}{2k+1} \binom{2k}{k} \frac{1}{2n-2k-1} \binom{2n-2k}{n-k} = -1, \ n=0, \]

\[ = \frac{\pi^{2n}}{2n (2n+1) \binom{2n}{n}}^{-1}, \ n \geq 1 \]

Here we first take \( 0 \leq z \leq \pi \), so that we may interchange sum and integral (take absolute values everywhere). Then we see \( \pi \) analyticity.

But see (2.52).
TABLE 3

(3.2) From (3.1) and D. (27)

$$\binom{2n}{n} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}^2 + \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k}^2 = \frac{1+(-1)^n}{2^n} \binom{n}{n/2}.$$ 

(3.3) From (3.1) and (4.6) with $\nu = n/2$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \frac{1+(-1)^n}{2^n} (-1)^{n/2} \binom{n}{n/2}.$$ 

$$\frac{1}{2} \left( \binom{2m}{m} \right)^2 + \left( \binom{m}{m} \right)^2 = \sum_{h=0}^{n} \binom{2h}{h} =$$ 

$$\leq m \sum_{k=m/2}^{n} \left( \binom{2m}{k} \right)^2 - \frac{1+(-1)^m}{2} \binom{2m}{m}.$$ 

$$\sum_{2h \leq m} \left( \binom{2m}{2h} \right)^2 - \frac{1+(-1)^m}{2^n} \binom{2m}{m}.$$
\[ \sum_{h \leq m/2} \left( \frac{2m+1}{2h} \right)^2 + \sum_{m/2 \leq h \leq m} \left( \frac{2m+1}{2h} \right)^2 = \frac{14(-1)^m}{2} \binom{2m+1}{m} \]

This relation is the same as (3.2) with \( n = 2m+1 \). One sees this by observing the
with \( n = 2m + 1 \) gives the relation (*).

(3.8) From (3.7) with \( n = 2m \),

\[
(-1)^m \binom{2m}{m} = \sum_{k=0}^{m} (-1)^k \binom{2m}{k} + \sum_{h=0}^{m} (-1)^{2m-h} \binom{2m}{2m-h} - (-1)^m \binom{2m}{m} = 2 \sum_{k=0}^{m} (-1)^k \binom{2m}{m} - (-1)^m \binom{2m}{m}.
\]

Functions (3.9) is equivalent with

\[
\sum_{j=0}^{m} \binom{u}{j}^2 (-2u-1)\binom{m-j}{m-j}(-1)^m - \binom{m+u}{m}.
\]

This is (6.31) in Gould (1972), with \( D(2y) \), or our (6.16) or (6.152) - with \( D(2y) \) for \( x = y = u \) in \( D_4 = \pi \) of Bateman and Van Vleck (1944).

(3.11) Both sides are polynomials in \( x \) of degree \( n \), so it suffices to prove (3.11) for \( x = m \in \{b_0, \ldots, n\} \).

The LHS, then is equal to

\[
\sum_{k=0}^{m} \binom{m}{m-k} \binom{y}{k} = \binom{y+m}{m}.
\]
\[ \Delta^n_{\gamma} \left( \begin{array}{c} y+m \\ n \end{array} \right) = \left( \begin{array}{c} y+m \\ m \end{array} \right) \]

By Eq. (47) the series converges absolutely for \(|z| < 1\), and also for \(|z| = 1\) when \(\text{Re}(\alpha + \beta) < 1\). By Eq. (24), (23), (11)

\[ \left( \frac{\alpha + k - 1}{k} \right) \left( \frac{\beta + k - 1}{k} \right) = \frac{\alpha^{(k)} \beta^{(k)}}{k! \cdot k!} = \frac{\alpha^{(k)} \beta^{(k)}}{\Gamma(k) \cdot k!} \]

and (3.12) follows by definition of the hypergeometric function, see G. K. P. (1989).

By Pochhammer's formula we have for \(|z| < 1\), \(0 < \text{Re} \beta < 1\) (G. Slater, Generalized Hypergeometric Functions, Camb. Univ. Pr. 1966)

\[ (2) \quad \phi(\alpha, \beta, z) = \frac{\Gamma(\beta) \Gamma(1-\beta)}{\Gamma(\beta) \Gamma(1-\beta)} \int_0^1 t^{\beta-1} (1-t)^{-\beta} (1-zt)^{-\alpha} \, dt \]

This relation also holds for \(|z| = 1\), \(z \neq 1\).
When \(0 < \text{Re} \beta < 1\), \(\text{Re}(\alpha + \beta) < 1\). This follows by letting \(z \to z_0\), with \(|z_0| = 1, z \neq 1\), noting the continuity in (1) when \(\text{Re}(\alpha + \beta) < 1\) and applying dominated convergence in (2). Note that \(0 < a < |1 - z^t| < b\) for \(0 \leq t \leq 1\) and \(z = \lambda z_0\) with \(\lambda \to 1\).

The second equality in (3.13) follows by the relation \(\Gamma(x) \Gamma(1-x) = \pi / \sin \pi x\) and the hint in (19). 

\[ g(\alpha, \beta, z) \to g(\alpha, \beta, 1) \] by continuity and the r.h.s. of (2) converges to

\[ \frac{\pi}{\sin \pi \alpha} \frac{\Gamma(1-\alpha - \beta)}{\Gamma(1-\beta)} \]

by monotone convergence and the beta integral \(I\). (33). The r.h.s. of (3.14) is analytic in \(\alpha\) and \(\beta\) for \(\text{Re}(\alpha + \beta) < 1\), when \(1/\Gamma(x) = 0\) for \(-x \in \mathbb{N}\). The r.h.s. also is analytic, since the convergence of the series is uniform in \(\alpha\) and \(\beta\) on compacta for \(\text{Re}(\alpha + \beta) < 1\), by \(B\). (47).

From (3.14) with \(\alpha = -\beta\) we obtain (3.15) of Borwein and Borwein (1987), p. 189.
For (3.19) we first take \( 0 < x < \beta < \frac{1}{2} \), so that our series converge absolutely and, below, we may interchange sums and integrals and also integrations, since everything is nonnegative. From (i) and (3.13)

(2) \[
\sum_{k=0}^{\infty} \left( -\beta \right)^{k} \frac{1}{x+k} = \int_{0}^{1} y^{-\beta} g(\beta, \beta; y) \, dy = \\
\pi^{-1} \sin \pi \beta \int_{0}^{1} y^{-\beta} \, dy \int_{0}^{1} t^{\beta-1} (1-t)^{-\beta} (1-\gamma t)^{-\beta} \, dt = \\
\pi^{-1} \sin \pi \beta \int_{0}^{1} t^{\beta-1} (1-t)^{-\beta} \, dt \int_{0}^{1} y^{-\beta} (1-\gamma t)^{-\beta} \, dy = \\
\pi^{-1} \sin \pi \beta \int_{0}^{1} t^{\beta-x-1} (1-t)^{-\beta} \, dt \int_{t}^{1} y^{-\beta} (1-\gamma t)^{-\beta} \, dv,
\]

\( x = 0 \) \quad \text{and} \quad x = 0 + \kappa

\[
\pi^{-1} \sin \pi \beta \int_{0}^{1} t^{\beta-x-1} (1-t)^{-\beta} \, dt \int_{0}^{t} y^{-\beta} (1-\gamma t)^{-\beta} \, dv = \\
\pi^{-1} \sin \pi \beta \int_{0}^{1} v^{\beta-x-1} (1-v)^{-\beta} \, dv \int_{0}^{t} t^{x-1} (1-t)^{-\beta} \, dt = \\
\pi^{-1} \sin \pi \beta \int_{0}^{1} v^{\beta-x-1} (1-v)^{-\beta} \, dv \int_{v}^{t} t^{x-1} (1-t)^{-\beta} \, dt = \\
\pi^{-1} \sin \pi \beta \int_{0}^{1} u^{\beta-x-1} (1-u)^{-\beta} \, du \int_{u}^{t} v^{x-1} (1-v)^{-\beta} \, dv,
\]

From (3) and (4) with the beta integral

\( D \) (3.3), and the relation \( \Gamma(z) \Gamma(1-z) = \pi \csc \pi z \).
\[ \sum_{k=0}^{\infty} \left( -\frac{\beta}{k} \right)^2 \left( \frac{1}{x+k} + \frac{1}{\beta-x+k} \right) = \]

\[ -1, \quad \beta - x - 1, \quad -\beta, \quad x - 1, \ldots, -\beta \]

which is (3.19). Now the l.h.s. of (5) converges uniformly w.r. to \( \beta \) on compacta in \( \beta: \text{Re} \beta < 1 \), outside \( \{ \beta: x - \beta \notin \mathbb{N} \} \), for fixed \( x \) with \( -x \notin \mathbb{N} \), and therefore it is analytic. The r.h.s. also is an analytic function of \( \beta \) for \( \text{Re} \beta < 1, x - \beta \notin \mathbb{N} \). So (3.19) holds in the domain indicated.

\[ \langle h = 0 \rangle \langle h \rangle = \binom{n}{1} \]

Differentiating this relation w.r. to \( x \) and...
\[ \sum_{h=1}^{n} \binom{n}{h}^2 \sum_{j=n-h+1}^{n} j^{-1} = \binom{2n}{n} \sum_{h=n+1}^{2n} h^{-1}, \]

and with \( H_0 \equiv 0 \) and (3.1) and D (27),

\[ \sum_{h=0}^{n} \binom{n}{h}^2 (H_n - H_{n-h}) = \binom{2n}{n} (H_{2n} - H_n), \]

\[ \binom{2n}{n} H_n = \sum_{k=1}^{n} \binom{n}{k} H_k = \binom{2n}{n} (H_{2n} - H_n), \]

which is (3.25).

Cf. Egorychev (1984), §2.4.5, Kaucky (1975), p. 302

(3.27) Induction on \( n \). For \( n=1 \) the relation holds trivially. For \( n \rightarrow n+1 \geq 2 \) from D (9) and D (11) we have

\[ \binom{\beta}{n+1}(-\binom{\beta}{n+1}) = (-1)^{n+1} \frac{\beta(\beta+1)\cdots(\beta+n)\beta(\beta-1)\cdots(\beta-n)}{(n+1)!} = \]

\[ (-1)^{n+1} \frac{n^{\alpha} \prod_{j=1}^{n} (1-j^{-2} \beta^2)}{(n+1)!}, \]

One also may apply (3.38) with \( a = b = n^2 \) putting \( h = n-k \).
(3.28) Induction on $n$. Trivial for $n=0$. For the step $n \rightarrow n+1$, with $D(14)$ and $D(7)$,

\[
\begin{align*}
\frac{(-1)^n}{2^{2n+1}} & \leq \frac{2(n+1)^2}{1+n} - \frac{1/2}{n+1} \\
\frac{(n+1)^2}{2^{2(n+1)+1}} & \leq \frac{2(n+1)^2}{1+n} - \frac{1/2}{n+1}
\end{align*}
\]

One also may apply (3.27) with $\beta = -1/2$, and $D(29), (30)$. 
\( \Sigma_{k=0}^{n} \binom{n}{k} (x+k)^{n-k} = \Sigma_{h=x}^{n} \binom{n}{h} (h)^{n-h} = \Sigma_{h=x}^{n} \binom{n}{h} (h)(n-h) = (x+n) \cdot (x+n-1) \cdot \ldots \cdot (x+1) \cdot x.

Also from Newton's interpolation formula \( G(41) \) with \( G(38) \)
\( (x+n) = \sum_{h=0}^{n} \Delta^{h} (x+h) / \binom{n}{h} = \sum_{h=0}^{n} (n+h-1)/ (n-h) \).

rial moment of number of colour runs.

(3.34) The first equality follows by putting \( k=n \) and the last one, with \( D(14) \), by canceling and rearranging factorials.
First take \( \text{Re} x > 0 \). Then the sequences

\[
\sum_{n=0}^{\infty} \binom{n}{k} x^n = \frac{1}{(1-x)^{k+1}}
\]
For their convolution $c_n$, $n \in \mathbb{Z}$, we have when $n \geq 0$

$$c_n = \sum_{k=-\infty}^{\infty} a_k b_{n-k} = \sum_{k=n}^{\infty} a_k b_{n-k} =$$

$$= \sum_{k=n}^{\infty} \binom{x}{k} \binom{x}{k-n}.$$

The sequence $c_n$ is symmetric about $n=0$ and we have

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta} = \sum_{k} a_k e^{ik\theta}, \sum_{k} b_k e^{ik\theta} =$$

$$(1 + e^{i\theta})^x (1 + e^{-i\theta})^x = 2^x (1 + \frac{1}{2} e^{i\theta} + \frac{1}{2} e^{-i\theta})^x =$$

$$= 2^x \sum_{r=0}^{\infty} \binom{x}{r} 2^{-r} (e^{i\theta} + e^{-i\theta})^r =$$

$$= 2^x \sum_{r=0}^{\infty} \binom{x}{r} 2^{-r} \sum_{k=0}^{r} \binom{r}{k} e^{(2k-r)i\theta} =$$

$$= 2^x \sum_{n=-\infty}^{\infty} e^{in\theta} \sum_{2k-n = n} \binom{x}{k} \binom{x}{k-n} 2^{-k}.$$

Changing the summation over $(k, r)$ into $n = 2k - r$ and $k$ is allowed since

$$\sum_{r=0}^{\infty} \binom{x}{r} 2^{-r} \sum_{k=0}^{r} \binom{r}{k} = \sum_{k=0}^{\infty} \binom{x}{k} < \infty.$$

For $n \geq 0$ we find

$$c_n = 2^x \sum_{k=n}^{\infty} \binom{x}{2k-n} \binom{x}{k-n} 2^{n-2k}.$$
or with (24)' with Stirling's formula, or Coffman (1965), §27. As all are analytic functions of \( x \) for \( \text{Re} x > -\frac{1}{2} \),

this is equal to

\[
\frac{1}{(m+1)^k} \frac{(2m)!}{(2e)^m} = \frac{m^k}{(2m)^{2k}} \cdot \left( \frac{2m}{2e} \right).
\]
\[
\frac{(-x-1)!}{n!(n+\varepsilon)!} \frac{x}{(-x-n-1)! (x-n-1)!} \frac{1}{(x-\varepsilon) \frac{(n+1)(n+\varepsilon+1)}{}} = \\
\frac{x}{x-\varepsilon} \frac{(-x-1)! (x-1)! (n+1)(n+\varepsilon+1)}{(n+1)! (n+\varepsilon+1)! (-x-n-1)! (x-n-1)!} = \\
\frac{x}{x-\varepsilon} \frac{(-x-1)! (x-1)! (n+1)(n+\varepsilon+1)! (-x-n-2)! (x-n-\varepsilon-2)!}{(n+1)! (n+\varepsilon)!} = \\
\frac{x}{x-\varepsilon} \frac{(-x-1)! (x-1)!}{(n+1)(n+\varepsilon+1)}.
\]

(3.38) For \( a < b \)

\[
(1+a) \Gamma(x) \Gamma(1-x) = \Gamma(x+a-b) \Gamma(1+b-a-x),
\]

and the first equality follows from (3.37) with \( n = a \), \( \varepsilon = b-a \), and \( D(14) \).

By \( D(14) \), the second equality is equivalent with
(3.40), (3.41). From $G(y)$, (5) with $q_k = \binom{x}{k}$, and $b_n = \binom{2n}{n}$ by (3.39) = I(26).
\[
\frac{(-x)! (x-\varepsilon)!}{(n-\varepsilon)! (x-\varepsilon)! (n-\varepsilon)! (-x-n+\varepsilon)!} \cdot \frac{(n-\varepsilon)(x-\varepsilon)}{n^\varepsilon (-x-n+\varepsilon)} = \\
\frac{n-\varepsilon}{(n-\varepsilon)! (x-\varepsilon)! \varepsilon! (-x-n+\varepsilon)!} \cdot \frac{(-x)! (x-\varepsilon)!}{n^\varepsilon (-x-n+\varepsilon)} \cdot \frac{(n-\varepsilon)(x-\varepsilon)}{n^\varepsilon (-x-n+\varepsilon)}
\]

with \( \varepsilon \) (3.14), and (3.4.2) follows with \( \varepsilon \) (3.14).

For \( \varepsilon \in \mathbb{Z} \), the relation follows by continuity.

Also from (3.115) with \( b = -a \), cf. Chu (1990).

(3.4.3) From (3.4.2) for \( 0 \leq \varepsilon \leq n-1 \), \( n \geq 2 \), \( x \notin \mathbb{Z} \),

\[
\sum_{k=0}^{\varepsilon} \frac{x!}{(x-k)!} \frac{(-x)!}{(n-k)!} = \sum_{k=0}^{\varepsilon} \frac{x!}{(x-k)!} \frac{(-x)!}{(n-k)!} + \sum_{k=0}^{\varepsilon} \frac{x!}{(x-k)!} \frac{(-x)!}{(n-k)!} = \\
\frac{n-\varepsilon}{n^x} \left( \frac{x-1}{\varepsilon} \right)^{-x} + \frac{n-1-\varepsilon}{n^x} \left( \frac{x-1}{n-\varepsilon} \right)^{-x} = \\
\left( \frac{x-1}{\varepsilon} \right)^{-x} \left( \frac{(-x)!}{(n-\varepsilon)! (-x+n+\varepsilon+1)!} \left( \frac{(-x+n+\varepsilon+1)}{n} + \frac{n-1-\varepsilon}{n-1} \right) \right)
\]
using (18) and (14). For \( x \in \mathbb{C} \), the relation follows by continuity.

One also might apply induction on \( r \), as in (3.42). Cf. also Chu (1990).

(3.44), (3.45) An urn contains \( a \geq 1 \) white and \( b \geq 0 \) black balls. They are drawn at random, one by one, without replacement. Let \( X_n \) be the number of white balls in the first \( n \leq a+b \) drawings and \( U_m \) the number of the drawing giving the \( m \)th white ball. Then for \( 1 \leq n \leq a+b, 1 \leq m \leq a, \)

\[
P(X_n = k) = \binom{a}{k} \binom{b}{n-k} \binom{a+b}{n}^{-1} = \]

\[
\binom{a}{m-1} \binom{a+b-j}{a-m} \binom{a+j}{a} \]

since for \( U_m = j \) we should have \( X_{j-1} = m-1 \).
and then a white ball at \( j \) from \( a-m+1 \)
white balls among \( a+b-j+1 \) balls. The
second equality follows with \( D \) (14), by
rearranging factorials. The relations
(3.44) and (3.45) now state that
\[
P(X_m \geq m) = P(U_m \leq n), \quad 1 \leq m \leq n.
\]

For \( a \in \mathbb{N} \), fixed, both sides of (3.44) are polyno-

mials in \( b \), so then (3.44) may be
extended to \( b \in C \). A similar remark
applies to a when \( b \in \mathbb{N}_0 \), by writing
\[
\binom{a+b-n}{a-k} = \binom{a+b-n}{b-n+k}, \quad \binom{a+b-j}{a-m} = \binom{a+b-j}{m+b-j}.
\]

A similar argument applies to (3.45). First
one should divide out as many factors
of \( \binom{a+b}{n} \) and \( \binom{a+b}{a} \) as possible.

(3.47), (3.48) With (3.39) the r.h.s. of (3.47) is
\[
\sum_{k=0}^{n} \binom{x}{k} \left( \frac{1}{y} \right)^{n-k} + \sum_{k=0}^{n} (-1)^k \binom{x}{k} \left( \frac{1}{y} \right)^{n-k} =
\]

(3.70) \[
\sum_{k=0}^{n} \binom{x}{k} \left( \frac{1}{y} \right)^{n-k} = \sum_{k=0}^{n} (-1)^k \binom{x}{k} \left( \frac{1}{y} \right)^{n-k}.
\]
\[ \sum_{2j+1 \leq n} \binom{x}{2j+1} \binom{y}{n-2j-1} \]

(3.49) - (3.52) For fixed \( m \) and \( y \), denote
the left-hand sides of (3.49) - (3.52) by \( q_i(x) \), \( i = 1, 2, 3, 4 \), respectively, and the right-hand sides by \( h_i(x) \), \( i = 1, 2, 3, 4 \).
Since \( h_i(x) \) is a polynomial in \( x \) of
degree \( 2m \), we have by Newton's interpolation
formula \( G(y) = C(x) \).

\[ q_i(x) = \sum_{k=0}^{2m} (x \Delta^k h_i(0)) \]

(3.49) - (3.52) follows.

From (3.39) = \( D(x) \)
and (3.50) follows from (3.49).

In the same way as above,

$$h_3(x) = \sum_{k=0}^{2m+1} \binom{x}{k} \Delta^k h_3(0),$$

and (3.51) follows. Also from (3.39) = D.126,

$$g_3(x) + g_y(x) = 2 \sum_{k=0}^{2m+1} \binom{x}{k} \frac{y}{2m+1-k} = 2 \binom{x+y}{2m+1} = h_3(x) + h_y(x),$$
and (3.52) follows from (3.51).

These relations also may be proved with generating functions, by bisection – see
(3.53)–(3.57) By taking \(x-y \in \mathbb{N}_0\), small, in
(3.49)–(3.52) the right-hand sides become simpler by (1 12).

\[ x = 2y+1 \quad \text{in (3.49)}, \quad \text{gives (3.57)}. \]

The relations (3.53)–(3.56) also may be derived by considering the generating
functions \( (1+z)^y \) and \( (1-z)^y \),
and \( (1+z)^x \) and \( (1-z)^x \),
for which see, for example, Riordan (1968), Ch. 4.3, Example 4; Egorychev
(1984), Ch. 2.4.7; Netto (1927), Ch. 13, \S\ 162, 163.

Still another method is combining

\[ (3.53) \] of respectively, obtaining and subtracting (3.39) with \( y = 2n - x \) and (3.70).

(3.61) We have \( (1-z)^x (1-z)^{-x} = (1+z+z^2)^x \), and for sufficiently small \(z\)

\[ (1-z)^x (1-z)^{-x} = \sum_{n=0}^{\infty} z^n \sum_{3hn+j=n} \binom{x}{h}(-1)^{h+j} = \]
\[ \sum_{k=0}^{n-1} \binom{n-1}{k} (u-k)^{\nu} + \sum_{l=0}^{n-1} \binom{n-1}{l} (u-1-h)^{\nu} = \binom{\nu}{u} + \binom{\nu+n-1}{u-1} = \binom{\nu+n}{u}, \]

with a generalization of \( D(18) \) that follows from \( D(16) \). The relation \( (3.62) \) is a curious extension of Vandermonde's convolution \( (3.39) = D(26) \).

\[ (3.63) \text{ With } D(13), \text{ also for } k=0 \text{ when we define } \binom{n}{-1} = 0, \]

\[ \binom{n}{k} - \binom{n}{k-1} = \frac{1}{n+1} \binom{n+1}{k} (n+1-2k). \]

So the l.h.s. of \( (3.63) \) is equal to
\[ a(n+1) (n+2) \left( \begin{array}{c} 2n \n+1 \end{array} \right) = \frac{1}{2} (2n+1)^{-1} \left( \begin{array}{c} 2n+2 \\ n+1 \end{array} \right) = \frac{1}{n+1} \left( \begin{array}{c} 2n \\ n \end{array} \right), \]

where in the last lines we applied (3.112) or (3.113) and D(13).
(3.66) By generating functions, see Theorem M1 from the relation
\[(1-z)^x (1+z)^y = (1-z^2)^x (1+z)^y - x \]

(3.67) From the convolution property (Theorem M1) of generating functions and the relation
\[(1-z)^x (1+z)^x = (1-z^2)^x,\]
or from (3.66) with \(y = x\),

(3.68), (3.69) From (3.66) and D (2y^a),
\[\sum_{k=0}^{n} (-1)^k \binom{y+1}{k} (n-k) = \sum_{2k \leq n} (-1)^k \binom{y+1}{k} (n-2k) = (-1)^n \sum_{2k \leq n} (-1)^k \binom{y+1}{k},\]
and (3.68), (3.69) follow with (1.3).

(3.70) Both sides of (3.70) are polynomials in \(x\) of degree \(\leq n\). So it is sufficient to prove
(3.70) for \(x = z\), \(z = 0, 1, \ldots, n\). Then from
D (2), D (13) and by rearranging factorials

\[\frac{c_n (z^n - c_n)}{n! n_n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (z-k).\]

By (3.67) this is zero for \(z = 2n+1\), as is the r.h.s.
of (3.70) for \( x = 2s + 1 \leq n \). For \( r = 2s \) we obtain by (3.67) and (13) and by (B.21)
\[
\frac{(2s)! (2n-2s)!}{n! n!} (-1)^s (n-s) = (-1)^s \frac{(2s)! (2n-2s)!}{n! n!} = (-1)^{n-s} \frac{(2s)! (2n-2s)!}{n! n!}.
\]

(3.72) From (3.66) with \( x = \frac{1}{2}, y = -\frac{1}{2} \). Then (1.3) and (B.13).

(3.73) With (1.3), D (14), and D (14)
\[
\sum_{i+j \leq n} (-1)^{i+j} \frac{x^i y^j}{(i! j!) (x-i-j)!} = \sum_{i=0}^{n} (-1)^i \left( \begin{array}{l} x \\ i \end{array} \right) \sum_{j=0}^{n-i} (-1)^j \left( \begin{array}{l} x-i \\ j \end{array} \right) = \sum_{i=0}^{n} (-1)^i \left( \begin{array}{l} x \\ i \end{array} \right) \left( \begin{array}{l} n-x \\ n-i \end{array} \right).
\]

With D (13) and (1.1)
\[
\sum_{i+j \leq n} (-1)^{i+j} \frac{x^i}{(i! j!)} (x-i-j)! = \sum_{r=0}^{n} (-1)^r \frac{x^r}{(x-r)!} \sum_{i+j=r} \frac{1}{i! j!} = \sum_{r=0}^{n} (-1)^r \left( \begin{array}{l} x \\ r \end{array} \right) \sum_{i=0}^{r} \frac{1}{i!} = \sum_{r=0}^{n} (-1)^r \left( \begin{array}{l} x \\ r \end{array} \right) 2^r.
\]
where changing the summation is justified since

\[
\sum_{h=0}^\infty \left| \binom{n}{h} \right| z^h \sum_{i=0}^{\infty} \binom{h+i-1}{i} |z|^{i+h} = \sum_{h=0}^\infty \left| \binom{n}{h} \right| z^h |1-|z||^{-h} < \infty.
\]

The second equality follows similarly with

\[
\sum_{n=0}^\infty q_n(x) z^n = \left( \frac{1-z}{1+z} \right)^{-x} = \left( 1 - \frac{z}{1+z} \right)^{-x},
\]

or by noting that \( q_n(-x) = (-1)^n q_n(x) \).

The sequence \( q \) is of convolution type and even a classical $\ast^n$ sequence. This follows from Chapter C, especially C (10), Theorems C, 3, 4, and 6. The delta operator is
\(-1\) \binom{m}{k} = \sum_{k=0}^{m} (-1) \binom{k}{k} \binom{2m-k}{m-k} \\
\sum_{k=0}^{m} (-1)^k \binom{x}{k} \binom{x}{2m-k} + \sum_{h=0}^{m} (-1)^{2m-h} \binom{2m-h}{h} \binom{x}{h} + (-1)^{m+1} \binom{x}{m} \\
= 2 \sum_{k=0}^{m} (-1)^k \binom{x}{k} \binom{x}{2m-k} + (-1)^{m+1} \binom{x}{m},

(3.76) For \( n \geq m \) the r.h.s. with D (12), by rearranging factorials and with (3.67) is equal to

\sum_{k=-m}^{m} (-1)^k \binom{2m}{m-k} \binom{2n}{n-k} =

\sum_{h=0}^{2m} (-1)^{h-m} \binom{2m}{2m-h} \binom{2n}{n+m-h} =

\frac{(2m)! (2n)!}{(m+n)!^2} \sum_{h=0}^{2m} (-1)^{h-m} \binom{m+n}{h} \binom{m+n}{2m-h} =

\frac{(2m)! (2n)!}{(m+n)!^2} \binom{m+n}{m} = \frac{(2m)! (2n)!}{m! n! (m+n)!}.

For \( n \leq m \) the relation follows by symmetry.
(3.81) With generating functions. From
\[(1 - z^y)^x (1 - z)^{-x} = (1 + z^y)^x (1 + z)^{-x}, \]
or
\[(1 - z^y)^x (1 + z)^{-x} = (1 + z^y)^x (1 - z)^{-x}, \]

(3.82), (3.83) From (3.77) with \( n = 2m \) and \( n = 2m+1 \), respectively, by putting \( k = m-h \).

(3.84a,b,c) From (3.73) and D (18) for \( n \geq 1 \),
\[
\sum_{k=0}^{n} (-2)^{r} \binom{x}{k} = \sum_{k=0}^{n} (-1)^{r} \binom{k}{r} \binom{n-1-x}{n-k} + \\
\sum_{k=0}^{n-l} (-1)^{r} \binom{k}{r} \binom{n-1-x}{n-1-k} = \\
\sum_{k=0}^{n} (-1)^{r} \binom{k}{r} \binom{n-1-x}{n-k} + \sum_{r=0}^{n-1} (-2)^{r} \binom{x}{r},
\]
and (3.84a) follows. It holds trivially for \( n = 0 \). From (3.39) for \( n \geq 1 \)
\[
\sum_{k=0}^{n} \binom{x}{k} \binom{n-1-x}{n-k} = \binom{n-1}{n} = 0.
\]
Adding (3.84a) to this relation and subtracting (3.84a) from it we obtain (3.84b) and (3.84c).

(3.85) From (3.51) with \( m = n \), \( x = n+1 \), \( y = n-1 \),
\[ \sum_{k=0}^{n} (-1)^{n-k} \binom{2u}{2k+1} \binom{n-u}{n-k} = \]

\[ -2 \sum_{j=0}^{n} (-1)^{n+u} \binom{n-u}{2j} \binom{2n+1-2j}{2n+1} \cdot \]

From (3.846) with \( n \) replaced by \( 2n+1 \)

The relation (3.85) follows from (\( \star \star \)) with \( x = n+u \) and from (\( \star \)).

The relation (3.85) is, by D(24), the same as (3.158)

\[ \sum_{k=0}^{n-1} (-1)^{n-k} \binom{2u}{2k+1} \binom{n-1-u}{n-1-k} = \]

\[ \sum_{k=0}^{n} (-1)^{n-k} \binom{2u}{2k+1} \binom{n-1-u}{n-k} - 2^{2n-1} \binom{n+1+u}{2n-1} \cdot \]

and (3.85) follows by applying D (14). For \( n=0 \) the relation (3.85) holds trivially.
\[ \sum_{k=0}^{n} (-1)^{n-k} \binom{2u}{2k} \binom{n-u}{n-k} = \]

\[ \sum_{k=0}^{n} \binom{n+u}{n-k} / \binom{n-u}{n-k} - 1 \]

Adding this relation to (3.73) we have

\[ \sum_{k=0}^{n} \binom{n+u}{n-k} / \binom{n-u}{n-k} = 1 + \sum_{k=0}^{\infty} (-2)^k \binom{n+u}{n-k} \]

Knowing that \( x = n+u \), we obtain

\[ \binom{n+u}{n-k} = \binom{n+u}{n-k} = 1 + \sum_{k=0}^{\infty} (-2)^k \binom{n+u}{n-k} \]

(3.88) For \( n=0 \) it holds trivially. For \( n \geq 1 \), from (3.87) and \( \Delta \) (18)

\[ \sum_{k=0}^{2n} (-2)^k \binom{n+u}{n-k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{2u}{2k} \binom{n-u}{n-k} + \]

\[ \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{2u}{2k} \binom{n-1-u}{n-1-k} = \]

\[ \sum_{k=0}^{n} (-1)^{n-k} \binom{2u}{2k} \binom{n-1-u}{n-k} - \sum_{k=0}^{2n-2} (-2)^k \binom{n+u}{n-k} \]

So, again with \( \Delta \) (18),
\[
\sum_{\varsigma=0}^{2n-2} (-1)^{\varsigma} \binom{n-1+\varsigma}{\varsigma} = \ldots
\]
A different proof of (3.89) is by taking generating functions w.r. to \( n \) of both sides of (3.89) and equating them.

Both sides of (3.89) are polynomials in \( h \) of degree \( n \). So, by Newton's interpolation formula \( C(1) \) they are equal when their \( n \)th differences at \( h=0 \) coincide for \( n = 0, 1, \ldots, n \). This may
\[ \binom{a}{n} \binom{a+b-n}{n} = \binom{n}{x} \binom{n}{1} \]  (3.90) From (3.89) with \( a = x \), \( b = n \), since

By D. (14)

\[ \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} x^{n-k} y^{k} = \sum_{h=0}^{n} \binom{n}{h} \binom{n-h}{n-h} x^{n-h} y^{h} \]

\[ n \leq n \quad \text{in} \quad \{n \leq h \leq n-1 \} \]

\[ y \leq i=0 \binom{i}{j} \binom{j}{i} \text{ for } 0 \leq i \leq n \]
\[ f^{(k)}(0) = \sum_{k=0}^{n-\varepsilon} \binom{n}{k} \binom{n-k}{k} \frac{(\varepsilon-1)!}{(n-k-\varepsilon)!} \, x^{n-k-\varepsilon} \, y^k \bigg|_{x=0} \]

\[ (-1)^{n-\varepsilon} \frac{n!}{(n-\varepsilon)!} \sum_{i=0}^{n-\varepsilon} (-1)^i \binom{n-\varepsilon}{i} (\varepsilon+i) = \]

\[ (-1)^{n-\varepsilon} \frac{n!}{(n-\varepsilon)!} \sum_{i=0}^{n-\varepsilon} \binom{n-\varepsilon}{i} (-\varepsilon-1) = \]

\[ (-1)^{n-\varepsilon} \frac{n!}{(n-\varepsilon)!} \left( \frac{n-\varepsilon-b-1}{n-\varepsilon} \right) = \gamma^{(n-\varepsilon)} \frac{n!}{(n-\varepsilon)!} (\varepsilon-b) = f^{(n)}(0). \]

See Egorychev (1984), Ch. 2.1.2, showing that there is a proof with generating functions (cf. the proof of (3.89)).

Also Knopp (1975), pp. 166, 170, where (3.91) is attributed, as in Gould (1972), to Ljunggren (1947). Cf. (3.178), (3.181).

(3.92) We apply $G(a,b)$ to $f(x) = \binom{x}{m}$ and $g(x) = z^x$ and then to $f(x) = \binom{x}{m}$, where $z > 0$. In the first case we obtain
\[
\sum_{k=0}^{n \wedge m} \binom{n}{k} \binom{x}{m-k} (z-1)^{n-k} z^{x+k}
\]
and in the second case,
\[
\sum_{k=0}^{n} \binom{n}{k} (z-1)^{k} z^{x} \binom{x+k}{m-n+k} =
\]
\[n=0\]

The formula \( C (x) \), for a general \( n \)th dif-

Dividing by \( z^{x} \), we obtain the equality
of the second, third and first member of
(3.92). Since these are polynomials in \( z \), the
equality holds for \( z \in \mathbb{C} \). First eq. also from (3.89).

(3.93) The first equality follows from (3.90)
with \( z = \overline{z} \), putting \( n - j = i \). The second
equality then follows by \( D (14) \), rearranging
factorials.

Ch. 4.3, Example 6, and Prob. 276, Nieuw
Arch (3), 19, 1971, 237–238. Also Monthly 99,
1992, p. 723.
\[
\left\langle \xi = 0 \right\rangle \left\langle j = 0 \right\rangle (\nu^n) (\nu^+)^n - \sum_{n=0}^{\infty} Z^n \sum_{i=0}^{\infty} \left( \frac{x}{z} \right)^i (\nu^+) \frac{Z^{2k}}{y^k (1 + y^{2k-1})} \]

\[
\sum_{n=0}^{\infty} \left| \left( \frac{x}{z} \right)^n \right| (\nu^+) \frac{Z^{2k}}{y^k (1 + y^{2k-1})} < \infty \]

for small \( z \). This proves the first equality.
\[
\sum_{k=0}^{\infty} |(x)^k y^{-k} u-v|^k \sum_{i=0}^{\infty} |(x-2)^i y^{-i} u+v|^2 z^{2k+i} \\
\sum_{k=0}^{\infty} |(x)^k y^{-k} u-v|^2k \sum_{i=0}^{\infty} (2|x|+2k+i-1) |y^{-i} u+v|^2 z^{2k+i} \leq -2|x|-2k
\]

A different proof of the first equality is by Newton's interpolation formula: \(C(ab) = G(h)\). Denoting the l.h.s. of (3.94) by \(f(x)\)

For \(x=0\) all terms except \(j=k\) vanish. When \(r<k\) there is no such term but this is accounted...
and \( \Delta^r f(0) = 0 \), \( 2r \leq n \), which proves the first equality. The relation \((***)\) follows by differentiating

\[
\sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k} = (u + v)^n
\]

\( n - \varepsilon \) times w.r. to \( u \). From \((**)\) we obtain with \( G \) \((22)\) the relation

\[
(***) \sum_{i=0}^{\infty} \frac{(-1)^{i}}{i!^2} \binom{n}{i} \binom{i}{k} \binom{i}{n-k} = \binom{n}{k} \binom{k}{n-k},
\]

\( \varepsilon \leq n \). See also \( F(91) \).

\((3.9b)\) From \((3.89)\) and \( D \) \((24)\)

\[
\sum_{k=0}^{n} \binom{x}{k} \binom{n-1-y}{n-k} t^k = \sum_{j=0}^{n} \binom{x}{j} \binom{x-y+n-1-j}{n-j} (-1)^j (t-1)^j
\]

\[
= \sum_{j=0}^{n} (-1)^{n-j} \binom{x}{j} \binom{x-y+n-1-j}{n-j} (t-1)^j
\]
\[ \sum_{k=0}^{\infty} \binom{x}{k} z^k z^k = \sum_{m=0}^{\infty} \binom{m+k-1}{m} z^m = \]

With the product property (Theorem M1) of generating functions we obtain, respectively, \((3.96)\) for \(t=2\), the first equality in \((3.97)\) and the second equality in \((3.97)\). For \(y=x\) we obtain \((3.98)\), either from \((3.97)\) or from the above generating function.
\[
\sum_{k=0}^{\infty} \frac{(-x)}{(k+1)} \left( \frac{z}{k+1} \right)^k \sum_{n=k}^{\infty} \binom{n}{k} (-\lambda z)^{n-k} = \\
\sum_{k=0}^{\infty} \frac{(-x)}{(k+1)} \left( \frac{z}{k+1} \right)^k \sum_{m=0}^{\infty} \binom{m+k}{m} (-\lambda z)^m = \\
\sum_{k=0}^{\infty} \frac{(-x)}{(k+1)} \left( \frac{z}{k+1} \right)^k \left( 1+\lambda z \right)^{k-1} = z^{-1} \left\{ \left( 1+ \frac{z}{1+\lambda z} \right)^x - 1 \right\} = \\
z^{-1} \left\{ \left( 1+\lambda z \right)^{-x} \left( 1+\lambda z+k \right)^{x} - 1 \right\} = \\

\sum_{k=0}^{\infty} z^n \sum_{k=0}^{n+1} \frac{(-x)}{(k)} \binom{x}{n-k} \left( \lambda z \right)^{n-k} - k-
\]

where interchanging summations is justified since

\[
\sum_{k=0}^{\infty} \left| \binom{x}{k} \right| \left| z \right|^k \sum_{n=k}^{\infty} \left| \frac{n}{n-k} \right| \left| \lambda z \right|^{n-k} = \\
\sum_{k=0}^{\infty} \left| \binom{x}{k} \right| \left| z \right|^k \left( 1-\lambda z \right)^{-k-1} < \infty
\]

for small \( z \).

The proof of (3.100) is similar and simpler.
\[ \sum_{k=0}^{\infty} \binom{k}{n} z^k = \sum_{k=0}^{\infty} \frac{(n)\cdots(n-k+1)}{k!} z^k = \frac{(n)\cdots(n-k+1)}{k!} \frac{1}{1+(\lambda+1)z} \]

See also C (185), (186).

(3.102) - (3.105). The relation (3.102) is (3.104) with \( x = 1 \) and (3.103) is (3.105) with \( x = 1 \).

To prove (3.102) and (3.104) we note that by B (16), B (17), and D (14)

\[
\binom{n - \frac{1}{2}}{k} \binom{n + \frac{1}{2}}{n - \frac{1}{2}} = \frac{4^{-k} (2n)! (n - k)!}{k! (2n - 2k)! n!} \frac{4^{n-k} (2n+1)! k!}{(n-k)!(2k+1)\cdots n!} =
\]

Now apply F (23a) - for \( x = 1 \) - see \( \Phi \) (30).

In the same way, from B (17) and D (14),
\[ \frac{n^{-n} (2n+1)!}{n! \cdot n! (2n+2)^{2k+1}} = \mu^{-n} \binom{2n+2}{n+1} \binom{2n+2}{2k+1}, \]

is in *Fib. Q. 34* (1), 1946, 90-91 (Prob. H-49). (3.106) - (3.109) The relation (3.106) is (3.108) with

To prove (3.106) and (3.108) we note that by B (16) and D. (14)

\[ \binom{n - \frac{1}{2}}{k} \binom{n - \frac{1}{2}}{n-k} = \]

\[ \mu^{-k} \frac{(2n)! \cdot (n-k)!}{k! \cdot (2n-2k)! \cdot n!} \frac{\mu^{-(n-k)} (2n)! \cdot k!}{(n-k)! \cdot (2k)! \cdot n!} = \mu^{-n} \left( \binom{2n}{k} / \binom{2n}{2k} \right). \]
\[(k)_n \binom{n-k}{2n+1} \frac{(2n)!}{(n-k)!} = \frac{n-k}{(n-k)(2n)!} k!
\]

\[\binom{n}{k}(2k)!
\]

and (3.107) follows with E (23), and (3.109) with \(\phi\), (31).

(3.111) From \(D (8), (9), (14)\) and (3.39) = \(D (26)\)

\[\sum_{h=0}^{\infty} \frac{(a-r)}{h} \frac{(n-r-h)}{h} = (a)_r \frac{(a+r-r)}{n-r}
\]

The second equality follows with \(D (8), (9), (14)\). For \(a \in \mathbb{N}_0, b \in \mathbb{N}_0, 1 \leq n \leq a+b\), we obtain the \(r\)th factorial moment of the hypergeometric probability distribution \(D (28\).

(3.112) Putting

\[\left( k - \frac{na}{a+b} \right)^2 = k(k-1) + k - 2k \frac{na}{a+b} + \frac{n^2 a^2}{(a+b)^2}
\]
For \( a \in \mathbb{N}, b \in \mathbb{N}, 1 \leq n \leq a+b \), the relation (3.112) gives the variance of the hypergeometric probability distribution \( D(26'a) \).

\[ r = 0 \quad \text{or} \quad D'(26) = (3.39) \]. For \( 1 \leq j \leq n-1 \) since

\[ ra - k(a+b) = (n-k)(a-k) - k(b-n+k), \]

the l.h.s. of (3.115) is equal to

\[ j \quad a! \quad b! \]

\[ \sum_{k=1}^{r} \frac{(k-1)! (a-k)!}{k! (a-k-1)!} \quad \frac{(n-k)! (b-n+k-1)!}{(n-k-1)! (b-n+k)!} - \]

\[ \sum_{h=0}^{n-1} \frac{a!}{h! (a-h-1)!} \quad \frac{b!}{(n-h-1)! (b-n+h)!} \]
\[ \frac{a!}{j!(a-j-1)!} \frac{b!}{(n-j-1)!(b-n+j)!} = \left( \frac{a}{b} \right)^j \left( \frac{a}{b} \right)^i \frac{a}{b} \]

Continuity for \( a \in C \), \( b \in C \).


(3.116) This is, apart from the factor \( \left( \frac{a+b}{n} \right)^{-1} \),
the absolute first central
moment of the hypergeometric probability
distribution \( D \) (26a). From the relation
(2x) in the proof of (1.89) and from (3.115)
the l.h.s. is equal to

\[ \frac{1}{a+b} \left( \frac{a}{a+b} \right)^j \left( \frac{b}{a+b} \right)^i \left( \frac{a+b}{n-j} \right). \]

(3.119) From (3.42) with \( x \) replaced by \( n \) and
\( m \) by \( m+n \) with \( m \geq 1 \) fixed

\[ \sum_{k=0}^{\infty} \binom{n}{k} \left( \frac{-x}{n+k} \right) = \sum_{h=0}^{\infty} \binom{x}{h} \left( \frac{-x}{m+n-h} \right) = \]

\[ \frac{m}{m+n} \left( \frac{x-1}{n} \right) \left( \frac{-x}{m} \right). \]

We may write this as \( \chi_n = \sum_{k=0}^{\infty} \binom{x}{n-k} x_k \).
\( n \in \mathbb{N}_0 \), with \( x_f = \left(\begin{array}{c} -x \\ m+k \end{array}\right) \). With the inverse pair \( IR(19) \) we then obtain the companion, in the sense of \( p. IR9 \), of the above relation:

\[
\left(\begin{array}{c} -x \\ m+n \end{array}\right) = \sum_{k=0}^{n} \binom{n}{k+1} \left(\begin{array}{c} -x \\ n-k \end{array}\right) y_k = \\
\sum_{k=1}^{n} \left(\begin{array}{c} -x_k \\ k \end{array}\right) \frac{m}{x_k-1} \left(\begin{array}{c} x_k-1 \\ x_k \end{array}\right) \\
+ \sum_{k=0}^{n} \binom{n}{k} \frac{-x_k}{x_k-1} \left(\begin{array}{c} x_k-1 \\ x_k \end{array}\right).
\]

\[
\frac{1}{2} m (y-m) (2y-1)^{-1} \binom{2y}{2m}, \; \gamma \neq \frac{1}{2}.
\]
(3.121) With \( G(z^3) \) the sum is equal to
\[
\frac{1}{2} \sum_{k=0}^{2m+1} \left( \begin{array}{c} 2m+1 \\ k-m-\frac{1}{2} \end{array} \right) (z^2)^{k} (2m+1-k).
\]

For \( r=0 \) we obtain (3.41). For \( r=1 \) we obtain (3.116) with \( n=2m+1 \) \( a=k \leq M \).

And (3.121) are related to the absolute central \( r \)th moment of a symmetric hypergeometric distribution, cf. \( D(2\alpha), (3.116), (3.111), (3.112) \).
(3.130) For $\tau > n$, both sides are zero by $D(12)$. For $\tau \leq n$, the l.h.s. by $D(13)$ and (1.1) is equal to

$$\sum_{k=\tau}^{n} \frac{n!}{(n-k)!} \frac{1}{(\tau-k)!} = (\tau) \sum_{h=0}^{n-\tau} (\tau-h) = (\tau)^{n-\tau}. $$

(3.132), (3.133) For $2\tau > n$ the left-hand sides are zero by $D(12)$. Proofs by generating functions, different from the proofs in Chapter 7, are as follows. By $D(25),$

$$\sum_{n=0}^{\infty} \sum_{2k\leq n} \frac{(n)}{2k} \frac{(k)}{n} z^n = \sum_{k=\tau}^{\infty} \frac{(\tau)}{n} \sum_{m=0}^{\infty} \frac{(m+2k)}{m} z^{m+2k} =$$

$$\sum_{k=\tau}^{\infty} \frac{(\tau)}{n} z^{2k} \frac{1}{1-z}^{2k-1} = \sum_{j=0}^{\infty} \frac{(\tau+j)}{n} z^{2j} (1-z)^{-2j-1} = \sum_{j=0}^{\infty} \frac{(\tau+j)}{n} z^{2j} (1-z)^{-2j-1} = \sum_{j=0}^{\infty} \frac{(\tau+j)}{n} z^{2j} (1-z)^{-2j-1} = \sum_{j=0}^{\infty} \frac{(\tau+j)}{n} z^{2j} (1-z)^{-2j-1} = \sum_{j=0}^{\infty} \frac{(\tau+j)}{n} z^{2j} (1-z)^{-2j-1}.$$
\[
\sum_{k=r}^{\infty} \binom{k}{r} \sum_{m=0}^{\infty} \binom{m+2k+1}{m} z^{m+2k} = \sum_{k=1}^{\infty} \frac{1}{k!} x_k \quad y_k - x_k - 2
\]

Write (3.133) as
\[
\sum_{k=0}^{n} \binom{n}{k} x_k = y_n, \quad n = 0, \ldots, n,
\]
with \( x_k = 2^{-n} \binom{n+1}{2k+1} \), \( y_n = 2^{-n} \binom{n-1}{n-1} \).

Let \( Y \) be the number of pairs \((X_i, X_{i+1})\), \(i = 0, \ldots, n-1\) with \( X_i = 0, X_{i+1} = 1 \) and \( A_j \) the event \( \{ X_j = 0, X_{j+1} = 1 \} \), \( j = 0, \ldots, n-1 \), so that \( Y \) is the number of events that occur. We now apply IR (17d). We have
otherwise, the number of good sequences \(i_1, \ldots, i_k\) is \(\binom{n-k}{k}\), see Sved (1982), Riordan (1958), Ch. 5.3.

\[
y_k = \frac{n}{n-k} \left( \frac{n-k}{k} \right)^{n-k} \xi, \quad \xi = 0, \ldots, n-1, \quad y_n = 0.
\]

The main relation is (13).

\[
y_k = \frac{\xi}{n-k} \left( \frac{n-k}{k} \right)^{n-k} \xi, \quad \xi = 0, \ldots, n-1, \quad n \geq 1. \text{ The relation (y) is given below as (3.49). Now consider the same proba-}
\]
bility model as above with the modification that the event $A_i = \{ X_i = 0, X_j = 1 \}$ also counts, i.e. the "letters" $X_1, \ldots, X_n$ are placed on the circumference of a circle. With this $A_i$, and with $A_1, \ldots, A_n$ as above, let $Z$ be the number of $A_i$ that occur, $i = 1, \ldots, n$. Again apply IR (170). We have, for $1 \leq i < \cdots < j \leq n$, that $P(\overline{A_i}, A_j, \ldots, A_k) = 2^{-k}$ if the points $i, \ldots, j$ on the circle are separated by at least

\begin{equation}
\mathcal{P}(Z = \varepsilon) = 2^{1-n} \binom{n}{\varepsilon}, \quad \varepsilon = 0, \ldots, n-1, \quad n \geq 1,
\end{equation}

Ch. 4, Problem 15; Ch. 6, Problem 10, Hewa (1985).

(3.134), (3.135) From B. (47) and D. (11) we see that the series in (3.135) converges absolutely for $Re y > \varepsilon$. We start from

\begin{equation}
\sum_{k=0}^{\infty} \left( \frac{y}{k+2} \right) = \left( 1 + \left( \frac{y}{k+2} \right) \right)^{1/2},
\end{equation}
\[
(2) \sum_{k=0}^{\infty} \binom{n}{k} (k/z)(1+z)^{k-\gamma} = \frac{1}{z} \sum_{k=0}^{\infty} D^{\gamma} (1+(1+z)^{\frac{k}{2}})^n.
\]

Now let \(z \to 0\) in (2), by extending \(A\). Then

For \(z \geq 0\), say, we have
\[
\sum_{1}^{n} \binom{n}{k} (1+z)^{k/2} = \left(1+\left(1+z\right)^{1/2}\right)^n.
\]

From (3) we have
\[
\left(1+\left(1+z\right)^{1/2}\right)^{n} = \sum_{k=0}^{\infty} c_k z^k,
\]
where \( c_n = \frac{n}{n-r} \binom{n}{r} 2^{n-r} \), \( n \neq r \),
\[
c_n = (-1)^{n-1} 2^{-n} = \lim_{y \to n} \frac{y}{y-n} \binom{y-n}{n} 2^{-n}, \quad n \neq 0
\]
c_0 = 1, \quad n = r = 0 , \text{ and } (3.134) \text{ follows. (with (2))}
\[
\sum_{i \leq n} \binom{n-i}{l-1} = (n \choose l).
\]

(3.137) From (3.39) and D. (12), since \( k \leq n \) in the sum,
\[
\sum_{2k \leq n+1} \binom{n+1}{2k} \binom{x+k}{n} = \\
\sum_{2k \leq n+1} \frac{x+1}{2} \binom{n+1-2k}{i} \binom{1-x}{n-i} = \\
\sum_{2i \leq n+1} \frac{x+1}{2} \binom{n+1-2i}{i} \binom{2x+2}{n+1} = \\
\frac{1}{2} \frac{n+1}{x+1} \binom{n+1}{n}.
\]

(3.138) We have, for small \(z\), with (1.50a)
\[
f_n(z) = \sum_{k=0}^{\infty} z^k \sum_{2k \leq n} \binom{n}{2k} \binom{k-\ell}{\ell} = \\
\sum_{2k \leq n} \binom{n}{2k} (1+z)^{k-\ell} = \\
\frac{1}{2} (1+z)^{\ell/2} \left\{ (1+(1+z)^{\frac{1}{2}})^n + (1-(1+z)^{\frac{1}{2}})^n \right\} = 
\]
\[ n-1 \leq \frac{1}{2}, \ldots, \frac{1}{2} \leq n \]

\[ \left( \begin{array}{c} n \ \\
\end{array} \right) \sum_{j=0}^{2^n} \binom{j}{n-j} y^j z^{n-j} = \sum_{n-1 \leq j \leq \infty} \binom{n-1-2\frac{j}{n}}{n-1} \frac{1}{n-2-j} \cdot \]

and (3.138) follows, again with D (24).

(3.139) The relation (3.139a) is trivial. Now let

\[ n \geq 1. \text{ With (1.50b)} \]

\[ \sum_{k=0}^{\infty} \frac{z^k}{2k+1} \sum_{2k+1 \leq n} \binom{n}{2k+1} \binom{k+1/2}{k} = \]
\[
\sum_{2k+1 \leq n} \left( \binom{n}{2k+1} \right) (1+z)^{k+1/2} = \\
\frac{1}{2} \left( 1 + (1+z)^{1/2} \right)^n - \frac{1}{2} \left( 1 - (1+z)^{1/2} \right)^n = \\
2^{n-1} \left( \frac{1}{2} + \frac{1}{2} (1+z)^{1/2} \right)^n + (-1)^{n+1} 2^{-n-1} z^n \left( \frac{1}{2} + \frac{1}{2} (1+z)^{1/2} \right)^{-n}.
\]

By C (128\(a\)) and D (25) this is equal to

\[ x \to n \quad x-n \in n \quad \ldots \quad \]

since the coefficients in C (128\(a\)) are the polynomials of convolution type \( q(x, -1) \) in \( x \).

The relations (3.139\(a\)) and B (139\(b\)) now follow.

For \( r > n \) we note that by D (24).
\[(\gamma+1) - n \left\{ \frac{1}{\kappa-n} \left( \frac{2\gamma-n-1}{\kappa} \right) + \frac{1}{\kappa} \left( \frac{2\gamma-n-1}{\kappa} \right) \right\} = \text{P}79\]

\[\Omega(\gamma) \frac{\gamma+1}{2\gamma-n} \left( \frac{2\gamma-n}{\kappa} \right) \]

(3.140), (3.141) With \(D\), we have for small \(z\)

\[1 \sum_{n=0}^{\infty} z^n \sum_{2k \leq n} \left( \begin{array}{c} n \\ 2k \end{array} \right) \left( \frac{a+k}{\kappa} \right) t^k = \]

\[\sum_{k=0}^{\infty} \left( \frac{a+k}{\kappa} \right) t^k z^{2k} \sum_{m=0}^{\infty} \left( \frac{m+2k}{m} \right) z^m = \]

\[\sum_{k=\gamma-a}^{\infty} \left( \frac{a+k}{\kappa} \right) t^k z^{2k} (1-z)^{-2k-1} \]

\[t^\gamma-a z^{2\gamma-2a} (1-z)^{2\gamma-1} \sum_{h=0}^{\infty} \left( \frac{r+h}{h} \right) t^h z^h (1-z)^{-2h} = \]

\[t^\gamma-a z^{2\gamma-2a} (1-z)^{2\gamma+1} \left( 1-2z+(1-t)z^2 \right) \]

For \(t = 1\) this gives, with the convolution property Th. M1 of generating functions,

\[\sum_{n=0}^{\infty} z^n \sum_{2k \leq n} \left( \begin{array}{c} n \\ 2k \end{array} \right) \left( \frac{a+k}{\kappa} \right) = \]

\[z^{2\gamma-2a} \sum_{j=0}^{\infty} \left( \frac{r+j}{j} \right)^2 d_j z^j \sum_{h=0}^{\infty} \left( \frac{2a+1}{h} \right) t^h z^h = \]

\[z^{2\gamma-2a} \sum_{m=0}^{\infty} z^m \sum_{j=0}^{m} \left( \frac{r+j}{j} \right)^2 d_j (2a+1) (m-j) \left( \cdot \right)^{-m-j} \]
\[
\sum_{n=2e-2a}^{\infty} \sum_{j=0}^{n+2a-2e-j} \binom{n+2a-2e-j}{j} \epsilon^{(-1)} \cdot \text{(expression)}
\]
\[
\sum_{j=\varepsilon-1}^{n-\varepsilon} \frac{n!}{(n-j)!} \frac{j!(j+\varepsilon)!}{(j+1-\varepsilon)!} = \frac{1}{(n+\varepsilon)} \sum_{i=0}^{n-\varepsilon+1} \binom{n+\varepsilon}{i} i(i+\varepsilon),
\]
\[ \sum_{k=0}^{n} \binom{n}{k} (1+z)^{x+\frac{1}{2} k} = (1+z)^x \left( 1 + \sqrt{1 + 4z} \right)^n. \]

With the product property of generating functions, Theorem M1, we obtain the same as above, by \( C(189) \). By writing

\[ \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \left( \begin{array}{c} x \cr m \end{array} \right)^n = \]

\[ \sum_{n=0}^{\infty} \binom{n}{m} \left( x - \frac{1}{4} k \right) \left( x - \frac{3}{4} n \right) \left( \begin{array}{c} m \cr n \end{array} \right) \left( \begin{array}{c} x \cr m \end{array} \right)^n \]
The coefficient of $x^n$ is $a^n/m!$ by D. (11), and D. (19). The assertion for $n=m$ follows from G. (24b).

$$
\sum_{h=0}^{n+\varepsilon} (-1)^{n-h} \binom{n+\varepsilon}{h} \binom{b-\alpha t + \varepsilon a}{m} = \\
- \sum_{h=0}^{n-\varepsilon} (-1)^{n-h} \binom{n+\varepsilon}{h} \binom{b-\alpha t + \varepsilon a}{m}.
$$

$$
\sum_{k=\varepsilon}^{n-\varepsilon} (x-k)!r! (k-\varepsilon)!
\binom{n-\varepsilon}{k} (-1)^{n-\varepsilon+k} (x-\varepsilon)^{n-\varepsilon+k} = (x)^\varepsilon (x-\varepsilon-1)^{n-\varepsilon}.
$$
(3.152), (3.153) From (1) in the proof of (3.140) and (3.141) with $t = -1$, by $D$ (25),

\[
(-1)^{-a} x^{-a-1} \sum_{j=0}^{\infty} (r+j)^j z^j \sum_{h=0}^{\infty} \frac{(\gamma a+j+1)^h}{h!} z^h = \sum_{m=0}^{\infty} \sum_{j=0}^{m} (r+j)\frac{(\gamma a+j+1)^{m-j}}{m!} z^j (e^1)^{m-j}
\]
The convolution property theorem is:

\[
\sum_{m=0}^{\infty} z^m \sum_{n=1}^{\infty} (-1)^{n-k} \binom{n}{k} \left( \frac{n}{u+2k} \right) =
\]

\[
\sum_{m=n}^{\infty} z^m \sum_{k=0}^{n} \binom{n}{k} (m-n-k) z^{m-n-k},
\]

and the last assertion follows. This relation also may be derived from (3.155) with (3.39).
\[ \sum_{m=n}^{2n} \binom{n}{m-n} 2^{2n-m} z^m. \]
The l.h.s. is equal to
\[ \sum_{h=0}^{n} (-1)^{n-h} \binom{n}{h} \binom{v-2n+2h}{m}. \]
The first and second equalities now follow from (3.154) with \( u = v-2n \). By putting \( \text{l.h.s.} \)

... from the last equality in (3.154) with \( u = m-v-1 \). The third assertion in (3.154) does not give anything new here (apply D (24)).

Direct proofs may be obtained from C (37) with \( f(x) = \binom{x}{m} \) and C (38), and
(3.154) For \( n \leq m \) from (3.154) with \( u = 1 \) and (3.142)
\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2k+1}{m} = \sum_{j=0}^{n} \binom{n}{j} \binom{j+1}{m-n} =
\]
\[
\frac{m+1}{n+1} \binom{n+1}{m-n} 2^{n-m}.
\]
For \( n > m \) the sum is zero also by (3.154) with \( u = 1 \), or by (24a).
From the last equality in (3.154) with \( u = 1 \), or by putting, with \( D \), (18),
\[
\binom{2k+1}{m} = \binom{2k}{m} + \binom{2k}{m-1},
\]
and applying (3.155) we see that for \( m \leq n+1 \) the sum in (3.157) is equal to

\[
\binom{n}{m} = \binom{n-1}{m-1}, \ldots, \binom{1}{0}, \binom{0}{0}.
\]

and \( m = n = 0 \).

follows from (22) and (27).
those with \( rn - rk + m \geq n \), i.e. \( k \leq (r-1)n + m \), and those with \( rn - rk + m < 0 \), or \( k > n + m/r \), i.e. \( k = n+1 \) since

\[
(-1) \binom{n}{n-k} = - \binom{n}{k}, \quad \text{with } U \text{ (24)}.
\]

The only nonzero terms in this sum are those with \( rn - rk - m < n \), or \( rk \leq m \). The two sets are disjoint since

\[
\text{something implies } (r-1)n - m < rn.
\]

The contribution to the sum of the terms
\[ k = n \text{ and } k = n+1 \text{ is, with D. (24)} \]

\[ (-1)^n (n+1) \left( \frac{-m}{n} \right) + (-1)^{n+1} \left( \frac{-m}{n} \right) = \]

\[ S = X_1 + \cdots + X_n. \text{ The sums in (3.163), are equal, respectively, to } \epsilon^n P(S=j) \]

\[ \text{and } \epsilon^n P(S \leq j). \text{ See Feller (1957), Ch. XII.7, Problems 18-20, with proofs by generating functions on } j, \text{ with an error in 19. (Cf. Monthly} \]

\[ \left( \begin{array}{c} n \varepsilon \leq j \varepsilon \end{array} \right) k \left( k \right) \left( \begin{array}{c} n-j \varepsilon \\varepsilon \end{array} \right) \begin{array}{c} n-j \varepsilon \\varepsilon \end{array} \begin{array}{c} j=1, \varepsilon = n(\varepsilon -1), \varepsilon \end{array} \]
replacement, \( M \leq na \). Let \( Y \) be the number of colours that are not in the sample. We apply the formulas of inclusion–exclusion in Feller (1957) in Ch. IV.1, IV.3, IV.5. See IR (17).

Let \( A_i \) be the event that colour \( i \) is not in the sample. Then \( Y \) is the number of \( A_i \) that occur. In Feller's notation \( S_0 = 1 \) and 

\[
S_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} P(A_{i_1}, A_{i_2}, \ldots, A_{i_k}) = 
\]

\[
1 (Y \geq m) = \binom{n}{m} \sum_{k=m}^{n} (-1)^{k-m} \frac{(m-1)!(k-1)!}{(n-m-k)!} = 
\]

\[
m \binom{n}{m} \binom{na}{m}^{-1} \sum_{h=0}^{n-m} (-1)^h \binom{n-m-h}{h} \binom{n-m}{h} \frac{1}{n-h} .
\]

For the interpretation of these formulas as \((n-m)^{th}\) differences see (8.147) and (9.2).
\[ \binom{m}{k} = \sum_{n=0}^{\infty} \binom{m}{n} \Delta^n \left( \frac{x}{m} \right) \]

With \( \Delta^n \),

\[ \Delta^n \left( \frac{x}{m} \right) = \sum_{k=n}^{m} (-1)^k \binom{m}{k} \binom{2m-k}{k-n} x^{k-2m} \]

For \( x = 2m-n \) only the term \( k = n \) is nonzero.
\[
\begin{align*}
\left( \binom{x}{m} \right)^2 &= \sum_{k=0}^{\infty} \binom{m}{k} k^2 \\
\text{and} \quad &0, \ (3.39) \quad \text{and} \quad \left( \binom{x}{m} \right)^2 \\
\end{align*}
\]

\[
\begin{align*}
\left< m=0 \right. \left. z \right> \Delta \left( \binom{x}{m} \right) &= \Delta \left( (1+z)^n \right) = \\
(1+z)^{-\frac{1}{2}} (1+z)^{\frac{1}{2} x} &= z^n \left( \frac{1-(1+z)^{-\frac{1}{2} x}}{-z} \right) (1+z)^{\frac{1}{2} x} \\
\end{align*}
\]

\[
\begin{align*}
\left< j=0 \right. \left. \frac{n}{n+2j} \right> \left( \binom{x}{j} \right)^2 &= z^n \left( \frac{1-(1+z)^{-\frac{1}{2} x}}{-z} \right) (1+z)^{\frac{1}{2} x} \\
(1+z)^{\frac{1}{2} x} \sum_{n=0}^{\infty} \frac{n}{n+2j} (2h-n)^{-\frac{1}{2} x} 2^{n-2h} z^n &= \\
\text{cf. (3.170), (3.471)}
\end{align*}
\]
\[ (3.1.69) \quad \sum_{m=0}^{\infty} z^m \Delta^n (\frac{x}{m}) = \Delta^n (1+z)^{\frac{x}{n}} = \]
\[ \left( (1+z)^x - 1 \right)^n (1+z)^x \sum_{i=0}^{\infty} \binom{n}{i} x^{n-i} z^n i = \]
(3.172) From (3.77) with \( x = m \), by \( D(24) \) and \( D(27) \). Netto (1927), Ch. 13, (28). Cf. (3.156).
(3.176) For \( \tau > n \), both sides are zero by D (12).

For \( \tau \leq n \), the L.H.S. is equal to

\[ \sum_{i=0}^{n} \binom{n}{i} \binom{n-i}{k} (z-1)^{M-k} \]

\( \equiv \)

\[ \sum_{i=0}^{n} \binom{n-i}{i} (z)^{M-i} \]

where we applied (3.39) = D (26) and (3.176).

A combinatorial interpretation of (3.181) generalizes Example 5 in Ch. 2 of Wilf (1990). We use probabilistic language and notation. A sample of size \( n \) without replacement is taken from a set of \( M+N \) objects labelled 1, 2, ..., \( M+N \). Let \( A_c \) be
be the event that object \( i \) is not in the sample, \( i = 1, \ldots, M \) and \( X \) the number of \( A_i \) that occur, i.e. \( X \) is the number of objects 1, \ldots, \( M \) that are not in the sample. We apply the inclusion-exclusion generating function identity

\[
\left\{ M+1, \ldots, M+N \right\} \text{ and } n-a \text{ objects from } \{1, \ldots, M\}.
\]

In order to have \( X = j \) we must have \( M-n+a = j \), or \( a = n+j-M \). Since \( 0 \leq a \leq N \) we should have \( j \geq M-n \), \( j \leq M+N-n \). The condition \( 0 \leq n-a \leq M \) entails the obvious \( 0 \leq M-j \leq M \). So

for \( 0 \leq j \leq M \), \( M-n \leq j \leq M+N-n \), and zero elsewhere. Also

\[
P(A_{i_1} \ldots A_{i_k}) = \binom{M+N-n}{n} \binom{M+N}{n}^{-1},
\]
\[
\sum_{j=[M-n]}^{M} \binom{M}{j} \binom{N}{n-M+j} z^j = 
\]

(3) \[
\sum_{k=0}^{n} \binom{n}{k} \binom{N-n}{n-k} (z-1)^k.
\]

Putting \( j = M-h \) the l.h.s. in (3) is equal to
\[
\sum_{h=0}^{n} \binom{M}{h} \binom{N}{n-h} z^{M-h}.
\]
(3.182) \[
\sum_{n=0}^{\infty} z^n \sum_{h=0}^{n} \binom{n}{h} \frac{(n+h)(y-1)^{n-h}}{(n+h)!/(h-h)!} = \\
= \sum_{r=0}^{\infty} \binom{2r}{r} H^{-r} \left( 2z(y+1) - z^r(y-1)^{r} \right) = \\
\sum_{r=0}^{\infty} \binom{2r}{r} H^{-r} \left( y+1 \right) \left( y \right) \left( \text{and (3.102) follows within (10).} \right)
\]
Twice we changed summation orders. This is followed by absolute convergence of the double series for small \( z \), hence \( z \)
With $D(19)$ and $D(13)$, the l.h.s. is equal to

\[
\sum_{i=1}^{\frac{n+\varepsilon}{2}} \begin{pmatrix} \frac{n+\varepsilon}{2} \\ i \end{pmatrix} \sum_{j=1}^{\frac{n+\varepsilon}{2}} \frac{(-1)^{j-i}}{j} \begin{pmatrix} \frac{n+\varepsilon}{2} - j \\ \frac{n+\varepsilon}{2} - i \end{pmatrix}
\]

\[
\sum_{i=1}^{\frac{n+\varepsilon}{2}} \frac{(-1)^{i}}{i} \begin{pmatrix} \frac{n+\varepsilon}{2} - i \\ \frac{n+\varepsilon}{2} - i \end{pmatrix}
\]
The last equality in (3.189) follows with (13).

From (3.186) we see that for \( z = 0 \) the l.h.s.
of (3.189) is equal to \( 2^n (1-yy)^n \) \( P_n((1-yy)^{1/2}) \),
but this does not help very much.

Granville (1989) studies the determinants of matrices constructed from numbers \( a_n \), where \( a_n \), for fixed \( x \), is the coefficient of \( x^n \) in \( f(x)^n \). We have.

and we may differentiate \( z \) times term by term. This follows from the theorem on power series by applying Leibniz's formula term by term. By taking principal values of nonintegral powers we may extend the \( z \)-domain. This may be extended still further when
\((-1)^n \sum_{j=0}^{n} \binom{2n-x}{j} \binom{n-x-1}{n-j} =
\)

\(\sum_{\varepsilon=0}^{n} \binom{n}{\varepsilon} \binom{n+\varepsilon}{\varepsilon} x^\varepsilon.\)
(3.193) With Newton's interpolation formula \( C(x_1) \) we have:

\[ \delta_n (\frac{1}{2} x - \frac{1}{2}) \sum_{\nu = 0}^{n} a_\nu (x) \]

\[ c_k = (-1)^n \Delta^k \left( \frac{1}{2} x - \frac{k}{2} \right) \bigg|_{x=0} = (-2)^n \left( \frac{x^n - k}{n} \right) \]

For \( x=n \) we obtain (3.179).

The identity (3.193), in a notation to be compared by \( D(1)-(14) \), was also proved as Problem 4795, Monthly 66, 1959, 320, by recurrence w.r. to \( n \). Since the coefficients in Newton's interpolation formula are unique,
$b \text{ and } b^{28}$: The $b$ is used for $b^{28}$.
and (3.203) follows with (3.202), (3.147) and (14).
\[
\frac{x + ra}{x + ma} \left( \begin{array}{c} x + ma \\ m \end{array} \right),
\]
and put \( x = y - \varepsilon \).

To the third member apply (3.206). In the sum then we must have \( k \leq m \). The factor \( \binom{n}{k} \) takes care of the case \( n \leq m \).

Again with (3.206) the last member is equal to
\[
\sum_{k=\lceil n-m \rceil}^{n} \binom{n}{k} (u-1)^k u^x \frac{x+k+(n-k)a}{x+k+ma} \left( m-n+k \right).
\]

noting \( G(22) \) and \( G(23) \). 

(3.212) Recall \( G(22) \) and the Vandermonde con-
The first equality in (3.218) follows from \( D(12) \). We have from \( D(25) \)

\[ \sum_{n=0}^{\infty} z^n \sum_{k=0}^{\infty} (\kappa)(n-k) = z^{\kappa+s} (1-z)^{-\kappa-s} \]

\[ \sum_{n=0}^{\infty} (\kappa+s+n+1)_{-s+n} \]

\[ \sum_{n=1}^{\infty} (n+1)_{-n} \]
Applying again the convolution property

\[ (-z)^{\varepsilon-a} (1+z)^{-\varepsilon-p-1} Z^{s-k} (l-z)^{-s-k} \]

where the sum is for \( \varepsilon-a \leq k \leq n-s+b \), so we must have \( n \geq \varepsilon+s-a-b \).
\[ (-1)^{r-a} z^{r+s-a-b} \sum_{j=0}^{\infty} \left( \frac{z + b + j}{j} \right)^{2j} = \]

\[
\Gamma(r-a \geq 1 + \frac{1}{2}(n+a+b-t-s)) \Gamma(n)
\]


Interpretation of (3.218): The number of samples of size \( r+s+1 \) without replacement from \( \{1, \ldots, n+1\} \) is \( \binom{n+1}{r+s+1} \). The number of such samples where the \( (r+1) \)th increasing order statistic is \( k+1 \), necessarily with \( r \leq k \leq n-s \), is \( \binom{k}{r} \binom{n-k}{s} \). Cf. David and Barton (1962), p. 11. For \( a = r+1 \) in (3.220), (3.221) see (3.529), (3.530).

\[
\binom{m}{r} \binom{m}{r} = \binom{m}{k-r} \binom{k-r}{r} = \binom{r+m}{n-r} \sum_{h=0}^{n-r} \binom{r+m+h}{h} = \binom{r+m}{m} \binom{m+n+1}{n-r}.
\]

Theorem stated in...
This is done by some computation, with

a relation found in the solution of Prob. 6550, Monthly 96, 1989, 655-657. Taking
\( x = \sqrt{1 + i} \) we see that the l.h.s. of (3.225)

\[(3.226) \text{ The l.h.s. converges absolutely for } |z| < 1. \text{ Then for small } z(1+y) \text{ with } (1394)\]

\[
\sum_{\ell=0}^{\infty} y^\ell \sum_{j=0}^{\infty} z^j \left( \binom{j}{\ell} \binom{\ell}{s} \right) = \\
\sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} z^j (1+y)^{\ell} = Z^{\ell/(1+y)} (1-z-zy)^{s-1} \\
= Z^s (1-z)^{-(s-1)} (1+y)^s \sum_{k=0}^{\infty} \binom{s+k}{k} z^k (1-z)^{-s} y^k,
\]
and the first equality follows with the product property of generating functions (Theorem M1).

\[
\sum_{k=0}^{\infty} \left( \frac{a+k}{a-1} \right) z^k = \sum_{i=1}^{\infty} \left( \frac{a+i-1}{i} \right) z^{i-1} = -1 \quad \text{for} \quad z \neq -1
\]

\[
\sum_{k=0}^{\infty} \left( \frac{a+k}{a-1} \right) z^k = \frac{1}{2z} \left\{ (1-z)^{-a} - (1+z)^{-a} \right\}
\]

\[
\left\{ \sum_{k=0}^{\infty} \left( \frac{a+2k}{a-1} \right) z^{2k} \right\}^2 = \frac{1}{2z} \left\{ (1-z)^{-a} - (1+z)^{-a} \right\}
\]
\[
\frac{1}{2z^2} \left\{ \sum_{j=0}^{\infty} \left( 2a + 2j - 1 \right) z^j - \sum_{j=0}^{\infty} \left( a + j - 1 \right) z^j \right\} = \\
\frac{1}{2} \sum_{m=0}^{\infty} \frac{(2a + 2m + 1)}{2m+2} z^{2m} - \frac{1}{2} \sum_{m=0}^{\infty} \frac{(a + m)}{m+1} z^{2m} = \\
\frac{1}{2} \sum_{m=0}^{\infty} \frac{(2a + 2m + 1)}{2a - 1} z^{2m} - \frac{1}{2} \sum_{m=0}^{\infty} \frac{(a + m)}{a - 1} z^{2m},
\]

and (3.227) follows with the convolution property of generating functions (Theorem M).
\sum_{n=0}^{\infty} z^n \leq n \binom{n-2}{j} \binom{\infty}{j},

and (3.340) follows. Changing the summation over \((n, j)\) into the summation over \((h, j)\) is justified since by B(45)

\langle h=0 | (x^2)_h \rangle = \sum_{j=0}^{\infty} \binom{\infty}{j} z^j \langle j=0 | (x^2)_j \rangle =

\sum_{n=0}^{\infty} z^n \langle 2x^n | h=\infty \rangle \langle h=\infty | \rangle z^n = |x|^{1+\frac{1}{2}h} \ldots

\[
\left\{ \frac{2y+1}{2m+1} - \frac{y}{y+m} \right\} \left( \frac{y+m}{2m} \right)^m.
\]

(3.346) From C(134a) for small \( z \geq 0 \)

\[
\sum_{n=0}^{\infty} (x+\frac{1}{n})^{n-n} \left( \frac{e^x}{e^x+1} \right)^{2x+1}
\]

(3.351) From D(18), (3.348) and (3.349) and D(14)

\[
\sum_{k=0}^{m} \frac{(2y+1)(y-\frac{1}{2}-k)}{(2k+1)(m-k)} = \sum_{k=0}^{m} \frac{(2y+1)(y-\frac{1}{2}-k)}{(2k+1)(m-k)}
\]

\[
\sum_{k=0}^{m} \frac{1}{2y \setminus \left( y-\frac{1}{2}-k \right)} = \left( y+m+\frac{1}{2} \right)^{2m+1}
\]
and (3.35) follows with the product property M, Theorem 1, of generating functions and (1.428). Cf. (3.353), (3.370), (3.430).

\[
\sum_{k=0}^{\infty} \binom{2k}{k} z^k (1-z)^{-2k-1} =
\]
\[(1-z)^{-1}(1-4z^2(1-z)^{-2})^{-\frac{1}{2}} = (1+z)^{-\frac{1}{2}}(1-3z)^{-\frac{1}{2}},\]

and (3.353) follows from (4.128) and the

\[(3.354) \quad \sum_{n=0}^{\infty} \frac{x^n}{x+\frac{1}{2}n} \left(\frac{x+\frac{1}{2}n}{n}\right)^2 n^{-\frac{1}{2}} z^n = \left(z + (1+z^2)^{\frac{1}{2}}\right)^{2x}.\]

By differentiation w.r.t. to \(z\)

\[\sum_{n=0}^{\infty} \frac{n}{x+\frac{1}{2}n} \left(\frac{x+\frac{1}{2}n}{n}\right)^2 n^{-\frac{1}{2}} z^{n-1} = \left(1-\frac{1}{2}z \right)^{-\frac{1}{2}} \left(1 + \left(1-z^2\right)^{\frac{1}{2}}\right)^{2x} = \]

\[\sum_{h=0}^{\infty} \left(\frac{2x}{h}\right) z^{h+1} (1+z^2)^{x-\frac{1}{2} - \frac{1}{2} h} = \]

\[\sum_{n=1}^{\infty} z^n \frac{\sum_{j \leq n-1} \left(\begin{array}{c} n-1 \times j \\ j \end{array}\right) \left(\begin{array}{c} n-1 \times 2 \\ j \end{array}\right)}{2^j} (n-2j) \left(\begin{array}{c} n \times 1 \\ j \end{array}\right),\]

and (3.354) follows. Changing summations may be justified in the same way as in the proof of (3.340).

Taking \(n=2m\) and then putting \(k=m-1-j\)
in (3.354) gives (3.350) with \( m-1 \).

Applying the same operations to (3.354) as applied above to (3.340) to derive

\[
\sum_{h=0}^{m-2} \binom{2m}{2h+2h} \binom{h}{h}.
\]

In the same way (3.359) follows from (3.132) with \( n = 2m+1 \).

Gould (Fib. Q. 10, 1972; 12, 1974) attributes these relations to Professor Motzkin.
$$\sum_{n=0}^{\infty} z^n \sum_{2j \leq n} (2x+1) (x - \frac{1}{2} n + j)$$

\begin{align*}
&\text{if } 10.311 y \geq 10.261
\end{align*}
(3.365) With $D(24)$ and Vandermonde's convolution $D(26)$ the l.h.s. is equal to
\[ n^\geq \binom{x}{x+n-y-1} \cdot \binom{y-x+y+n-1}{n-y} 
\]

Let $X$ have the hypergeometric distribution $D(26^a)$ with parameters $a, b, n$.
\[ P(X = m) = \binom{a}{m} \binom{b}{n-m} \binom{a+b}{n}^{-1}, \quad m = 0, \ldots, n. \]
From (3.111) and $D(11)$ we see that $X$ has binomial moments
\[ \mathbb{E}[X^r] = \binom{a}{r} \binom{a+b-r}{n-r} \binom{a+b}{n}^{-1}, \quad r \leq n. \]

Then from $IR(17^6)$ we have, with $D(14)$, rearranging factorials,
which is a combinatorial proof of (3.365).

Since \((y-x)\) is a polynomial in \(x\) of degree \(n\), we could also use Newton's

\[
\binom{n}{x} = \binom{n}{x} = \sum_{k=0}^{n} (-1)^k \binom{n-k}{x-1} = (-1)^n \binom{n-x-1}{n} = \binom{x}{n}.
\]

\[
(u-x)! 
\binom{u}{u} 
\binom{2+n}{x+u} 
\binom{x}{x} 
\]
equally likely (Bose–Einstein statistics).

Let $X$ be the number of cells that contain balls. Let $X$ be the number

\[
E \left\{ \left( \frac{X}{k} \right) \right\} = \binom{N}{k} \left( \frac{N+m-ks-1}{N-1} \right) \binom{N+m-1}{m}, \; ks \leq m.
\]

(Choose $i_1, \ldots, i_k$. Put $s$ balls into each of them and distribute $m-ks$ balls over all $N$ cells). With TV (176) and D (14)

\[
\Phi (x) = \left( \frac{N+m-1}{N} \right)^{-1} \left( ^{N+m-1}C_{(N)k} \right) \binom{N}{x} (N-1)^{-1}.
\]
\[
\sum_{k=0}^{\infty} (-1)^{k} \binom{2k}{k} z^k (1-z)^{-k-1} = (1-z)^{-1} \left( 1 + yz (1-z)^{-1} \right)^{-1/2} = (1-z)^{-1/2} (1+3z)^{-1/2},
\]
and with the product property Theorem M1 of generating functions and again. [1, 42] the
\[ \lim_{k \to \infty} \left| (k) \right| \leq 1 - (1 - 2i) < \infty. \]
(3.377) With $D(2y)$ and the Vandermonde convolution $D(2b) = (3.39)$ the l.h.s. is equal to
\[ \sum_{k=0}^{n} \binom{n}{k} (-y-1)^k = \binom{x-y-1}{n}. \]

(3.378) With (3.377) the l.h.s. is equal to
\[ -\sum_{h=1}^{n+1} \binom{n+1-h}{h} \binom{y-1+h}{h} = \binom{x-y}{n+1} + \binom{x}{n+1}. \]

\[ \sum_{j=0}^{\kappa} (-1)^j \frac{(y)^j}{j!(y-\kappa)! (\kappa-j)!} = (\frac{y}{\kappa}) \sum_{j=0}^{\kappa} \binom{\kappa}{j} (-1)^j. \]

(3.382)-(3.384) With $D(2y)$ the l.h.s. of (3.382) is equal to
\[ \sum_{k \leq m} (-1)^k \binom{-\frac{1}{2}m-1}{k} \binom{2m}{m-2k}. \]

With (3.66) where $x = -\frac{1}{2}m-1$, $y = \frac{3}{2}m-1$, $n = m$ and then $D(\frac{y}{2})$, $D(\frac{y}{2})$ (1.1) and (1.66) this is equal to...
\[
\sum_{k=0}^{m} (-1)^k \binom{-\frac{1}{2} m - 1}{k} \left( \frac{3}{2} m - k \right) = \\
\sum_{k=0}^{m} \binom{\frac{1}{2} m + k}{k} \left( \frac{3}{2} m - k \right) = \\
\sum_{k=0}^{m} \binom{\frac{3}{2} m}{k} 2^k.
\]
\[(3.385) \quad A(z) = \sum_{n=0}^{\infty} z^{2n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{n-k} \binom{x+2k}{2k} = \sum_{k=0}^{\infty} \binom{y+2k}{2k} \sum_{m=0}^{\infty} (-1)^m \binom{m}{m} z^{2m+2k} \]
\[
= \frac{1}{2} \left\{ (1-z)^{-y-1} + (1+z)^{-y-1} \right\} (1-z^2)^x, \]

from \(D(25)\) and \(G(8)\). Changing the summation order may be justified by applying \(B(50)\) and \(B(45)\).

For \(y = x\) we have

\[A(z) = \frac{1}{2} (1-z)^{-x} (1+z)^x + \frac{1}{2} (1+z)^{-x} (1-z)^x = \]
\[
= \frac{1}{2} \sum_{m=0}^{\infty} z^m \sum_{k=0}^{m} \binom{m}{k} + \frac{1}{2} \sum_{m=0}^{\infty} z^m (-1)^m \sum_{k=0}^{m} \binom{m}{k} = \]
\[
\sum_{k=0}^{\infty} (-1)^{n-k} \binom{n-k}{h-k} \binom{x+2k}{2k} = \sum_{k=0}^{\infty} \binom{x}{k} \]

For \(y = x+1\)

\[A(z) = \frac{1}{2} (1-z)^{-x} (1+z)^x + \frac{1}{2} (1+z)^{-x} (1-z)^x = \]
\[
= \frac{1}{2} \left\{ (1-z)^x+2 + (1-z)^x \right\} = \]
\[
= \sum_{j=0}^{\infty} (j+1) z^j \sum_{h=0}^{\infty} \binom{x+2}{2h} z^{2h} \]

So with Theorem M1
\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{x+2k}{2k} = \sum_{h=0}^{n} (n+1-h) \binom{x+2}{2h}.
\]

So
\[
\sum_{k=0}^{n} (1-x)^{n-k} (x_1)^k \sqrt{x+2}^{k-1} = \binom{x}{n}.
\]

\[x = 2n', y = 1, \text{ with } (1.5) \text{ and } (1.73). \]

\[
\frac{1}{2} \left\{ (1-z)^{-y-1} - (1+z)^{-y-1} \right\} (1-z^2)^x.
\]

\[
\sum_{n=0}^{\infty} z^{2n+1} \sum_{k=0}^{2n+1} \binom{x}{k}, \quad \text{so}
\]
\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n-k}{k} \binom{x+2k+1}{2k+1} = \sum_{k=0}^{\frac{2n+1}{2}} \binom{x}{k}.
\]

For \( y = x+1 \)

\[
B(z) = \frac{1}{2} \left\{ (1-z)^{-x-2} - (1+z)^{-x-2} \right\} (1-z^2)^x = \frac{1}{2} \left\{ (1-z^2)^{-x-2} \right\} \left\{ (1+z)^{x+2} - (1-z)^{x+2} \right\} = z \sum_{j=0}^{\infty} \frac{z^{2j}}{(j+1)!} \sum_{i=0}^{\infty} \frac{(x+2i) z^{2i}}{2i+1}.
\]

So

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n-k}{k} \binom{x+2k+2}{2k+1} = \sum_{h=0}^{n} \binom{n+h}{h} \binom{x+2}{2h+1}.
\]

\[
\frac{1}{2} (1+z)^{x} - \frac{1}{2} (1-z)^{x} = \sum_{n=0}^{\infty} \sum_{n=0}^{2n+1} \binom{x}{2n+1}.
\]

So

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n-k}{k} \binom{x+2k}{2k+1} = \binom{x}{2n+1}.
\]
\[(3.392) \text{ With } D(14) \text{ the l.h.s. is equal to} \]
\[
\frac{y^1}{x^1(y-x)!} \sum_{j=0}^{\infty} \frac{x^j}{j!(x-j)!} - x^j = (\frac{y}{x})(1+x)^x.
\]

Cf. Feller (1957), Ch. II.12, Prob. 2, Bender et.a. (1990), Lemma 1.

\[(3.393) \text{ From (3.89) with } a = b = x, z = -1 \text{ and then (3.67). The generating function w.r. to } n \text{ is equal to} \]
\[
\sum_{k=0}^{\infty} \binom{x}{k} (-2)^k \sum_{n=k}^{\infty} \binom{2x-k}{n-k} z^n = \sum_{k=0}^{\infty} \binom{x}{k} (-2z)^k (1+z)^{2x-k} = (1+z)^x \left(1 - \frac{2z}{1+z}\right)^x = (1 - z^2)^x.
\]

From (3.393) one obtains Egorychev (1984) §2.4.8, (d), by putting \(n = m - i = k\) in the latter sum.

\[(3.395) \text{ The generating function w.r. to } n \text{ of the l.h.s. is} \]
\[
\sum_{k=0}^{\infty} \binom{x}{k} 4^k \sum_{n=k}^{\infty} \binom{y+k}{n-k} z^n = \sum_{k=0}^{\infty} \binom{x}{k} (4z)^k (1+z)^{y+k} = (1+z)^y (1+yz (1+z))^x.
\]
\[(1+z)^Y (1+2z)^{2x}\]

and the first equality follows with the product property of generating functions. Interchanging the summations over \(n\) and \(k\) is allowed since \(B(45)\)

\[\sum_{k=0}^{\infty} \left(\frac{x}{k}\right) \frac{1}{k!} z^k \sum_{m=0}^{\infty} \binom{1+y+k+m-1}{m} z^m =\]

The second equality follows from (3.89) with \(a = 2x, b = y, z = 2\), or by taking the generating function w.r. to \(n\) of the l.h.s., in the same way as above.
\[ \sum_{j=0}^{\infty} \binom{\lambda}{j} \sum_{m=0}^{\infty} \binom{j}{m} z^{-m-j} = \]

\[ \sum_{j=0}^{\infty} \binom{\lambda}{j} z^j (1+\frac{1}{2}z)^j = (1+z)^{2\lambda}. \]

(3.398) - (3.404)

1. \( F(z) = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{x}{k} \left( \frac{y+n+k}{n-k} \right) \lambda^k \]

\[ \sum_{k=0}^{\infty} \binom{x}{k} \left( \lambda z \right)^k \sum_{m=0}^{\infty} \binom{y+2k+m}{m} z^m = \]

\[ \sum_{k=0}^{\infty} \binom{x}{k} \left( \lambda z \right)^k (1-z)^{-y-2k-1} \]

Where we may change the summation order and the absolute convergence follows since by \( B(50) \) for small \( z \)

\[ \sum_{k=0}^{\infty} |\binom{x}{k}| |\lambda z|^k \sum_{m=0}^{\infty} \left| \frac{y+2k+m}{m} \right| |z|^m \leq \]

\[ \sum_{k=0}^{\infty} |\binom{x}{k}| |\lambda z|^k \sum_{m=0}^{\infty} \left( \frac{|y|+2k+m}{m} \right) |z|^m = \]

\[ \sum_{k=0}^{\infty} |\binom{x}{k}| |\lambda z|^k (1-|z|)^{-|y|-2k-1} < \infty. \]
\[
\sum_{m=0}^{\infty} \binom{X}{m} \sum_{k=0}^{m} \binom{m}{k} (1-n)^{m-k} z^{m+k} = -\infty \quad -n < \lambda < \lambda - 2k
\]

(3.398) follows with (3.94).
For \( \lambda = 2 \) we have from (1)

and (3.400) follows. For \( y = -2x - 1 \), \( \lambda = 2 \) we have

since then

\[
F(z) = (1-z)^2 z (1+z)^2 z = (1-z^2)^2 z.
\]
For $\lambda = 3$, $y = -3x - 1$ we obtain (3.404) by
\[
F(z) = (1-z)^x (1+z+z^2)^x = (1-z^3)^x.
\]
From (3.401) with $x = -n$, $n = 2m+1$ one derives (c) in Egorychev (1984), p. 2.4.7.

\[
(3.407) \quad \mathcal{D}^n x^n (1-x)^n = \mathcal{D}^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{n+k} = \\
\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n+k)_n x^k = \\
n! \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n+k)_n x^k.
\]
These polynomials are called 'Shifted Legendre' in Borwein and Borwein (1987), p. 366.

\[
(3.414) \quad \text{The l.h.s. is equal to} \\
\sum_{k=0}^{n-\varepsilon} (-1)^k \binom{n}{n-k} (n+k+\varepsilon)_2^{n-\varepsilon-k} = \\
\sum_{h=0}^{n-\varepsilon} (-1)^{n-h} \binom{n}{h} \binom{2n-h}{n+\varepsilon} 2^h = \\
\sum_{h=0}^{n-\varepsilon} (-1)^{n-h} \binom{n}{h} \binom{2n-h}{n+\varepsilon} 2^h.
\]
With $G(22)$, $G(28)$ and $G(46)$ this is equal to
\[
(-1) \Delta^n 2^n x^{\binom{2n-x}{n+\varepsilon}} \bigg|_{x=0} = \
\]
\[ (-1)^k \sum_{k=0}^{n} \binom{n}{k} x^k (-1)^{n-k} E^x \left( \frac{x-x-n+k}{k+c} \right) \bigg|_{x=0} = \]
\[ (-1)^{n+c} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n}{k+c} = \]
\[ (-1)^{n+c} \sum_{k=0}^{n-k} (-1)^k \binom{n}{k} \binom{n-k}{n-k}, \]

The number of formulas is derived by computing generating functions such as
\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} q_k(x) \lambda^{n-k}, \]
where \( q_k \in \mathbb{N} \) is a sequence of polynomials of convolution type. The sequence \( q_k \) then is specialized e.g. to \( q_k(x) = \binom{x}{k} \).

So (3.415) - (3.417), (3.459) - (3.463) and (3.563) below fit into a general framework. These relations also may be proved by taking the (specific) generating function w.r. to \( n \).
From (3.411) with $D(4)$ by canceling and rearranging factorials. The generating function \( w.r.t. n \) is

\[
2 \sum_{k=0}^{\infty} \binom{y}{k} z^m \left(1 + 2z\right)^y = \frac{1}{1 + z} \left(\frac{1}{1 + 2z} - 1\right)^y = (1 + z)^{2y}
\]

and \( R \leq 2M+N \) identical coins, to be put into the cells. Cells 1, \ldots, \( M \) may receive two, one or no coins, cells \( M+1, \ldots, M+N \) one or no coin. A single coin in a cell may show heads or tails, of two coins in a cell one should show heads and one tails (Pauli exclusion principle). How many visibly different configurations are possible (Bose–Einstein statistics)?

First choose \( j \) cells from \( \{1, \ldots, M\} \) and put two coins in each of them. So \( 2j \leq R \). Then choose \( R-2j \) from the remaining \( M+N-j \) cells and put a coin (either heads or tails) into each of them. The total number of configurations is

\[
\sum_{2j \leq R} \binom{M}{j} \binom{M+N-j}{R-2j} z^{R-2j}
\]

An urn contains \( p \) pairs of balls and \( q \) single balls. The balls in a pair e.g., may have the same colour, but all \( 2p+q \) balls are different. A random sample of size \( n \) without replacement is taken from the urn. Let \( X, Y, Z \), respectively, be the number of pairs, "half pairs" and singles in the sample. We have

\[
X + Y \leq p, \quad Z \leq q, \quad 2X + Y + Z = n,
\]

\[
\binom{2p+q}{n} P(X = h, Y = i, Z = j) = \binom{p}{h} \binom{p-h}{i} \binom{q}{j} 2^i,
\]

with \( h+i \leq p \), \( j \leq q \), \( 2h+i+j = n \).

[First choose \( h \) out of \( p \) pairs, then \( i \) out of the remaining \( p-h \) pairs and one ball out of each of these \( i \) pairs, finally \( j \) out of the \( q \) singles. The second equality with \( D(n) \) by canceling and rearranging factorials]

When \( q = 0 \) we have \( Z = 0 \), \( 2X + Y = n \) and

\[
\binom{2p}{h} P(X = h) = \binom{p}{h} \binom{p-h}{n-2h} 2^{n-2h},
\]

and (3.425) states that the sum of the nonzero \( P(X = h) \) is 1.

We have with (3.89)
\[
\binom{2p+2}{n} P(X = h) = \\
\binom{p}{h} \sum_{i=0}^{n-2h} \binom{p-h}{i} \binom{q}{n-2h-i} x^i = \\
\binom{p}{h} \sum_{j=0}^{n-2h} \binom{p-h}{j} \binom{p+q-h-j}{n-2h-j}, \quad h \leq p, \quad 2h \leq n.
\]

Also
\[
\binom{2p+2}{n} P(Y = i) = \sum_{2h \leq n-i} \binom{p-i}{h} \binom{q}{n-i-2h}
\]

We have
\[
P(Z = j) = \binom{q}{j} \frac{\binom{2p}{n-j}}{\binom{n}{n-j}} \left(\binom{2p+2}{n}\right)^{-1},
\]
ef. D (26a). This also is found from (i) with
\[
P(Z = j) = \sum_{2h \leq n-j} P(X = h, Y = n-2h-j, Z = j)
\]
by applying (3.425), giving a combinatorial interpretation (or proof) of this relation.

Cf. Feller (1957), Ch. II.10, Prob. 26, Ch. IV.6, Prob. 1;
For $e=1, c=z, t=-2$ we obtain

\[
\sum_{n=0}^{\infty} \frac{(1+z)^n}{(1+z)^e + t z^2} \]

\[
= \sum (x \sqrt{y-z})
\]
\[
\sum_{k=0}^{n} \binom{x}{k} \binom{y-2x}{n-k} (-y)^k = \binom{x}{m}, \quad n = 2m,
\]
\[
= 0, \quad n \text{ odd}.
\]

From \(k = 1, c = 2, t = -y\) in (1):
\[
G(z) = (1+z)^{y-2x} (1-z)^{2x},
\]
\[
\sum_{k=0}^{n} \binom{x}{k} \binom{y-2x}{n-k} (-y)^k = \binom{-1}{n} \binom{2x}{m}, \quad n = 2m,
\]
\[
= 0, \quad n \text{ odd},
\]

Taking \(k = 1, c = 2, t = -3\) in (1) we find
\[
G(z) = (1+z)^{y-3x} (1+z^3)^x,
\]
so
\[
\sum_{k=0}^{n} \binom{x}{k} \binom{y-2x}{n-k} (-3)^k = \sum_{3j \leq n} \binom{x}{j} \binom{y-3x}{n-3j}.
\]

In particular
\[
\sum_{k=0}^{n} \binom{x}{k} \binom{3x-2k}{n-k} (-3)^k = \binom{x}{m}, \quad n = 3m,
\]
\[
= 0, \quad n = 3m+1, \quad n = 3m+2.
\]
Some special cases of \( r = 2 \) were derived earlier, e.g. (3.77), (3.371), (3.425).

More examples of special cases might be found, e.g. \( r = 2 = 5, t = -1 \). Then by (1)

\[
G(z) = (1 + z)^{y - 3x} \left(1 + 3z(1+z)\right)^x = \sum_{h=0}^{n} \binom{x}{h} \binom{y - 3x + h}{n - h} 3^h.
\]
From Newton's interpolation formula \( G(y) \) and (3.155):

\[
\binom{2x}{M} = \sum_{k=0}^{M} \binom{x}{k} \Delta^k \binom{2x}{M} \bigg|_{x=0} = \\
\sum_{\frac{1}{2}M \leq k \leq M} \binom{x}{k} (\frac{k}{M-k})^{2M-M}.
\]

For \( M = 2m \) and \( M = 2m+1 \), respectively,

\[(3.428), (3.429)\]

(3.430), (3.431). \( \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k} (\frac{1}{2})^k x^k = \)

\[
\sum_{k=0}^{\infty} \binom{2k}{k} x^k \sum_{m=0}^{\infty} \binom{m+k}{m} z^{m+k} = \\
\sum_{k=0}^{\infty} \binom{2k}{k} x^k z^k (1-z)^{-k-1} = \\
(1-z)^{-\frac{1}{2}} (1-z-4xz)^{-\frac{1}{2}},
\]

with (1.428) = \( B(13) + D(20) \). Interchanging summations is justified, more generally in \( (3.434) \). For \( x = -\frac{1}{2y} \) and \( x = -\frac{1}{2} \), the above generating function reduces to \( (1-z)^{-\frac{1}{2}} \) and \( (1-z^2)^{-\frac{1}{2}} \), respectively, and (3.430) and (3.431) follow, again with (1.428) = \( B(13) + D(20) \).

See Wilf (1990), Ch. 4.3; Greene and Knuth (1982), p. 7; Frohne (1994). Cf. \( (3.352), (3.353), (3.370), (3.432)-(3.434), \) & Gorychev (1984), (2.77).
From (3.432), (3.433)

\[
\sum_{n=\ell}^{\infty} z^n \sum_{k=\ell}^{n} \binom{n}{k} \binom{2k}{k-\ell} x^k =
\]

\[
\sum_{k=\ell}^{\infty} \binom{2k}{k-\ell} x^k z^k (1-z)^{-k-\ell-1} =
\]

\[
x^\ell z^\ell (1-z)^{-\ell-1} \sum_{h=0}^{\infty} \binom{2\ell+2h}{h} x^h z^h (1-z)^{-h} =
\]

\[
x^\ell z^\ell (1-z)^{-\frac{1}{2}} (1-z-yxz)^{-\frac{1}{2}} \left( \frac{\sqrt{1-z} - \sqrt{1-z-4yzz}}{2xz} \right)^{2\ell}.
\]

Changing the summation order is justified,

\[
\sum_{k=\ell}^{n} \binom{n}{k} \binom{2k}{k-\ell} (-\ell)_k (-\ell)_x^n (2n)^k = (-1)^\ell y^{-\ell} (n-\ell),
\]
\[
(-1)^{r} \sum_{j=0}^{\infty} \binom{r+2j}{j} \left( \frac{z}{2} \right)^{r+2j},
\]

By \( C_{n}(111a) \) and (3.433) follows \( \text{cf. (3.352), (3.363), (3.370), (3.434)}. \)

(3.434) Following Riordan (1968), Ch.2, Prob.4,5.

\[
\begin{align*}
\langle n, w, x \rangle &= \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} (k-x)^w, \ w \geq 0.
\end{align*}
\]

We also define

\[
\begin{align*}
c_n(u, \alpha, x) &= \sum_{k=0}^{n} (-1)^k \binom{n+u}{n-k} (2k) \alpha^k x^k.
\end{align*}
\]

So

\[
\begin{align*}
(1) \quad B_n(0, x) &= B_n(x), \quad c_n(0, \alpha, x) = B_n(\alpha, x).
\end{align*}
\]

In (3.430) – (3.433) we proved

\[
\begin{align*}
(2) \quad B_n(1/4) &= 4^{-m} \binom{2n}{m},
\quad (3) \quad B_n(1/2) &= \binom{2m}{m} 4^{-m}, \ n = 2m; \quad = 0, \ n \text{ odd};
\end{align*}
\]
(6) \[ \sum_{n=1}^{\infty} c_n(u, t, x) z^n = \frac{1}{1 - \left( \frac{-1}{z} \right)^n} \]

Changing the summation order is justified since by B(50) and the absolute convergence in C(111a)

\[ \sum_{k=0}^{\infty} \left| x^k \left( \frac{z^k}{k-1} \right) \sum_{m=0}^{\infty} \left| \frac{(m+k+1)}{m} \right| \right| z \right|^{m+k} \leq \]

\[ \sum_{k=0}^{\infty} \left| x^k \right| \left( \frac{z^k}{k-1} \right) \left| (-1)^k \right| < \infty \]

From (6)
\[ \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \binom{2k}{k} \frac{1}{4^k} = \frac{(2n+1)(2n)}{(n)_n} \frac{1}{4^n} \]

\[ \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \binom{2k}{k} \frac{1}{2^k} = \frac{(2m+1)(2m)}{(m)_m} \frac{1}{2^m} \]

\[ (-xz)^{u-2-1} \sum_{h=0}^{\infty} (\frac{2h+2\kappa}{\kappa}) (-xz)^h (1-z)^{-n} = \]

\[ \frac{1}{u-2-1} \cdot \frac{1}{u x z - \frac{1}{2} \sqrt{1 + 4 x z \kappa (1-z)^{-1}}} \]
\[ \left( -xz \right)^{u} \left( 1-z \right)^{-\frac{1}{2}} \left( 1-z+4xz \right)^{-\frac{1}{2}} \left\{ \frac{\sqrt{1-z} - \sqrt{1-z+4xz}}{2xz} \right\}^{x}. \]

(13) \( \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{k} (n+u) \binom{n-u}{k}}{(n-k)} x^{k} \]

so with Theorem M1 and \((1,428) = B(13) + D(20), \)

(14) \( \sum_{k=0}^{n} (-1)^{k} \binom{n+u}{n-k} \binom{2k}{k} x^{k} \]

. \( n < n \binom{2k}{u+1} \binom{n-2k}{n-1} \)

(15) \( \sum_{k=0}^{\infty} \binom{2k}{k} \binom{n}{n-2k} \binom{2k-1}{n-2k-1} \)

F. \( \sum_{k=0}^{\infty} \binom{2k}{k} \binom{n}{n-2k} \binom{2k-1}{n-2k-1} \)
\[ (-1)^{m-h} \sum_{m=0}^{\infty} \sum_{h=0}^{m} \binom{2\varepsilon + 2h}{h} H^{-h} (\nu + m - h - 1) \]

So

\[ (-1)^{\nu} \sum_{m=0}^{\infty} \binom{\nu + \nu \sqrt{n} \nu}{\nu} \nu, -\nu \]
\[ \sum_{n=0}^{\infty} z^n \binom{n+u}{z} (r^k)^{-n-k} = \phi_{1-n-u}(\frac{z}{r}) \]
\[ \left\langle \sum_{n=0}^{\infty} x^n \right\rangle = (1 - x)^{-1} \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

So
\[ \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n x^n}{n!^2} \]

(24) \[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{2^k} = \]
\[ \sum_{n=0}^{\infty} b_n(x) z^n = \left\{ \frac{(1-z)^2 + 4xz(1-z)}{(1-z+2xz)^2} \right\}^{-\frac{1}{2}} \]

\[ = \left\{ \frac{1 - 4xz^2 (1-z+2xz)^{-2}}{(1-z+2xz)^2} \right\}^{-\frac{1}{2}} \]

\[ \sum_{k=0}^{\infty} \binom{2k}{k} x^k z^k z^{2k} (1-z+2xz)^{-2k-1} = \sum_{j=0}^{\infty} \binom{2j}{j} (2j+1) z^j (1-2x)^j = \]

\[ n=0 \quad 2k \leq n \quad \binom{n}{k} \]

So

\[ \sum_{n=0}^{\infty} \binom{n}{k} \left( \frac{2}{3} \right)^{-n} \left( \frac{4x-1}{4x-2} \right) z^n = \]

\[ k \mod \text{mod of } 2k = (n) \]
\[\sum_{n=0}^{\infty} \left( \frac{1}{2h + n} \right)^2 \leq h \sum_{n=h}^{\infty} \left( \frac{1}{n} \right)^2 < \pi^2 \frac{h}{2} \]

So

\[n \to \infty, \quad h \to 0^+\]
\[(3.439) - (3.441) \text{ follows from (3.425) with } D(iy) \text{ by canceling and rearranging factorials. Taking } y = m \text{ and } n = m - e \text{ in (3.439) we find (3.440).}

Putting } y = m, n = m + s \text{ in (3.439) we obtain (3.441). In order that } (n - 2k) \neq 0 \text{ for these values in (3.441) we should have } 2k \geq s. \text{ Because of the two first factors in (3.441) we even may restrict the sum to } s \leq k \leq (m + s)/2. \text{ Cf. Math. Student 15, 1947, 93–100; Prob. E3258, Monthly 96, 1989, 651–652;}

\[
\sum_{k=0}^{n} \binom{n}{k} x^k (x+1)^h = \sum_{n \geq 1} \binom{n}{k} (2x)^h (2x)^{n-h} = \prod_{j=1}^{n} (2x+1)^j = \prod_{j=1}^{n} (2x+1)^j
\]

\[
\sum_{k=0}^{n} x^k (x+1)^{n-k} \sum_{j+k \leq n} \frac{(2k)! (2n)!}{j! (n-j)! (j-k)!} = \sum_{j+k \leq n} \frac{(2k)! (2n)!}{j! (n-j)! (j-k)!}
\]

Without Legendre: The l.h.s. is equal to
\[ \sum_{r=0}^{n} x^r (1+x)^n = \sum_{k=0}^{\text{even}(n-r)} (\binom{n}{2k}) \binom{2k}{k} \binom{n-2k}{r-k} = \]
\[ \sum_{r=0}^{n} x^r \binom{n-r}{r} \frac{x^{n-r}}{r!} \frac{1}{n-r} \]
\[ \sum_{r=0}^{n} \binom{n}{r} x^r (1+x)^n \]

\[(3.446), (3.447) \) Take \( x = n = 2m+1 \) in (3.94).
In the third member then put \( h = m-f \) and cancel and rearrange, factorials, using D(13). The equality of third and first member then is (3.446).

The proof of (3.447) is similar with \( x = n = 2m \).
Cf. Gould (1972e), (3.166), (3.167).
Changing the summation order is allowed since by $B(45)$ and $D(25)$

$$
\sum_{k=0}^{\infty} \frac{1}{(2k)!} y^k z^k \sum_{m=0}^{\infty} \left| \binom{y-2k}{m} \right| z^m
$$

Similarly,

$$
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{\infty} \frac{1}{(2k+1)} \left( \frac{y-2k-1}{n-k} \right) z^{2k+1} =
$$
\[ \sum_{k=0}^{\infty} \left( \frac{y}{2k+1} \right) z^{2k+1} z^k (1+z)^{y-2k-1} = \]

\[ \frac{1}{2} z^{-\frac{1}{2}} (1+z)^y \left\{ \left( 1 + \frac{\sqrt{z}}{1+z} \right)^y - \left( 1 - \frac{\sqrt{z}}{1+z} \right)^y \right\} = \]

\[ \frac{1}{2} z^{-\frac{1}{2}} (1+\sqrt{z})^y - \frac{1}{2} z^{-\frac{1}{2}} (1-\sqrt{z})^y = \]


(3.450) With \( \Phi(z) \), by canceling and rearranging factorials the l.h.s. is equal to

\[ \frac{(n)!}{(n-k)!} \cdot n-1-2k \]
\[
\sum_{i=0}^{m} \left( \frac{m}{m} \right) (-1)^{m-i} (\gamma+i) \cdot (x+i+1) \cdot (x+i+2) \cdot \ldots \cdot (x+i+n)
\]
(3.459) - (3.463). In Chapter C, (181) - (221) a number of formulas is derived from some general generating functions. This implies that (3.459) - (3.463) may be derived directly by taking the generating function w.r. to \( n \).

(3.461) With \( D(n) \), by canceling and rearranging factorials, the l.h.s. is equal to

\[
\sum_{k=0}^{n} \binom{n+1}{n+1-k} \frac{n-2k}{2^n}
\]

(3.464) Put \( y + n = y + k + n - k \). The l.h.s. is equal to

\[
\sum_{k=0}^{n} \binom{x}{k} \binom{y+k}{n-k} y^k + \sum_{k=0}^{n-1} \binom{x}{k} \binom{y-1+k}{n-1-k} y^k =
\]
\[ \sum_{k=0}^{n} \binom{k}{n-k} \binom{n-1-k}{k} = \]
\[ \sum_{k=0}^{n} \binom{2x}{k} \binom{y+2x-k}{n-k} + \sum_{h=1}^{n} \binom{2x}{h-1} \binom{y+2x-h}{n-h} = \]
\[ \sum_{k=L}^{n} \binom{2x+1}{L} \binom{y+2x-k}{n-L} , \quad n \geq 1 , \]

From the first or second line, one may proceed differently, by \((3.395)\) or adding terms differently, to obtain other

\[ (-1)^{x+y-1} \sum_{h=1}^{n+1} \binom{x+1}{h} \binom{n-y-1}{n+1-h} = \]
\[ \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \binom{n+k}{k} k^{-1} = \]

Proof by recurrence, applying \( D(n) \).

In the second line of his recurrence one should apply \( D(24) \) and Vandermonde's convolution, \( D(18) \).
\[
\frac{1}{2} z^{-\frac{1}{2}} \left\{ \left( \sqrt{1+z} + \sqrt{z} \right)^{2x} - \left( \sqrt{1+z} - \sqrt{z} \right)^{2x} \right\},
\]

and from the binomial series and \( G(q) \)

\[
\begin{align*}
&\left(1 + \frac{1}{z} \right)^x \left( \frac{1 + \frac{1}{z}}{z} \right)^{\frac{1}{2}} \left\{ \left(1 + \sqrt{\frac{z}{1+z}} \right)^{2x} - \left(1 - \sqrt{\frac{z}{1+z}} \right)^{2x} \right\} = \\
&\sum_{k=0}^{m} \binom{2x}{2k+1} \binom{m-k}{m-k} = \\
\end{align*}
\]

and (3.468) follows from (3.345).

\((3.469), (3.470)\) From Newton's interpolation formula \( G(41) \)

\[
\binom{2x+1}{M} = \sum_{k=0}^{M} \binom{x}{k} \Delta^k \left( \frac{2x+1}{M} \right) \bigg|_{x=0}
\]
This is zero for \( 2k+1 < M \). So for \( M = 2m \)

\[
\binom{2x+1}{2m+1} = \sum_{k=m}^{2m+1} \binom{x}{k} \frac{2m+k}{k+1} \left( \frac{k+1}{2m+1-k} \right)^{2k-2m-1}.
\]

Putting here \( k-m = h \) we find (3.469) and

(3.470). Cf. Riordan (1968). Ch. 1, Prob. 16
so that the l.h.s. of (3.473) is \((-4)^n \binom{x/2}{n}\).

\[
\sum_{n=0}^{\infty} \frac{-x}{-x+2n} \left( \frac{-x+2n}{n} \right)^z n,
\]

so that the l.h.s. of (3.473) is \(-x (-x+2n)^z \binom{-x+2n}{n}\)
(3.477) By $D(n+2x), \text{ and Vandermonde's convolution } D(2x)$ the l.h.s. is equal to

$$\sum_{j=1}^{n} j^{-1} \sum_{k=j}^{n} \binom{n}{k} (-n)^{k} =$$

$$\sum_{j=1}^{n} j^{-1} \sum_{k=0}^{n-j} \binom{n}{k} (-n)^{k} =$$

$$\frac{1}{n} \sum_{j=1}^{n} \binom{n-1}{n-j} (-n)^{j} = \frac{1}{n} \left\{ \binom{n}{n} - \binom{n-1}{n} \right\} =$$

$$\frac{1}{n} (-1)^{n}.$$

Cf. Gould (1961a)

(3.478) From (3.477) and (3.411) for $n \geq 2$ we see that the l.h.s. is equal to

$$1 + \sum_{k=1}^{n} (-1)^{k} \binom{n}{k} \sum_{j=1}^{n+k-1} j^{-1} =$$

$$\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{n+k-1}{k+1} \frac{1}{k+1} +$$

$$\sum_{k=1}^{n} (-1)^{k} \binom{n}{k} \binom{n+k-1}{k} \sum_{j=1}^{k} j^{-1} =$$

$$\frac{1}{n-1} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{n-1+k}{n-2} + \frac{1}{n} (-1)^{n} = \frac{1}{n} (-1)^{n}.$$

Cf. Gould (1961a); Riordan (1968), Ch.2, Prob. 17, p.24.
\[ \sum_{h=0}^{n-1} \sum_{k=0}^{\lfloor \frac{h}{2} \rfloor} (-1)^k \binom{k}{\frac{h-k}{2}} = 0 \text{ for } 0 \leq h \leq n-3. \]

From (3.202) with \( b = n-1, a = -1, m = n-1 \) and

See Gould (1961a); Monthly 66, 1959, 517; Ch. 2, Prob. 17, p. 85; Riordan (1968), Prob. 4805.
\((-1)^{n-m} \binom{-m-1-x}{n-m} = \binom{x+n}{n-m}\).

(3.486) With \(D(2y)\) and Vandermonde's convolution \(D(2b)\) the l.h.s. is equal to
\[\sum_{k=r}^{n} \binom{k}{n-k}(y-k) = (y-r) \sum_{k=r}^{n} \binom{k}{k-r}(n-k) = \binom{n-r}{r-1}\binom{n-r-1}{n-r-1}.\]

Ranging factorials we see that both summands are equal to
and canceling and rearranging factorials. That $S_{mn} = S_{nm}$ also follows from the generating function:

$$
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} S_{mn} x^m y^n = (1-x)^{-1}(1-y)^{-1}(1-x-y)^{-u-1}.
$$

$$
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (x-k) (1+w)^y k+1 =
$$
\[ \sum_{k=0}^{\infty} (1+w)^{y+k+1} \sum_{n=k}^{\infty} z^n \binom{x-k}{n-k} = \]
\[ \sum_{k=0}^{\infty} (1+w)^{y+k+1} z^k (1+z)^{x-k} = (1+w)^{y+1} (1+z)^{x+1} (1-wz)^{-1}, \]

\[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left| w \right|^m \left| z \right|^n \sum_{k=0}^{n} \frac{(x-k)}{(n-k)} \left| \left( \frac{y+k+1}{m} \right) \right| \leq \]
\[ \sum_{n=0}^{\infty} \left| z \right|^n \sum_{k=0}^{n} \frac{(x-k)}{(n-k)} \sum_{m=0}^{\infty} \left| w \right|^m \left| y+k+m \right| = \]
\[ \sum_{n=0}^{\infty} \left| z \right|^n \sum_{k=0}^{n} \frac{(x-k)}{(n-k)} \left( 1-\left| w \right| \right)^{-1}\left| y+k-1 \right| \]
\[ < \sum_{\kappa=0}^{\infty} \left( \frac{-\left| y-1 \right| - k}{\left| y+k+\varepsilon-1 \right|} \right) > \]

In the same way

\[ (1+w) \quad (1+z) \quad (1-wz) \]

Also
\( \cdot \cdot \cdot \sqrt{X^+} \left| \sqrt{X^+} \right| \left( \sqrt{X^+} \right)^{-1} \)

(3.490) The first equality follows from (1.23) by canceling and rearranging factorials. The second one follows by differentiation term by term of the series (1.428), ab-

\( X \) and \( Y \) be the numbers of white and black balls in the first \( n \) drawings; \( Z \) be the number of the drawing where the
\[ P(U_m = j) = \binom{a}{m-1} \binom{b}{j-m} \binom{a+b}{j-1}^{-1} \frac{a-m+1}{a+b-j+1} = \]
The author does not have an analytical proof. Trying (3.220) for each single sum does not help: This would mean a simple expression for \( P(X_{k+s+1} \geq s+1) \). And the conditions for (3.220) are not satisfied.

\[
(3.493) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} t^n z^r = \sum_{j=0}^{n} \binom{2j}{j} \binom{j}{r} y^j = \sum_{n=0}^{\infty} t^n \sum_{j=0}^{n} \binom{2j}{j} y^j (1+z)^j =
\]

\[
(-t) \sum_{j=0}^{\infty} \binom{2j}{j} y^j (1+z)^j =
\]

\[
(1-t)^{\frac{1}{2}} (1-yyt-yyzt)^{-\frac{1}{2}} [\text{with (3.488)}] = (1-t)^{\frac{1}{2}} (1-yyt)^{-\frac{1}{2}} (1-yytz (1-yyt)^{-\frac{1}{2}})^{-\frac{1}{2}}
\]

\[
2 \sum_{r=0}^{\infty} \binom{2r}{r} y^r t^r z^r (1-t)^{\frac{1}{2}} \sum_{h=0}^{\infty} \left(\frac{-\xi}{h}\right)^h (-yy)^h t^n =
\]

\[
i \leq n \text{, } n \geq 1 \text{, } \xi = n-x (x \neq 0), t
\]
\[
\binom{n+1}{n+1-i} (n+1-i)! = \binom{n}{i} \binom{n+1-i}{0} (n+1-i)! \quad .
\]

This may be shown by \( D(13), D(24) \) and \( B(14) \).

Simply: \( j-i=k \), \( D(24) \) and \( B(16) \).

(1111) For fixed \( m \perp 1 \) and \( \mu \) and small \( \tau \),

\[
\sum_{k=m}^{\infty} \binom{k}{m} (uz)^k \left( 1-z \right)^{-t-\lambda k-k-1} = \\
\sum_{h=0}^{\infty} \binom{m+h}{h} (uz)^{m+h} \left( 1-z \right)^{-t-1-(\lambda+1)(m+h)} = \\
(uz)^m \left( 1-z \right)^{\lambda-t} \left\{ \left( 1-z \right)^{\lambda+1} - uz \right\}^{-m-1},
\]

where changing the summation order is allowed since by \( B(50) \) and \( D(25) \)

\[
\sum_{k=m}^{\infty} \sum_{\tau=0}^{\infty} \sum_{n=0}^{N} \binom{n}{m} \frac{(k)!}{(N+t+\lambda k)!} \left( 1-z \right)^{-t-\lambda k-k-1} - \\
\quad \sum_{k=m}^{\infty} \sum_{\tau=0}^{\infty} \sum_{n=0}^{N} \binom{n}{m} \frac{(k)!}{(N+t+\lambda k)!} \left( 1-z \right)^{-t-1-(\lambda+1)(m+h)} = \\
\quad (uz)^m \left( 1-z \right)^{\lambda-t} \left\{ \left( 1-z \right)^{\lambda+1} - uz \right\}^{-m-1},
\]

\[
\} \sum_{k=m}^{\infty} \sum_{\tau=0}^{\infty} \sum_{n=0}^{N} \binom{n}{m} \frac{(k)!}{(N+t+\lambda k)!} \left( 1-z \right)^{-t-\lambda k-k-1} = \\
\quad \sum_{k=m}^{\infty} \sum_{\tau=0}^{\infty} \sum_{n=0}^{N} \binom{n}{m} \frac{(k)!}{(N+t+\lambda k)!} \left( 1-z \right)^{-t-1-(\lambda+1)(m+h)} = \\
\quad (uz)^m \left( 1-z \right)^{\lambda-t} \left\{ \left( 1-z \right)^{\lambda+1} - uz \right\}^{-m-1},
\]
\[
\sum_{k=m}^{\infty} \binom{k}{m} |u_z|^k \sum_{n=0}^{\infty} \left( \left| \frac{t}{|z|} + \lambda \right|^{k+k+n} \right) |z|^n = \\
\sum_{k=m}^{\infty} \binom{k}{m} |u_z|^k \left( 1 - |z| \right)^{-|t| - |\lambda|^{k+k-1}} \\
\times \left( h+m \right)_{l-m} \left( 1-1 \right)_{l-m}^{-|t|-(\lambda+1)(h+m)-1}
\]

The relation (2) also holds, trivially, for \( N = m \).
For \( t = \lambda = 1, \, u = -y \) the relation (1) becomes, with \( D(25) \),
\[
(\wedge \ldots \wedge m \ldots \wedge -2m \ldots \wedge 0)
\]
where we applied (3), with \( n \) replaced by \( m \) and \( m \) by \( m-1 \). This relation also holds for \( N \geq 1, \, m = 0 \), as is seen from
\[
\sum_{k=0}^{N} \binom{N+k}{2k} \frac{N}{N+k} (-y)^k =
\]
\[ \sum_{k=0}^{N} \binom{N+k}{2k} (-y)^k - \sum_{k=1}^{N} \binom{k}{1} \binom{N+k}{2k} \frac{(-y)^k}{N+k} = (-1)^N \binom{2N+1}{N} - 4 (-1)^N \binom{N+1}{N} \frac{1}{N+1} = (-1)^N, \]

the sense of \( p. IR7 \) of the above relations. A slightly modified version of the inverse pair \( IR(16) \), a so-called transposed pair,

\[(5^a) \quad y_m = \sum_{k=m}^{N} (-1)^k \binom{m}{k} x_k, \quad m = 0, \ldots, N, \]

if and only if
\[ (7) \sum_{k=m}^{N} \binom{k}{m} \binom{N+k+1}{2k+1} \frac{2N+1}{N+k+1} (-y)^k = \]
\[ (-1)^N \binom{N+m}{2m} H^m, \quad m \leq N. \]

The relation (3) is its own companion by (5), and so is (4).

From \( t = 0, \lambda = 1, u = -x \) we have in (1), with \( \mathcal{D}(25) \)

\[ F(z) = (-2z)^m (1-z) (1+z^2)^{-m-1} = \]
\[ = \sum_{j=0}^{\infty} \binom{m+j}{m} (-1)^{m+j} \frac{1}{(m+2j)} \]
\[ \times 2^m (-1)^{m+j} \binom{m+j}{m+j} \quad , \quad N - m = 2j, \quad m \leq N. \]

\[ \vdots \quad n = 0, 1, 2, \ldots \]

\[ F(z) = (-2z)^m (1+z^2)^{-m-1} = \]
(9) \[ \sum_{k=m}^{N} \binom{k}{m} \binom{N+k+1}{2k+1} (-2)^k = 0, \quad N-m \text{ odd,} \]
\[ = 2^m (-1)^{m+j} \binom{m+j}{m}, \quad N-m = 2j, \quad m \leq N. \]

From (9), in the same ways as (4) from (3), for \(1 \leq m \leq N,\)

(10) \[ \sum_{k=m}^{N} \binom{k}{m} \binom{N+k}{2k} (N+k)^{-l} (-2)^k = \]
\[ = -\frac{1}{m} \sum_{h=m-1}^{N-1} \binom{h}{m-1} \binom{N+h}{2h+1} (-2)^h = 0, \quad N-m \text{ odd,} \]
\[ = -\frac{1}{m} 2^{m-1} (-1)^{m-1+j} \binom{m-1+j}{m-1} = \]
\[ = (m-1) (-1)^{m+j} \binom{m+j}{m+j} \frac{1}{m}, \quad N-m = 2j. \]

As is seen from
\[ \sum_{k=0}^{N} \binom{N+k}{2k} \frac{N}{N+k} (-2)^k = \]
\[ \sum_{k=0}^{N} \binom{N+k}{2k} (-2)^k - \sum_{k=1}^{N} \binom{k}{k} \binom{N+k}{2} (N+k)^{-l} (-2)^k, \]
and applying (8) with \(m=0\) and (10) with \(m=1.\)

Or use \(\Phi(123) = (1, 340)\) and \(\Phi(97).\)

We may write (9) in the form \((5^{a})\) with
\[ x_k = \binom{N+k+1}{2k+1} 2^k, \quad y_m = 0, \quad N-m \text{ odd,} \]
\[ y_{N-h} = 2^{-n} (\binom{N-h+j}{j} \binom{N-h+j}{j}) = \]

\[ \sum_{x_j \leq n} (-1)^j \binom{x-j}{n-x_j} \binom{x-j}{j} 2^{n-x_j} = \]

\[ \sum_{x_j \leq n} (-1)^j \binom{n-j}{j} \binom{x-j}{n-j} 2^{n-x_j} = \binom{2x-n+1}{n}, \]

where the second member is obtained with
\( D(\beta) \), canceling and rearranging factorials.

We also may prove (14) with generating functions. We will apply \( C(\alpha, \beta) \):

\[
\sum_{n=0}^{\infty} z^n \sum_{j \leq n} (-1)^j \binom{x-j}{n-xj} \binom{x-j}{j} x^{n-j} =
\]

\[
\sum_{j=0}^{\infty} (-1)^j \binom{x-j}{j} z^j (1+2z)^{-1} =
\]

\[
(1+2z)^x \left( 1 - 4 \frac{z^2}{(1+2z)^2} \right)^{-\frac{1}{2}} \left\{ \frac{1}{2} + 1 \sqrt{1 - 4 \frac{z^2}{(1+2z)^2}} \right\}^{+1} =
\]

\[
\frac{1}{(1+2z)^x - 4z^2} \left( \frac{1}{1+z} + \frac{1}{\sqrt{(1+2z)^x - 4z^2}} \right)^{+1} =
\]


Application of the inverse pair (5) to (8).

\[ \gamma = -\infty, \quad u = 7, \quad \alpha = \infty \]
\[ F(z) = (yz)^m (1-z)^{m-t-1} (1-z)^{-2m-2} \]

The identities (2), (3), (4), and (6) are in §2.14 of Egorychev, who, following Davis (1962?), calls them Moriarty identities. But see...
(3.356) - (3.359). An extensive discussion of (2) - (7) and their connection with (3.132), (3.133) is in Gould (1972b).

(3.496), (3.497). References as in (3.132), (3.133) and Ascher (1974); Egorychev (1984), § 2.4.8, (m).

(3.498) This is equal to \( n! P(W_n = j) \), where \( W_n \) is the number of husbands next to
(3.506) With $D(2y)$ and Vandermonde's convolution $D(2b)$ the l.h.s. is equal to

$$(-1)^n \sum_{k=0}^{n} \binom{-x-1}{k} \binom{n-y-1}{n-k} =$$

$$(-1)^n \binom{n-x-y-2}{n} = \binom{x+y+1}{n}.$$

With $x = \alpha - 1$, $y = \beta + n - 1$ the relation (3.506) becomes

showing, for $\alpha > 0$, $\beta > 0$, that the probabilities in the Polya–Eggenberger distribution sum to 1. See Johnson and Kotz

(3.507) dy $\times \Phi (y)$ the r.h.s. is equal to

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{-z-1}{k} \binom{-z-1}{n-k},$$

Now apply (3.67).

(3.508) The first equality follows from Theorem C12, with $D(2y)$, since $\binom{x-n}{n}$, $n \in \mathbb{N}_0$, is a
\[ \sum_{k=0}^{\infty} \binom{x-k}{k} z^k + \sum_{j=0}^{\infty} \binom{j}{j} z^j = \]

see C(44). It also follows from the equality of the first and third member. From the convolution property (Theorem M1) of generating functions and C(11a)

\[ \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \binom{x+2k}{y+2n-2k} = \]

\[ \sum_{n=0}^{\infty} \binom{x-k}{k} z^k \]
\[
\sum_{k=0}^{\infty} \binom{x-k}{k} z^k (1+z)^k = \\
\left\{1 + yz(1+z)\right\}^{-\frac{1}{2}} \left(\frac{1}{x} + \frac{1}{x} \sqrt{1+yz(1+z)}\right)^{x+y}.
\]

\[(-y)^n \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k \binom{n-k}{n-k} \quad \text{and} \]
\[(-y)^n \sum_{k=0}^{n} (-1)^k \left(-\frac{1}{2}\right)^k \binom{n-k}{n-k}.\]

functions w.r. to \(n\) are, by (1.428),
\[
(1-4z)^{-\frac{1}{2}} \quad \text{and} \quad (1-16z^2)^{-\frac{1}{2}}.
\]
(3.511) and (3.512) as in G(10) and D(11).

\[
(\frac{y+M}{M}) = \sum_{2i+1 \leq M} \binom{y+i}{2i+1} \binom{M-i-1}{M-2i-1}
\]

\[
+ \sum_{2j \leq M} \binom{y+j}{2j} \binom{M-j}{M-2j},
\]

and (3.517) follows with D(27).

From (3.517) with \(y = N\) we have

\[
\sum_{2j \leq M} \binom{M-j}{j} \binom{N+j}{2j}.
\]
Here we may restrict the summations to \( i \leq N-1 \) and \( j \leq N \) when \( M \) is large, say \( M \geq 2N \). This means that (3.518) holds for \( x = M \epsilon \mathbb{N} \), \( M > 2N \), and since both sides of (3.518) are polynomials in \( x \) of degree \( N \), this identity holds for \( x \in \mathbb{C} \). Cf. Probl. E3439, Monthly 100 (1973), 188.

(3.519) For \( y = N \), \( x = M \) the r.h. sides of (3.517) and (3.518) are equal. Subtracting the l.h. sides one finds (3.519). The lower summation bound is found by considering \( M = 2m \) and \( M = 2m+1 \). For \( M \geq 2N \) a similar relation may be found.

(3.520), (3.521). The proofs are similar to the proofs of (3.517) \( - \) (3.518). From \( G(21) \) with \( n = M \), \( \alpha = M \), \( b = 0 \) and \( q(x) = \binom{x}{m} \) with \( G(38) \)

\[
\sum_{2j \leq M} \binom{y+j-1}{2j} \binom{M-j}{M-2j},
\]

and (3.520) follows with \( D(27) \). For \( y = N \) we have from (3.520)

\[
\binom{N+M}{N} = \binom{N+M}{M} =
\]
\[
\sum_{2i+1 \leq M} \binom{M-i}{i+1} \binom{N+i}{2i+1}
\]

For large \( N \), we may restrict these summations to \( i \leq N-1 \) and \( j \leq N-1 \), when \( N \geq 1 \). For \( N = 0 \) we do not obtain the empty sum, since there is a non-zero term for \( j = 0 \). Since both sides of (3.521) are polynomials in \( x \) of degree \( N \) and (3.521) holds for large

(3.522) For \( y = N \), \( x = M \) the r.h.s. sides of (3.520) and (3.521) are equal. Subtracting the l.h.s. sides, one finds (3.522). The lower

(3.524), (3.525) These relations are equivalent. Denoting the l.h.s. of (3.524) by \( f(x) \) we have by Newton's interpolation formula, G(41), since \( f \) is a polynomial of degree \( \leq n \), noting G(38),

\[
\sum_{j=0}^{\infty} \frac{x^j}{j!} \left( \frac{t}{x} \right)^{\frac{k}{2}} \]
\( \binom{Y}{n-j} \binom{-n+j+Y}{j-\ell} = \binom{n-\ell}{j-\ell} \binom{Y}{n-\ell}, \)

with \( Y \not\in \mathbb{Z} \) to avoid infinite factorials.

Letting \( Y \Rightarrow \lceil Y/2 \rceil \) we see that the l.h.s.
of (3.15.25) is equal to

\[
\sum_{j=\ell}^{n} \binom{n-\ell}{j-\ell} \binom{\lceil Y/2 \rceil}{n-\ell} = \sum_{i=0}^{n-\ell} \binom{n-\ell}{i} \binom{\lceil Y/2 \rceil}{n-\ell}
\]

For \( n = 2s \) this is equal, by \( D(12), D(18) \) and

\[
\sum_{i=0}^{\infty} \binom{n-2s-1}{s+h} + \sum_{h=1}^{\infty} \binom{n-2s-1}{2h-1} (s+h) = 0
\]

\[
\sum_{h=0}^{\infty} \binom{n-2s}{s+h} (s+h) = \binom{2s+1}{n-2s-1} = \binom{s}{n-\ell}
\]
for the delta operator $E^{-a}_{\Delta}$,

\[ S_{\tau}(x, y) = \sum_{j=0}^{\tau} \binom{\tau}{j} \frac{x+j}{y+\tau a-j} \left( y+\tau a-j \right) + \sum_{j=0}^{\tau-1} \binom{\tau-1}{j} \frac{\tau a-j}{y+\tau a-j} \left( y+\tau a-j \right) = \]

\[ \binom{x+y+\tau a}{\tau} + a \sum_{j=0}^{\tau-1} \binom{\tau-1}{j} \left( y+\tau a-j \right) \binom{y+\tau a-j}{\tau-1-j} = \]

\[ \binom{x+y+\tau a}{\tau} + a S_{\tau-1}(x+u, y-u+\tau a). \]

The relation (3.528) is derived as $\Phi(\tau)$ in Chapter $\Phi$ from identities for Fibonacci-like polynomials. It is different from (3.527) with $a=u=-1$, $x=n$, $y=m$ which gives
\[
\sum_{j=0}^{k} \binom{n-j}{j} \binom{m-k+j}{n-j} + \sum_{i=0}^{k-1} \binom{n-1-i}{i} \binom{m-k+i}{n-1-i} = \binom{n+m-k}{k}.
\]

e.g. terms with \(n-j < 0\) when \(k \geq n\),

Cf. Riordan (1968), Ch. 2, Prob. 7a.

(3.529), (3.530) With \(D(25)\)

\[
\sum_{n=0}^{\infty} Z^n \sum_{k=0}^{n} \binom{a+k}{k+1} \binom{b+n-k}{n-k+1} =
\]

\[
= \left\{ \left( \frac{a-b}{(1-Z)^2} \right) - \left( \frac{-a}{(1-Z)^2} \right) - \left( \frac{-b}{(1-Z)^2} \right) + 1 \right\} =
\]

\[
\sum_{n=0}^{\infty} Z^n \left\{ \left( \binom{a+b+n+1}{n+2} - \binom{a+n+1}{n+2} - \binom{b+n+1}{n+2} \right) \right\},
\]

\[
\sum_{n=0}^{\infty} Z^n \sum_{k=0}^{n} (-1)^k \binom{a+k}{k+1} \binom{b+n-k}{n-k+1} =
\]
\[-z^{-a}(1+z)^{-a} - 1 \right) \left( (1-z)^{-a} - 1 \right) =
\[-z^{-a} \{ (1-z^2)^{-a} - (1+z)^{-a} - (1-z)^{-a} + 1 \} =

For \( a, b \in \mathbb{N} \), we may consider (3.529) and (3.530) as special cases of (3.530).

(3.531), (3.532) From (1.14), for \( x \notin \mathbb{Z} \), we have by canceling and rearranging factorials (and then by (3.531) and (3.532))
\[
\sum_{i=0}^{\tau} (-1)^i \binom{x-i}{i} (x-i) = \sum_{i=0}^{\tau} (-1)^i \binom{\tau}{i} (x-i) = \Delta^{\tau} (x-\tau) = 1
\]

When \( x = n \geq \tau \) we have \( n-i \geq 0 \) in the sum of (3.531) and we may restrict summation to \( i \leq n/2 \). Then \( n-2i \geq 0 \) and we may restrict summation further to \( i \leq n-\tau/2 \). For \( \tau > n \) this argument fails and the sum in (3.532) is empty.
(3.53.8) From (1.428) and the convolution property (Theorem 1.1) of generating functions
\[
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} w^k v^{n-k} =
\]
\[
(1-4u)z^{-\frac{1}{2}} (1-4uvz)^{-\frac{1}{2}}
\]
\[
\left(1 - 4(u+v)z + 16uvz^2 \right)^{-\frac{1}{2}} =
\]
\[
\sum_{n=0}^{\infty} \binom{2n}{n} (u+v)z - 4uvz^2 \right)^{\frac{1}{2}}
\]
\[
\sum_{r=0}^{\infty} \binom{2r}{r} \sum_{j=0}^{r} \binom{r}{j} (-4uv)^j (u+v)^{r-j} z^{r+j} =
\]
\[
\sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n-j}{j} \binom{2n-2j}{n-j} (-4uv)^j (u+v)^{n-2j}
\]

Ring absolute values in the triplet. This gives a finite result, by going backwards. By (1.428) and D (25) the above generating function is also equal to
\[
\sum_{k=0}^{\infty} \binom{2k}{k} (u-v)^{2k} z^{2k} (1-2(u+v)z)^{-\frac{1}{2}} =
\]
\[
\sum_{k=0}^{\infty} \binom{2k}{k} (u-v)^{2k} z^{2k} (1-2(u+v)z)^{-\frac{1}{2}} =
\]
\[
\sum_{k=0}^{\infty} \binom{2k}{k} (u-v)^{2k} \sum_{j=0}^{\infty} \binom{2k+j}{j} z^j (u+v)^j z^{2k+j} =
\]
\[ \sum_{n=0}^{\infty} z^n \sum_{2k \leq n} \binom{2k}{k} \binom{n}{2k} 2^{n-2k} (u-v)^k (u+v)^{n-2k}, \]

which proves the second equality. Changing the summation order is justified by taking absolute values in the fourth line (for small \( z \)) and working backwards. See also (3.442) - (3.448).


Kriordan (1968), ch. 4, Probl. 13.

(3.542) From D(14), by canceling and regrouping factorials, we see that the l.h.s. is equal to

\[ \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n} x^k = \sum_{h=0}^{n} \binom{n}{h} \binom{2n-h}{n} x^{n-h}, \]

which by (3.178) or (3.90) with \( x = n \) equals
(3.543) By (3.542) with $x = -\frac{1}{2}$ the l.h.s. is equal to

$$\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} = \sum_{k=0}^{n} (-1)^{k} (-1)^{k} \binom{n}{k},$$

Now apply (3.67).

Or, with generating functions, see (1.428),

$$\sum_{n=0}^{\infty} z^{n} \sum_{k=0}^{n} (-1)^{k} \binom{n+k}{2k} \binom{2k}{k} z^{n-k} =$$

$$\sum_{k=0}^{\infty} (-1)^{k} \binom{2k}{k} \sum_{m=0}^{\infty} \binom{m+2k}{m} z^{m} =$$

$$\sum_{k=0}^{\infty} (-1)^{k} \binom{2k}{k} z^{k} (1-2z)^{-2k-1} =$$

$$\frac{1}{(1-2z)^{-1} \left(1 + yz (1-2z)^{-2} \right)^{-\frac{1}{2}}} = (1 + yz)^{-\frac{1}{2}}.$$  

(cf. (3.158), (3.180) and Riordan (1968), Ch. 2, Problem 11. Also Grosswald (1953), Carlitz (1956).

(3.544) With $B(x)$, $D(x)$ and Vandermonde's convolution $D(2x)$ the l.h.s. is equal to

$$(-1)^{n} \sum_{x}^{n} (-\frac{1}{2}) \binom{n-x-1}{x}.$$
functions to the identity

\[(1 - 4x)^{-m-\frac{1}{2}} = (1 - 4x)^{-m-\frac{1}{2}} (1 - 4x)^{-1}\]

we obtain (3.547), with B(17).

Cf. Riordan (1968), Ch.4, Example 2. A direct proof of (3.547) with induction on \(n\).

With C(1286)

1. \[\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\frac{x-k}{y-k})^n = \sum_{k=0}^{\infty} (-1)^k \binom{y-k}{k} \sum_{m=0}^{\infty} \binom{x-k}{m} y^m z^{m+k} = (1+yz)^x \sum_{k=0}^{\infty} \binom{y-k}{k} \left(\frac{-z}{1+yz}\right)^k = (1+yz)^x \left(1-\frac{yz}{1+yz}\right)^{-\frac{1}{2}} \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{yz}{1+yz}}\right)^{1+y} \]

Interchanging summations is justified since by B(48) and C(1286) for small \(z\).
\[ \sum_{n=0}^{\infty} |z^n| \sum_{k=0}^{n} \frac{(x-k)(y-k)}{(n-k)!} y^{n-k} = \]
\[ \sum_{k=0}^{\infty} \frac{|(y-k)|}{|z|^k} \sum_{m=0}^{\infty} \frac{|(x-k)|}{|z|^m} y^m \leq \]
\[ \sum_{k=0}^{\infty} \frac{|(y-k)|}{|z|^k} \sum_{m=0}^{\infty} \frac{|(x-k)|}{|z|^m} (|x|+k+m-1) y^m = \]
\[ \sum_{k=0}^{\infty} \frac{|(y-k)|}{|z|^k} (1-|y|z)^{-|x|-k} < \infty. \]

The first equality in (3.548) follows from (1) and Corollary (28a). Writing (1) as
\[ \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} (-1)^k \frac{(x-k)(y-k)}{(n-k)!} y^{n-k} = \]

we obtain the second equality with \( z = y \).

Taking \( y = 2x \), we obtain from the first equality in (3.548)
\[ \sum_{k=0}^{n} (-1)^k \frac{(x-k)(2x-k)}{(n-k)!} y^{n-k} = \frac{2x+1}{2x+1-n} \frac{(2x+1-n)}{n}. \]

With \( y = 2x+1 \) in the third member of (3.548)
\[ \sum_{k=0}^{n} (-1)^k \frac{(x-k)(2x+1-k)}{(n-k)!} y^{n-k} = \frac{(2x+1-n)}{n}. \]

Identities for other special values of \( y \), e.g. \( y = 2x-1 \), are easily found.
Cf. Riordan (1968), Ch. 6, Prob. 18.

Take \( y = 2 \times + z \) in the second equation.

Other special values of \( y \), see (3.548).

\[
\sum_{n=0}^{\infty} z^n \sum_{2k \leq n} (-1)^k \binom{x+k}{k} \binom{n-k}{k} z^{-n-k} =
\]

\[
\sum_{k=0}^{\infty} \binom{-1}{k} \binom{x}{k} z^{-x-k} = (1-xz)^{-2x-2}.
\]

\[
\sum_{n=0}^{\infty} z^n \sum_{j \leq n} (-1)^j \binom{x}{j} \binom{n+1}{2j+1} =
\]

\[
\sum_{j=0}^{\infty} (-1)^j \binom{x}{j} \sum_{m=0}^{\infty} \binom{m+2j+1}{m} z^{m+2j} =
\]

\[
\sum_{j=0}^{\infty} (-1)^j \binom{x}{j} z^j (1-z)^{-2j-2} =
\]

\[
(1-z)^{-2} \left\{ 1 - z^2 (1-z)^{-2} \right\}^x = (1-z)^{-2x-2} (1-xz)^x.
\]
The second equality follows with the product property (Theorem M1) of generating functions. See (25) in Gould (1973), with $x = n$. 
(3.555) From (1.428) and
\[ \sum_{k=0}^{\infty} k \binom{2k}{k} z^k = 2z (1-yz)^{-3/2}, \]
to be derived from (1.428) by differentiation, we have with the convolution property (Theorem M1) of generating functions

(3.556) With (1.430), (1.428) and the convolu-

\[ \sum_{k=0}^{\infty} \binom{2k}{k} (-x)^k \sum_{m=0}^{\infty} \binom{m+k+1}{m} (yz)^{m+1} = \]
\[ \sum_{k=0}^{\infty} \binom{2k}{k} (-yz)z^k (1-yz)^{-k-2} = \]
\[ (1-yz)^{-2}(1+16xz(1-yz))^{-1/2} \]
\[ = (1-yz)^{-3/2}(1-4z+16xz)^{-1/2} \]
The identity also may be derived from (3.434). See (14) in the proof of (3.434).
(3.557) With $B(16)$, $B(13)$ and Vandermonde's convolution the l.h.s. is equal to
\[
(-1)^{n+1} n \sum_{k=0}^{n} \binom{n}{k} \binom{1}{n-k} = (-1)^{n+1} n \binom{n}{n}.
\]

(3.558) From (3.202) with $a = 1$, $b = n - \frac{1}{2}$, $m = n$, from $D(24)$ and then $B(21)$ with $k = n$, $m = k$, we have
\[
x^{-1} \left( \frac{x+n}{n} \right)^{-1} \left( \frac{x+n-\frac{1}{2}}{n} \right) = \\
\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{n-k-\frac{1}{2}}{n} \frac{1}{x+k} = \\
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{k-\frac{1}{2}}{n} \frac{1}{x+k} = \\
4^{-n} \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{x+k}.
\]

See Riordan (1968), Ch. 3, Prob. 9, with $B(16)$.

(3.559) With $B(15)$, $B(13)$ and $D(18)$ the l.h.s. for $n \geq 1$ is
\[
(-1)^{n-1} 4^n \sum_{k=0}^{n} \left( \binom{1}{2} \right) \binom{1}{n-k} \frac{1}{x+k} = 
\]
\[ (-1)^n \sum_{k=0}^{n} \left( \frac{1}{k} \right) \left( \frac{1}{n-k} \right) \frac{1}{x+k} = \]

\[ \sum_{k=0}^{n} \left( \frac{1}{k} \right) \left( \frac{1}{n-k} \right) \frac{1}{x+k} = \]

\[ \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{1}{x+2n+1} \]

\[ (\arcsin z)^2. \]
From (2.58)
\[ \sum_{n=0}^{\infty} \frac{(n+1)^{-2}}{n+1} \left( \frac{2n+2}{n+1} \right)^{-1} \frac{1}{(n+1)^{2n+2}} = z \left( \text{arcsin} z \right)^{2}, \]

\[ (3) \sum_{n=0}^{\infty} \frac{(n+1)^{-2}}{n+1} \left( \frac{2n+2}{n+1} \right)^{-1} \frac{1}{2^{n+1}} \left( \frac{2n+1}{n+1} \right)^{2n+1} = (yz) \left( \text{arcsin} z \right)^{2}, \]

and (3.560) follows from (2) and (3).

From (1) and (1.428) and then (3)
\[ \sum_{n=0}^{\infty} \frac{z^{2n}}{2^{n+1}} \sum_{k=0}^{n} \frac{1}{2^{k+1}} \left( \frac{k}{k} \right) \left( \frac{2n-k}{n-k} \right) = (1-yz^{2})^{1/2} (2z)^{-1} \text{arcsin} z = \frac{1}{8z} \frac{d}{dz} \left( \text{arcsin} z \right)^{2} = \sum_{n=0}^{\infty} \frac{1}{n+1} \left( \frac{2n+2}{n+1} \right)^{-1} \frac{1}{2^{n+1}} \frac{z^{2n}}{2^{n+1}}, \]

and (3.561) follows. (First equality with B(30)).

This identity also follows from (3.558) with \( x = \frac{1}{2} \) and B(30). See also the proof of (4.7).

(3.562) With B(15) and Vandermonde's convolution D(26) the l.h.s. is equal to
\[ (-1)^{n} 4^{n} \sum_{k=0}^{n} \binom{1/2}{k} \binom{1/2}{n-k} = (-4)^{n} \binom{1}{n}. \]

Cf. Riordan (1968), Ch.3, Prob. 8.
(3.563) Since \( \frac{1}{k+1} \binom{2k}{k} = \frac{1}{1+2k} \binom{1+2k}{k} \), we also may see \( C(197) \) as a special case of \( C(91) \) for the Gould polynomials with \( a=2 \).

The proof of \( C(197) \) was by a general generating function. Direct computation of the generating function w.r. to \( n \) of \( C(197) \) may be done with (3.428) and (3.426). The identity was also proved as \( C(126) \).

(3.564) Descending or ascending induction on \( r \in \{0,1,\ldots,n\} \) for fixed \( n \). The identity holds for \( r=n \) and for \( r=0 \) it coincides with (3.563). The induction step may be carried out with the application of \( \mathcal{D}(13) \).


(3.565) With \( \mathcal{D}(13) \), by canceling and rearranging factorials, and then (3.186) or Si(5)—and Si(5) the l.h.s. is equal to

\[
\sum_{2h \leq n} (-1)^h \binom{n}{k} \binom{2n-2k}{n} (1+2x)^{n-2k} =
\]

\[2^n \mathcal{P}_n(1+2x) = 2^n \sum_{h=0}^{n} \binom{n}{h} \binom{n+h}{n} x^h.
\]

See also (3.178), (3.443), (3.542).
(3.56g) With $B(15)$ and $B(13)$ and then $(3.42)$ with $x = \frac{1}{2}$, $y$ replaced by $n$ and $n$ by $m+n$, the left-hand side is equal to

\[
(-4)^{m+n} \sum_{k=0}^{n} \binom{n-k}{\frac{1}{2}} \binom{m+k}{\frac{1}{2}} = (-4)^{m+n} \sum_{h=0}^{n} \binom{h}{rac{1}{2}} \binom{m+n-h}{\frac{1}{2}}
\]

\[
(-4)^{m+n} \frac{m}{m+n} \binom{-\frac{1}{2}}{n} \binom{-\frac{1}{2}}{m} = \frac{m}{m+n} \binom{2n}{n} \binom{2m}{m}
\]

and (3.57a) follows with (3.17) and (3.18).

(3.57b). With $D(13)$ the left-hand side is equal to

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{n} (x+k)^{-1}
\]

Now (3.202) and $D(xy)$.

Cf. Ranjan Roy (1987), (3.6), and Graham, Knuth, and Patashnik, Ch. 5.2, Prob. 8.

See also (3.57e).
\[ (3.572) \text{ With } D(13) \text{ the l.h.s. is equal to } \]
\[ \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{n+1+k}{n-1} (x+k)^{-1}. \]

Now (3.202) and D(24).


\[ (3.573) \text{ With } D(14) \text{ the l.h.s. is equal to } \]
\[ \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{n+y+k}{n} (x+k)^{-1}. \]

Now (3.202) and D(24).

\[ (3.574) \text{ With } (1.430) \text{ or } B(14) \text{ and with } A(1.438) \]
\[ \sum_{k=0}^{\infty} \left( \frac{2^{k}}{k!} \right) (2k+1)^{-2} z^{k} = \frac{1}{2\sqrt{z}} \text{ arcsin } 2\sqrt{z}. \]

With the convolution property of generating functions (Theorem M1) the generating function w.r. to \( m \) of the l.h.s. of (3.574) is
\[ (1-4z)^{-3/2} (2\sqrt{z})^{-1} \text{ arcsin } 2\sqrt{z}. \]

\[ \text{From (2.53)} \]
\[ x^{-1} (1-x^2)^{-3/2} \text{ arcsin } x = - (1-x^2)^{-1} \]
\[ + \sum_{k=1}^{\infty} \left( \frac{2^{k}}{k!} \right) \frac{1}{4^{k}} x^{2k-2}. \]
and (3.574) follows. See also the proof of (4.6).

(3.575) With $B(15)$, $D(18)$, $B(13)$, (3.561) and $D(13)$ the l.h.s. for $n \geq 1$ is equal to

$$
\sum_{k=0}^{n} \frac{1}{2k+1} \binom{2k}{k} (-1)^{n-k-1} 4^{n-k} \binom{1/2}{n-k} =
$$

$$
\sum_{k=0}^{n} \frac{1}{2k+1} \binom{k}{n-k} \cdot \frac{1}{n-k} \binom{2n-2k}{n-k} + \frac{1}{2n} \cdot \frac{1}{2n+1} \binom{n}{n} .
$$
\[ \sum_{k=0}^{\infty} \frac{x^{x-k}}{k!} (x-k)^k z^k (1+z)^k = \left( \frac{1}{2} + \frac{1}{2} \sqrt{1+yz(1+z)} \right)^x = (1+z)^x. \]

And \( k \geq m-k \).
\[ (-1)^n \sum_{k=0}^{n} \frac{x}{x-k} \binom{x-k}{k} \binom{n-x-1+k}{n-k} = (-1)^n \frac{(n-1)}{n}. \]

One also might apply \( B(31) \) with \( a = -1 \) and then \( (3.526) \).

(3.580) These sums were computed for \( r = 0, 1, 2 \) by recurrence in Riordan (1968), Ch. 1, Example 8 and Problem 14. Our proof is based on the general identity \( G(73) \) or \( G(75) \). With \( D(24) \) we see that the sum in \( (3.580) \) is equal to

\[ \sum_{k=0}^{m} (2k+1) \binom{x-1-k}{m-k} \frac{(x+k)!}{(m+k+1)! (x-m)!} = \]

\[ (x-m)^{-1} \sum_{k=0}^{m} \binom{k+\frac{1}{2}}{m-k} (x-1-k) (x+k) (x-m) (x-m)^{-1} \binom{k+\frac{1}{2}}{m-k} (m-k) (m+k+1). \]

With \( D(24) \) this is equal to

\[ (1) \binom{m-x}{1} \sum_{k=0}^{m} \binom{k+\frac{1}{2}}{m-k} (m-x) (m-x) (m-k) (m+k+1), \]

and with \( G(73) \) or directly by \( G(75) \) it is equal to

\[ (2) \binom{m-x}{1} \sum_{k=0}^{2m+1} \binom{k-m-\frac{1}{2}}{k} (m-x) (m-x) (2m+1-k). \]

For \( r = 0 \) we find from (2), with Vandermonde's
For \( r = 1 \), we see from (1) and (3.121) with 
\( y = m - x \), or from (2) and (3.116) that the sum in (3.580) is equal to 
\( \frac{(2m+1)(2x+1)}{(2x-2m)(2x-2m+1)} \binom{2x}{2m+1} = \binom{2x+1}{2m} \).

For \( r = 2 \), we find from (1) and (3.121) 
with \( D(24) \) or from (2), (3.112) and \( D(24) \) that the sum in (3.580) is equal to

\[
\binom{2m+1}{2m} = \binom{2x+1}{2m}.
\]

(3.581), (3.582).

These sums satisfy the 'binomial' recurrence
\[
\binom{n}{m} = \binom{n-1}{m} + \binom{n}{m-1}, \quad n \geq 1, \quad m \geq 1,
\]

that was studied in G(76) = G(90).

A special case of (3.581), viz. \( r = 1 \), is in
Ranjan Roy (1987); Knuth (1961), Vol. 1,
Wilf (1989); Wilf (1990), p.112, 137; Graham,
Knuth and Patashnik (1989), Ch.5.2, Prob. 7, p.182.

The sum in (3.581) may be restricted to
\( 0 \leq k \leq n - m \), for \( n \geq m \). For \( r = 0 \) the identity
(3.581) follows from (1.3). One sees immediately that 
\( \binom{n}{m} = \delta_{nm} \).

For \( n \geq 1 \) we have

\[
\frac{1}{\varepsilon_{k+1}} = \sum_{k=0}^{m} (-1)^k \binom{n}{k} \frac{(n+2k)!}{n!(2k)!}.
\]
\[ n^{-1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{n} = 0 \]

By (3.147), the identity (3.581) then follows from \( G(26) \).

For (3.582): It is easily seen that \( S_{1m} = \frac{1}{2} \delta_{m,0} \) and \( S_{10} = e (\alpha + \gamma) \). For \( n \geq 2 \), with \( D(n) \) and (3.147):

\[
S_{nm} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{n+3k}{n! (2k)! (2k+1)! (2k+2)!} \]

\[
= \frac{1}{n(n-1)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+3k}{n-2} = 0
\]

Then in the notation of \( G(77) = G(86) \)

\[
\sum_{n=0}^{\infty} S_{nm} z^n = V_m(z) = \frac{1 + \frac{ez}{2\alpha \beta + y}}{(z + \frac{ez}{2\alpha \beta + y})} z^m (1-z)^{-m}
\]

\[
\frac{1}{2} \sum_{n=m}^{\infty} (n-1) z^n + \frac{ez}{2\alpha \beta + y} \sum_{n=m+1}^{\infty} \frac{(n-1)^2}{(m-1)!} z^n, \ m \geq 1,
\]

and (3.582) for \( n \geq 2, \ m \geq 1 \) follows.

(3.583) - (3.585) These relations are consequences of the theory in pp. 567-94, where an exposition and extension of the papers of Xingrong Ma (1998) and Carlitz (1978) was given. Some of the identities below are in Xingrong Ma. We have
(3.583): Equate the coefficients of \( y^m \) in both sides of \( z(t) \) after expanding \((t+1)^{n-3t}\) by the binomial theorem.

The sum in (3.583) may be restricted (further) to \( k \leq n-m \). With \( \mathcal{D}(14) \), by canceling and rearranging factorials, the \( l.h.s. \) for \( m \leq n \) is equal to

\[
\frac{n}{n-m} \sum_k (-1)^k \binom{n-m}{k} \binom{n-1-2k}{n-m-1},
\]

with \( 3k \leq n \), \( 2k \leq m \), \( k \leq n-m \). The restriction \( k \leq n-m \) now is cared for by the factor \( \binom{n-m}{k} \).

Note that \( 2k \leq n \) when \( k \leq \lfloor n/2 \rfloor \). So we obtain

\[
(3) \sum_{k \leq \frac{n}{2}} (-1)^k \binom{n-m}{k} \binom{n-1-2k}{n-m-1} = \binom{n-m}{m}, \quad m \leq n,
\]

whereas the sum with \( k \leq n-m \) is zero by (3.147).

(3.584): Put \( a=1 \), \( b=x \) in \( \Phi(375) \), \( a \times y = -x \) in \( \Phi(375) \), and equate coefficients of \( x^m \) in both sides, after expanding \((1-x)^{n-3t}\) by the binomial theorem.

The sum in (3.584) may be restricted (further) to \( 2k \leq n-m \). With \( \mathcal{D}(14) \), by canceling and rearranging factorials, the \( l.h.s. \), with \( 1 \leq m \leq n \), is equal to

\[
\frac{n}{m} \sum_k (-1)^k \binom{m}{k} \binom{n-1-2k}{m-1},
\]
with \( k \leq m, \ 3k \leq n, \ 2k \leq n-m \). The restriction \( k \leq m \) now is cared for by the factor \( \binom{m}{k} \).

So we have

\[ \sum_{k \leq \frac{n-m}{3}} (-1)^k \binom{m}{k} \binom{n-1-2k}{m-1} = \binom{m}{n-m}, \ 1 \leq m \leq n, \]

whereas the sum with \( k \leq m \) is zero by (3.147).

\[ (3.585) \quad \text{Take } c = (1-x)^{-\theta}, \ A = (1-x)^{1-\theta} \text{ in } \Omega_{(2.75)}. \text{ Then the l.h.s. becomes} \]

\[ \sum_{3k \leq n} \frac{n}{n-2k} \binom{n-2k}{k} (1-x)^{k-n} \theta \frac{n-3k}{x^{n-3k}} = \]

\[ \sum_{3k \leq n} \frac{n}{n-2k} \binom{n-2k}{k} \sum_{j=0}^{\infty} \binom{n-3k+j-1}{j} x^{n-3k+j} = \]

\[ \sum_{3k \leq n} \frac{n}{n-2k} \binom{n-2k}{k} \sum_{\varepsilon = n-3k}^{\infty} \left( \varepsilon + 2k + n(\theta-1) - 1 \right) x^\varepsilon = \]

\[ \sum_{\varepsilon = 0}^{\infty} x^\varepsilon \sum_{[\varepsilon-\varepsilon] \leq 3k \leq n} \frac{n}{n-2k} \binom{n-2k}{k} (\varepsilon + 2k + n(\theta-1) - 1) x^{\varepsilon+2k+n(\theta-1)-1}. \]

The r.h.s. becomes

\[ (1-x)^{-n\theta} + \sum_{2k \leq n} (-1)^{n-k} \frac{n}{n-k} \binom{n-k}{k} (1-x)^{n(1-\theta)-k} \]

\[ \sum_{\varepsilon = 0}^{\infty} \binom{n+\varepsilon-1}{\varepsilon} x^{\varepsilon+2k+n(\theta-1)-1} \]

\[ + \sum_{\varepsilon = 0}^{\infty} x^{\varepsilon} \sum_{2k \leq n} (-1)^{n-k} \frac{n}{n-k} \binom{n-k}{k} (k+n(\theta-1)+\varepsilon-1). \]
and the conditions $k \geq m$ and $2k \leq n$ are used for the binomial.
(3.6.6) This is \( C(x) \) with \( \xi_n(x) = \binom{x}{n} \) and
\[
\xi_n \approx \lim_{x \to 0} x^{-1} \binom{x}{n} = \frac{(-1)^{k+1}}{k}, \text{ see (2.6) and after (2.4)}.
\]

(3.6.7) With \( P(y), D(24) \) and (3.5.93) the l.h.s. is equal to
\[
(-1)^m \sum_{j=0}^{m} \frac{1}{1+j^6+j} \binom{1+j^6-j}{j} \begin{pmatrix} -x-1-j^6+j \\ m-j \end{pmatrix} =
\]
\[
(-1)^m \left( \frac{-x}{m} \right) = \binom{x+m-1}{m}.
\]

(3.6.8) Induction on \( n \) for the first equality.
Denoting the l.h.s. by \( S_n(x, y) \) we have for the step \( n \to n+1 \) by C(91)
\[
S_n(x, y) = \sum_{k=0}^{n} \left( \frac{x}{x+kz} + \frac{kz}{x+kz} \right) \binom{x+kz}{k} \binom{y-kz}{n-k} =
\]
\[
\binom{x+y}{n} + z \sum_{k=1}^{n} \binom{x-1+kz}{k-1} \binom{y-kz}{n-k} =
\]
\[
\binom{x+y}{n} + z \sum_{h=0}^{n-1} \binom{x-1+z+hz}{h} \binom{y-z-hz}{n-1-h} =
\]
\[
\binom{x+y}{n} + z \sum_{n-1}^{x-1+z, y-z} (x-1+z, y-z) =
\]
\[
\binom{x+y}{n} + \sum_{k=0}^{n-1} \binom{x+y-1-k}{n-1-k} = \sum_{h=0}^{n} \binom{x+y-h}{n-h} z^h.
\]
The second equality is (1.32a). One also might apply $D(24)$ to the factors in the l.h.s. and then use the first equality.


(3.613) With (3.612) the l.h.s. is equal to

$$
\sum_{k=0}^{n} \binom{n+k}{n-k} (z-1)^k = \sum_{h=0}^{n} \binom{n+1}{h} (z-1)^{n-h}
$$

(3.614) With $D(24)$ and C(91) the l.h.s. is equal to

$$
(-1)^{n-1} \sum_{k=0}^{n-1} \frac{(-1)^{n-k}}{1-kb} \left( \binom{k}{n-1-k} \right)
= (-1)^{n-1} \binom{n-1-nb}{n-1} = \binom{nb+n-1}{n-1}.
$$

Or, apply $D(24)$ to the second factor in the sum, and then (3.607). Cf. Engelberg (1965), Gould and Kaufky (1960).
(3.621) = \Phi(78). \text{ With (3.612) for } z = -1 \text{ the l.h.s. is equal to:}

\sum_{h=0}^{\infty} \binom{x+y-1}{h}(-1)^{h} = \sum_{h=0}^{\infty} \binom{x+y-1}{h}(-1)^{h} = \binom{x+y}{1}.

\text{Cf. Riordan (1968), p.75, Prob. 7.}

(3.623) \text{ With the convolution property of generating functions and C(1289), C(1286) }

\sum_{h=0}^{\infty} z^{h} \sum_{k=0}^{n} \binom{x-k}{k} \binom{y-n+k}{n-k} =

\frac{1}{1+yz} \left( \frac{1}{2} + \frac{1}{2} \sqrt{1+yz} \right)^{x+y+z} =

\sum_{h=0}^{\infty} z^{h} \sum_{k=0}^{n} \frac{x+y+z}{x+y+2-k} \binom{x+y+2-k}{k}(-1)^{n-k}.

\text{Cf. (3.633), (3.645)}

(3.632) \text{ With C(85) the l.h.s. is equal to:}

\sum_{h=0}^{n-r-s} \binom{r+2h}{h} \frac{s}{\Phi(n-r-s-h)+s} \binom{z(n-r-s-h)+s}{n-r-s-h} =

\frac{r+s}{2n-r-s} \binom{2n-r-s}{n-r-s} = \frac{r+s}{2n-r-s} \binom{2n-r-s}{n-r-s}.
With the convolution property of generating functions and \( C(n^a), C(108) \)

\[
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{x+2k}{n-k} \binom{y+2n-2k}{n-k} =
\]

\[
(1-4z)^{-1/2} \left\{ (2z)^{-1/2} \binom{1-\sqrt{1-yz}}{x+y} \right\}^{x+y}
\]

\[
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \frac{x+y}{x+y+2k} \binom{x+y+2k}{k} y^{n-k}
\]

Cf. (3.623), (3.645).

(3.634) With \( D(24), D(13) \) and \( C(91) \) the l.h.s. of (a) is

\[
\sum_{j=0}^{n} (-\varepsilon-1+2j) \frac{1}{1+2n-2j} \binom{1+2n-2j}{n-j} =
\]

\[
\sum_{h=0}^{n} \frac{1}{1+2h} \binom{1+2h}{h} (-\varepsilon-1+2n-2h) \binom{2n-2h}{n} = (2n-\varepsilon)
\]

We prove (b) by induction on \( \varepsilon \). The relation is easily seen to hold for \( \varepsilon=0 \) and \( \varepsilon=1 \), with \( n \geq \varepsilon \). Denoting the l.h.s. by \( S(n, \varepsilon) \) we have by \( D(13) \), since \( \varepsilon-1 \geq \varepsilon/2 \) for \( \varepsilon \geq 2 \)

\[
S(n, \varepsilon) = \sum_{j=0}^{\varepsilon-1} (-1)^j \binom{\varepsilon-j}{j} \frac{1}{n-j+1} \binom{2n-2j}{n-j} =
\]

\[
\sum_{j=0}^{\varepsilon-1} (-1)^j \binom{\varepsilon-1-j}{j} \frac{1}{n-j+1} \binom{2n-2j}{n-j} +
\]

\[
\sum_{j=1}^{\varepsilon-1} (-1)^j \binom{\varepsilon-1-j}{j-1} \frac{1}{n-j+1} \binom{2n-2j}{n-j} =
\]
\[
\sum_{j=0}^{[r-1/2]} (-1)^j \binom{r-1-j}{j} \frac{1}{n-j+1} \binom{2n-2j}{n-j} + \sum_{i=0}^{[r-2/2]} (-1)^i \binom{r-2-i}{i} = S(n, 2-1) = S(n-1, r-2) \\
= \frac{r}{n+1} \binom{2n-r+1}{n} - \frac{r-1}{n} \binom{2n-r}{n-1} = \frac{r+1}{n+1} \binom{2n-r}{n} .
\]

For \([r/2] \leq n \leq r\) the l.h.s. of (a) and (b) coincide. For \(n = r\) both r.h.s. sides are equal to 1. For \(r/2 \leq n < r\), \(a > 0\), both r.h.s. sides are zero.

See Riordan (1968), Ch. IV, Prob. 1, on Ballot numbers.

(3.635) From C. (91) with \(a = 2\), since
\[
\frac{1}{k+1} \binom{2k}{k} = \frac{1}{1+2k} \binom{1+2k}{k}.
\]

Given as Jonah’s formula by Hilton and Pedersen (1990).

(Cf. Nieuw Arch. (5), 4, 2003, no 1, 92-93.)
With the convolution formula for generating functions and \( C_{18y}^{3}, C_{18y}^{3} \),

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n}{k} (-1)^{n-k} z^{n-k} = (1 + z)^{-2}
\]

the analogue for \( a = 1/6 \) of (3, 623)

\[\sum_{n=0}^{\infty} \binom{n}{h} \frac{x^{n+h}}{x+y+1} \]

This is the analogue for \( a = 1/6 \) of (3, 623).

Now apply (3, 136).

With \( D(y) \) the L.H.S. is equal to

\[\sum_{n=0}^{\infty} \frac{1}{n+m+1} \cdot \begin{pmatrix} m+2 \\ 2k \end{pmatrix} \cdot \begin{pmatrix} (m+2) \\ k+1 \end{pmatrix} \]

Now apply (3, 137).

(3.646)

\[\sum_{n=0}^{\infty} \frac{1}{n+m+2} \sum_{1 \leq k \leq n \leq m} \binom{m+k+1}{n+k+1} \binom{m+k+1}{n+k+1} \]

(3.645)

With the convolution formula for generating functions and \( C_{18y}^{3}, C_{18y}^{3} \),
For $n \geq 1$ by (3.14) and rearrangement of factorials, the summand is equal to 
\[ \frac{2x}{n} \binom{n}{xj} \binom{x-1+j}{n-1}. \]
\textbf{TABLE 4}

(4.1) With $D(n)$, by canceling and rearranging factorials, and with (1.3.17) the l.h.s. is equal to

\[ x^{-1} \sum_{k=1}^{n} (x-k)^{-1} = x^{-1} \sum_{k=1}^{n} (x-n+k)^{-1} \]

\[
\binom{n}{2} \binom{n}{n-1} \quad \binom{n}{1} \binom{n}{n-1} = n
\]

One from (4.2) with $z = n$.

An urn contains $a$ white and $b$ red balls. The balls are drawn randomly, one by one, without replacement. Let $X$ be the number of the drawing giving the first white ball. Then

(4.2) Induction w.r. to $n$, using $D(n)$.

Or: both sides are polynomials in $z$ of degree $n$ and there is equality for $z = 0, 1, \ldots, n$ by (4.1).

For $z = x+1$ the l.h.s. reduces to

\[ (x+1) \sum_{k=0}^{n} (x+1-k)^{-1} \]
For the same turn as in (4.1) the identity (4.2) illustrates that, when \( n \leq b+1 \),

\[
\sum_{a}^{n} P(X = k) - P(X \leq n) = \sum_{a}^{n} (n-k)!((-x-2))! \frac{(-z)^{a}}{(-z-1)!} \frac{(n-x-1)!}{(n-x-1)!} \frac{(-z-1)!}{(-z-1)!} \frac{1}{n-z+x+1}.
\]

By (4.2) this proves the convergence. For \( z \in \mathbb{N} \) the series has finitely many nonzero terms. For \( z \in \mathbb{N} \) and \( b < n \), similarly

so that the convergence is absolute for \( \text{Re}(x-z) < -1 \).

(Cf. Erdélyi e.a. (1953), I §2.5.3,(15); Nörlund
(4.4) With $D(\mu)$, canceling and rearranging factorials, and then (3.506) we find that the l.h.s. is equal to

$$\frac{n! \cdot \mu^x}{x!} \frac{(x-\mu-n)!}{n!} \sum_{k=0}^{n} \frac{(\mu+k)}{k} \frac{x-\mu-k}{n-k} =$$

**White Ball. Then**

$$P(\gamma = h) = \frac{a}{a-1} \frac{(a-1)!}{b!} \frac{1}{(a+b-1)!}$$,
(4.6), (4.7) With the convolution property of generating functions (Theorem M1)

\[ \sum_{n=0}^{\infty} -n \binom{n}{x} \sum_{\nu} \binom{n}{\nu} (x, 1/y, y, y) \]

For \( x = \frac{3}{2}, y = -\frac{1}{2} \), this is equal, with \( B(\nu), \nu \in \mathbb{Z} \), and \( 1/2, 1/2 \), to

\[ \sum_{n=0}^{\infty} \binom{n}{\nu} (-1)^{n-\nu} \sum_{\nu=1}^{n-1} \binom{n-1}{\nu-1} 2^{n-1} \]

and (4.6) follows. When \( x = \frac{1}{2}, y = -\frac{3}{2} \),

\[ \sum_{k=0}^{n} \left( \frac{3}{2} \right) \binom{n}{n-k} x^{k} = y_{n}, \quad n \in \mathbb{N}, \]
\[ 1 + 4^{n-1} \sum_{k=1}^{\frac{1}{2}n} \binom{2n-2k+1}{n-k} \binom{n-1}{k-1} (x^{n-1}) = 4^n \binom{2n}{n}^{-1}. \]

which is (3.574). So (4.6) and (3.574) are companions by IR (19), \( a = \frac{3}{2} \), in the sense of p. IR 7. So it suffices to prove one of them.
\[ \sum_{k=0}^{n} \binom{n}{k} x_k = y_n, \quad n \in \mathbb{N}_0, \]

for, by B(13) and B(14), with
\[ \sum_{k=0}^{n} \frac{1}{2k+1} \binom{2k}{k} \binom{2n-2k}{n-k} = 2^n (2n+1)^{-1} \binom{2n}{n}, \]
which is (3.561). So (4.7) and (3.561) are equivalent in the case of \( l = T \).

(7.4) or, with \( D(11) \),
\[ \sum_{k=2}^{n} (\binom{n}{k})(y)^{-1} = \sum_{j=1}^{n-1} j (y+1-j)^{-1}, \]

or, with \( D(11) \),
\[ n \geq 2, \quad \ldots \quad n-2, \ldots \]

The partial fractions expansion of the r.h.s. is
\[ \sum_{k=0}^{n-2} A_k (y-\alpha)^{-1}, \]
where for \( \alpha \leq n-2 \)
A different proof starts with the beta integral \( P(34) \). For \( \Re y > n-2 \) the L.H.S. equals

\[
(y+1) \sum_{k=1}^{n} \binom{n}{k} \int_{0}^{1} t^{k-1} (1-t)^{y-k+1} \, dt =
\]

\[
(y+1) \int_{0}^{1} t^{y+1-n} \left\{ (1-t)^{y+1-n} - (1-t)^{y+1} \right\} \, dt =
\]

\[
(y+1) \lim_{\varepsilon \downarrow 0} \int_{0}^{1} t^{y+1-n} \left\{ (1-t)^{y+1-n} - (1-t)^{y+1} \right\} \, dt =
\]

\[
(y+1) \lim_{\varepsilon \downarrow 0} \left\{ \frac{\Gamma(\varepsilon) \Gamma(y-n+2)}{\Gamma(y-n+2+\varepsilon)} - \frac{\Gamma(\varepsilon) \Gamma(y+2)}{\Gamma(y+2+\varepsilon)} \right\} =
\]

\[
\lim_{\varepsilon \downarrow 0} \Gamma(\varepsilon) \frac{1}{\Gamma(y-n+2)} \cdot \frac{1}{\Gamma(y+2)}
\]
\[
\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{\Gamma'(z)}{\Gamma(z)} + z^{-1}
\]

and by induction
\[
\frac{\Gamma'(z+m)}{\Gamma(z+m)} - \frac{\Gamma'(z)}{\Gamma(z)} = \sum_{i=0}^{m-1} (z+i)^{-1}.
\]

Inserting this into the last formula gives
\[
\sum_{i=1}^{n} \left( \frac{1}{i} \right) (y+1)^{-1} = \sum_{i=1}^{n-1} \frac{1}{y+1}.
\]
(4.11) With $D(y)$ the l.h.s. is equal to

$$\frac{n!x!x!}{(n+2x)!} \sum_{k=0}^{n} (-1)^k \left(\frac{x+k}{k}\right)\left(\frac{x+n-k}{n-k}\right).$$

Now apply (3.507).

$$(x+1) \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^1 t^{k+1} (1-t)^{x-n-k-1} \, dt =$$
\[(x+1) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_0^x t^k (1-t)^{n-k} \, dt = \]
\[
\frac{x+1}{x+1-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x-n}{k}^{-1}.
\]

All three members of (4.13) are rational functions of \(x\), so the identities hold outside the poles. These are a subset of \(\{0, 1, \ldots, 2n-1\}\). In the remaining points of \(\{0, 1, \ldots, 2n-1\}\), the relations hold in the sense of limits.

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} (2n)^{-1} = \frac{2n+1}{n+1} \sum_{k=0}^{n} (-1)^k =
\]
starts with the beta integral $D(3y)$:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2n+1}{2k}\right)^{-1} =$$

$$(2n+2) \int_0^1 (1-t)(1-\lambda t)^n \, dt,$$

where the integral may be evaluated by putting $1-t = \frac{1}{2} (1-\lambda t)$.

From (4,12) with $x=2n+1$, by $D(13)$,

$$(-1)^k \binom{n}{k} (2k+1) =$$

$$(2n+2) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_0^1 t^{2k+1} (1-t)^{2n-2k} \, dt =$$

where the integral may be evaluated by putting $t = \frac{1}{2} (1-\lambda t)$. 
We may prove (4.17) from (4.12) with \( x = 2n + z \), but the \( \beta \) integral \( \int D(34) \) gives a slightly simpler proof:

\[
\int_0^1 \frac{d^m \Gamma (1/2 - i x)}{d i x} (1 - x)^{z - n/2} dx
\]

where the integral may be computed by partial integration.

The identity (4.17) is a simple case of taking an even integer near \( 2n \) for \( x = \ln(4.12) \).

A more difficult case is (4.18) below where

\[
(4.18), (4.19) \quad \text{The last terms on the left in (4.12) and (4.13) are singular for } x = 2n
\]

and \( x = 2n - 1 \), respectively. Here we find the sums of the non-singular terms by a limiting operation. Trying to find a direct derivation by applying the \( \beta \) integral \( \int D(34) \) to the inverse binomial
\[
- \binom{n-1}{k/n}^\times \nabla^{-1}
\]

\[
-(2n+1) \left( \sum_{h=0}^{n} 2^h \binom{n}{h} (2n-x-1)^{-h} \right) \sum_{i=0}^{h} (2n-x-1-i)^{-1}
\]

This is the first equality in (4.18). The second one follows with (10.209).
\[
\frac{x+1}{x-2n+1} \left\{ \sum_{h=0}^{n} \binom{n}{h} \left( \frac{2n-x-2}{h} \right)^{-1} 2^h + (-1)^{n+1} \binom{x+1}{2n}^{-1} \right\}.
\]

\[
(1-z) \sum_{k=0}^{\infty} \frac{x+1}{x+1-k} \left( -z^2 \right)^k (1-xz)^{-k-1} =
\]
\[ (1 - z) = \sum_{m=0}^{\infty} \frac{(-1)^{m}}{2k} \binom{2k+1}{k} \binom{k}{k} z^{2k} \]

and the first equality in (4.20) follows. The second one follows with (D13). Note, that the second sum in the second member may be extended to \( 2k \leq n \). The missing term, only present when \( n \) is even, is zero when \( n \geq 1 \).

In the same way, from (3.7) and D(25),

\[
\sum_{n=1}^{\infty} z^{n} \sum_{2k \leq n-1} (-1)^{k} \binom{n}{2k+1} \binom{k}{k} z^{k} =
\]

\[
\sum_{k=0}^{\infty} (-1)^{k} \binom{k}{k} \left( \sum_{m=0}^{\infty} \binom{m+2k+1}{m} z^{m+2k+1} \right) =
\]

\[
\sum_{k=0}^{\infty} (-1)^{k} \binom{k}{k} \frac{2k+1}{(-z)^{2k}} =
\]

\[
\sum_{k=0}^{\infty} (-1)^{k} \frac{x+1}{z^{2k+1}} \binom{x+1}{k} =
\]

and (4.21) follows.
With the beta integral $D(34)$ and $G(8)$ or $G(10)$

$$A \sim \sum_{m=0}^{\infty} \frac{1}{(2m)!!} \left( \frac{1}{2} \right)^m \left( -1 \right)^m \left( m-k \right)$$

With the new integration variable $x$, where $t^m = \sin x$, $x \in [0, \pi/2]$

$$B_m = 2(m+1) \int_0^{\pi/2} \text{Im} (\cos x + i \sin x)^{2m+1} \cos x \, dx =$$
\[
\sum_{h=0}^{m} (-1)^h \binom{2m+1}{2h+1} \binom{m}{h} = B_m,
\]

If (4.14) and (4.15) suggest that there might be a simple proof by recurrence of (4.22) and (4.23) but no one was found by the author. But see Probl. 10494. Monthly 164, 1997, 371-372 and Nemes e.a. (1997).

Trying to derive (4.22)-(4.25) as special cases of (4.20) and (4.21) we found the following identities:
From (4.20) with $n = 2m$, $x = m$ and (4.22)

$$A_m = \sum_{k=0}^{m} (-1)^k \frac{m+1}{m+1-k} \frac{2m}{2m-k} \binom{2m-k}{k} 2^{2m-1-2k},$$

(1) $$\sum_{h=0}^{m} (-1)^h \frac{m+1}{m+h} \frac{m}{2h} \binom{m+h}{2h} 2^h = \frac{1+(-1)^m}{2-2m},$$

$$B_m = \sum_{h=0}^{m} (-1)^h \frac{m+1}{m+1-k} \binom{2m-k}{k} 2^{2m-2k},$$

(3) $$\sum_{h=1}^{m} (-1)^{m-h} \frac{m+1}{h+1} \binom{m+h-1}{2h-1} 2^{2h-1} = \begin{cases} m (m-1)^{-1}, & m \text{ even;} \\ 1+m^{-1}, & m \text{ odd.} \end{cases}$$
\[(4) \sum_{h=0}^{m} \frac{(-1)^{m-h}}{h+1} \frac{2m+1}{m+1} \left(\frac{m+h+1}{2h+1}\right) y^h = (-1)^m + (2m-1)(-1)^m - 1, \quad m \geq 1.\]

The identities (1)-(4) are listed in Table 1.

This does not prove (4.26) but instead...
(4.28) Putting \( y = -z \) we may write \( \Phi(z) \) as
\[
(\tau + z) \sum_{3k \leq n} \frac{(-1)^k}{k!} \binom{n-2k}{k} \tau^{2k} (z-\tau)^{n-3k}
\]

\[
= \sum_{3k \leq n} \frac{(-1)^k}{k!} \left( \frac{(2k+1)!}{(n-k+2)!} + 2 \frac{(2k)!}{(n-k+1)!} \right)
\]
(1) \( (x+1) \sum_{k=0}^{n} \binom{n}{k} \frac{t^k}{t^k (1-t)} \ dt = \)
\( (x+1) \int_{0}^{1} (1 + t)^{n-1} (1-t + at)^n \ dt = \)
\( (x+1) \sum_{k=0}^{n} \binom{n}{k} \frac{(x-n)!}{(x-n+k+1)} (a-1)^k = \)

By canceling and rearranging factorials in the l.h.s., with Di4, and then putting \( h = n-k \) we obtain the third member in (4.33). The second equality also may be proved with (1.318).

For \( a = 2, x = 2n-1, \ n \geq 1, \) we have with (1.5)
\[ \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(2n-1)_k} \frac{1}{(2n-1)} = \sum_{k=0}^{n} \binom{n}{k} \]
\[
\binom{2n+1}{n} \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} \frac{1}{(2n+2)(n+1)!} = \frac{1}{4} \binom{2n+2}{n+1} - 1.
\]

\[
2n \int_0^1 (1+t)(1-t^2)^{n-1} \, dt = n \int_0^1 (1+y^{-1/2})(1-y)^{n-1} \, dy = \quad n \geq 1, \text{ with } B(2n).
\]

\[
\frac{(2n)!}{(2n)!} \left( \sum_{k=0}^{n} \frac{1}{(n-k)!} \right)^2 = \frac{(2n)!}{(2n)!} \left( \sum_{k=0}^{n} \frac{1}{k!} \right)^2 = \frac{n! n!}{(2n)!} q^n \quad \text{with (1.319)}.
\]

See Egorychev (1984), (2.7), and Gupta, Math. Student 22 (1954) 105-107.
(4.48) Writing \( x - 2k = x - k - n + n - k \) we see from \( D(14) \), \( G(22) \) and \( G(32) \) that the l.h.s.

\[ \text{etc. (1997)} \]

(4.49) With \( D(14) \) and \( D(18) \) the l.h.s. is equal to
\[
\frac{x! \cdot n!}{(x+n+1)^n} \sum_{k=0}^{n} \binom{x+n+1}{k} (x+n+1-2k) = \\
(x+n)^{-1} \sum_{k=0}^{n} \binom{x+n+1}{k} \\
- 2 \frac{x! \cdot n!}{(x+n+1)^n} \sum_{k=1}^{n} \frac{(x+n+1)!}{(k-1)! (x+n+1-k)!} = \\
(x+n)^{-1} \left\{ \sum_{k=0}^{n} \binom{x+n}{k} + \sum_{k=1}^{n} \binom{x+n}{k-1} - 2 \sum_{k=1}^{n} \binom{x+n}{k-1} \right\} \\
\text{n...n-1...}\_1
\]

(4.50) For \( \Re \lambda > -1 \), the l.h.s. is equal (by the beta integral) to

\[
(n+1) \int_0^1 t^\lambda (1-t^2)^{n-\lambda} dt = \frac{1}{2} (n+1) \int_0^1 u^{\frac{\lambda}{2}-\frac{1}{2}} (1-u)^{n-\lambda} du = \\
\text{...} \int_0^1 \left(\frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right)\right) \Gamma(n+1) = \sqrt{\pi} \Gamma(n+1)
\]
(4.54) With \( D(n) \) the l.h.s. is equal to:

\[
(x+n)^{-1} \sum_{k=0}^{n} \binom{n}{k} \frac{n+x}{n} = (x+n)^{-1} \sum_{k=0}^{n} \binom{n}{k} \frac{n+x}{n}.
\]

Equivalence:

\[
x \sum_{k=0}^{n} \binom{n}{k} \int_{0}^{1} t^{x-1} (1-t)^{k} \, dt = (\ast)
\]

\[
x \int_{0}^{1} t^{x-1} (2-t)^{n} \, dt =
\]

\[
x \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{j} \binom{n}{j} (x+j) \cdot 2^{n-j},
\]

for \( \Re x > 0 \) and then for \( -x \notin \{0, \ldots, n\} \)

\[
x \sum_{k=0}^{n} \binom{n}{k} \int_{0}^{1} \alpha =
\]

\[
x \int_{0}^{1} (1+t)^{n} (1-t)^{x-1} \, dt =
\]

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\[
\binom{x}{k} \binom{x+k}{k}^{-1} = \frac{(-x-1)!}{x!} \frac{(k-x-1)!}{(k+x)!} (-1)^k \sim \\
(-1)^k \frac{(-x-1)!}{x!} k^{2x-1},
\]

(1) \[ y \int_{0}^{1} (1 + t)^x (1 - t)^{y-1} \, dy, \]

\[
\sum_{\kappa=0}^{1} \int_{0}^{1} (1-t)^{y-x-1} \, dt < \infty. 
\]
\[ x \int_0^1 (1+t)(1-t)^{x-1} \, dt. \text{ So} \]

\[ \sum_{k=1}^\infty \frac{1}{k^x} = x \int_0^1 (1-t)^{x-1} \, dt. \]

\[ 2 \times \int_0^1 (1-u) (u^{\gamma+1}) \, du = \frac{1 + \frac{\gamma}{2}}{\Gamma(x+\frac{1}{2}),} \]

with the beta integral $B(\gamma, \delta)$. For $\gamma > 0$ the first equality follows by analyticity since the LHS of (4.53) converges and hence to a sum of absolutely...
We note that both sides of (4.56) are rational functions of $x$.

The identity (4.56) is the companion, in

(4.57) With $D(2y)$ the l.h.s. may be written as

$$\sum_{k=0}^{\infty} \binom{n}{k} \binom{-x-1}{k}^{-1}.$$ 

Now apply (4.3).
$$\sum_{j=0}^{\infty} (-1)^j \binom{\lambda}{j} (\lambda - j)$$

Cf. Riordan (1968), Ch. 1, Prob. 6.

Writing (4.58) as

$$\sum_{x=0}^{n-\epsilon} \sum_{k=0}^{\infty} (-1)^{n-\epsilon-k} \binom{n-\epsilon}{k} \frac{a}{a+k} = (-1)^{n-\epsilon} \binom{a+n-\epsilon}{n-\epsilon}^{-1}$$
and determining the $n$th difference of the l.h.s. at $x = 0$ by $G(yb)$ and $(4.56)$.
(4.60) The first equality follows with $\mathcal{D}(14)$. With the beta integral $\mathcal{D}(34)$ the l.h.s. is equal to

$$\int_0^1 t^{n-1} (1 + a - at) \, dt =$$

$$\sum_{n=0}^{\infty} \frac{(n)}{n!} \cdot \frac{1}{(1+n)^n} \cdot \int_0^1 t^{n+x-1} \, dt =$$

$$\sum_{k=0}^{\infty} a^k z^k (1-z)^{-x-k-1} = (1-z)^{-x} (1-z-a z)^{-1},$$

with $\mathcal{B}(13)$ and with (4.2) the l.h.s. is equal to
\[ \sum_{k=0}^{m} \left( \frac{1}{2n+2k+2} \right)^{-1} \]

\[ = \sum_{h=0}^{n} \left( \frac{3/2}{m+h} \right) (-y)^h \]

\text{(4.66)} \text{A generalization of (4.56). With } D(8), D(14), G(22), G(49), \text{ and } D(24) \text{ the l.h.s. for } \varepsilon \leq n \text{ is equal to}

\[ \sum_{k=1}^{n} (-1)^k \frac{n!}{(k-1)! (n-k)!} \left( \frac{n}{y+k} \right)^{-1} \]
\[ \begin{array}{c}
\frac{n!}{(n-i)!} \sum_{h=0}^{n-i} (-1)^h \binom{n-i}{h} (\gamma + h + i)^{-1} \\
\end{array} \]
(4.71) With B(13), (4.41) and (4.8) the l.h.s. is equal to

\[- \sum_{k=1}^{n} \frac{1}{k} \left( \frac{n}{k} \right)^{\frac{3}{2}} = 2 \sum_{k=1}^{n} \left( \frac{n}{k} \right) \left( \frac{1}{k-1} \right)^{\frac{3}{2}} =

2 \sum_{j=0}^{n-1} (2j+1)^{\frac{3}{2}}.

(4.72) The l.h.s. is equal to

\[ \frac{1}{2} \left( \frac{2n+1}{n} \right)^{\frac{3}{2}} \Delta^{n+1} (2x+1)^{\frac{3}{2}} \]

With B(27) this is equal to

\[ H^{2n+1} \frac{1}{2n+2} \left( \frac{2n+1}{n} \right)^{\frac{3}{2}} \left( \frac{2n+2}{n+1} \right)^{\frac{3}{2}} = H^{2n} \frac{1}{n+1} \left( \frac{2n+1}{n} \right)^{\frac{3}{2}} \]

\[ (n+1) H^{2n} \left( \frac{2n+1}{n} \right)^{\frac{3}{2}} \left( \frac{2n}{n} \right)^{\frac{3}{2}} \]

The last equality follows with B(30).

(4.76) - (4.79) With D(13), by canceling and rearranging factorials, we see that the l.h.s. of (4.76) is equal to
\[ (2n)^{-1} \sum_{k=1}^{n} \binom{2n}{n-k} k^{-1}, \]

and (3.76) follows from (1.104). The identities (3.77) - (3.79) follow in the same way from (1.105), (1.106) and (1.107).

(4.81) Differentiate (4.56) with application to \( \mu_R(\beta), \)

(4.83), (4.84). Since with \( G(24^a) \)
\[ \Delta^n \frac{b^m - x^m}{b-x} = \Delta^n \sum_{i=0}^{m-1} b^i x^{m-1-i} = 0, \quad 1 \leq m \leq n, \]
we have with \( G(48), \) also for \( m = 0, \)

(1) \[ \Delta^n \frac{x^m}{b-x} = \Delta^n \frac{b^m}{b-x} = \frac{b^m}{b-x-n} \binom{b-x}{n}^{-1}. \]

\[ \sum_{k=0}^{n} \binom{n}{k} (b-x-n)^{-1} \binom{b-x-1}{n-k}^{-1} \Delta^k x^m, \]
(4.85) In Chapter 1R, from two general lower triangular matrices that are each other’s inverses the relation

\[ \sum_{k=0}^{n-1} \frac{\binom{n}{k}}{a+n+1} \left( \binom{n-1}{k} \right) \alpha. \]

$X^{(\nu,0)} = 1$, we see that the l.h.s. is equal to

$$\sum_{k=0}^{n} \frac{n!}{(n-k)! (n+\alpha) (\ell, k)} - \sum_{k=0}^{n-1} \frac{n!}{(n-k)! (n+\alpha) (\ell, k+1)}$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)! (n+\alpha) (\ell, k)} - \sum_{h=1}^{n} \frac{n!}{(n-h)! (n+\alpha) (\ell, h)}$$

$$= 1, \quad n \geq 1. \quad (\text{trivial for } n = 0).$$
Since
\[ |(j)(j+a+1)^{-1}| = \left(\frac{j}{a+1}\right) \frac{\Gamma(a+1)}{(j+a+1)!} \sim \frac{\Gamma(a+1)}{(j+a+1)!} \sim e^{-j} \text{ Re } a, \]
the l.h.s. of (4.89) converges absolutely.

\[(a+1) \int_0^1 t^x (1-t)^{a-x-1} \, dt = \frac{(a+1) \Gamma(x) \Gamma(a-x-1)}{\Gamma(a+1)} \]

For \( x = \frac{a}{a-1} \),

\[
\int_0^1 \frac{dt}{(1-t)^{a-1}} = \int_0^1 \frac{t^{x-1}}{(1-t)^{x-1}} \, dt
\]
\[ \frac{a_{n+1}}{n+1} - \frac{a_n}{n+a+1} - 2h = 0 \text{ for } h = 0, h \in \{n, n+2, n+4, \ldots\} \]

For \( n = 2 \) we obtain a special case of (4.4).
with $E(\delta^2, D_{i,j})$ and (4.56).

This also may be done the other way round.

\[
\sum_{n=0}^{\infty} (-1)^{n} \binom{n}{i} \binom{n}{j} = \sum_{n=0}^{\infty} (-1)^{n} (n-i)(n-j) \binom{n}{i} \binom{n}{j}
\]
The \( \eta \)th difference at \( x=0 \) of the l.h.s. then may be found with (5.46) and (5.100). Cf. (4.58) and p. 1R 61-65.

\[(4.97) \quad \text{For } -\beta \notin \mathbb{N}, -\alpha -\beta \notin \mathbb{N}, \text{ the l.h.s. satisfies}
\]
\[
\sum_{j=0}^{\infty} \frac{r^j \Gamma(\beta+j-1) \Gamma(r+j-1) \Gamma(\alpha+\beta-1)}{j! \Gamma(\beta-1) \Gamma(\alpha+\beta+r+j-1)} \sim c^j \Gamma(-\alpha-1)
\]
as \( j \to \infty \). So the series converges absolutely for \( \Re \alpha > 0 \).

When \( \alpha > 0, \beta > 0, \) by the beta integral \( \int_0^1 t^{\beta-1} (1-t)^{\alpha-1} \, dt \) and by \( D(25) \) the l.h.s. is equal to

\[
2 \sum_{j=0}^{\infty} \frac{r^j \Gamma(\beta+j-1) \Gamma(r+j-1) \Gamma(\alpha+\beta-1)}{j! \Gamma(\beta-1) \Gamma(\alpha+\beta+r+j-1)}
\]

\[
2 \int_0^1 t^{\beta-1} (1-t)^{\alpha-1} \, dt = \frac{r^2 \Gamma(\alpha-1) \Gamma(\beta-1)}{(r+\alpha-1) \Gamma(\alpha+\beta)} = \left( \frac{r \Gamma(\alpha)}{\Gamma(\alpha+\beta)} \right),
\]

where interchanging sum and integral is allowed since everything is nonnegative. For other values of \( \alpha \) and \( \beta \) the relation

\[
\int_0^1 t^{\beta-1} (1-t)^{\alpha-1} \, dt = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)} = \left( \frac{r \Gamma(\alpha)}{\Gamma(\alpha+\beta)} \right),
\]
(4.98). Write \((3.575)\) with \(B(15)\) as

\[
\sum_{k=0}^{n} (-4)^{n-k} \binom{n}{n-k} \left( \frac{1}{2} \right)^{n-k} \binom{n-k}{k} = Y_n,
\]

with \(Y_0 = 1\), \(Y_n = 4^n \{2n \cdot (2n+1) \binom{2n}{n}\}^{-1}, n \geq 1\).

From TD(15) and Lemma TD9 with

\[
\binom{n-k}{k} \binom{n}{n-k}
\]

form an inverse pair, i.e. the lower triangular matrices \(b_{nk}\) and \(b_{nk}^T\), with \(a_{nk} = b_{nk} = 0, \ k > n, \) are each other's inverses. So we have, with \(B(15)\),

\[
-(2n+1)^{-1} \binom{2n}{n} = \sum_{k=0}^{n} (-4)^{n-k} \binom{n-k}{k} \frac{1}{2^{n-k}} Y_k =
\]

\[
\sum_{k=0}^{n} \binom{2n-2k}{k} Y_k =
\]

and (4.98) follows. So (4.98) and (3.575) are companions in the sense of IR7. The author does not have a proof without appeal to companions.
\[ \sum_{j=0}^{n} \frac{1}{x+j+1} \int_0^1 t^{n-j} (1-t)^{x-j} \left\{ (1-t)^{j+1} + t^{j+1} \right\} \, dt = \]

\[ \binom{x}{2k \leq n} \frac{1}{x^{n+1-k}} \binom{n+1-k}{k} \left[ \begin{array}{c} x+k \end{array} \right] \left[ \begin{array}{c} n-k \end{array} \right] = \]

\[ \leq n+1 \binom{n-k}{k} k! (x+k-n)! \]

For the remaining \( x \)-values: All members are rational functions of \( x \).
(5.2) Staver (1947) gives, with only a suggestion of a proof, a recurrence for the l.h.s. denoted by $S_2$, of (5.2), viz.

and states that this proves the first equality in (5.2).

From (5.1) with $x = n$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \frac{2(n+1)^2}{n+2} \sum_{j=0}^{n} \frac{1}{n-j} \binom{2n-j+2}{n-j}^{-1} =$$

$$2(n+1)(n+2)^{-1} \sum_{h=1}^{n} \left( \binom{n-h+1}{h} \right)^{-1} \binom{n+2+h}{h}^{-1}.$$

$$\sum_{h=0}^{n} \binom{n}{h} = \binom{n+1}{n+1} \frac{1}{n+1}.$$

$$n \sum_{h=0}^{n} \binom{n}{h} = \binom{n+1}{n+1}. $$
Abbreviations of journal and author names.

Fib. Q.: The Fibonacci Quarterly.
Nieuw Arch. (3) 5: Nieuw Archief voor Wiskunde, 3rd series, Vol. 5.
Wisk. Opg.: Wiskundige Opgaven met Oplossingen, uitgegeven door het Wiskundig Genootschap te Amsterdam. (Dutch).


REFERENCES


Ascher, M., A combinatorial identity. Fib. Q. 12, 1974, 186-188.


Baron,G. e.a., The number of spanning trees in the square of a cycle. Fib.Q. 23, 1985,258-264.


REFERENCES


Bernstein, L., Zeros of the functions f(n) =...... J. Number Theory 6, 1974, 264-270.

Bessel-Hagen, E., Repertorium der hoheren Mathematik. Leipzig 1929.


Bicknell-Johnson, M. and C.P. Spears, Classes of identities for the generalized Fibonacci numbers G(n)=G(n-1) + G(n-c) from matrices with constant value determinant. Fib.Q. 34, 1996, 121-128.


REFERENCES


Boyer, C.B., Pascal's formula for sums of powers of the integers. Scripta Math. 9, 1943, 237-244.


REFERENCES


REFERENCES


Dilcher, K., A generalization of Fibonacci polynomials and a representation of Gegenbauer


Dixon, A.C., On the sum of the cubes of the coefficients in a certain expansion by the binomial theorem. Messenger Math. 20, 1891, 79-80.


Dörrie, H., Einführung in die Funktionentheorie. R. Oldenbourg, München 1951.


REFERENCES


REFERENCES


Frame, J.S., Mean deviation of the binomial distribution. Monthly 52, 1945, 377-379.


Freitag, H.T. and P. Filipponi, Division of Fibonacci numbers by k. Fib.Q. 37, 1999, 128-134.


Gessel, I. and D. Sturtevant, A combinatorial proof of Saalschütz's theorem.
REFERENCES


Gould, H.W., A new convolution formula and some new orthogonality relations for inversion
REFERENCES

of series. Duke 29, 1962b, 393-404


REFERENCES


REFERENCES 14


Hock, J.L. and R.B. McQuistan, The occupancy degeneracy for \( \lambda \)-Bell particles on a saturated \( \lambda \times W \) lattice space. Fib.Q. 21, 1983, 196-202.

Hoggatt Jr., V.E., Fibonacci and Lucas numbers. The Fibonacci Association, Univ.of Santa Clara, 1969.

Hoggatt Jr., V.E., Some special Fibonacci and Lucas generating functions. Fib.Q. 9, 1971, 121-133.


Hoggatt Jr., V.E. and M. Bicknell, Pascal, Catalan and general sequence convolution arrays in a matrix. Fib.Q. 14, 1976, 135-143.

REFERENCES


Hsu, L.C. and E.L. Tan, A refinement of De Bruyn's formula for \( \sum a_k \). Fib.Q. 38, 2000, 56-59.

REFERENCES


REFERENCES 17


Kaucky, J., Remarque à un travail de P. Turán. Mat. -fyz. časop. 12, 1962, 212-216. (Slovak)


Kluyver, J. C., Over de ontwikkeling van eene functie in eene faculteitenreeks. Nieuw Arch. (2) 4, 1900, 74-82. (Dutch).


Koornwinder, T. H., Clebsch-Gordan coefficients for $SU(2)$ and Hahn polynomials. Nieuw Arch. (3) 29, 1981, 140-155.


Lang, W., On polynomials related to powers of the generating function of Catalan's numbers. Fib. Q. 38, 2000, 408-419.


Ljunggren, W., Problems 100, 132, etc. Nordisk Mat. Tidskr. 5, 1957.


Ma: See Xinrong.


REFERENCES

Fib.Q. 28, 1990b, 3-10.


Mate, J., A kinai matematika történetének egy problémájárol. Mat. Lapok 7, 1956, 112-113.


Mohsen: See Pourahmadi.


REFERENCES 21


Nguyen-Huu: See Bong.


Nörlund,N.E., Differenzenrechnung. Springer 1924.


REFERENCES 22


Pólya, G., On the number of certain lattice polygons. J.C. Th. 6, 1969, 102-105.


Pólya, G. and G. L. Alexander, Gaussian binomial coefficients. Elem. der Math. 26/5, 1971,
REFERENCES


Pourahmadi, Mohsen, Taylor expansion of $\exp(\sum a \cdot z )$ and some applications. Monthly 91, 1984, 303-307.


REFERENCES


Schuh, F., Wonderlijke Problemen, 1943. (Dutch).


REFERENCES


Stewart, I. See Mathematical Recreations.


REFERENCES


Wenchang. See Chu.


Whenpeng: See Zhang.


Williams, H.C., Fibonacci numbers obtained from Pascal’s triangle with generalizations. Fib. Q. 10, 1972, 405-412.

Woan: See Wen-Jin.


Yang: See Kung-Wei.


Zucker, I.J., On the series $\sum \left( \frac{1}{k} \right)^{-n}$ and related sums. J. Number Th. 20, 1985, 92-102.

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Binomial Identities

With Old-Fashioned Proofs

Part II
INTRODUCTION

This work is the continuation of Binomial Identities with Old-fashioned Proofs, \textit{PART I}, by the same author. It contains the Tables 6, 7, 8, 10 with sums of products of three and four binomial coefficients or their inverses. Arrangement is similar to \textit{PART I}, now restricted to Introduction (pages II - I16), Tables (pages T1-T103) and Proofs, remarks and references (pages T1 - T175), reference pages Ref1 - Ref5.

The set of Tables is preceded by a small list giving some of the table entries to help locating an identity when present.

Table 8 lists identities connected to the theory of Fibonacci numbers, found in
Dilcher (2000): Table 10 is a small collection of identities with more than three binomial coefficients.

The identities in PART II are more difficult to prove than those in PART I. For a number of them we made use of hypergeometric functions. So a number of hypergeometric identities is collected in the Introduction. Here more should be possible. For many identities with three or more binomial coefficients manipulation of formulas by computer, as mentioned in the introduction to PART I, will be preferable.

The same notation as in PART I is used, and also the results of the
I2a

General chapters of PART I.

References to authors are mainly in
the reference list of PART I or else in
the reference list at the end of PART II.

Formulas are referred to in the same
way as in PART I, e.g. (17) is formula
(17) in Chapter C (of PART I), (3.48) and (7.16)
refer to formulas 48 and 16 in Table 3
(of PART I), and Table 7 (of PART II).

Numbering the pages starts anew, with
I1, I2, ... and then T1, T2, ... for the set
of tables and P1, P2, ... for the chapter
of proofs. Proof of an identity in a table
is in Chapter P, under the same number.
Part II, in particular Tables 6, 6 and 7, contains a number of identities whose proofs all 

\[ \text{generated by the } E^a, \text{ where } E^a f(x) \]

\( \text{(often having a polynomial in } x) \) we obtain
proof by operator identity. For a justification see pp.15-18 below.

Interpreting \( \Delta \) by \(-\) we have, since

\[
\Delta + I = E,
\]

Applying both sides to the function \( f(x) \) we obtain, by \( G(38) \), the identity \( C(14) \)

\[
_\binom{n}{m} x^m (x + M) \quad \text{we have with} \; G(40)
\]

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{x+k}{x+k+M} \binom{x+k+M}{m} =
\]
Replacing $z$ by $E^{-1}$ in (3.178) we have
\[
\sum_{k=0}^{n} \binom{n}{k} E^{-k} = \sum_{j=0}^{n} \binom{n}{j} \binom{2n-j}{n} (-1)^{j} E^{-j} \Delta^{j}.
\]

Application of both sides here to the function $\binom{x}{M}$ gives
\[
\sum_{k=0}^{n} \binom{n}{k} \binom{x-k}{M} = \sum_{j=0}^{n \wedge M} \binom{n}{j} \binom{2n-j}{n} \binom{x-j}{M-j}.
\]

These and similar operator identities may be applied to the function $1/\binom{x}{M}$.

We have
\[
\binom{x}{M}^{-1} (x+1)^{-1} \left[ \frac{x}{m+1} \right]^{-1} = \frac{1}{m+1}, \quad m+1 \geq 1.
\]

This relation is proved easily by induction on $i$.

The two operator identities on p. 14, applied to the function $1/\binom{x}{M}$ then give
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{M}{M+k} \binom{x+k}{M+k}^{-1} = \sum_{j=0}^{n} (-1)^{j} \binom{n}{j} \binom{2n-j}{n} \binom{x+j}{M+j}^{-1}, \quad M \geq 1,
\]
and
(7.3.0) $\sum_{k=0}^{n} \binom{n}{k}^2 (x+k)^{-1}$

Many variants of identities in Table 6 could be proved in this way and added to Table 7. The evaluation of

**Justification**

We will show that there is an isomorphism between the set of polynomials in a complex variable and the set of polynomials in an operator-valued variable. More precisely, let $P$ be the set of polynomials with complex coefficients in the variable $z \in \mathbb{C}$ seen as functions $\mathbb{C} \to \mathbb{C}$. Let $\mathcal{A}$ be the commutative algebra.
of operators generated by the operators \( E^a, a \in \mathbb{C} \), operating, say, on functions in the set of all polynomials, functions \( u \) of \( \mathbb{C} \), i.e. all functions \( \sum_{i=0}^{m} a_i u^i, a_i \in \mathbb{C} \), with sum and product (composition). Define
must have \[ \sum_{i=0}^{m} a_i \Delta^i (x) = \sum_{i=0}^{m} a_i (x^{M-i}) \]
Product:

\[ \psi \left( \sum_{i=0}^{m} a_i z^i \cdot \sum_{j=0}^{n} b_j z^j \right) = \]

\[ \psi \left( \sum_{r=0}^{m+n} \sum_{i+j=r} a_i b_j \right) = \]

\[ \sum_{r=0}^{m+n} \sum_{i+j=r} a_i b_j \cdot \]

\[ (\sum_{i=0}^{m} a_i z^i) (\sum_{j=0}^{n} b_j z^j) = \]

\[ \psi (\sum_{i=0}^{m} a_i z^i) \psi (\sum_{j=0}^{n} b_j z^j) . \]

5° Linearity: Similar.
lated in terms of hypergeometric functions — or even proved by hypergeometric theory.
We only use a small part of this theory, mainly for notation, i.e. to formulate part of the identities in hypergeometric terms.

The Gauss hypergeometric function \( \genfrac{[}{]}{0pt}{}{2}{1} \) is defined by

\[
(1) \quad F(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+k-1)}{\beta(\beta+1)\ldots(\beta+k-1)} \frac{z^k}{k!} c(k).
\]

\( a^{(0)} = 1, \quad a^{(k)} = a(a+1)\ldots(a+k-1), \quad k \geq 1. \)

The series converges absolutely for \( |z| < 1 \) and also for \( |z| = 1, \ Re \ c > Re \ (\alpha+b) \).

When, \( \alpha = -n \) or \( b = -n \) all terms in

we may write \( (1) \) as

\[
(2) \quad F(\alpha, b; c; z) = \sum_{k=0}^{\infty} \frac{(\alpha+k-1)(b+k-1)}{k!} \frac{z^k}{c+k-1}.
\]
With \( D(2y) \) we may write (1) also as

\[
(3) F^{(a,b;c;z)} = \sum_{k=0}^{\infty} (-1)^k \binom{a}{k} \binom{b}{k} \binom{c+k-1}{k} z^k,
\]

\[
(4) F^{(a,b;c;z)} = \sum_{k=0}^{\infty} (-1)^k \binom{a+k-1}{k} \binom{-b}{k} \binom{c+k-1}{k} z^k,
\]

\[
(5) F^{(a,b;c;z)} = \sum_{k=0}^{\infty} (-1)^k \binom{a+k-1}{k} \binom{b+k-1}{k} \binom{-c}{k} z^k,
\]

\[
(6) F^{(a,b;c;z)} = \sum_{k=0}^{\infty} \binom{-a}{k} \binom{-b}{k} \binom{c+k-1}{k} z^k,
\]

\[
(7) F^{(a,b;c;z)} = \sum_{k=0}^{\infty} \binom{a+k-1}{k} \binom{-b}{k} \binom{-c}{k} z^k,
\]

\[
(8) F^{(a,b;c;z)} = \sum_{k=0}^{\infty} \binom{a+k-1}{k} \binom{-b}{k} \binom{-c}{k} z^k,
\]

\[
(9) F^{(a,b;c;z)} = \sum_{k=0}^{\infty} (-1)^k \binom{-a}{k} \binom{-b}{k} \binom{-c}{k} z^k.
\]

This shows that in Table 7 below we could have many trivial variants of certain identities. Also

\[
(10) F^{(a,b;c;z)} = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} \binom{b+k-1}{k} \binom{c+k-1}{k} z^k,
\]

\[
(11) F^{(a,b;c;z)} = \sum_{k=0}^{\infty} \binom{n}{k} \binom{-b}{k} \binom{c+k-1}{k} z^k,
\]
\[ \text{For } a = -n \text{ we obtain} \]

\[
\binom{n}{c} \binom{n}{c} = (c - b)^n \binom{n}{c^n}, -c \notin \mathbb{N}.
\]

Known as Vandermonde's theorem,

\[
\binom{n}{c} \binom{n}{c} = \binom{n}{c} \binom{n}{c} \binom{n}{c} = (1-z)^{c-a-b} \binom{n}{c} \binom{n}{c} \binom{n}{c}.
\]

Euler's identity. Slater (1966), (1.5.12), (1.3.15); G.K.P. Prob. 5.28, p. 233.
\[ (17) \quad (1-z)^{-a} F(a, b; c; -z, (1-z)^{i-1}) = \frac{1}{\Gamma(a, c-b; c; z)}. \]

Pfaff’s theorem, G.K.P. (1988), (5.101), Slater (1966), (1.7.1.3).

\[ (18) \quad \frac{\Gamma(a, b; 1+a-b; -1)}{(a-b)! (a/2)! / a! (\frac{1}{2} a-b)!} = -\frac{a}{2} \notin \mathbb{N}, \]

Kummer’s theorem, Slater (1966), (1.7.16), where \( z^{-a} \) should be deleted. Also G.K.P. 1988, (5.94). \( Re \, b < \frac{1}{2} (a+b) \).

\[ (19) \quad F(a, 1-a; c; \frac{1}{2}) = \frac{\Gamma(c/2) \Gamma(\frac{1}{2} c + \frac{1}{2})}{\Gamma(\frac{1}{2} c + a) \Gamma(\frac{1}{2} c + c - \frac{1}{2} a)} \]

\( -c \notin \mathbb{N} \). Bailey’s theorem, Slater (1966), (1.9.1.8), cf. G.K.P. (1988), (5.102).

\[ (20) \quad F(a, b; \frac{1}{2}, (a+b+1); \frac{1}{2}) = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} + \frac{1}{2} a + \frac{1}{2} b) \sqrt{\Gamma(\frac{1}{2} + \frac{1}{2} a) \Gamma(\frac{1}{2} + \frac{1}{2} b)}}{\Gamma(\frac{1}{2} + \frac{1}{2} a + \frac{1}{2} b)}. \]

Gauss’ second summation theorem.
(21) Pochhammer's formula, Slater (1966), (16.6):

\[ _2F_1 (a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \, dt, \]

\[ |z| < 1, \text{ Re } b > 0, \text{ Re } (c-b) > 0. \]

By expansion of \((1-zt)^{-a}\) into a binomial series, the identity may be extended to \(z = -1\) by dominated convergence when \(-1 < z < 0\) and \(z \downarrow -1\).
Hypergeometric functions and Fibonacci.

In his paper, Hypergeometric Functions and Fibonacci numbers, E16, D38, 2000, 342-363 Karl Dilcher expresses Fibonacci and Lucas numbers as values of hypergeometric functions. Many of his proofs apply hypergeometric transformations such as (16) and (17) above but also more complicated ones.

Our interest is the opposite one: evaluating hypergeometric binomial sums in terms of Fibonacci or Lucas numbers or related polynomials. Our proofs are different. We express the hypergeometric binomial sum in terms of the Fibonacci-like or Lucas-like polynomials $F_n$ and $L_n$ defined in Part I, Chapter F, using the theory of that chapter. Or we start from direct summations when available.

Special values of the argument $z$ in the polynomials or the general sum then lead to expressions in terms of $F_n$ and $L_n$, the Fibonacci and Lucas numbers.

Note that our notation is different from Dilcher's which is the one of The Fibonacci Quarterly. We define

$$F_n = F_{n-1} + F_{n-2}, \quad L_n = L_{n-1} + L_{n-2}, \quad F_0 = F_1 = 1,$$

$$L_0 = 2, \quad L_1 = 1.$$ See PART I, Ch. F.

$$\Phi_n(x) = \Phi_{n-1}(x) + x \Phi_{n-2}(x), \quad \Phi_0(x) = \Phi_1(x) = 1.$$
\[
\Lambda_n(x) = \Lambda_{n-1}(x) + x \Lambda_{n-2}(x),
\]
\[
\Lambda_0(x) = 2, \quad \Lambda_1(x) = 1.
\]

Since our derivations of a number of Dilcher's identities are based on the theory of Chapters 6 and 7 in PART I, we collect a set of them in a special table, viz. Table 8. Special cases are included in the proofs and remarks to Table 8 (pages 71).

The method of proof used here might not be strong enough to prove all of Dilcher's identities.
Introduction to tables

The ordering of the tables is similar to the one in PART I. There are four tables:

Table 6, sums with three binomial coefficients ( ) ( ) ( )

Table 7, sums with three binomial coefficients ( ) ( ) ( )

Table 8, hypergeometric binomial sums related to Fibonacci- and Lucas type numbers and polynomials.

Table 10, remnants

Entries in a table have a formula number, e.g. (7.42) for formula 42 in Table 7. Proofs of identities and also remarks and references are in Chapter P, each one under the same number as the entry in the tables.

To indicate the arrangements of entries in a table we give a short list of entries, with their numbers.
Examples of entries into tables

(6.3) \( \sum_{n=0}^{m} \left( \begin{array}{c} n \\ m \end{array} \right)^{2} \)

(6.28) \( \sum_{n} (-1)^{k} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{p-k}{b-n+k} \)
\[ (6.227) \quad \frac{\Lambda M}{2\pi \leq n ( i + \frac{1}{2} ) ( i + \frac{3}{2} ) ( i + \frac{5}{2} ) \cdots ( i + M ) ! } \]

\[ \phi_n ( x ) = \frac{1}{(i + h)^{1/4} \sqrt{2 \pi}} \exp \left( \frac{-x^2}{1 + h} \right) \]

\[ (7.7) \quad \sum_{i=0}^{\infty} \left( \frac{\alpha n}{\lambda} \right)^{2i} \left( \frac{\mu \eta}{\nu} \right)^{2i-1} \]
(8.51) \[ _2 F_1 (-n, -\frac{1}{2} - n; -2n; z) \]

(10.2) \[ \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} \binom{\infty}{n} \]

(10.7) \[ \sum_{j=0}^{n} (c-a-b)(a-c)(b-c) (-c)^{-1} \]
\textbf{TABLE 6}

\begin{align*}
(6.1) \quad & \sum_{k=0}^{n} \left( \binom{n}{k} \right)^3 \\
(6.2) \quad & \sum_{j \leq \min \{ m, n \}} \binom{m}{j} \binom{n}{j} \binom{n}{j} \text{ as trigonometric integral: Bragg (1999)} \\
(6.3) \quad & \sum_{i=0}^{m} \binom{n}{i} \binom{n}{m-i}^2 \\
(6.4) \quad & \sum_{2j \leq n} \binom{n}{2j} \binom{n}{j}^2 \text{ as trigonometric integral: Bragg (1999)} \\
(6.6) \quad & \sum_{k=0}^{n} \binom{n}{k} \binom{m}{n-k} \binom{m}{k} (-1)^k = 0, \text{ n odd,} \\
& = \binom{n}{n/2} \binom{m + n/2}{n} (-1)^{n/2}, \text{ n even, from (6.4)} \\
& \text{with } x = -1.
\end{align*}
\[ (6.7) \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} (-x)^k = \frac{3}{\pi} \left[ \frac{1}{x} \right]^{-3} \cos \frac{1}{2} \pi x, \]

when \( \Re x < 7/3 \). This is (15) in Dougall (1906), 

\[ (6.8) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \eta \right)^3 = (-1)^m (3m)! (m!)^{-3}, n = 2m, \]

\[ = 0, \quad n \text{ odd, (Dixon's formula)} \]

\[ (6.9) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n-1}{k} \left( \frac{x}{k} \right) = \text{See } (6.267), \]

\[ (6.10) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{x}{k} \right) \left( \frac{x}{n-k} \right) = 0, \quad n \text{ odd}, \]

\[ = \left( \frac{x}{m} \right) \left( \frac{-x-1}{m} \right) = (-1)^m \left( \frac{x}{m} \right) \left( \frac{x+m}{m} \right) = (-1)^m \binom{2m}{m} \left( \frac{x+m}{2m} \right), \]

\[ n = 2m. \]

\[ (6.11) \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \left( \frac{x}{k} \right) \left( \frac{y}{k} \right) \left( \frac{-x-y-1}{m-k} \right) = \left( \frac{x+m}{m} \right) \left( \frac{y+m}{m} \right), \quad \text{Eq. to } (6.166) \text{ with } D(2y). \]

\[ (6.12) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{x-y}{k} \right) \left( \frac{x+y}{n-k} \right) = \]

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{x-y+k}{k} \right) \left( \frac{x+y+n-k}{n-k} \right). \]
\[ \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left(\frac{y-x+k-1}{n}\right) \left(\frac{x+y+k}{n}\right) = \]

\[ \sum_{j=0}^{n} (-1)^{j} \binom{n}{j} \left(\frac{u+n-j}{n}\right) \left(\frac{x+j}{m}\right) . \]

(6.13) \[ \sum_{k=0}^{n} \sum_{u=0}^{n-1} (-1)^{k} \binom{n}{k} \left(\frac{u}{u+n-k}\right) \binom{x}{m-k} = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{2n-k}{n} \binom{x+k}{m} . \]

From (6.20) with \( a = b = n \)

(6.14) \[ \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{2n}{n-k} \binom{x}{m-k} = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{2n-k}{n} \binom{x+k}{m} . \]

(6.15) \[ \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{2n}{2n-k} \binom{2n-k}{n} = \sum_{k=0}^{n} \binom{2n}{k} \binom{4n-k}{2n} \binom{2n+k}{n} = \sum_{j=0}^{n} (-1)^{j} \binom{2n}{j} \binom{4n-j}{3n} \binom{2n+j}{2n} . \]

(6.16) \[ \frac{2x}{n} \left(\frac{2x}{x}\right) \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{2x-n}{x-k} = \begin{cases} 0, & n \text{ odd} \\ (-1)^{m} \binom{2m}{m} \binom{x+m}{2m}, & n = 2m . \end{cases} \]

Notation as in D(16).
(6.17) \[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x}{n-k} \binom{\frac{t}{n-k}}{\binom{\frac{t}{n}}{n-k}}^2 = \]

(6.18) \[ \sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{\frac{t}{n-k}}{\binom{\frac{t}{n}}{n-k}}^2 = \]

\[ \sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{\frac{t}{n-k}}{\binom{\frac{t}{n}}{n-k}}^2. \]

(6.19) \[ \sum_{k \leq \frac{n}{2}} \binom{n}{2k} \binom{x}{k} \binom{\frac{t}{n}}{\binom{\frac{t}{n}}{n-k}} = \]

\[ (-1)^n \sum_{j=0}^{n} \binom{n-j}{j} \binom{\frac{t}{n}}{\binom{\frac{t}{n}}{n-j}} 2^{n-2j} \]

(6.20) \[ \sum_{k=0}^{n \Lambda M} (-1)^k \binom{\alpha}{k} \binom{\beta}{n-k} \binom{x}{M} = \]

\[ \sum_{j=0}^{n} (-1)^j \binom{\alpha}{j} \binom{\alpha+j}{n-j} \binom{x+j}{M} \]

(6.21) \[ \sum_{k=0}^{n \Lambda M} \binom{\alpha}{k} \binom{\alpha}{n-k} \binom{x}{M} = \]

\[ \sum_{j \leq \frac{n}{2} \Lambda M} \binom{\alpha}{j} \binom{\alpha-j}{n-j} \binom{x+n-2j}{M-j} \]

\[ \sum_{j \leq \frac{n}{2} \Lambda M} \binom{\alpha}{j} \binom{\alpha-j}{n-2j} \binom{x+n-2j}{M-j}. \]
\[(6.22) \sum_{h=0}^{n \wedge M} \frac{(-1)^{h} \binom{n}{h} (\frac{a}{n-h})(x)}{(M-h)} = \]

\[= \sum_{k=0}^{n \wedge M} \frac{(-a-1)}{k} \left( \frac{a}{n-k} \right) \left( \frac{x+n-k}{M-k} \right) \]

\[(6.23) \sum_{k=0}^{n \wedge N} (-1)^{k} \binom{m+u+1-k}{m-k} \left( \frac{x}{N-k} \right) = \]

\[= \sum_{j=0}^{m} (-1)^{j} \binom{u+j}{j} \left( \frac{x+j}{N} \right) \]

\[= \sum_{i=0}^{m \wedge N} (-1)^{i} \binom{m+u+1-i}{m-i} \left( \frac{x+m-i}{N-i} \right) \]

\[(6.24) \sum_{j=0}^{n \wedge M} (-1)^{j+1} \left( \frac{a}{h} \right) \left( \frac{-a}{n+1-h} \right) \left( \frac{x}{M-h} \right) = \]

\[(6.25) \sum_{k=\lfloor \frac{x}{z} \rfloor}^{n} (-1)^{k} \left( \frac{z}{k} \right) \left( \frac{2n-2z}{n-k} \right)^{2} = \]

\[\frac{(-1)^{\tau} (\tau-\zeta)! (2n-\tau)! (2n-2\zeta)!}{\tau! n! n! (n-\zeta)! (n-\zeta)!}, \quad \tau \leq n, \]

\[(6.26) \sum_{k=0}^{n \wedge (j-z)} (-1)^{k} \left( \frac{n}{k} \right) \left( \frac{-x+j-1}{k+z} \right) \left( \frac{x}{j-z-k} \right) = \]

\[= (-1)^{\zeta} \left( \frac{j+n}{\zeta} \right) \left( \frac{x}{\zeta} \right), \quad \zeta \leq j. \]
\[(6.27) \sum_{k=0}^{n} (-1)^{k} \left( \begin{array}{c} n \\vspace{0.5em} \end{array} \right)^{k} \left( \begin{array}{c} n+\varepsilon-1 \\vspace{0.5em} \end{array} \right)^{k} \left( \begin{array}{c} x-\varepsilon-k \\vspace{0.5em} \end{array} \right) = \]

\[-(-1)^{\varepsilon} \left( \begin{array}{c} j+\varepsilon \\vspace{0.5em} \end{array} \right) \left( \begin{array}{c} x \\vspace{0.5em} \end{array} \right), \varepsilon \leq j \]

\[(6.28) \sum_{k=0}^{m} (-1)^{k} \left( \begin{array}{c} n \\vspace{0.5em} \end{array} \right)^{k} \left( \begin{array}{c} n+\varepsilon \\vspace{0.5em} \end{array} \right)^{k} \left( \begin{array}{c} q-k \\vspace{0.5em} \end{array} \right) \left( \begin{array}{c} p-n+k \\vspace{0.5em} \end{array} \right) = \]

\[-(-1)^{q-\varepsilon} \left( \begin{array}{c} n+\varepsilon \\vspace{0.5em} \end{array} \right)^{q} \left( \begin{array}{c} q-\varepsilon \\vspace{0.5em} \end{array} \right)^{q} \left( \begin{array}{c} p-\varepsilon \\vspace{0.5em} \end{array} \right) \text{ when} \]

\[p+q=n=2\varepsilon, \varepsilon \in \mathbb{N}_{p}, \text{ and } \varepsilon \leq q, \varepsilon \leq p. \text{ Otherwise the sum vanishes.} \]

\[p,q \in \mathbb{N}_{0}. \]

\[(6.29) \sum_{k=0}^{p+q} (-1)^{k} \left( \begin{array}{c} p+q \\vspace{0.5em} \end{array} \right)^{k} \left( \begin{array}{c} a \\vspace{0.5em} \end{array} \right)^{k} \left( \begin{array}{c} b \\vspace{0.5em} \end{array} \right)^{k} \left( \begin{array}{c} -a-b-1 \\vspace{0.5em} \end{array} \right) = \]

\[-(a+q)(b+p), p,q \in \mathbb{N}_{0}. \]

\[(6.30) \sum_{h} (-1)^{h} \left( \begin{array}{c} u+v \\vspace{0.5em} \end{array} \right)^{h} \left( \begin{array}{c} v+u \\vspace{0.5em} \end{array} \right)^{h} \left( \begin{array}{c} u+v \\vspace{0.5em} \end{array} \right)^{h} \left( \begin{array}{c} v+u \\vspace{0.5em} \end{array} \right)^{h} = \]

\[\frac{(u+v+w)!}{u!v!w!}, u, v, w \in \mathbb{N}_{0}, |h| \leq u \wedge v \wedge w. \]

\[(6.31) \sum_{h} (-1)^{h} \left( \begin{array}{c} u+v \\vspace{0.5em} \end{array} \right)^{h} \left( \begin{array}{c} v+w \\vspace{0.5em} \end{array} \right)^{h} \left( \begin{array}{c} w+v \\vspace{0.5em} \end{array} \right)^{h} \left( \begin{array}{c} u+w \\vspace{0.5em} \end{array} \right)^{h} = \]

\[\frac{(u+v+w)!}{u!v!w!}, u, v, w \in \mathbb{N}_{0}, |h| \leq u \wedge v \wedge w. \]
\[ (6.32) \sum (-1)^k \binom{2u+k}{u+k} \binom{2v+k}{v+k} \binom{2w+k}{w+k} = \]
\[ \frac{(2u)! (2v)! (2w)!}{(u+v+w)!} \frac{(u+v+w+k)!}{(u+v+k)! (v+w+k)! (w+u+k)!} \]
\[ |k| \leq u \land v \land w. \]

\[ (6.33) \sum_k (-1)^k \binom{2n}{k} \binom{2x}{n+k} \binom{2z}{z+n+k} = \]
\[ (-1)^n \frac{(2n)! (2x)! (2z)!}{(n+x)! (n+z)! (x+z)!} \frac{(n+x+z)!}{n! x! z!} \]
\[ [n-x] \lor [n-z] \leq k \leq 2n \land (x+n) \land (z+n), \ x, z \in \mathbb{N}. \]

\[ (6.34) \sum_{h=[m-N]}^{n \land m} (-1)^{m-h} \binom{m}{h} \binom{a}{n-h} \binom{x}{N-m+h} = \]
\[ \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{m+a-j}{n} \binom{x+j}{N-j}. \]

\[ (6.35) \sum_{h=[2] \lor [5]} \binom{t+s}{h} \binom{x}{t+s-h} \binom{5}{s-h} \binom{x}{h} \]
\[ \text{See (6.136) with } k = t+s-h \text{ in } y^{th} \text{ member.} \]

\[ (6.36) \sum_{h=-n}^{n} (-1)^h \binom{2x}{x-h} \binom{2n}{n-h} \]
\[ \text{See (6.33) with } x = z \text{ and } n - k = h. \]
\[(6.37) \sum_{i=[n-1]}^{m} (-1)^i \binom{n}{i} \binom{u}{m-i} (r-n+i), \]

see (6.128).
\[ (6.40) \sum_{k=0}^{n} \left( \frac{n}{k} \right) \left( \frac{n}{j} \right) x^k = \sum_{2j \leq n} \left( \frac{n}{2j} \right) \left( \frac{n}{j} \right) x^j (1+x)^{n-2j} \]

\[ (6.41) \sum_{k=0}^{n} \left( \frac{n}{k} \right) \left( \frac{m}{n-k} \right) \left( \frac{m}{k} \right) x^k = \sum_{0 \leq k \leq m \land \frac{n}{2} \leq k} \left( \frac{2}{k} \right) \left( \frac{m+k}{m-k} \right) \left( \frac{m-k}{n-2k} \right) x^k (1+x)^{n-2k} \]

\[ (6.42) \sum_{k=0}^{\infty} \left( -x \right)^k \frac{x^{1+k}}{k} = \frac{\sin \pi x}{\pi x}, \quad \text{Re} x < \frac{1}{3} \]

\[ (6.43) \sum_{k=0}^{\infty} \left( -1 \right)^k \frac{x^{2+k}}{k} \left( -x \right)^3 = \frac{\sin \pi x}{\pi x} \]

\[ \left( \frac{x-1}{2} \right)! \left( \frac{3x-1}{2} \right)! \left( \frac{-x-1}{2} \right)! \left( \frac{-x-3}{2} \right)! \frac{\sin \pi x}{\pi x} \]

(6.42) and (6.43) are (16) and (18) in Duggal (1906). Cf. (6.27).

\[ (6.44) \sum_{j=0}^{m} \left( \frac{x}{j} \right) \left( \frac{x+m-j}{n-j} \right) \left( \frac{n-j}{m-j} \right) (m-2j) = 0, \quad m \leq \xi, \]

\[ (6.45) \sum_{j=0}^{m} \left( \frac{x}{\xi-j} \right) \left( \frac{x+m-j}{\xi-j} \right) \left( \frac{\xi}{m-j} \right) (m-2j) = 0, \quad m \leq \xi. \]
The identities (6.50) and (6.51) have a proof by operator identity (cf. I3, I8), similar to the proofs of (6.20) and (6.21), with the only difference that the operator identity now is applied to a function of the form
\[ x(x+mc)^{-1}(\frac{x+mc}{m}) \].

We then use the relation, cf. C(40),
\[ \Delta x(x+mc)^{-1}(\frac{x+mc}{m}) = \frac{x+2c}{x+mc}(x+mc) \],
\[ \forall m \geq 0, \forall e > m. \]

\[ (6.50) \sum_{k=0}^{N} (-1)^k \binom{a}{k} \binom{b}{n-k} \frac{x+k\epsilon}{x+Mc} \frac{x+Mc}{(x+m\epsilon)} = \]
\[ \sum_{j=0}^{n} (-1)^j \binom{a}{j} \binom{a+j}{n-j} \frac{x+j}{x+j+Mc} \frac{x+j+Mc}{M} \]

In (3.89) replace \( z \) by \( -\Delta \) and apply the resulting operator identity to the function \( x(x+Mc)^{-1}(\frac{x+Mc}{m}) \).
\[
(6.51) \quad \sum_{k=0}^{n \wedge M} \binom{n}{k} \binom{a}{n-k} \frac{x+k\alpha}{x+M\alpha} \left(\frac{x+M\alpha}{M-k}\right) = \\
\sum_{j \leq \frac{n \wedge M}{2}} \binom{n-j}{j} \binom{a}{n-j} \frac{x+n-2j+j\alpha}{x+n-2j+M\alpha} \left(\frac{x+n-2j+M\alpha}{M-j}\right)
\]

In (3.94) with \( \tau = 1 \) replace \( u \) by \( \Delta \) so that \( utv \) is replaced by \( E \) and apply the resulting operator identity to the function \( x(x+M\alpha)^{-1}(x+M\alpha) \).

Variants of the identities (6.14), (6.13), (6.12), (6.34), (6.23) and (6.19) may be derived in a similar way.
\[(6.60) \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} \binom{k}{r} = \binom{x}{r} \binom{x+y-r}{n-r}, \quad r \leq n. \]

\[(6.61) \sum_{k=0}^{n} \binom{u}{k} \binom{v}{n-k} \binom{k}{m} = \sum_{j=0}^{\min(n,M)} \binom{u}{j} \binom{u+v-j}{n-j} \binom{x}{M-j} \]

\[(6.62) \sum_{k=0}^{n} \binom{u}{k} \binom{v}{n-k} \binom{x+k}{m} = 0, \quad u + v = M < n , \]

\[= \binom{x}{M-n} \binom{x+v+n}{n}, \quad u + v = M \geq n. \]

\[(6.63) \sum_{j=0}^{n} \binom{-a}{j} \binom{b}{n-j} \binom{c+j}{e} = \binom{-a}{n} \binom{c}{e-n} = (-1)^n \binom{b}{n} \binom{c}{e-n}, \quad b-a = n-1, \]

\[(\text{with } \binom{0}{k} = 0, \quad -k \in \mathbb{N}^+) \quad . \]

\[(6.64) (-1)^n \sum_{j=0}^{n} \binom{n-b-e-1}{j} \binom{b}{n-j} \binom{c+j}{e} = \sum_{h=\lfloor n-e \rfloor}^{n} \binom{e}{h} \binom{b}{n-h} \binom{c+h}{e-n+h} \]
\[
\begin{align*}
\text{(6.65)} & \quad \sum_{k=0}^{n} \binom{k}{k} \binom{n+k}{n} = (n) (n) \\
\text{From (6.62) with } u = \tau, \nu = n, \quad \binom{n}{k} \binom{\nu}{\nu/k} \left( \begin{array}{c} x-k \\ \frac{\nu}{\nu/k} \\ \frac{\nu}{\nu/k} \end{array} \right) = \\
\sum_{j=0}^{nM} (-1)^j \binom{j}{j} \binom{\nu}{n-j} \binom{x-j}{n-j} \\
\text{(6.67)} & \quad \sum_{k=0}^{n} \binom{k}{k} \binom{\nu}{\nu/k} \left( \begin{array}{c} x-k \\ \nu \end{array} \right) = 0, \quad \nu + \nu = M < n, \quad \chi = \left( \begin{array}{c} x-n \\ M-n \end{array} \right) \left( \begin{array}{c} x-n \\ n \end{array} \right), \quad \nu + \nu = M \geq n, \quad \frac{n^\nu}{\nu} \\
\text{(6.68)} & \quad \sum_{k=0}^{n\lambda \nu} \binom{n}{\nu/k} \binom{\nu}{k} \left( \begin{array}{c} y+n \nu - k \\ n+k \end{array} \right) = \\
\sum_{j=0}^{M \nu} (-1)^j \binom{\nu}{j} \binom{n+x-j}{n-j} \\
\text{(6.69)} & \quad \sum_{k=0}^{n} \binom{k}{k} \binom{\nu}{\nu/k} \left( \begin{array}{c} x+k \\ \nu \end{array} \right) = \\
\sum_{j=0}^{nM} (-1)^j \binom{a}{j} \binom{-a-1}{n-j} \binom{x+n-h}{n-h} \binom{a}{h} \binom{\nu}{\nu/h} \left( \begin{array}{c} x+h \\ h \end{array} \right) = \\
\text{From (6.67) with } u = \tau, \nu = n, \quad \chi = y+M. \\
\text{(6.70)} & \quad \sum_{h=0}^{n} \binom{\nu}{h} \binom{\nu-h}{h} \binom{x+h}{M} = \\
\sum_{j=0}^{M \nu} (-1)^j \binom{a}{j} \binom{-a-1}{n-h} \binom{x+n-h}{n-h} \\
\end{align*}
\]
\[
(6.71) \sum_{k=0}^{m} \binom{\text{u}}{k} \binom{n}{m-k} \binom{x-k}{n} = \sum_{k=0}^{m} \binom{x-\text{u}}{\text{k}} \binom{\text{m} - \text{k}}{\text{m} - \text{k}} \binom{x-\text{k}}{n} \\
\sum_{k=0}^{m} \binom{x-\text{u}}{\text{k}} \binom{\text{m} - \text{k}}{\text{m} - \text{k}} \binom{x-\text{k}}{n} = \\
\sum_{\text{k}=0}^{m} (-1)^{\text{k}} \binom{x-\text{u}}{\text{k}} \binom{x-\text{k}}{n} \binom{x-\text{k}}{n}.
\]

\[
(6.72) \sum_{k=0}^{m} \binom{\text{u} - 1}{k} \binom{\text{m} + \text{u} + 1}{\text{m} - \text{k}} \binom{x+\text{k}}{N} = \\
\sum_{i=0}^{m} \binom{-\text{u} - 1}{i} \binom{x}{\text{N}-\text{i}} = \\
\sum_{j=0}^{m} \binom{-1}{j} \binom{\text{m} + \text{u} + 1}{\text{j}} \binom{x + \text{m} - \text{j}}{\text{N} - \text{j}}.
\]

For \(\text{N} \leq \text{m} : \left( \frac{x-\text{u}-1}{\text{N}} \right) \)

\[
(6.73) \sum_{k=0}^{m} \binom{\text{u} - 1}{k} \binom{\text{m} + \text{u} + 1}{\text{m} - \text{k}} \binom{x-\text{k}}{N} = \\
\sum_{i=0}^{m} \binom{\text{u} + \text{i}}{i} \binom{x-\text{i}}{\text{N}-\text{i}} = \\
\sum_{j=0}^{m} \binom{\text{m} + \text{u} + 1}{\text{j}} \binom{x-\text{m}}{\text{N}-\text{j}}.
\]

For \(\text{N} \leq \text{m} : \left( \frac{x+\text{u}+1}{\text{N}} \right)
\( (6.74) \quad \sum_{h=0}^{n} \binom{n}{h} \binom{a}{n+1-h} \left( \frac{x+h}{M} \right) = \)

\( (6.75) \quad \sum_{h=0}^{n} \binom{m}{h} \binom{a}{n-h} \left( \frac{x+h-m}{N} \right) = \)

\[ \sum_{j=0}^{m} \binom{m}{j} \binom{m+a-j}{n} \left( \frac{x-j}{N-j} \right) \]

\( (6.76) \quad \sum_{h=0}^{n} \binom{m}{h} \binom{a}{n-h} \left( \frac{x+m-h}{N} \right) = \)

\[ \sum_{j=0}^{m} \binom{m}{j} \binom{m+a-j}{n} \left( \frac{x}{N-j} \right) \]

\( (6.77) \quad \sum_{k=0}^{n} \binom{n+1/2}{k} \binom{n+1/2}{n-k} \left( \frac{x+k}{M} \right) = \)

\[ (2n) \sum_{k=0}^{M} \binom{2n-k}{k} \binom{2n-M}{M-k} z^k \]

For \( M \leq n \):

\[ (2n) \sum_{k=0}^{M} \binom{2n-k}{k} \binom{2n-M}{M-k} z^k = \]

\[ (2n) \sum_{k=0}^{M} \binom{2n-k}{k} \binom{2n-M+2x-k}{M-x} \]
\[ (6.78) \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) (x+k) = \]

\[ (2n+1) \sum_{k=0}^{n} \binom{n-1}{k} \binom{n-1-k}{M-k} x^{-k} \]

For \( M \leq n \):

\[ (2n+1) x^{-M} \sum_{k=0}^{M} \binom{2n}{k} \binom{2n+1-M}{M-k} x^{-k} = \]

\[ (2n+1) x^{-M} \sum_{k=0}^{M} \binom{2n}{k} \binom{2n+1-M+2x-k}{M-k} \]

\[ (6.79) \sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) (x+k) = \]

\[ \frac{1}{2} (2n) \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n}{k} \binom{x}{M-k} x^{-k}, \quad n \geq 1, \]

For \( 1 \leq M \leq n \):

\[ \frac{1}{2} (2n) x^{-M} \sum_{k=0}^{M} \binom{2n}{k} \frac{2n-k}{2n-M} \binom{2n}{M-k} 2^k = \]

\[ (2n) x^{-M} \sum_{k=0}^{M} \binom{2n}{k} \frac{n+x-k}{2n+2x-M-k} \binom{2n+2x-M-k}{M-k} = \]

\[ \frac{1}{2} (2n) \sum_{k=0}^{M} \binom{2x}{k} \binom{2n-M+2x-k}{M-k} \]

(The last one also for \( M=0, \ n \geq 1 \)).
\[(b.80)\sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{m} =\]

\[= \binom{2n}{m} \sum_{k=0}^{M} \frac{2n+1}{2n-M+1} \binom{2n+1-k}{k} \binom{2n-M+1}{M-k} (-1)^k\]

For \(M \leq n:\)

\[= \binom{2n}{m} \sum_{k=0}^{M} \binom{2n+1}{k} \frac{2n+1+2n-2k}{2n-M+1+2n-k} \binom{2n-M+1+2n-k}{M-k} (-1)^k\]

\[= \binom{2n}{m} \sum_{k=0}^{M} \binom{2n+1}{k} \binom{2n+1-M+2n-k}{M-k} (-1)^k\]

\[(6.81)\sum_{k=0}^{n} \binom{n}{k} \binom{x+k}{m} =\]

\[\sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{m} \binom{x}{M-k}\]

\[(6.82)\sum_{k=0}^{n} \binom{n}{k} \binom{3n+k}{2n} = \sum_{k=0}^{n} \binom{n}{k} \binom{2n}{2n-k}\]

\[\gamma_4 = \binom{3n}{n}\]

\[(6.83)\sum_{k=0}^{n} \binom{n}{k} \binom{x+k}{2n} = \binom{x}{n}\]

\[(6.84)\sum_{k=0}^{n} \binom{n}{k} \binom{y+2n-k}{2n} = \binom{y+n}{n}\]
\[
(6.85) \sum_{k=0}^{n} \binom{n}{k} \binom{a}{k+1} \binom{x+k+1}{\frac{\alpha}{M}} =
\]

\[
(6.86) \sum_{k=d+\ell}^{n} \binom{x}{k-d} \binom{y}{n-k} \binom{k-d}{\ell} =
\]

\[
\binom{x}{\ell} \binom{x+y-\ell}{n-\ell-d}, \quad d \in \mathbb{N}_0, \quad d+\ell \leq n.
\]

\[
(6.87) \sum_{k=\ell-d}^{n} \binom{x}{k} \binom{y}{n-k} \binom{k+d}{\ell} =
\]

\[
\binom{x}{\ell} \binom{x+y-\ell}{n-\ell+d}, \quad d \in \mathbb{N}_0, \quad 0 \leq \ell-d \leq n.
\]

\[
(6.88) \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k+d} \binom{k+d}{\ell} =
\]

\[
\binom{x}{\ell} \binom{n+x-\ell}{n+d-\ell}, \quad d \in \mathbb{N}_0, \quad \ell \leq d.
\]

\[
(6.89) \sum_{k=d}^{b} \binom{b}{k} \binom{c}{k-d} \binom{a+k}{b+c} =
\]

\[
\binom{a}{d} \binom{a+d}{c+d}, \quad b, c, d \in \mathbb{N}_0, \quad d \leq b.
\]
(6.90) \( \sum_{k=0}^{c-d} \binom{b}{k} \binom{c}{k+d} \binom{d}{k} = \binom{a}{d} \binom{b+1}{d'}, \quad b, c, d \in \mathbb{N}_0, \quad d \leq c. \)

(6.91) \( \sum_{k=0}^{d} \binom{b}{k} \binom{c}{k} \binom{a+k}{b+c} = \binom{a}{b+c-d} \binom{a-c+d}{d}, \quad d \leq b+c. \)

(6.92) \( \sum_{k=m}^{m+n} \binom{N}{k} \binom{k}{m} \binom{m-n}{k-n} = \binom{N}{n} \binom{N}{m} \)

(6.93) \( \sum_{h=0}^{m \wedge n} \binom{N}{h+m \wedge n} \binom{m \wedge n}{h} \binom{h+m \wedge n}{m \wedge n} = \binom{N}{n} \binom{N}{m} \)

(6.94) \( \sum_{k=0}^{n} \binom{x}{k} \binom{n-x}{n-k} \binom{a+b}{m} = \binom{a+b}{m} \), \( m \leq n. \quad \text{From G(55) with } q(x) = \binom{a+b}{m}. \)

(6.95) \( \sum_{2k \leq n} (-1)^k \binom{n}{2k} \binom{-1/2}{k} \binom{x+k}{m} = \sum_{j=0}^{n \wedge m} (-1)^j \binom{n-j}{j} \binom{-1/2}{j} \binom{x}{m-j} 2^{n-j} \).
\begin{align}
(6.96) \sum_{\lambda k \leq n} (-1)^k \binom{n}{2k} \left(-\frac{1}{2}\right)^k \binom{x-k}{M-k} = T_{19} \\
\sum_{j=0}^{n-M} \binom{n-j}{j} \left(-\frac{1}{2}\right)^j \binom{x-j}{M-j} 2^{n-2j} \\
(6.97) \sum_{\left[n-M\right] \leq 2k \leq n} (-1)^{n-k} \binom{n}{k} \binom{2n-2k}{n} \binom{x}{M-n+2k} = \\
\sum_{h=0}^{n} (-1)^{h} \binom{n}{h} \binom{n+h}{h} \binom{x+h}{M} 2^{n-h} \\
(6.98) \sum_{i \leq MA \left(z+n \lambda \right)} \binom{n+i}{i} \binom{2n+i-2-i}{n} \binom{x}{M-i} = \\
\sum_{\lambda i \leq n} \binom{n+i}{i} \binom{n-i}{i} \binom{x+i+i}{M} \lambda^{-i} \\
(6.99) \sum_{k=0}^{n} \binom{x}{k} \binom{-x}{n-k} \binom{x+i-k}{z} = \\
\sum_{i=0}^{\kappa} \binom{\kappa}{i} \binom{x}{i} \binom{-i}{n} .
\end{align}
$$\sum_{h=0}^{M \wedge n} \binom{n}{h} \binom{n+h}{n} (M-h)^{n-h}$$

See (6.131).
\[ \chi \leq M, \quad \chi = \nu, \quad \tau > \nu. \]

Examples:

\[ (6.110) \sum_{k=0}^{n} \binom{\nu}{k} \binom{\nu}{n-k} \frac{x+k}{x+k+M} \binom{x+k+M}{M} = \]

\[ \frac{x}{x+MC} \frac{(x+MC)(x+MC-v+n)}{(M-n)_{\nu+1}} \left\{ 1 + \frac{nc4}{x+MC-v+n} \right\}. \]

\[ \psi(v,x) = \frac{\nu}{\nu+1} \left\{ 1 \right\}. \]
From (6.110), the r.h.s. then is

\[ \sum_{h=M}^{n M} \left( \frac{1}{M-j} \right) \frac{x+j}{x+Mc} \left( \frac{x+Mc}{u} \right) \]

By canceling and rearranging factorials, see \( \tilde{D}(14) \), the above sum is equal to

\[ \prod_{i=0}^{n} \left( \frac{x+Mc}{u} \right)^{-1} \]

(6.67) Other example:

(6.112) \( \sum_{k=0}^{m} \left( -u-1 \right) \frac{x+k}{x+k+Mc} \left( x+k+Mc \right)^{N} \)

(6.83) \( \sum_{i=0}^{m} \left( -u-1 \right) \frac{x+i}{x+i+Mc} \left( x+i+Mc \right)^{N} \)

We start with (x) in the proof of (6.23), p. 79:

\[ \sum_{i=0}^{m} \left( -u-1 \right) \frac{x+i}{x+i+Mc} \left( x+i+Mc \right)^{N} \]
\[
(6.121) \sum_{k=m}^{n} (-1)^{k-m} C_{k}^{m} (u+k\alpha) (k) = \\
\sum_{m-m \leq M, \ n-m} (-1)^{n-m} C_{m-n+m}^{m} (u+m) (M-n+m),
\]

\[
(6.122) \sum_{k=0}^{\infty} (-1)^{k} C_{k}^{\infty} (u) (m) (z^{-k}) = \\
\sum_{m \leq n} \left( \eta \setminus \{u\} \setminus \omega + k, \right)
\]

\[
\sum_{k=[n-m]}^{\infty} (-1)^{k} C_{k}^{\infty} (k) (k-k) (m-n+k).
\]
see (6.14).

\[
(6.127) \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-k}{n} = (-1)^{n/2} \frac{(2n)!}{(n!)^2}, \quad n = 2m; \quad = 0, \ n \ odd.
\]

From (6.14) with \( x = M = n \) and (6.8),

\[
(6.128) \quad A(n, m, r, u, v) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{u+k}{m} \binom{v+n-k}{r} = 0.
\]

Identities and special cases,

\[
(6.129) \quad B(n, m, r, u, v) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{u+k}{m} \binom{v+n-k}{r}.
\]
\[
(6.132) \sum_{k \leq \frac{n}{2}} (-1)^k \binom{n}{k} \binom{2n - 2k}{n} \binom{n}{k} = \\
\sum (-1)^j \binom{m+j}{m} \binom{n-j-1}{n-m} j^{l-j+k+l}
\]
Kedlaya and Ng, Monthly 105, 1993, 819-824.
\[ \prod_{j=0}^{M-1} \binom{\tilde{\tau}}{\tilde{r}}(\tilde{r}-k)(\tilde{N}-\tilde{s}-k) = \]

\[ = -s \Lambda(N-s) \text{ for } k \geq 1 \]
Preceding proofs may be varied by replacing the function \( f(x) \) by \( x \left( x+mc \right)^{-1} \). The relation

\[ \Delta^r \left( \frac{1}{M} \right) = \left( \frac{1}{M-1} \right), \quad r \leq M, \]

then is replaced by \( G(y) \):

\[ \Delta^r x \left( x+mc \right)^{-1} \left( \frac{x+Mc}{M} \right) = \frac{x+Mc}{x+mc} \left( \frac{x+Mc}{M} \right), \quad r \leq M. \]

Examples:

(6.150) \[ \sum_{k=m}^{n} (-1)^{k-m} \binom{n}{k} \binom{k}{m} \frac{u+ka}{u+ka+Mc} \left( \frac{u+ka+Mc}{M} \right) = \]

\[ (-1)^{n-m} \binom{n}{m} \Delta^{n-m} \frac{u+ax}{u+ax+Mc} \left( \frac{u+ax+Mc}{M} \right) \Bigg|_{x=m}, \quad m \leq n. \]

(6.151) \[ \sum_{k=2}^{n} (-1)^{k} \binom{n}{k} \binom{2n-2k}{n-k} \frac{x+k}{x+k+Mc} \left( \frac{x+k+Mc}{M} \right) = \]

\[ \sum_{h=0}^{n} \binom{n}{h} \binom{n+h}{h} \frac{x+hc}{x+Mc} \left( \frac{x+Mc}{h-h} \right)^2 \frac{2^{n-h}}{h}. \]
From (6.166) with \( x \) and \( y \) both replaced by \( x - m \). This is (6.31) in Gould (1972). 

\[
(6.168) \quad \sum_{k=0}^{m} \binom{x}{k} \binom{x-k}{m-k} = \binom{x}{m} \binom{x+m}{m}
\]

\[
(6.170) \quad \sum_{j=0}^{\infty} \frac{(n)_k (n'-j)_k}{(m+j)_k} = \binom{m+x+n}{n} \binom{m+x+y}{m+n}.
\]
(6.173) \[ \sum_{k=0}^{\infty} (\binom{w-x}{k})(m-k)(n-k) > \]

from (6.172) with \( u = w-x \), \( v=x \). See also (6.71).

(6.12H) \[ \sum_{k=0}^{n} \binom{n}{k} (a_i / a_1) (x+n-k) = \]
\begin{align*}
\text{(6.178)} \quad \sum_{k=0}^{m+n} \binom{n}{k} \binom{m+n-k}{m-k} \binom{m+n-k}{n-k} &= T_{33} \\
\sum_{k=0}^{m+n} \binom{m}{k} \binom{m+n-k}{n-k} \binom{m+n-k}{m+n-k} &= \binom{x}{m} \binom{x}{n} \\
\text{(6.179)} \quad \sum_{k=0}^{n} \binom{a}{p-k} \binom{b}{q-k} \binom{a+b+k}{k} &= (p+q) (\frac{a}{q}) (\frac{b}{p}) , \quad p, q \in \mathbb{N} \quad \text{From (6.28) with D}(l^2y) \\
\text{(6.180)} \quad \sum_{k=0}^{n+x} \binom{n}{k} \binom{y+z+k}{n-k} \binom{z+n-k}{x-k} &= \\
\binom{x+z}{x} \binom{y+z}{n} \\
\text{(6.181)} \quad \sum_{k} \binom{n}{k} \binom{\nu}{m-k} \binom{\nu+k}{x-n+k} = \\
\sum_{k} \binom{n}{k} \binom{\nu}{x-k} \binom{\nu+k}{m-n+k} = \quad \text{See (6.12)} \\
\text{(6.182)} \quad \sum_{k=0}^{n} \binom{x+1}{k} \binom{x+a-1}{k} \binom{x-k}{n-k} = \\
\binom{x+a-1}{n+a-1} , \quad a \in \mathbb{N} \\
\text{(6.12)} \quad \sum_{k=0}^{2m+2} r_{k} (-1)^{j+k} \binom{j}{k} \\
\end{align*}
\[
\binom{b,10.5}{k=0} = \sum_{k=0}^{b,17.2} (-1)^k \binom{N}{2j} \binom{x+j}{M-j},
\]

\[
\sum_{j \leq \frac{N}{2}} \binom{N}{2j} \binom{x+j}{M-j}.
\]

(6.184) \( F(a, b, m, n) = F(b, a, n, m) \) with

\[
F(a, b, m, n) = \sum_{j=0}^{\min(a+m+n-2j)} \binom{a+m+n-2j}{n-j} \binom{a+n}{j} \binom{b+m}{m-j}.
\]
\[ (6.190) \sum_{h=0}^{\log_{M} n} \binom{n}{h} \binom{n+h}{h} (\frac{x}{M-h}) z^{n-h}. \]

See (6.131)
\[(6.196) \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} (M-k)^x 2^{-x} \]
\[(6.203) \sum_{h=0}^{m \wedge N} \binom{m+h+1}{m-h} (2^{h+2}) (h+1) (N-h) \frac{1}{2^{h}} = \sum_{h=0}^{m} \binom{m}{m-h} (2^{h+2}) (-1)^{h} (N-h) \frac{1}{h} \]

\[\sum \binom{n+2-i}{i} \binom{12-i}{M-i} \binom{1}{M-h} \]

(see pp. 13-14) have a variant by applying the operator identity to the function \(x (x+Mc)^{-1} \binom{x}{M} \) instead of \(\binom{x}{M} \).

We then use the relation, cf. G. (40),

\[x \sim \frac{1}{(x+Mc)^{-1} \binom{x}{M} - \text{cont.} (x+Mc)} \]
\[ \sum_{j=0}^{n} \left( \sum_{j} \sum_{k} n / x+j+i \right) \]

with (4).

\[ \sum_{h=0}^{n} \left( n \right) \left( n-n \right) \]

By application of this identity to the
(6.212) \[
\sum_{j=0}^{n} (-1)^{j} \binom{a}{j} \binom{a+b-j}{n-j} \binom{x+j}{M-j}.
\]

See (6.20).

(6.214) \[
\sum_{j=0}^{n} (-1)^{j} \binom{m}{j} \binom{m+a-j}{n} \binom{x-j}{N-j}.
\]

See (6.25).

(6.216) \[
\sum_{k=a}^{n} \binom{k+a-1}{k} \binom{2n-k}{n-k} \binom{k}{a} = \binom{2n-1}{n} \binom{n}{a}, \quad a \in \{1, 2, \ldots, n\}.
\]
\[ (n/2) \sum_{k=0}^{[n/2]} (n+2k+1)/(k+1)/(j) \]

\[ \lambda \equiv (m \mid n-m) \]
\[ (6.226) \sum_{h=0}^{n} (-1)^h \binom{n}{h} \binom{n+h}{h} \left( \frac{x+h}{M} \right)^2 h^{n-h}. \]


\[ (6.227) \sum_{k \leq \lambda} \binom{n+\varepsilon}{k+i} \binom{n-\varepsilon}{i} \left( \frac{x+i+k}{M} \right)^{-i}. \]

See (6.98).

\[ (6.228) \sum_{h=0}^{M} \binom{2h+m}{h} \binom{2h+2m}{m} \left( \frac{x}{M} \right) (-1)^h = \binom{2m}{m} \left( \frac{x}{m-1/2} \right). \]

\[ (6.229) \sum_{j \leq n} \binom{n-j}{j} \left( \frac{-1/2}{n-j} \right) \left( \frac{x+j}{M} \right)^2 n^{-2j}. \]

See (6.19).

\[ (6.230) \sum_{i=0}^{n} \binom{n+2-i}{i} \binom{1/2}{n+2-i} \left( \frac{x+i}{M} \right) (-2)^{n+1-2i}. \]

See (6.204).

\[ (6.231) \sum_{i=0}^{n} \binom{n+2-i}{i} \binom{1/2}{n+2-i} \left( \frac{x-i}{M} \right) (-2)^{n+1-2i}. \]

See (6.266).

\[ (6.232) \sum_{\lambda \leq i \leq n/2} \binom{n}{\lambda} \binom{2i}{i} \binom{i}{\lambda} 2^{n-\lambda i} = \binom{n}{\lambda} \binom{2n-2\lambda i}{n}, \quad 2\lambda \leq n. \]
$$\sum_{k=0}^{n} (-1)^{n-k} \binom{2k}{k} \binom{-n-k}{n-k} \binom{x+k}{M-k} y^{-k} $$

See (6.193).

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{2k}{k} \binom{-n-k}{n-k} \binom{x+k}{M-k} y^{-k} $$

See (6.198) and below (6.269). Also (6.284)

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2k}{k} \binom{x+k}{M-k} y^{-k} =$$

$$\binom{n+k}{2k} (x+2k)_k y^{-k}$$
\[ (6.240) \sum_{2k \leq n} \binom{n}{2k} \frac{1}{k+1} \binom{2k}{k} \frac{1}{M} (x+k+1)^{-k-1} = \]

\[ \sum_{i=0}^{n \wedge M} (-1)^{n-i} \binom{n+2-i}{i} \binom{1}{n+2-i} \binom{x}{M} 2^{n+1-2i} \]

\[ (6.241) \sum_{j=0}^{n} (-1)^{j} \binom{j}{a-j} \binom{a+b-j}{n-j} \frac{x+j}{x+j+MC} \binom{n}{M} . \]

See (6.50).

Identities proved by operator identity (see pp. 13-18) admit a variant by applying the operator identity to the function \( x (x+MC)^{-1} (x+MC)^{-1} \) instead of \( x (M) \). We then use the relation

\[ \Delta^x (x+MC)^{-1} (x+MC) = \frac{x+MC}{x+MC} - \frac{x+MC}{M} \]

\( \varepsilon \leq M \); \( \varepsilon = 0 \), \( \varepsilon > M \), see G (40).

**Examples**

(6.242) In (x), for \( u+n = 2\sigma \), in the proof of (6.217) replace \( z \) by \( E \) and apply the resulting operator identity to the function \( x (x+MC)^{-1} (x+MC) \) to give

\[ \sum_{h=0}^{n} (-1)^{h} \binom{h}{u-h} \frac{v}{n-h} \frac{x+h}{x+h+MC} \frac{(x+h+MC)}{M} = \]

\[ \sum_{j=0}^{n \wedge M} (-1)^{j} \binom{j}{u+j} \frac{v}{n-j} \frac{x+1}{x+MC} \frac{(x+MC)}{M-j}, u+n = 2\sigma \]
In (6.243), replace $t$ by $E^{-1}y$ to get

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{n-k} (\frac{2}{n})^k y^{-k} E^{-1} =$$

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{2}{k} \binom{n-k}{n-k} y^{-k} E^{-1} \Delta^k.$$

Applying both sides to the function $x (x+Mc)^{-1} (x+Mc)$ shows that

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{n-k} (\frac{2}{n})^k (x-k) y^{-k} =$$

$$\sum_{k=0}^{n+k} (-1)^{n-k} \binom{2}{k} \binom{n-k}{n-k} \frac{x-k+Mc}{x-k+Mc} \frac{x-k+Mc}{x-k+Mc} y^{-k}.$$
\[(6.253) \sum_{k=0}^{n} (-1)^{n-k} \binom{x+y+k}{n-k} \binom{x+k}{k} \binom{y+k}{k} = \binom{x}{n} \binom{y}{n}.\]

\[(6.254) \sum_{k=0}^{m+\xi} (-1)^{m+\xi-k} \binom{m+\xi}{k} \binom{m+k}{k} \binom{\xi+k}{k} = \binom{m+\xi}{\xi}.\]

From (6.125) with \(n = m+\xi, \alpha = \xi, \zeta = m, \beta = d = 1\). For \(m = \xi\) we obtain (6.36) in Gould 1972.

\[(6.255) \sum_{k=0}^{\frac{n}{2}} (-1)^{k} \binom{n}{k} \binom{\frac{y+k}{2}}{\frac{y}{2}} \binom{\frac{y+n-k}{2}}{\frac{y}{2}} = 0, \ n \ odd, \]
\[= \binom{\frac{y}{m}}{\frac{y+1}{m}} = (-1)^{\frac{m}{2}} \binom{\frac{y}{m}}{\frac{y}{m}} \binom{\frac{y+m}{2}}{\frac{y+m}{2}} = (-1)^{\frac{m}{2}} \binom{2m}{m} \binom{\frac{y+m}{2}}{\frac{y+m}{2}}.\]

\[(6.256) \sum_{k=0}^{n} (-1)^{n-k} \binom{x-k}{n-k} \binom{x}{k} \binom{x+k}{k} = \binom{x^2}{n}.\]

\[(6.257) \sum_{2k \leq n} (-1)^{k} \binom{n-k}{k} \binom{2n-2k}{n-k} \binom{n-2k}{\xi} = \sum_{2k \leq n} \binom{n}{k} \binom{n+\xi}{\xi} \binom{n-k}{\xi}. \ From \ (6.132) \ since \]
\[\binom{n-k}{k} \binom{2n-2k}{n-k} = \binom{n}{k} \binom{2n-2k}{n}, \ 2k \leq n.\]
\[
(6.258) \sum_{h=\left[\frac{b-a}{2}\right]}^{n} (-1)^h (M)^{h} (2h-2h) (a+b-h) (a-b-h) = \\
(6.259) \sum_{h=\left[\frac{b-a}{2}\right]}^{n} (-1)^h \binom{M}{h} (2h-2h) (a+b-h) (a-b-h) = 0, \quad a, b \in \mathbb{Z}.
\]
\[
(6.260) \sum_{k=0}^{n} (-1)^k \binom{2k}{k} \binom{2n-2k}{n-k} = 0, \quad n \text{ odd}; \\
= (2m)^2, \quad n = 2m.
\]
\[
(6.261) \sum_{k=0}^{n} (-1)^k \binom{n+k}{n-k} \binom{2k}{k} \binom{2n-k}{n} = 0, \quad n \text{ odd}; \\
= (-1)^{m} \binom{3m}{m}! \left( \frac{m!}{3} \right), \quad n = 2m.
\]
From (6.127) since \( \binom{n+k}{n-k} \binom{2k}{k} = \binom{n}{k} \binom{n+k}{n} \)
\[
\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \frac{1}{2^k} (x-k-1) y^{-k-1} = T_{51}
\]

(6.269)

\[
\sum_{k=0}^{n} \binom{2n}{n-k} \frac{(2n-2k)}{2^k} (x+k) y^{-k}. \text{ See (6.198)}
\]

we obtain similar relations by replacing \( t \) with \( E/\gamma \) and \( E^{-\Delta}/\gamma \).
\[
\begin{align*}
(6.277) & \sum_{k=0}^{M} \binom{2}{k} \binom{x-k}{M-k} y^{-k} = \\
& \sum_{2j \leq M} \binom{2j}{j} \binom{x-2j}{M-2j} y^{-2j}, \\
(6.278) & \sum_{k=0}^{M} \binom{2}{k} \binom{x+k+1}{M-k} (-1/4)^k = \\
& \sum_{2j \leq M} \binom{2j}{j} \binom{x-2j}{M-2j} y^{-2j}, \\
(6.279) & \sum_{j=0}^{[n/2]} (n-j)(2n-2j)/(n-2j) (-4)^j = \\
& \binom{2\varepsilon}{\varepsilon} \binom{2n-2\varepsilon}{n-\varepsilon}, \quad \varepsilon \leq n.
\end{align*}
\]
\[
(7.1) \sum_{k=0}^{\infty} (-1)^{k} \binom{n}{k} \binom{-a}{k} \binom{-b}{k} \binom{-c}{k}^{-1} = F(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(a) \Gamma(b) \Gamma(c-a-b)}
\]

\[
\sum_{k=0}^{\infty} \binom{-a}{k} \binom{-b}{k} \binom{a-b+1}{k}^{-1} = \frac{\Gamma(a+b+1)}{\Gamma(a) \Gamma(b) \Gamma(a+b+1-k)}
\]

Kummer's theorem, Slater (1966), (1.7.16), where \(x^a\) should be deleted. Also G. K. P. (1988), (5.94).
\[
\sum_{k=0}^{\frac{n}{2}} \binom{n}{k} \binom{n-k}{(n-\nu-1)/2} = 0, \quad n \text{ odd},
\]

\[
1 \quad m/(\nu) (\nu)^{-1} \quad n = 2m, \quad \nu \in \{0, \ldots, n-1\}.
\]
\[(7.9) \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{p+q+n}{p+k}^{-1}\]

\[= 0, \ p, q \in \mathbb{N}_0, \ p + q < n, \quad \text{cf. (7.11)}\]

\[= (-1)^p \binom{2n}{n}^{-1}, \ p, q \in \mathbb{N}_0, \ p + q = n.\]

\[(7.10) \text{Let } S(p, q) = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{p+q+n}{p+k}^{-1}, \ p, q \in \mathbb{N}_0. \text{ Then}\]

\[S(p, q) = (-1)^n S(q, p), \quad S(p, p) = 0, \ n \text{ odd.}\]

\[(7.11) \ S = S(M, u, v, w, q) = \sum_{k=0}^{q} (-1)^{k} \binom{u}{k} \binom{v}{m-k} \binom{w}{q+k}^{-1}, \ q \in \mathbb{N}_0.\]

Identity and special cases.

\[(7.12) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2a(x+k)}{2r}^{-1} \]

\[= 0, \ n > r; \quad (y \alpha)^r \binom{2r}{r}^{-1}, \ n = r.\]

\[(7.13) \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} \binom{x+k}{m}^{-1} = \]

\[\sum_{j=0}^{n} (-1)^{j} \binom{a}{j} \binom{a+b-j}{n-j} \frac{m}{m+j} \binom{x+j}{m+j}^{-1}, \ m \geq 1.\]
$(a^2 - \omega^2) / a; \; (\frac{1}{2} a - \omega) / 2$

Kummer's theorem, see Slater (1966), (17.1.6), where $2^a$ should be deleted. Also G.K. F.
\[(7.18) \sum_{j=0}^{n} \binom{n}{j} \binom{a}{j} \binom{b}{j}^{-1} (-2)^{j} = \]

\[\sum_{k=0}^{n} (-1)^{k} \binom{a}{k} (b-a) (b-k) 2^{k} \]

\[\sum_{k=0}^{n} (-1)^{n-k} \binom{b-a}{k} (b-k) 2^{k} \]

\[(7.19) \sum_{j=0}^{n} \binom{n}{j} \binom{a}{j} \binom{2a}{j}^{-1} (-2)^{j} = 0, \text{ } n \text{ odd,} \]

\[= (-1)^{m} \binom{a}{m} \binom{2a}{2m}^{-1}, \text{ } n = 2m. \]

From (7.18) with \(b = 2a\) and (3.67),

\[(7.20) \sum_{2h \leq n} \binom{n}{2h} (-1)^{h} \binom{a}{h} (\frac{2a}{2h})^{-1} = 2^{n} (\binom{n}{n}). \]

\[(7.21) \sum_{k=0}^{\infty} (-1)^{k} \binom{-a}{k} (-k) (\frac{1}{2} a + b + k) \binom{-\frac{1}{2}}{k} \]

\[\frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} a + \frac{1}{2} b) / \Gamma(\frac{1}{2} + \frac{1}{2} a) \Gamma(\frac{1}{2} + \frac{1}{2} b)}{2F1(a, b; \frac{1}{2} a + b + 1; \frac{1}{2})} \]

Gauss's second summation theorem, Slater (1966), (17.1.9).
(7.22) \[ \sum_{k=0}^{\infty} \frac{(-1)^k (-a)(-b)((-a-\frac{1}{2}b-\frac{3}{2})^{k-1}}{(2k+1)!} \] 

\[ \frac{2F_1(a, b; \frac{1}{2}a+\frac{1}{2}b+\frac{3}{2}; \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2}b+\frac{3}{2})} \]

\[ \sum_{j=0}^{2\epsilon-1} (-1)^j \left( \begin{array}{c} 2\epsilon-1 \\ j \end{array} \right) \Gamma(\frac{1}{2}a+\frac{1}{2}j) / \Gamma(\frac{1}{2}b+\frac{1}{2}j-\epsilon+1) \]

\( \lambda \epsilon \in N \), \( \Re a > 0 \), \( \Re (b-a) > 2\epsilon - 1 \)

(7.23) \[ \sum_{k=0}^{\infty} \frac{(-1)^k (-a)((-b)((-\frac{1}{2}a-\frac{1}{2}b-1)^{-1}}{(2k+1)!} \] 

\[ \frac{2F_1(a, b; \frac{1}{2}a+\frac{1}{2}b+1; \frac{1}{2})}{\Gamma(\frac{1}{2}) (b-a)} \left\{ \frac{\Gamma(\frac{1}{2}a)}{\Gamma(\frac{1}{2}b)} - \frac{\Gamma(\frac{1}{2}a+\frac{1}{2})}{\Gamma(\frac{1}{2}b+\frac{1}{2})} \right\} \]

\( \Re a > 0 \), \( \Re (b-a) > 0 \)

(7.24) \[ \sum_{k=0}^{\infty} \frac{(-1)^k (-a)((-b)((-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2})^{k-1}}{(2k+1)!} \] 

\[ \frac{2F_1(a, b; \frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}; \frac{1}{2})}{\Gamma(\frac{1}{2})} \]
\begin{equation}
(7.26) \quad \frac{\varpi \nu_+ n}{\nu} \sum_{k=0}^{\nu} (-1)^k \binom{\nu}{k} \binom{m-k}{\nu-k} \frac{n^\nu}{\nu+k} = \binom{\nu}{m}, \quad m \leq n.
\end{equation}

\begin{equation}
(7.27) \quad \sum_{k=0}^{\nu} (-1)^k \binom{\nu}{k} \binom{m}{k} \frac{1}{\nu+k} = \frac{1}{\nu} \frac{(x+m)(m^n)^{-1}}{(m+n)^{-1}}, \quad n \leq m \leq 2n.
\end{equation}
\[
\sum_{j=1}^{n} \frac{1}{j^{n+1}}, \quad 1 \leq n \leq m \leq 2n.
\]

\[
(7.29) \quad \sum_{k=0}^{n} \left(\sum_{i=0}^{n-1} \frac{1}{\omega - i}\right) = \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \binom{n}{k} \binom{\omega}{k} \binom{\omega-1}{k-1}\right).
\]
\[\sum_{k=0}^{n} \binom{n}{k} \binom{x+k}{k}^{-1} = F(-n, -n; x+1; 1)\]

\[= \binom{x+n}{n}\binom{x+n}{n}^{-1}, \text{ from (7.3) with } D(24).\]

\[\sum_{k=0}^{n} \binom{n}{k} \binom{-n-1}{n-k} \binom{x+k}{k}^{-1} = \text{ cf. (7.81)}\]

\[(-1)^n \sum_{k=0}^{n} \binom{n}{k} \frac{x}{x+k}, \quad x \notin \mathbb{N}^+.\]

\[\sum_{k=0}^{n} \binom{2n+1}{n-k} \binom{y}{k} \binom{y+k+1}{k}^{-1} = \]

\[4^n \binom{y+n+1/2}{n} \binom{y+n+1}{n}^{-1}, \quad y \in \mathbb{N}^+.\]

\[\frac{y! (y+1)!}{(2y+1)!} \frac{(2y+2n+1)!}{(y+n)! (y+n+1)!}, \quad y \notin \mathbb{N}^+.\]
\[
(7.46) \sum_{k=0}^{n} \frac{1}{k} \left( \frac{1}{k-1} \right) \left( \frac{y}{n-k} \right) \left( \frac{n-k}{k} \right)^{-1} = 
\]

\[
\frac{1}{y-x} \left\{ \left( \begin{array}{c} y \\ n \end{array} \right) - \left( \begin{array}{c} x \\ n \end{array} \right) \right\}, \quad y \neq x, \quad n \geq 1.
\]

\[
(7.47) \sum_{k=1}^{n} \frac{1}{k} \left( \frac{1}{k-1} \right) \left( \frac{x}{n-k} \right) \left( \frac{n-k}{k} \right)^{-1} = 
\]

\[
\left( \begin{array}{c} x \\ n \end{array} \right) \sum_{i=0}^{n-1} \frac{1}{x-i}, \quad n \geq 1. \quad \text{From (7.46) and B(38).}
\]

\[
(7.48) \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \frac{2n-k}{n} \right) \frac{2k}{x+k} \left( \frac{k}{k} \right)^{-1} = 
\]

\[
\sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{2k}{x+k}.
\]

\[
(7.49) \sum_{2k \leq n} \left( -\frac{1}{4} \right)^{k} \left( \frac{2k}{k} \right) \left( \frac{1}{n+k} \right) \left( \frac{1}{2k+1} \right) = 
\]

\[
0, \quad p, q \in \mathbb{N}_{0}, \quad n+p \text{ even, } p+2q < n
\]

\[
(-1)^{q} \frac{2^{n} n! n!}{(2n+1)!}, \quad p, q \in \mathbb{N}_{0}, \quad n+p \text{ even, } p+2q = n
\]

\[
(7.50) \sum_{2k \leq n} \left( -\frac{1}{4} \right)^{k} \left( \frac{2k}{k} \right) \left( \frac{1}{m+n} \right) \left( \frac{1}{2k+1} \right) = 
\]

\[
\frac{n!}{(n+m+1)!} \sum_{2k \leq n} \left( -\frac{1}{4} \right)^{k} \left( \frac{1}{2k+1} \right) \left( \frac{1}{2m+2n+1} \right)
\]
\[ = 0, \text{ } m+n \text{ even, } m < n, \]

\[ = \frac{m! \cdot n!}{(n+m+1) \cdot \left(\frac{1}{2} m + \frac{1}{2} n\right)} \cdot \left(\frac{1}{2} m - \frac{1}{2} n\right)^{m+n} \text{ even, } m \geq n. \]

\[
(7.51) \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} \binom{x+k}{x} \left(\sum_{j=1}^{k} \frac{1}{x+j}\right)^{-1} = \sum_{j=1}^{n} \frac{1}{x+j},
\]

\[
\left(\frac{x+n+1}{n}\right) \left(\frac{x+n+1}{n}\right)^{-1} \left\{ \sum_{j=1}^{n} \frac{1}{x+j} \right\} - \left\{ \frac{\sum_{j=1}^{n} \frac{1}{x+j}}{x+n+1+j} \right\}.
\]

Differentiate (7.39) w.r.t. to \( x \), with \( B(39) \).

See Gould (1960).

\[
(7.52) \sum_{k=0}^{\infty} \left(-\frac{u}{2}\right)^k \binom{u/2}{k} \binom{2k}{k} \left(2 \cdot \sin x\right)^{2k} = \]

\[
2 \int \left(\frac{u}{2}, -\frac{u}{2}; \frac{1}{2}; \sin^2 x\right) = \cos u x, \]

\[-\frac{\pi}{2} \leq x < \frac{\pi}{2}.\]

\[
(7.53) \sum_{k=0}^{\infty} \left(-\frac{u}{2}\right)^k \binom{u/2}{k} \binom{2k}{k} \left(2 \cdot \sin x\right)^{2k} = \]

\[
2 \int \left(\frac{u}{2} + \frac{1}{2}, -\frac{u}{2} + \frac{1}{2}; \frac{1}{2}; \sin^2 x\right) = \frac{\cos u x}{\cos x}, \]

\[-\frac{\pi}{2} \leq x < \frac{\pi}{2}.\]
\[
\sum_{k=0}^{\infty} \left( -\frac{u_l}{k} - \frac{1}{2} \right) \left( \frac{u_r}{k} + \frac{1}{2} \right) \frac{1}{2k+1} \binom{2k}{k}^{-1} (2\sin x)^{2k} \\
= 2F \left( -\frac{u_l}{2}, -\frac{u_r}{2}; \frac{3}{2}; \sin^2 x \right)
\]

\[
\sin ux / u \sin x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}
\]
\[
(7.61) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{\beta+k}{m-n+k} \binom{\nu+k}{k}^{\beta+n+k-1} = \\
\left(\frac{\beta - \nu}{m} \right) \left(\frac{\nu+n}{n} \right)^{\beta+n}, \quad m \leq n
\]

See the proof of (7.26)
\[
(7.62) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{u+x+k}{k} \binom{\nu+x+k}{k}^{\beta+n+k-1} = \\
(-1)^{n} F\left(-n, u+x+1; \nu+x+1; 1\right) = \\
\left(\frac{\nu}{n} \right)^{\beta+n} \left(\frac{\nu+x+n}{n} \right)^{\beta+n}.
\]

From I(10), I(24) and (7.3).
\[
(7.63) \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \binom{\beta+n+k}{k} \left(-\frac{1}{2}\right)^{k} = \\
F\left(-n, n+1; \beta+n+1; \frac{1}{2}\right) = \\
\frac{(\beta/2)!(\beta+n)!}{(\beta+2)!}.
\]

\[\binom{-1/2}{n} = \frac{1}{\Gamma(1/2)} \frac{1}{(x+n)^{n+1/2} (x+n-1/2)^{n}} = \frac{(2n+2x)! x! x!}{x! (x+1)! \cdots (x+n-1)!} y^{-n}\]
\[
(7.68) \sum_{k=0}^{n} \binom{2n}{n+k} \binom{2k}{k} \binom{2n+2k}{n+k}^{-1} \frac{1}{2n+2k+1} = T_{75}
\]

\[
\binom{4.9}{k=0} (-1)^{k} \binom{k}{k} \binom{n+k}{n+k}^{2n+2k+1} = \begin{align*}
\text{for } \frac{2n}{n+1/4} & > 1 \nonumber \end{align*}
\]

\[
(7.7) \sum_{k=0}^{n} \binom{n}{k} \binom{k}{k} \binom{x+k}{x+k}^{-1} = \frac{1}{(x-1/2)^{1/2}} \left( -\frac{x-1}{2} \right)^{1/2} \frac{1}{x} \varepsilon \Delta N.
\]

\[
\binom{x-1/2}{n} \left(-\frac{3}{2}\right)^{-1} = \frac{(x-1/2)(2n)^{-1}(-4)^{n}}{2n+1}.
\]
\[
(7.81) \sum_{k=0}^{n} \binom{n-k}{k} \binom{2n-2k}{2n-k} \frac{1}{(k+1)^{-\frac{1}{2}}} 
= \gamma_n \quad \text{as } n \to \infty
\]

\[
(7.92) \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{k} \binom{\frac{1}{2}}{k} \binom{\frac{1}{2}}{n-k} \frac{1}{(k+1)^{\frac{1}{2}}} 
= \gamma_{2k} \quad \text{as } n \to \infty
\]

\[
\left(\frac{16}{27}\right)^{\frac{n}{2}} \Gamma(\beta) \Gamma(3\beta) \left(\Gamma(2\beta)\right)^{-2}, \quad \beta > 0.
\]
As remarked in the introduction to Part II, p. 115, Karl Dilcher in his paper Hypergeometric functions and Fibonacci numbers, Fib. Q. 38(4), 2000, 342–363 expresses Fibonacci and Lucas numbers as values of hypergeometric functions. Many of his proofs apply hypergeometric transformations such as I(16) and I(17) but also more complicated ones.

Our interest is in the opposite direction: evaluating hypergeometric binomial sums in terms of Fibonacci or Lucas numbers or related polynomials. Our proofs are different. We express the hypergeometric binomial sum in terms of the Fibonacci-like or Lucas-like polynomials \( \Phi \) and \( \Lambda \) defined in Chapter \( \Phi \), using the theory of that chapter. Special values in the polynomials of the argument \( z \) then lead to evaluations in terms of the Fibonacci or Lucas numbers \( F \) and \( L \) or as numerical values.

Because of the application of the theory in Chapters \( F \) and \( \Phi \) a number of Dilcher's identities is collected in this table. Special cases are included in the Proofs and Remarks (pages P) to Table 8.
polyominals are defined, respectively, by

\begin{align}
(8.3) &= \Phi(5) \\
\Phi_n(x) &= \Phi_{n-1}(x) + \frac{x}{2^n-1} \\
&\text{for } x \in \mathbb{C}, \quad n \geq 2 \in \mathbb{N}.
\end{align}

\begin{align}
(8.4) &= \Phi(6) \\
\Phi_0(x) &= \Phi_1(x) = 1, \quad x \in \mathbb{C}
\end{align}
\[(8.5) = \mathcal{D}(7) \quad \Lambda_n(x) = \Lambda_{n-1}(x) + x \Lambda_{n-2}(x), \quad x \in \mathbb{C}, \quad n \geq 2,\]

\[(8.6) = \mathcal{D}(8) \quad \Lambda_x(x) = 2, \quad \Lambda_y(x) = 1, \quad x \in \mathbb{C}. \quad \Phi_n(1) = F_n,\]

\[(8.9) = \mathcal{F}(13) \quad L_n = c_1^n + c_2^n, \quad n \in \mathbb{Z}.\]

\[(8.10) = \mathcal{D}(18) \quad c_1(x) = \frac{1 + \frac{1}{2} \sqrt{1 + 4x}}{1}, \quad c_2(x) = \frac{1 - \frac{1}{2} \sqrt{1 + 4x}}{1},\]

\[(8.11) = \mathcal{D}(26) \quad \Phi_n(x) = (1 + 4x)^{-\frac{1}{2}} \left\{ c_1^{n/2}(x) - c_2^{n/2}(x) \right\}, \quad n \in \mathbb{Z},\]

\[x \neq -\frac{1}{4}, \quad x \neq 0 \text{ when } n < 0.\]

\[\pm \frac{1}{n} (-1/4) = x, \quad n \in \mathbb{Z}.\]

\[(8.14) = \mathcal{D}(96) \quad \Phi_n(-1) = \frac{2}{\sqrt{3}} \sin \left( \frac{(n+1)\pi}{3} \right),\]

\[\Lambda_n(-1) = 2 \cos \frac{n\pi}{3}, \quad n \in \mathbb{Z}.\]
\[(8.16) = \Phi(q) \quad \Phi_n(2) = \frac{1}{3} \left( 2^{n+1} + (-1)^n \right),
\]
\[\Lambda_n(2) = 2^n + (-1)^n, \quad n \in \mathbb{Z}.\]
\[(8.20) \quad (i-2)^n \Lambda_{2n} \left( \frac{1}{i-2} \right) = \]
\[
\left( (i-2)^n \left( \frac{1}{i-2} + \frac{1}{2} \sqrt{1+\frac{4}{(i-2)}} \right)^{2n} + \left( \frac{1}{2} \sqrt{1+\frac{4}{(i-2)}} \right)^{2n} \right) = \]
\[
\left( \frac{\sqrt{i-2} + \sqrt{i+2}}{2} \right)^{2n} + \left( \frac{\sqrt{i-2} - \sqrt{i+2}}{2} \right)^{2n} = \]
\[
\left( \frac{i + i \sqrt{5}}{2} \right)^n + \left( \frac{i - i \sqrt{5}}{2} \right)^n = i^n L_n. \]

*(Complex conjugate of (8.20)*)

\[
(8.21) \quad (i+2)^n \Lambda_{2n} \left( \frac{-1}{i+2} \right) = i^n L_n. \]

\[
(8.22) = \Phi (1/3^a) \quad \Phi_{2m+1} (-1/3) = (-1)^m \Phi_m (-1), \]
\[
\Phi_{2m+1} (-1) = (-1)^m \Phi_m (-1). \]

\[
(8.23) = \Phi (1/4) \quad \Phi_{2m} (-1/5) = \]
\[
(1 - 4/5)^{-1/2} \left\{ \left( \frac{1 + \sqrt{1-4/5}}{2} \right)^{2m+1} - \left( \frac{1 - \sqrt{1-4/5}}{2} \right)^{2m+1} \right\} = \]
\[
5^{-m} \left\{ \left( \frac{\sqrt{5} + 1}{2} \right)^{2m+1} - \left( \frac{\sqrt{5} - 1}{2} \right)^{2m+1} \right\} = 5^{-m} L_{2m+1}. \]

Similarly:
\[(8.27) \quad 5^m \Phi_{2m} \left( \frac{1}{15} \right) = \]

\[\frac{1}{3} \left\{ \left( \frac{3 + \sqrt{5}}{2} \right)^{ \Phi_m + 2} + \left( \frac{3 - \sqrt{5}}{2} \right)^{ \Phi_m + 2} \right\} = \frac{1}{3} \left[ \Phi_{m+2} \right] \]

\[\left( 1 - \frac{4}{9} \right)^{\frac{1}{2}} \left\{ \left( \frac{1 + \sqrt{1 - 4/9}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{1 - 4/9}}{2} \right)^{n+1} \right\} = \]
\[
\left( \frac{3+\sqrt{5}}{2} \right)^{n+1} - \left( \frac{3-\sqrt{5}}{2} \right)^{n+1} \right\} = T_{07}
\]

Similarly,
\[
(8.32) \quad \Lambda_n (-\frac{1}{4}) = 3^{-n} L_{2n},
\]

For \( \sqrt{9+\sqrt{5}} \quad \sqrt{9} \cdot \sqrt{5} \)

\[
(8.36) \quad \gamma_{2m}^3 \left( \Phi_{\mu m+1} \left( -\frac{1}{\sqrt{2}} \right) = \frac{1}{2} \gamma_{2m}^3 \right) = L_{\mu m+1};
\]

\[
(8.38) \quad \left( -\frac{1}{2} \right) \Phi_{\mu m+3} \left( -\frac{1}{\sqrt{2}} \right) = 5^{1/2} F_{2m+1}.
\]
Proof of (8.33) - (8.38). We have no proof by Binet's theorem. So we have recourse to the recurrences (8.1) - (8.5) for $F_n$, $L_n$, $Q_n$ and $\Lambda_n$. By the theorem on p. 595 the sequences $\Phi_{ym+i}(x)$ and $\Phi_{ym+j}(x)$, $m \in \mathbb{Z}$, satisfy the recurrence

$$U_m(x) = \Lambda_y(x) U_{m-1}(x) - (xy)^4 U_{m-2}(x).$$

So $\Lambda_{ym+i}(-1/y_k)$ and $\Phi_{ym+j}(-1/y_k)$ satisfy

$$U_m = \Lambda_y(-1/y_k) U_{m-1} - y_k^{-4} U_{m-2}, \quad k = 1, 2.$$

It follows that the l.h. sides of (8.33) - (8.38) satisfy the recurrence

$$w_m = y_k^2 \Lambda_y(-1/y_k) w_{m-1} - w_{m-2}, \quad k \in \mathbb{Z}.$$

We have, see p. 595, with $y_k = 2 \pm \sqrt{5}$,

$$y_k^2 \Lambda_y(-1/y_k) = y_k^2 - 4y_k + 2 = 3, \quad k = 1, 2.$$

So the recurrence (8.39) is the same as

$$w_m = 3w_{m-1} - w_{m-2}. $$
\[ X_m = \Lambda \left( 1 \right) X_{m-1} - X_{m-2} = \]

\[ \Lambda \left( 1 \right) X_{m-1} - X_{m-2} = 3 \left( X_{m-1} - X_{m-2} \right) \cdot \]

Since both sides in (8.22) \((8.30)\) are equal for some \(m = s\) and \(m = s+1\).

We list both sides of (8.33) - (8.38) for such \(m\) values. This proves (8.33) - (8.38)

\[
\binom{m}{0} \cdot \binom{m}{1} \sim \frac{1}{\gamma_2} \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) = \frac{1}{\gamma_2} \left( \gamma_1 - \gamma_2 \right) = -\frac{\gamma_2}{\gamma_1} \]

\[ = -5^{1/2} E_2 \cdot \]

\[
\binom{m-2}{0} \cdot \binom{m-2}{1} \sim -\frac{1}{\gamma_2} \left( -\frac{1}{\gamma_2} \right) = -\frac{1}{\gamma_2} \left( -\frac{1}{\gamma_2} \right) = 2 = L_0 \cdot \]

\[ \binom{m-2}{0} \cdot \binom{m-2}{1} \sim -\frac{1}{\gamma_2} \gamma_2 = \frac{\gamma_2}{\gamma_2} = 1 \sim 1 \cdot \]

\[ \sim \frac{1}{\gamma_2} \gamma_2 = \frac{1}{\gamma_2} \cdot \gamma_2 = 1 \sim 1 \cdot \]
\[ (8.3.6) \quad m = 0: \quad \Phi_{\frac{\lambda}{2}} (-\frac{1}{\gamma}) = \Phi_{\frac{\lambda}{2}} (-\frac{1}{\gamma_2}) = 1 = L_1. \]

\[ m = 1: \quad \gamma_1 \Phi_{\frac{\lambda}{3}} (-\frac{1}{\gamma}) = \gamma_1 - 4 \gamma_1 + 3 = y = L_3. \]

\[ \gamma_2 \Phi_{\frac{\lambda}{5}} (-\frac{1}{\gamma_2}) = \gamma_2 - 4 \gamma_2 + 3 = y = L_3. \]

\[ (8.3.7) \quad m = 0: \quad -\gamma_2 \Phi_{\frac{\lambda}{3}} (-\frac{1}{\gamma_2}) = -\gamma_2 (1 - \frac{2}{\gamma_2}) = -\gamma_2 + 2 = \sqrt{5} = 5^{\frac{1}{2}} F_1. \]

\[ m = 1: \quad (-\gamma_1) \Phi_{\frac{\lambda}{3}} (-\frac{1}{\gamma}) = 0, \quad 5^{\frac{1}{2}} F_1 = 0. \]

\[ (8.3.8) \quad m = 0: \quad -\gamma_2 \Phi_{\frac{\lambda}{3}} (-\frac{1}{\gamma_2}) = -\gamma_2 (1 - \frac{2}{\gamma_2}) = -\gamma_2 + 2 = \sqrt{5} = 5^{\frac{1}{2}} F_1. \]

\[ m = 1: \quad (-\gamma_1) \Phi_{\frac{\lambda}{3}} (-\frac{1}{\gamma_2}) = 0, \quad 5^{\frac{1}{2}} F_1 = 0. \]
\[
\sum_{k=0}^{\infty} \binom{u}{k} \binom{u-1/2}{k} \binom{k+1/2}{k}^{-1} z^{2k} =
\]

Special cases: see p. P139

(8.43) \[
{}_{2}F_{1}(-u, \frac{1}{2} - u, \frac{1}{2}; z^{2}) =
\sum_{k=0}^{\infty} \binom{u}{k} \binom{u-1/2}{k} \binom{k-1/2}{k}^{-1} z^{2k} =
\sum_{k=0}^{\infty} \binom{u}{k} \binom{1}{u-1/2} \binom{2k}{1}^{-1} z^{2k}
\]
\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (\frac{2}{3})^k (-\frac{4}{3})^k (1-\frac{2}{3})^k = \frac{1}{4(u+1)z} \left\{ (1+z)^{2u+2} - (1-z)^{2u+2} \right\}
\]
\[
(8.47) \quad _{2} F_{1} \left( -n, n+1; \frac{1}{2}; z \right) = \\
\sum_{k=0}^{n} (-1)^{k} \binom{n+1}{k} \binom{n+k+1}{k} \left( \frac{1}{2} \right)^{k} z^{k} \\
= \sum_{k=0}^{n} \frac{(-1)^{k}}{n+k+1} \binom{n+k+1}{2k+1} (-yz)^{k} = \\
\frac{(-yz)^{n-1}}{n} \Phi_{2n-1} \left( -\frac{1}{yz} \right), \quad n \geq 1.
\]

\[
(8.48) \quad _{2} F_{1} \left( -n, n+1; \frac{3}{2}; z \right) = \\
\sum_{k=0}^{n} (-1)^{k} \binom{n+1}{k} \binom{n+k}{k} \left( \frac{3}{2} \right)^{k} z^{k} \\
= \sum_{k=0}^{n} \frac{1}{n+k+1} \binom{n+k}{2k+1} (-yz)^{k} = \\
\frac{(-yz)^{n}}{2n+1} \Phi_{2n+1} \left( -\frac{1}{yz} \right), \quad n \geq 0.
\]

\[
(8.49) \quad _{2} F_{1} \left( -n, n+1; \frac{1}{2}; z \right) = \\
\sum_{k=0}^{n} \binom{n+k}{k} \binom{n+k+1}{k} \left( k - \frac{1}{2} \right)^{-1} (-z)^{k} = \\
\]
\[
\sum_{k=0}^{n} \binom{n+k}{k} \left( -\frac{1}{2} \right)^k (-yz)^k = (-yz)^n \Phi_{2n} \left( -\frac{1}{y} z \right),
\]

(8.50) \[
\sum_{k=0}^{n} \binom{n+k-1}{k} \left( -\frac{1}{2} \right)^k (-z)^k = \frac{1}{2} (-yz)^n \Lambda_{2n} \left( -\frac{1}{y} z \right), \quad n \geq 1.
\]

(8.51) \[
\sum_{k=0}^{n} \binom{n}{k} \left( \frac{1}{2} \right)^k (-n-k)^{-1} \left( -\frac{1}{y} z \right)^k = \Lambda_{2n+1} \left( -\frac{1}{y} z \right).
\]

(8.52) \[
\sum_{k=0}^{n-1} \binom{n-1}{k} \left( \frac{1}{2} \right)^k \left( -\frac{1}{y} z \right)^k = \Phi_{2n-1} \left( -\frac{1}{y} z \right), \quad n \geq 1.
\]
\[ (8.53) \quad {}_2F_1 \left( a, -a; \frac{1}{2}; -z^2 \right) = \quad \]

\[ \sum_{k=0}^\infty \left( a+k-1 \right) \left( a \right) \binom{a}{k} \binom{a-1}{k} \zeta^{2k} = \quad \]

\[ \sum_{k=0}^\infty \frac{a}{a+k} \binom{a+k}{2k} (2z)^{2k} = \quad \]

\[ \frac{1}{2} \left( z + \sqrt{1 + z^2} \right)^{2a} + \frac{1}{2} \left( -z + \sqrt{1 + z^2} \right)^{2a}, \quad |z| < 1. \]

\[ (8.54) \quad {}_2F_1 \left( a, 1-a; \frac{3}{2}; -z^2 \right) = \quad \]

\[ \sum_{k=0}^\infty \left( a+k-1 \right) \left( a-1 \right) \binom{a-1}{k} \binom{a+k}{k} \zeta^{2k} = \quad \]

\[ \sum_{k=0}^\infty \frac{1}{a+k} \binom{a+k}{2k+1} (2z)^{2k} = \quad \]

\[ \frac{1}{2z(2a-1)} \left[ \left( z + \sqrt{1 + z^2} \right)^{2a-1} - \left( -z + \sqrt{1 + z^2} \right)^{2a-1} \right]. \quad |z| < 1, \text{ last equality } a \neq \frac{1}{2}. \]

\[ (8.55) \quad {}_2F_1 \left( a, 1-a; \frac{1}{2}; -z^2 \right) = \quad \]

\[ \sum_{k=0}^\infty \left( a+k-1 \right) \left( a-1 \right) \binom{a-1}{k} \binom{a+k}{k} \zeta^{2k} = \quad \]

\[ \sum_{k=0}^\infty \frac{a+k-1}{2k} (2z)^{2k} = \quad \]
\[
\frac{1}{2} \left( \frac{1}{z} \right)^{-\frac{1}{2}} \left[ \left( \frac{1}{z} + \frac{1}{z^2} \right)^{\frac{1}{2}} + \left( -\frac{1}{z} + \frac{1}{z^2} \right)^{\frac{1}{2}} \right]^{2a-1} = \left( \frac{1}{z} \right)^{-\frac{1}{2}} \left( \frac{1}{2} \sqrt{1 - \frac{1}{z^2}} \right)^{2a+1} \]
\[
|z| < 1.
\]

(8.5.6) \[ \sqrt{F} \left( -a, -a + \frac{1}{2}; z \right) = \sum_{k=0}^{\infty} \frac{a}{k!} \frac{(a-\frac{1}{2}) \cdots (2a-2k)\cdots}{k!} \left( \frac{1}{z} \right)^k = \sum_{k=0}^{\infty} \frac{2a-k}{k!} \left( -\frac{1}{2} z \right)^k = \frac{1}{\left( 1 - \frac{1}{2} \sqrt{1 - z} \right)^{2a+1}} \]

\[ |z| < 1, \quad a \not\in \mathbb{N}_0. \]
TABLE 10 REMNANTS

(10.1) \[ \sum_{k=0}^{n} \binom{n}{k} k \binom{n}{k}^3 = \frac{n}{2} \sum_{k=0}^{n} \binom{n}{k}^4 \]

(10.2) \[ \sum_{k=0}^{3n} (-1)^k \binom{3n}{k} \left( \frac{x+3n-k}{n} \right)^3 = \Delta_{3n}^3(x) = \frac{(3n)!}{(n!)^3} \]

(10.3) \[ \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} \sum_{j=0}^{k} (-1)^j \binom{n}{j}^2 \binom{k}{j} = \sum_{k=0}^{n} \binom{n}{k}^4 \]

Special case of general summation formula:

(10.4) \[ \sum_{k=0}^{3n} (-1)^k \binom{3n+1}{k} \left( \frac{x+3n-k}{n} \right)^3 = (-1)^n \binom{x-1}{n}^3 \]

(10.5) \[ \sum_{k=0}^{m+r} (-1)^k \binom{m+r}{k}^{-1} \sum_{j=0}^{k} \left( \binom{m}{j} \binom{r}{k-j} \right)^2 = S(m, r) \]

\[ = 0, \quad m \neq r; \quad = 1, \quad m = r. \]

(10.6) \[ \sum_{k=0}^{m+r} (-1)^k \sum_{j=0}^{k} \binom{m}{j} \binom{r}{k-j} = 0, \quad \text{m or r odd,} \]

\[ = (-1)^{m+r} \binom{2p}{p} \binom{2q}{q}, \quad m = 2p, \quad r = 2q, \]

\[ p \in \mathbb{N}, \quad q \in \mathbb{N}. \]
\[(10.7) \sum_{j=0}^{n} \left( \frac{c-a-b}{n-j} \right) \left( \frac{a-c}{j} \right) \left( \frac{b-c}{j} \right) \left( \frac{-c}{j} \right)^{-1} = \left( \frac{-a}{n} \right) \left( \frac{-b}{n} \right) \left( \frac{-c}{n} \right)^{-1}, \]

\[(10.8) \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} \binom{y}{k} \binom{z}{k} \left( \frac{x+y+n}{k} \right)^{-1} = \left( \frac{x+z+n}{n} \right) \left( \frac{y+n}{n} \right) \left( \frac{x+y+n}{n} \right)^{-1}, \]

\[(10.9) \sum_{k=0}^{n} \binom{x}{k} \binom{y}{k} \binom{z}{n-k} \binom{x+y+z}{k}^{-1} = \left( \frac{x+z}{n} \right) \left( \frac{y+z}{n} \right) \left( \frac{x+y+z}{n} \right)^{-1}, \]

\[(10.10) \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} \binom{y}{k} \binom{z}{n+k} \left( \frac{1}{y+k} \right)^{-1} = \frac{1}{2} + \frac{1}{2} \left( \frac{2n+2y}{2n} \right) \left( \frac{n+y}{n} \right)^{-2}, \]

\[(10.11) \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{y}{k} \binom{z}{n+k} \left( \frac{1}{y+k} \right)^{-1} = \frac{1}{2} + \frac{1}{2} \left( \frac{y+n}{n} \right)^{-1}. \]
\[(10.12) \sum_{k=0}^{n} \binom{n}{k} \binom{y-1}{k} \left(\frac{y+k}{k}\right)^{-1} \binom{n+1+k}{k}^{-1} = \]
\[
\frac{1}{2} \frac{(n+y)^{-1}}{n} \left(\frac{n+y}{n+1}\right)^{-1} \left(\frac{2n+2y}{2n+1}\right) .
\]

\[(10.13) \text{When } a + b = c + d \]
\[
(ab - cd) \sum_{k=0}^{n} \binom{a+k}{k} \binom{b+k}{k} \left(\frac{c+k+1}{k}\right)^{-1} \left(\frac{d+k+1}{k}\right)^{-1} = \\
\binom{a+n+1}{n+1} \binom{b+n+1}{n+1} \left(\frac{c+n+1}{n+1}\right)^{-1} \left(\frac{d+n+1}{n+1}\right)^{-1} .
\]
\[(6.1) \text{ and } (6.3). \text{ From } S(3) \text{ we have for the Legendre polynomial } P_n\]

\[
(1-x^2)^n P_n \left( \frac{1+x}{1-x} \right) = -\sum_{k=0}^{n} \binom{n}{k} (2x)^k \frac{2x^n}{1-x} \sum_{k=0}^{n-1} \binom{n}{k} (2x)^k \frac{2x^n}{1-x} \]

\[
(1+x)^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} = (1+x)^n \sum_{h=0}^{n} \binom{n}{h} x^h \]

\[
\sum_{m=0}^{2n} x^m \sum_{i=0}^{m} \binom{n}{i} \binom{n}{m-i} \]

So \(\sum_{i=0}^{m} \binom{n}{i} \binom{n}{m-i}\) is the coefficient of \(x^m\) in \((1-x^2)^n P_n (1+x)/(1-x)\). So we must have

(a) \(\sum_{i=0}^{m} \binom{n}{i} \binom{n}{m-i} = 0\) \(m > 2n\)

This follows also from the fact that the nonzero terms in the sum (a) satisfy

\([m-n] \leq i \leq m n\).

For \(n=m\) we see that (6.1) is the coefficient of \(x^n\) in \((1-x^2)^n P_n (1+x)/(1-x)\)

\(\text{Cf. } (6.40). \)

(6.8) and (6.10). The identity (6.8) follows from (6.10) with \( n = 0 \). A short direct proof: Ekhad (1990). See also (6.40) with \( x = -1 \).

The two last equalities in (6.10) follow with \( D(24) \) and \( D(14) \).

Denoting the l.h.s. of (6.10) by \( S_n(x) \), we have

\[
S_n(x) = 2^{n-h} \binom{n-h}{h} \binom{x}{h} = (-1)^h S_n(x),
\]

so that \( S_n(x) = 0 \) for \( n \) odd.

The difficult part was proved by Gessel and Stanton (1985) using the following lemma. Let \( f(x,y) \) be a Laurent series containing finitely many terms with one or two negative exponents. Then

\[
C_{00} f \left(\frac{x}{1+y}, \frac{y}{1+x}\right) = C_{00} (1-xy)^{-1} f(x,y).
\]

Here, \( C_{pq} g(x,y) \) denotes the coefficient of \( x^p y^q \) in \( g(x,y) \).

For \( f(x,y) = x^{-p} y^{-q} (1-xy)^{-n} (x-y)^m \) with \( p \in \mathbb{N}_0 \), we have

\[
C_{00} f \left(\frac{x}{1+y}, \frac{y}{1+x}\right) = C_{00} x^{-p} y^{-q} (1+x)^{-k} (1+y)^{-k} (x-y)^{-n}.
\]
\[ \left( 1 + x \right)^b \left( 1 + y \right)^p (x - y)^n = \]

\[ C_{pp} \sum_{i=0}^{p} \sum_{j=0}^{b} \binom{p}{i} \binom{b}{j} \sum_{k+h=n} (-1)^k \frac{n!}{k! h!} x^{i+h} y^{j+k} = \]

\[ \sum_{k+h=n, k \leq p, h \leq b} (-1)^k \binom{b}{k-p} \binom{p}{h} \frac{n!}{k! h!} = \]

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{p}{p-k} \binom{p}{n-k} \] Also,

\[ C_{oo} (1-xy)^{-1} f(x,y) = C_{oo} x^{-k} y^{-p} (1-xy)^{n-1} (x-y)^n = \]

\[ C_{pp} (1-xy)^{-n-1} (x-y)^n = \]

\[ C_{oo} \sum_{i=0}^{\infty} \binom{n+i}{n} x^i y^i \sum_{k+h=n} (-1)^k \frac{n!}{k! h!} x^h y^k. \]

Here we have to take \( i+h = i+k = p \). So we must have \( h = k \), so \( n \) should be even. For \( n = 2m \) we have \( h = k = m \), \( i = p - m \) and

\[ C_{oo} (1-xy)^{-1} f(x,y) = (-1)^m \binom{m+p}{m} \binom{2m}{m}, \quad p \geq m, \]

and also for \( p < m \) since then there is no \( i \) with \( i = p - m \). The second equality in (6.10) now follows for \( x = p \in \mathbb{N} \) and then for \( x \in \mathbb{C} \) since both sides are polynomials.
Other proofs are in Ljunggren (1947), Knacke (1903), Stever (1947). See also Egorovchuk (1984), Kvamsdal (1942), Fjeldstad (1954).

The proofs of some other identities are based on (6.10), e.g. (6.10) \Rightarrow (6.11).
For (6.12) see also (6.40) with \( x = -1 \), giving finally a different proof.

\[(6.12)\] The last equality is (C2). From \( G(28) \) and \( G(38) \) we have with \( D(24) \)
\[
\Delta^n_y \left( \frac{y-x-1}{n} \right) \left( \frac{x+y}{n} \right) = \\
\sum_{k=0}^{n} \binom{n}{k} \left( \frac{y-x-1}{n-k} \right) \left( \frac{x+y+k}{k} \right) = \\
\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left( \frac{x-y+n-k}{n-k} \right) \left( \frac{x+y+k}{k} \right) = \\
\sum_{h=0}^{n} (-1)^h \binom{n}{h} \left( \frac{x-y+h}{n-h} \right) \left( \frac{x+y+n-h}{n-h} \right),
\]
and also
\[
\Delta^n_y \left( \frac{x+y}{n} \right) \left( \frac{y-x-1}{n} \right) = \\
\sum_{k=0}^{n} \binom{n}{k} \left( \frac{x+y}{n-k} \right) \left( \frac{y+k-x-1}{k} \right) = 
\]
\[ \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} u \\ k \end{array} \right) z^k = \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j \end{array} \right) (n+j-1) (z-1)^j. \]

When we replace \( z \) by \(-\Delta\) we obtain
\[ \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \left( \begin{array}{c} u \\ k \end{array} \right) \Delta^k = \sum_{j=0}^{n} (-1)^j \left( \begin{array}{c} n \\ j \end{array} \right) (n+j-1) E^j. \]

Application of these operators to the function \( \binom{x}{M} \) proves (6.13), see G(38).

(6.15) The first equality follows from (6.14) by replacing \( n \) by \( 2n \) and taking \( x=2n \), \( M=n \).

Equality of first and third member follows by taking \( a=b=x=M=2n \) in (6.20).

The identity (6.15) is given in Gould (1972) as (6.25).
(6.16) This is (6.29) in Gould (1972).

With \( D(16) \) and \( D(14) \) the l.h.s. is equal to

\[ \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} \binom{x}{n-k}. \]

Now (6.10).

(6.17) The identity (6.17), with \( \alpha = u + t \), may be written as

\[ \sum_{k=0}^{n} (-1)^{k} \binom{u+t+1}{k} \binom{-t-1}{n-k} = \sum_{k=0}^{n} (-1)^{k} \binom{u}{k} \binom{t}{n-k}. \]

By Newton's interpolation formula (6.41) the l.h.s. in (\( \star \)) may be written as

\[ \sum_{k=0}^{n} c_k \binom{u}{k}, \]

where, with (6.38),

\[ c_k = \Delta^e \binom{u}{k} \sum_{k=0}^{n} (-1)^{k} \binom{u+t+1}{k} \binom{-t-1}{n-k} |_{u=0} = \sum_{k=0}^{n} (-1)^{k} \binom{u+t+1}{k} \binom{-t-1}{n-k} = \sum_{h=0}^{n-r} (-1)^{h+r} \binom{x}{h} \binom{2t+1}{n-r-h} = \sum_{j=0}^{n-r} (-1)^{n-j} \binom{-t-1}{j} \binom{2t+1}{n-r-j}. \]

From (6.11) with \( m = n-e, x = y = -t-1 \) and \( D(24) \)
\[ c_t = (-1)^t \left( -t - 1 + n - \frac{\alpha}{2} \right)^\frac{\alpha}{2} = (-1)^t \left( \frac{t}{n - \frac{\alpha}{2}} \right)^2. \]

\[ (6.18) \text{ From } (6.17) \text{ with } a + t + 1 = x, \]
\[ a - t = y. \]

\[ (6.19) \text{ Proof by operator identity, see } p. \text{II-18. } \]

\[ \sum_{2k \leq n} \left( \begin{array}{c} n \\ 2k \end{array} \right) \left( \frac{n}{2k} \right)^{-1/2} z^k = \]
\[ (-1)^n \sum_{j=0}^{n} \left( \begin{array}{c} n-j \\ j \end{array} \right) \left( \frac{n-j}{n-j} \right) 2^{n-j} (z+1)^j, \]

replace \( z \) by \( \Delta \) and apply the resulting operator identity to the function \( (x)_m \).

\[ (6.20) \text{ Proof by operator identity, see } p. \text{II-18. } \]

\[ \sum_{k=0}^n \left( \begin{array}{c} \alpha \\ k \end{array} \right) (n-k)^{b-k} z^k = \sum_{j=0}^n \left( \begin{array}{c} \alpha \end{array} \right) \left( \frac{a+b-j}{n-j} \right) (z-1)^j \]

replace \( z \) by \(-\Delta\), giving

\[ \sum_{k=0}^n (-1)^k \left( \begin{array}{c} \alpha \\ k \end{array} \right) (n-k) \Delta^k = \sum_{j=0}^n (-1)^j \left( \begin{array}{c} j \end{array} \right) \left( \frac{a+b-j}{n-j} \right) E^j. \]

Applying this identity to the function \( (x)_m \),
we find \( (6.20) \) with \( G(38) \).
(6.21) Proof by operator identity, see pp. 13-18. In (3.94):

\[ \sum_{k=0}^{n} \binom{n}{k} \binom{a}{n-k} u^k v^{n-k} = \]

\[ \sum_{j \leq n} \binom{n-j}{j} \binom{a}{n-j} (uv)^j (u+v)^{n-2j} = \]

\[ \sum_{j \leq n} \binom{a}{j} \binom{a-j}{n-2j} (uv)^j (u+v)^{n-2j} \]

Replace \( u \) by \( \Delta \) and \( v \) by \( I \) and apply the resulting operator identity to the function \( f_M \).

(6.22) Proof by operator identity, see pp. 13-18. In (3.100):

\[ \sum_{h=0}^{n} \binom{n}{h} \binom{a}{n-h} (-\lambda)^h = \]

\[ \sum_{k=0}^{n} \binom{-a-1}{k} \binom{a}{n-k} \lambda^k (\lambda+1)^{n-k} \]

Replace \( \lambda \) by \( \Delta \) and apply the resulting operator identity to the function \( f_M \).

See (38).

(6.23) The first equality follows from (6.20) with \( n=m, a=-u-1, b=m+u+1, M=N \) and from D(24).

A proof by operator identity, see pp. 13-18, starts from the relation
\[ \sum_{k=0}^{m} \binom{u+k}{k} \binom{m+u+1}{m-k} z^k = \]

\[ \sum_{j=0}^{m} \binom{u+j}{j} (1-z)^j = \sum_{i=0}^{m} \binom{m+u+1}{m-i} z^i (1-z)^{m-i} \]

Proof: The second member is

\[ \sum_{j=0}^{m} \binom{u+j}{j} \sum_{k=0}^{j} \binom{j}{k} (-1)^k z^k = \]

\[ \sum_{k=0}^{m} (-1)^k z^k \sum_{j=k}^{m} \binom{u+j}{j} \binom{j}{k} = \]

\[ \sum_{k=0}^{m} (-1)^k \binom{u+k}{k} \binom{u+m-1}{m-k} z^k, \]

with (3.487) or by canceling and rearranging factorials and applying (1.317).

The third member is equal to

\[ \sum_{i=0}^{m} \binom{m+u+1}{m-i} \sum_{h=0}^{m-i} \binom{m-i}{h} (-1)^h z^i h = \]

\[ \sum_{k=0}^{m} z^k \sum_{i+h=k} \binom{m+u+1}{m-i} \binom{m-i}{h} \binom{i}{h} = \]

by canceling and rearranging factors.

Replace \( z \) by \(-1\) in \((\cdot )\) and apply the resulting operator identity to the function \((\cdot )\).
(6.25) The l.h.s. may be written as
\[
\sum_{k=1}^{n} \left( \frac{a_k}{k} \right) \left( \frac{2n-2\epsilon+k}{n-2\epsilon+k} \right)^x =
\]
\[
\sum_{k=1}^{\frac{n}{2}} \left( \frac{a_k}{k} \right) \left( \frac{a_n-2\epsilon}{n-2\epsilon+k} \right)^x
\]

For \( \epsilon < n \) the change of summation boundary is trivial. For \( \epsilon > n \) we must have \( n-2\epsilon+k \leq 2n-2\epsilon \), i.e., \( k \leq n \).

The identity now follows from (6.33) with \( n = x, x = x = n-x \).

(6.26) With D(24), D(14) and Vandermonde's convolution the l.h.s. is equal to
\[
(-1)^x \binom{x}{j} \sum_{k=0}^{j-x} \binom{n}{k} \left( \frac{j-k}{j-\epsilon} \right) = (-1)^x \binom{x}{j} \binom{n+j}{j-\epsilon}
\]

(6.27) By D(24) we may write (6.27) as
\[
\sum_{k=0}^{j-x} \binom{n}{k} \frac{x+k}{x} \left( \frac{x-n-\epsilon}{j-\epsilon} \right) = \binom{j+n}{j-\epsilon} (x).
\]

Since both sides here are polynomials in \( x \) of degree \( \leq j \), it is sufficient to prove this identity for \( x = \{0, 1, \ldots, j\} \).

For \( x = p < \epsilon \), both sides vanish by D(12).
For $x = z$ the l.h.s. equals

$$\sum_{k=0}^{j-z} \binom{n}{k} \binom{j-n-k}{j-z-k} = \binom{j}{j-z} = \delta_{j-z}.$$  

By Vandermonde's Convolution (26), so that it is equal to the r.h.s. of (x).

For $x = p$ with $z < p \leq j$ the l.h.s. of (x), by Vandermonde's Convolution (26) and by (3.111) is equal to

$$\sum_{k=0}^{j-z} \binom{n}{k} \binom{p-z-n}{j-z-k} \binom{p+k}{p} =$$

$$\sum_{k=0}^{j-z} \binom{n}{k} \binom{p-z-n}{j-z-k} \sum_{i=0}^{p-z} \binom{k}{i} \binom{p-i}{p-z-i} =$$

$$\sum_{i=0}^{p-z} (p-i) \sum_{k=0}^{j-z} \binom{n}{k} \binom{p-z-n}{j-z-k} \binom{k}{i} =$$

$$\sum_{i=0}^{p-z} (p+i) \binom{n}{i} \binom{p-z-i}{j-z-i}$$

For $p \leq j$ the third factor here is zero.

For $p = j$ the last sum equals

$$\sum_{i=0}^{j-z} \binom{n}{i} \binom{j}{k+i} = \sum_{i=0}^{j-z} \binom{n}{i} \binom{j-z-i}{j-i} = \binom{j+n}{j-i} \binom{j}{i}.$$
(6.28), (6.29) Proof by Gessel and Stanton (1986). In the lemma on p. 129,

\[ f(x, y) = x^{-b} y^{-q} (1+x)^a (1+y)^b (x-y)^n \]

with \( p, q \in \mathbb{N} \). Then

\[ f\left(\frac{x}{1+y}, \frac{y}{1+x}\right) = x^{-b} y^{-a+q} (1+x)^a (1+y)^b (x-y)^n \]

and the lemma gives

\[ (1) \ C_{pq}^n (1+x)^{a+q} (1+y)^{b+p} (x-y)^n = C_{pq} (x-y)^n (1-x-y)^{-a-b-n} \]

For (6.28) we take \( a = b = 0 \) in (1), giving

\[ (2) \ C_{pq} (1+x)^q (1+y)^p (x-y)^n = C_{pq} (x-y)^n (1-x-y)^{-n-1} \]

We have

\[ (1+x)^q (1+y)^p (x-y)^n = \sum_{i=0}^{q} \sum_{j=0}^{p} \sum_{k=0}^{n} \binom{q}{i} \binom{p}{j} (-1)^k (n) x^{i+n-k} y^{j+k} \]

For \( C_{pq} \) we must have \( i+n-k = p, j+k = q \), so \( n-k \leq p, k \leq q \). So
(3) \[ C_{pq} \left(1+x\right)^q \left(1+y\right)^p \left(x-y\right)^n = \sum_{k=\left[n-p\right]^+}^{q+n} (-1)^k \binom{n}{k} \binom{p}{q-k} \binom{q}{p-n+k}, \]

where the sum may be empty.

Also, with \( D(25) \),
\[ (x-y)^n \left(1-xy\right)^{n-1} = \sum_{i=0}^{\infty} \binom{n+i}{i} \sum_{\alpha=0}^{n-i} (-1)^{n-i} \frac{(i+x)^\alpha}{\alpha!} \frac{1}{\beta^i}, \]

For the coefficient of \( x^k y^l \) we must have \( i+\beta = p \), \( i+\alpha = q \). So a necessary condition for \( C_{pq} \neq 0 \) is

(4) \[ p+q-n = 2r \quad \text{with} \quad r \in \mathbb{N}_0, \]

and then \( i=r, \beta = p-r, \alpha = q-r \). Therefore, also necessary for \( C_{pq} \neq 0 \) is

(4\textsuperscript{a}) \[ p \geq r, \quad q \geq r, \quad \text{So} \]

(5) \[ C_{pq} \left(1+x\right)^q \left(1+y\right)^p \left(x-y\right)^n = (-1)^{q-r} \binom{n+r}{r} \frac{n!}{(q-r)! (p-r)!}, \]

under (4) and (4\textsuperscript{a}) and zero otherwise.

Then (6.28) follows from (2), (3) and (5).
For (6.29) we take \( n = 0 \) in (1):

\[
(6) \quad C_{pq}^{\alpha+q} (1+x)^{\alpha} (1+y)^{b+1} = C_{pq}^{\alpha} (1+x)^{\alpha} (1+y)^{b} (1-xy)^{-a-b-1}
\]

With \( D(25) \)

\[
(1+x)^{\alpha} (1+y)^{b} (1-xy)^{-a-b-1} = \sum_{i} \sum_{j} \sum_{k} \frac{(\alpha) (b)}{\alpha} (-a-b-1) (-1)^{i+k+j+k} x^{i+k} y^{j+k}
\]

For the coefficient of \( x^i y^j \) we must have \( i + k = p \), \( j + k = q \). So \( k \leq p \) and

\[
(7) \quad C_{pq}^{\alpha} (1+x)^{\alpha} (1+y)^{b} (1-xy)^{-a-b-1} = \sum_{k=0}^{p \land q} (-1)^{k} (p-k)(q-k) (-a-b-1)
\]

Since \( C_{pq}^{\alpha} (1+x)^{\alpha} (1+y)^{b+1} = \left( \frac{\alpha+q}{p} \right) \left( \frac{b+1}{q} \right) \)

the identity (6.29) follows with (6) and (7).

For (6.29) see Gould (1977b), Kreweras (1967).

(6.30), (6.31) In (6.28) take \( p = u + t \), \( q = v + r \) where \( u, v, w, t, u, v, w \in \mathbb{N}_0 \) and \( u + v + w = n \). Then \( p + q = n + 2r \) and \( t \leq q, t \leq p \). So the conditions for nonzero sum in (6.28) are satisfied and we have

\[
(6.30), (6.31)
\]
\[
\sum_{k} \frac{(-1)^{k}}{(u+k)\binom{k}{u+k} (v+k)\binom{k}{v+k} \binom{v}{u+v+k}} = \frac{1}{u!v!} \frac{(u+v+k)_k}{(u+v)_k} \]

where the summation bounds are,

from (6.2): \( k \geq 0, k \leq u+v, k \leq u+v+2 \),

and from the last two factors in the summand: \( k \geq u, k \leq 2v \). Now put \( h = k-u \), then we obtain (6.30), the summation bounds now being \( |h| \leq u, |h| \leq v, |h| \leq 2 \). A short proof of (6.30): Ekhad (1989).

(6.31) is a version of (6.30) with somewhat lesser cyclical symmetry.

The summands of (6.30) and (6.31) are identical, as is seen by writing the binomial coefficients as quotients of factorials, see D(14).

(6.32) follows from (6.31) since

\[
\binom{2u+k}{u+k} \binom{2v+k}{v+k} \binom{2r+k}{r+k} = \frac{(2u)_r (2v)_r (2r)_r}{(u+v)_r (u+v+2)_r (2r+1)}
\]

\[
\cdot \frac{(u+v)_r (v+r)_r (u+k)_r (v+k)_r (r+k)_r}{(u+k)_r (v+k)_r (r+k)_r}, \text{ of G.K.R. (1989)}
\]

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(6.33) From (6.32) by \( u = n, \ v = x, \ z = z \) by putting \( h = n - k \).

This is (6.1) in Gould (1972) which suggests \( x \in \mathbb{C}, \ z \in \mathbb{C} \) with notation as in (16).

It is (3.20) in Roy (1987) where it is proved with Dixon’s identity for the hypergeometric function \( \phi_2 \).


(6.34) Proof by operator identity, see (3.18).

In (3.181):

\[
\sum_{h=0}^{m} \binom{m}{h} \binom{\alpha}{n-h} z^{m-h} = \sum_{j=0}^{m} \binom{m}{j} \binom{m+a-j}{n} (z-1)^j,
\]

replace \( z \) by \( -\Delta \), so that \( z-1 \) is replaced by \( -E \). Application of this operator identity to the function \( \binom{x}{n} \) gives (6.34), by (38).
(6.40) With \( D(13) \) the r.h.s. is equal to

\[
\sum_{i=0}^{\left(\begin{array}{c}n\vspace{12pt}i\end{array}\right)} \frac{(n+i)!}{(j!)^3(n-2j)!} \sum_{i+j+1\leq n} \left(\begin{array}{c}n-2j\vspace{12pt}i\end{array}\right) x^i =
\]

\[
\sum_{\tau=0}^{n} x^\tau \sum_{j=0}^{n_\tau \wedge (n-\tau)} \frac{(n+j)!}{(j!)^3(n-j)!(n-\tau-j)!} =
\]

\[
\sum_{\tau=0}^{n} x^\tau \sum_{j=0}^{n_\tau \wedge (n-\tau)} \left(\begin{array}{c}n-\tau\vspace{12pt}j\end{array}\right) \left(\begin{array}{c}n-\tau\vspace{12pt}j\end{array}\right) \left(\begin{array}{c}n+j\vspace{12pt}j\end{array}\right),
\]

and (6.40) follows with (6.165).

Cf. Egorychev (1984), § 5.2.2, (5.107) and Riordan (1968), p. 41, exere. 2.

(6.41) With \( D(13) \) the r.h.s. is equal to

\[
\sum_{\left[\begin{array}{c}n-m\vspace{12pt}k\end{array}\right] \leq k \leq m \wedge \frac{n}{2}} \frac{(m+k)!}{k!k!(n-2k)!(m-k)!} \sum_{i=0}^{n-2k} \left(\begin{array}{c}n\vspace{12pt}i\end{array}\right) x^i =
\]

\[
\sum_{\tau=0}^{n} x^\tau \sum_{\left[\begin{array}{c}n-m\vspace{12pt}k\end{array}\right] \leq k \leq m \wedge \frac{n}{2}} \frac{(m+k)!}{k!k!(m-n+k)!i!(n-2k-i)!} =
\]
\[ \sum_{\tau=0}^{\tau} \binom{n}{\tau} \sum_{k} \binom{\tau(k)}{k} \left( \frac{n-k}{n} \right) \left( \frac{m+k}{n} \right), \]

where \( k \geq [n-m] \), \( k \leq m \), \( k \leq \frac{n}{\tau} \), \( k \leq \tau \), \( k \leq n-\tau \).

Since \( k \leq \tau \) and \( k \leq n-\tau \) imply \( 2k \leq n \), the condition \( 2k \leq n \) may be omitted and

\[ \text{Coeff } x^\tau = \binom{n}{\tau} \sum_{k} \binom{\tau(k)}{k} \left( \frac{n-k}{n} \right) \left( \frac{m+k}{n} \right), \]

where \([n-m] \leq k \leq m \) \( \wedge \) \( (n-\tau) \).

When \( \tau > m \) we have \( n-\tau < n-m \leq [n-m] \), so that the sum is empty and

\[ \text{Coeff } x^\tau = 0 = \binom{n}{\tau} \binom{m}{\tau} \binom{m}{\tau}. \]

When \( \tau \leq m \) we have

\[ \text{Coeff } x^\tau = \binom{n}{\tau} \sum_{k=0}^{n-\tau} \binom{\tau(k)}{k} \left( \frac{n-k}{n} \right) \left( \frac{m+k}{n} \right) \]

since the conditions \( k < \tau \) and \( k \geq n-m \) are implied by the binomial coefficient.

Then from (6.62) with \( n \) replaced by \( n-\tau \) and \( u = \tau \), \( v = n-\tau \), \( x = m \), \( M = n \),

\[ \text{Coeff } x^\tau = \binom{n}{\tau} \binom{m}{\tau} \binom{m}{n-\tau} \]

Cf. Egorychev (1984), §57.9, (5.110).
(6.44), (6.45) By rearranging factorials the l.h.s. of (6.44) is
\[ \frac{\pi ! (x + r - m)!}{x!} \sum_{j=0}^{m} \binom{x}{j} \binom{X}{m-j} \frac{m-2j}{(r-m+j)! (r-j)!} \]

Similarly, the l.h.s. of (6.45) is
\[ \sum_{j=0}^{m} \binom{x+m-r}{j} \frac{x! \cdot r!}{(r-j)! (x-r+j)!} \frac{m-2j}{(m-j)! (r-m+j)!} = \frac{x! \cdot r!}{(x+m-r)! (x+m-r)!} \sum_{j=0}^{m} \binom{x+m-r}{j} \binom{x+m-r}{m-2j} \frac{(x+m-r)! (x+m-r)!}{(r-m+j)! (r-j)!} \]

equal to
\[ \sum_{i=0}^{m} \binom{x+m-r}{m-i} \binom{x+m-r}{i} \frac{2i-m}{(r-m+i)! (r-i)!} \]
This identity is equivalent to

\[(3.111) \text{ since } (k)_{k} = r! \binom{k}{r}.\]

For \(r > n\) the sum on the left is empty.

(6.61) Proof by operator identity (see (3-18))

In (3.89): 

\[
\sum_{k=0}^{n} \frac{(u)}{(k)} (n-k) z^k = \sum_{j=0}^{n} \frac{(u)}{(j)} \frac{(u+v-j)}{(n-j)} (z-1)^j
\]

replace \(z\) by \(E\) and apply the resulting operator identity to the function \((x)^n\) to Vandermonde, \(D(\alpha)\), to \((x + k)^M\) and then (6.60)

(6.62) By (6.61) since \(u+v=M\) the l.h.s.

is equal to

\[
\sum_{j=0}^{n} \frac{(u)}{(j)} \frac{(M-j)}{(n-j)} (x)^j.
\]

For \(M < n\) all terms here are zero. For \(M \geq n\) this is equal to

\[
\binom{x}{M-n} \sum_{j=0}^{n} \frac{(u)}{(j)} \frac{(x+n-M)}{(n-j)}.
\]

by \(D(\nu)\). Now (6.62) follows with Vandermonde's convolution \(D(\alpha)\) since \(u+v=M\).
The proof also may start with Dirichlet's formula (6.12) and (6.60). The l.h.s. is equal to
\[
\sum_{k=0}^{n} \binom{k}{n} \binom{n-k}{x} \sum_{j=0}^{M} \binom{k}{j} \binom{M-j}{x-j} =
\sum_{j=0}^{M} \binom{x}{M-j} \sum_{k=0}^{n} \binom{k}{j} \binom{n-k}{x-j} \binom{k}{j} =
\sum_{j=0}^{M} \binom{x}{M-j} \binom{n-j}{j} \binom{\mu+j-\epsilon-j}{\nu-j}.
\]


\textit{cf.} (6.67)

(6.63) With (6.61) the l.h.s. is equal to
\[
\sum_{j=0}^{n} \binom{-\alpha}{j} \binom{-\alpha+\beta-\epsilon-j}{n-j} \binom{c}{j} =
\sum_{j=0}^{n} \binom{-\alpha}{j} \binom{n-1-j}{n-j} \binom{c}{j}.
\]

All terms with \( j < n \) vanish.

(6.64) The l.h.s. is equal to

\[ (-1)^n \sum_{i=0}^{n} \binom{\lambda}{i} \binom{n-b-e-1}{n-i} (\varepsilon)^{i}. \]

With (3.20) where \( u = b, v = n-b-e-1, x = c+n, M = \varepsilon \), this is equal to

\[ \sum_{j=0}^{n} \binom{n}{j} \binom{n-e-1-j}{n-j} (\varepsilon)^{j}. \]

\[ \sum_{h=[u-e]}^{n} \binom{h}{u-e} \binom{n-h}{n-h} (\varepsilon)^{n-h}. \]


(6.66), (6.67) Proof by operator identity (see pp. 13-14) of (6.72): In (3.89):  

\[ \sum_{k=0}^{n} \binom{\mu}{k} \binom{\nu}{n-k} z^k = \sum_{j=0}^{n} \binom{\mu}{j} \binom{\mu+\nu-j}{n-j} (z-1)^j. \]

Replace \( z \) by \( E^{-1} \). Then \( z-1 \) is replaced by \( -E^{-1} \Delta \). Application of the operator identity to the function \( \binom{\lambda}{M} \) gives (6.66), see G (38).

When \( u+v = M \) the r.h.s. of (6.66) is

...
\[ \sum_{j=0}^{n \wedge M} (-1)^j \binom{u}{j} \binom{M-j}{n-j} \binom{x-j}{M-j} \]

When \( M < n \), all terms here vanish. When \( M \geq n \), this sum is equal to
\[ \binom{x-n}{M-n} \sum_{j=0}^{n} (-1)^j \binom{u}{j} \binom{x-j}{n-j} , \]
and (6.67) follows with (3.365), or with \( D(24) \) and Vandermonde's convolution
\( D(26) \), cf. (16.62).

We may reduce (6.66) and (6.67) to (6.61) and (6.62) by \( D(24) \):
\[ \binom{x-k}{M} = (-1)^{M-k} \binom{M-x+k-1}{M} . \]

(6.69) From (6.61) with \( n = m, u = n, \)
\( v = x, x = M = n \).
For a direct proof by operator identity (see pp. 13–132) we take (3.192):
\[ \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n} z^k = \sum_{j=0}^{n} \binom{n}{j} z^j (1+z)^{n-j} , \]
where we replace \( z \) by \( A \). Then apply the resulting operator identity to the function \( \binom{x}{m} \), noting (6.138).
\[(6.70) \text{ Proof by operator identity, see pp. 13-18.}\]

\[\sum_{k=0}^{n} \binom{n}{k} \binom{a}{k} (-\lambda)^{n-k} = \]

\[\sum_{k=0}^{n} \binom{a-1}{k} \binom{a}{n-k} \lambda^k (\lambda+1)^{n-k}\]

Replace \(\lambda\) by \(-E\) and apply the resulting operator identity to the function \(\binom{x}{m}\). Then put \(k = n - h\).

This relation looks like a special case of (6.82), (6.61), (6.66) or (6.85) but we could not derive it from one of these.

\[(6.71) \text{ Double generating function}\]

\[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z^m w^n \sum_{k=0}^{m} \binom{n}{k} \binom{a}{m-k} (x-k) = \]

\[\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{a}{k} (x-k) \binom{x-k}{n} z^k (1+z)^n = \]

\[\sum_{k=0}^{\infty} \binom{a}{k} z^k \left(1 + w + wz\right)^{x-k} = \]

\[\left(1 + w + wz\right)^{x} \left(1 + z \left(1 + w + wz\right)^{-1}\right)^{u} = \]

\[\left(1 + w + wz\right)^{x-u} \left(1 + w\right)^{u} \left(1 + z\right)^{u},\]
Also
\[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z^m w^n \sum_{k=0}^{\infty} \frac{(x-u) \binom{u}{m-k} \binom{x-k}{n-k}}{k!} = \]
\[ \sum_{k=0}^{\infty} \binom{x-u}{k} \sum_{m=k}^{\infty} \sum_{n=k}^{\infty} z^m w^n \binom{u}{m-k} \binom{x-k}{n-k} = \]
\[ \sum_{k=0}^{\infty} \binom{x-u}{k} z^k w^k (1+z)^u (1+w)^x = (1+z)^u (1+w)^x \left( \frac{1}{1+wz(1+w)^{-1}} \right)^x = \]
\[ (1+z)^u (1+w)^x \left( 1 + wz(1+w)^{-1} \right)^{x-u} = \]
\[ (1+z)^u (1+w)^x (1+w+wz)^{x-u}. \]

Finally
\[ (1+z)^u (1+w)^x (1+w+wz)^{x-u} = \]
\[ (1+z)^x (1+w)^x \left( 1 - z(1+z)^{-1}(1+w)^{-1} \right) = \]
\[ \sum_{k=0}^{\infty} \binom{x-u}{k} (-1)^k z^k (1+z)^x (1+w)^x = \]
\[ \sum_{k=0}^{\infty} \binom{x-u}{k} (-1)^k z^k \binom{x-k}{i} \sum_{n=0}^{\infty} \binom{x-k}{n} w^n = \]
\[ \sum_{k=0}^{\infty} z^k w^n \sum_{i=0}^{\infty} \binom{x-u}{k} \binom{x-k}{i} (x-u)(x-k)(x-k) = \]
\[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z^m w^n \sum_{i=0}^{m} \binom{x-u}{k} \binom{x-k}{m-i} \binom{x-k}{n} = \]
\[ \sum_{n=0}^{\infty} z^n w^n \sum_{i=0}^{m} \binom{x-u}{k} \binom{x-k}{m-i} \binom{x-k}{n} \binom{x-k}{n} = \]
\[ \text{See (4) in Gould (1961a).} \]
(6.72), (6.73) We start with (x) in the proof of (6.73), p. 79:

\[ \sum_{k=0}^{m} (-1)^{k} \binom{m+k+1}{m-k} z^k = \]

\[ \sum_{i=0}^{m} \binom{m+i}{i} (1-z)^i = \sum_{i=0}^{m} \binom{m+i+1}{m-i} z^i (1-z)^{m-i}. \]

Proofs by operator identity, see p. 13-18. In (x) replace \( z \) by \( E \). Then \( 1-z \) is replaced by \( -\Delta \). Applying this operator identity to the function \( (x) \) we obtain (6.72) with \( G(28) \). In the third sum, put \( j = m-i \). Then (3.365) in a similar way we obtain (6.73) when \( z \) is replaced by \( E \).

For \( n \leq m \) apply (3.365).


\[ \sum_{h=0}^{n} \binom{n}{h} \left( \frac{a}{n+1-h} \right) (-\lambda)^h \]

\[ \sum_{h=0}^{n+1} \binom{n+1-h}{h} \left( \frac{a}{n+1-h} \right) \lambda = \sum_{h=0}^{n+1} \binom{n+1-h}{h} \left( \frac{a}{n+1-h} \right) \lambda^{h+1} \]

replace \( \lambda \) by \( -E \), so \( \lambda+1 \) by \( -\Delta \), and apply the resulting operator identity to the function \( (x) \), see \( G(38) \).
(6.75), (6.76) Proof by operator identity, see pp. 13-18, in (3.181):

\[ \sum_{h=0}^{n} \binom{m}{h} \binom{n}{h-h} z^{m-h} = \]

\[ \sum_{j=0}^{m} \binom{m}{j} \binom{m+a-j}{n} (z-1)^j, \]

replace \( z \) by \( E^{-1} \) and apply the resulting operator identity to the function \( \binom{x}{n} \).

Then (75) follows.

Replacing \( z \) by \( E \) gives (6.76).


From (3.104) and \( \Phi (6d) \)

\[ \sum_{k=0}^{n} (n-k) \binom{n+k}{n-k} (1+yz)^k = \]

\[ \left( \begin{array}{c} 2n \end{array} \right) \Phi_{2n} (z) = \left( \begin{array}{c} 2n \end{array} \right) \sum_{k=0}^{n} \left( \begin{array}{c} 2n-k \end{array} \right) z^k \]

Replacing \( z \) by \( \Delta/4 \) and applying the resulting operator identity to the function \( \binom{x}{n} \) we obtain the first equality in (6.77), see G (38).

When \( M \leq n \) the second member in (6.77) is equal to
\[
(2^n) \sum_{k=0}^{M} \binom{2n-k}{k} \binom{x}{M-k} y^{-k} =
\]
\[
\frac{(2^n)}{n} y^{-M} \sum_{h=0}^{M} \binom{x}{h} \binom{2n-M+h}{M-h} y^{h}
\]

The proof of (6.78) is similar. From \((3, 105)\) and \((\Phi(68))\)
\[
\sum_{k=0}^{n} \binom{n+1/2}{k} \binom{n+1/2}{n-k} (1+yz)^k =
\]
\[
\left(\frac{2n+1}{n}\right) \Phi_{2n+1}\left(\frac{z}{y}\right) = \left(\frac{2n+1}{n}\right) \sum_{k=0}^{n} \binom{2n+1-k}{k} z^k.
\]

Replace \(z\) by \(\Delta/y\) and apply the resulting operator identity to the function \(x^{\frac{1}{2}}(\frac{x}{M})^{\frac{1}{2}}\).

For \(M \leq n\) the second member is equal to
\[
\left(\frac{2n+1}{n}\right) \sum_{k=0}^{M} \binom{2n+1-k}{k} \binom{x^{\frac{1}{2}}(\frac{x}{M})^{\frac{1}{2}}}{M-k} y^{-k} =
\]
\[
\left(\frac{2n+1}{n}\right) y^{-M} \sum_{h=0}^{M} \binom{x^{\frac{1}{2}}(\frac{x}{M})^{\frac{1}{2}}}{h} \binom{2n+1-M+h}{M-h} y^{h},
\]
and the other equalities in (6.78) follow from (3.395).

For (6.79) we start from (3.108) and \(\Phi(69)\).
\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{k} (1+yz)^k = \frac{1}{2} \left( \frac{2n}{n} \right)^n A_{\frac{2n}{n}}(z)
\]

\[\begin{align*}
&= \frac{1}{2} \left( \frac{2n}{n} \right)^n \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} z^k, \quad n \geq 1.
\end{align*}\]

Replacing \( z \) by \( \frac{\Delta}{y} \), and applying the resulting operator identity to the function \( f(M) \) gives the first equality in (6.79).

For \( 1 \leq M \leq n \), with \( 2n = 2n-k + k \), the second member of (6.79) is equal to

\[\begin{align*}
\frac{1}{2} \left( \frac{2n}{n} \right)^n \sum_{k=0}^{M} \binom{2n-k}{k} (M-k) y^{-k} + \\
\frac{1}{2} \left( \frac{2n}{n} \right)^n \sum_{k=1}^{M} \binom{2n-k-1}{k-1} (M-k) y^{-k} = \\
\frac{1}{2} \left( \frac{2n}{n} \right)^n y^{-M} \sum_{j=0}^{M} \binom{x}{j} \binom{2n-M+j}{M-j} y^j + \\
\frac{1}{2} \left( \frac{2n}{n} \right)^n y^{-M} \sum_{j=0}^{M-1} \binom{x}{j} \binom{2n-M-1+j}{M-1-j} y^j \\
\text{(x)}
\end{align*}\]

With (3.395) this is equal to

\[\begin{align*}
\frac{1}{2} \left( \frac{2n}{n} \right)^n y^{-M} \sum_{k=0}^{M} \binom{2x}{k} \binom{2n-M}{M-k} z^k + \\
\frac{1}{2} \left( \frac{2n}{n} \right)^n y^{-M} \sum_{k=0}^{M-1} \binom{2x}{k} \binom{2n-M-1}{M-1-k} z^k
\end{align*}\]
\[ \frac{1}{2} \binom{2n}{n} y^{-m} \sum_{k=0}^{M} \frac{2n-k}{2n-M} \binom{2n-M}{m-k} 2^k. \]

With (3.395) (x) is also equal to
\[ \frac{1}{2} \binom{2n}{n} y^{-m} \sum_{k=0}^{M} \binom{2n-M+2x-k}{m-k} + \]
\[ \frac{1}{2} \binom{2n}{n} y^{-m} \sum_{k=0}^{M-1} \binom{2n-M-1+2x-k}{m-1-k} \Rightarrow \]
\[ \binom{2n}{n} y^{-m} \sum_{k=0}^{M} \binom{2x}{k} \frac{n+x-k}{2n+2x-M-k} \binom{2n+2x-M-k}{M-k}. \]

When \( 1 \leq M \leq n \) the second member of (6.79) may be written
\[ \frac{1}{2} \binom{2n}{n} y^{-m} \sum_{h=0}^{M} \frac{2n}{2n-M+h} \binom{2n-M+h}{M-h} (x) y^h \]

From (3.464) with \( y = 2n-M \) and \( n \) replaced by \( M \) the last equality in (6.79) now follows.

For (6.80) we start from (3.109) and (6.9):
\[ \sum_{k=0}^{n} \binom{n+1/2}{k} \binom{n-1/2}{n-k} (1+zy)^k = \binom{2n}{n} \Delta_{2n+1}(z) = \]
\[ \binom{2n}{n} \sum_{k=0}^{n} \frac{2n+1}{2n+1-k} \binom{2n+1-k}{k} z^k. \]
Replacing $z$ by $\Delta/y$ and applying the resulting identity to the function $(x)$ we obtain the first equality in (6.80). For $1 \leq M \leq n$ the second member of (6.80) is equal to

$$(2n) \sum_{k=0}^{M} \binom{2n+1-k}{k} (x^{-k}) y^{-k} +$$

$$(2n) \sum_{k=1}^{M} \binom{2n-k}{k-1} (x^{-k}) y^{-k} =$$

$$(2n) y^{-M} \sum_{j=0}^{M} \binom{M}{j} \binom{2n-M+1+j}{M-j} y^{j} +$$

$$(2n) y^{-M} \sum_{j=0}^{M-1} \binom{M-1}{j} \binom{2n-M+j}{M-1-j} y^{j}$$

With (3.395) this is equal to

$$(2n) y^{-M} \sum_{k=0}^{M} \binom{2x}{k} (\frac{2n-M+1}{M-k}) 2^{k} +$$

$$(2n) y^{-M} \sum_{k=0}^{M-1} \binom{2x}{k} (\frac{2n-M}{M-1-k}) 2^{k} =$$

$$(2n) y^{-M} \sum_{k=0}^{M} \binom{2x}{k} \frac{2n-k+1}{2n-M+1} \binom{2n-M+1}{M-k} 2^{k}$$

With (3.395) (**) is also equal to
\[ \left( \frac{2n}{n} \right)^{-M} \sum_{k=0}^{M} \left( \frac{2^k}{k} \right) \left( \frac{2n-M+2x-k}{M-k} \right) + \]

\[ \left( \frac{2n}{n} \right)^{-M} \sum_{k=0}^{M-1} \left( \frac{2^k}{k} \right) \left( \frac{2n-M+2x-k}{M-1-k} \right) = \]

\[ \left( \frac{2n}{n} \right)^{-M} \sum_{k=0}^{M} \left( \frac{2^k}{k} \right) \left( \frac{2n+1-M+2x-k}{2n+1-M+k} \right) \left( \frac{2n+1-M+2x-k}{M-k} \right). \]

The second member of (6.80) may be written as (1 \leq M \leq n)

\[ \left( \frac{2n}{n} \right)^{-M} \sum_{h=0}^{2n+1-M+h} \left( \frac{2n+1-M+h}{M-h} \right) \left( \frac{2n+1-M+h}{h} \right)^{-h}. \]

With (3.464), where \( y = 2n+1-M \) this is equal to

\[ \sum_{k=0}^{M} \left( \frac{2x+1}{k} \right) \left( \frac{2n+1-M+2x-k}{M-k} \right) \left( \frac{2n}{n} \right)^{-M}. \]

For \( M=0 \) the identity (6.80) follows trivially.

(6.81) - (6.84) We note that the equality of first and third member in (6.82) is (6.83) with \( x = 3n \) and that (6.84) is (6.83) with \( x = y+n \) and \( k = n-n \).

The identity (6.81) is (6.61) with \( u = v = n \).
From (6.65), with \( r = n \), \( x = 3n \), and
from (6.68), with \( r = n \), \( y = 2n \), we
obtain (6.82).

The relation (6.83) is (6.65), with \( r = n \),
or we may start from (6.81), with \( M = 2n \).
By canceling and rearranging factorials
the r.h.s. is equal to
\[
\left( \begin{array}{c} x \\ n \end{array} \right) \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (-1)^{n-k} (x-n) = \left( \begin{array}{c} x \\ n \end{array} \right).
\]

\[ (6.85) \text{ Proof by operator identity, see } \]
\[
\sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{a}{k+1} (-1)^{n-k} = \sum_{h=0}^{n+1} \left( \begin{array}{c} n+1 \\ h \end{array} \right) (-1)^{n+1-h} \frac{a}{h} (-a)^{h} \\
\sum_{h=0}^{n+1} \left( \begin{array}{c} n+1 \\ h \end{array} \right) \frac{a}{h} (-a)^{h} \Delta^{h}.
\]

Replace \( \lambda \) by \(-E\). Since \(-E + 1 = E\Delta\)
this gives
\[
\sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \frac{a}{k+1} \Delta^{k-n} = \sum_{h=0}^{n+1} (-1)^{n+1-h} \left( \begin{array}{c} n+1 \\ h \end{array} \right) \frac{a}{h} (-a)^{h} \Delta^{h}.
\]
Multiplying both sides by \( E^{n+1} \) and
applying the resulting identity to the
function \( \left( \begin{array}{c} x \\ M \end{array} \right) \) gives (6.85).
(6.86) With $D_{14}$, canceling and rearranging factorials, and then Vandermonde's convolution $D_{26}$ the l.h.s. is equal to

\[
\binom{x}{r} \sum_{k=r-d}^{n} \frac{Y^{k} (x-r)}{(h-k)(k-d-r)} =
\]

\[
\binom{x}{r} \sum_{h=0}^{n-r+d} \frac{Y^{h}}{(h)} (n-r+d-h) = \binom{x}{r} \binom{x+y-r}{n-d-r}.
\]

(6.87), (6.88) For $0 \leq r-d \leq n$ the l.h.s. of (6.87) is equal to

\[
\binom{x}{r} \sum_{k=r-d}^{n} \frac{Y^{k} (x-r)}{(h-k)(k+d-r)} =
\]

\[
\binom{x}{r} \sum_{h=0}^{n-r+d} \frac{Y^{h}}{(h)} (n-r+d-h) = \binom{x}{r} \binom{x+y-r}{n-r+d},
\]

with $D_{14}$, canceling and rearranging factorials, and Vandermonde's convolution $D_{26}$.

When $r < d$ the above proof fails since then $k$ runs from 0 to $n$ and not from $r-d$ to $n$. We then still have (6.88).

Since both sides of (6.88) are polynomials in $x$, it is sufficient to prove it for $x = p \in \mathbb{N}$ with $p > nd$. Then
\[ \sum_{k=0}^{n} \binom{n}{k} \binom{p}{k+d} \left( \frac{k+d}{\nu_c} \right)^{p} = \]

\[ \sum_{k=0}^{p-d} \binom{n}{k} \binom{p}{k+d} \left( \frac{k+d}{\nu_c} \right) = \]

\[ \binom{p}{\nu_c} \sum_{k=0}^{p-d} \binom{n}{k} \binom{p-r}{p-d-k} = \]

\[ \binom{p}{\nu_c} \binom{n+p-r}{p-d} = \binom{p}{\nu_c} \binom{n+p-r}{n+d-r}, \]

with \( D(14) \) and Vandermonde's Convolution \( D(26) \).

\[ (6.89) = (6.91) \] These extensions of Nanjundiah's formula \( (6.62) \) are due to Bizley (1970).

With Vandermonde's Convolution \( D(26) \) and \( (6.86) \) the l.h.s. of \( (6.89) \) is equal to

\[ \sum_{k=d}^{b} \binom{k}{d} \binom{c}{k-d} \sum_{i=0}^{b+c} \binom{k-d}{i} \binom{a+d}{i} = \]

\[ \sum_{i=0}^{b+c} \binom{a+d}{b+c-i} \sum_{k=d+i}^{b} \binom{c}{k-d} \binom{b}{i} \binom{k-d}{i} = \]

\[ \sum_{i=0}^{b-d} \binom{a+d}{b+c-i} \binom{c}{i} \binom{c+b-i}{b-i-d} = \]
\[
\sum_{i=0}^{b-d} \frac{(a+d)(c)(a-c)}{(c+d)(b-d-i)} = (a+d)(b-d)
\]

where in the last lines D(\(i\)) and D(\(i\)) were applied.

For the proof of (6.90) we apply Vandermonde's convolution and D(i). The d.h.s.

is equal to

\[
L = \sum_{k=0}^{b+c} \frac{(b)(c)}{(k)(k+d)} \sum_{i=0}^{b+c-i} \frac{(k+d)(a-d)}{(i)(b+c-i)} =
\]

\[
\sum_{i=0}^{b+c-i} \sum_{k=0}^{c-d} \frac{(b)(c)}{(k)(k+d)} (i)
\]

A necessary condition for a term in the sum over \(k\) to be nonzero is \(i \leq k+d \leq c\).

So

\[
L = \sum_{i=0}^{c} \frac{(a-d)}{(b+c-i)(i)} \sum_{k=0}^{c-d} \frac{(b)(c)}{(k)(k+d)} (i)
\]

\[
\sum_{i=0}^{c} \frac{(a-d)}{(b+c-i)(i)} \sum_{k=0}^{c-d} \frac{(b)(c-i)}{(k)(c-d-k)}
\]

Since \((c-i)(c-d-k) = 0\) for \(k < i-d\),

\[
L = \sum_{i=0}^{c} \frac{(a-d)}{(b+c-i)(i)} \sum_{k=0}^{c-d} \frac{(b)}{(k)(c-d-k)} =
\]

\[
\sum_{i=0}^{c} \frac{(a-d)}{(b+c-i)(i)} (b+c-i)(c-d)
\]
\[
\sum_{i=0}^{n} \binom{c}{i} \binom{a-d}{i} \binom{b+c-i}{c-d} = \sum_{i=0}^{b+d} \frac{(a-d)!}{(a-d)!} \binom{c}{i} \binom{a-c}{b+d-i} = \frac{(a-d)(a)}{(c-d)(b+d)}.
\]

With Vandermonde's convolution, \(D(26)\) and \((6.60)\) the l.h.s. of \((6.91)\) is equal to
\[
L = \sum_{k=0}^{d} \binom{b}{k} \binom{c}{d-k} \sum_{i=0}^{b+c} \frac{(a)}{i} \binom{k}{i} (b+c-i) = \sum_{i=0}^{b+c} \frac{(a)}{i} \sum_{k=i}^{d} \binom{b}{k} \binom{c}{d-k} (k) = \sum_{i=0}^{d} \binom{b+c}{i} \binom{a}{b} (b+c-i) \binom{d}{d-i}.
\]

It follows that \(L = 0\) for \(d > b+c\). For \(d \leq b+c\) we then have
\[
L = \sum_{i=0}^{d} \frac{d! b^a}{(b+c-d)! (a-c)!} \sum_{i=0}^{d} \frac{(d)}{i} \binom{a-c}{b-i} = \frac{a! b!}{(b+c-d)! d! (a-c)!} \sum_{i=0}^{d} \binom{d}{i} \binom{a-c}{b-i} = \frac{a! b!}{(b+c-d)! d! (a-c)!} \sum_{i=0}^{d} \binom{d}{i} \binom{a-c}{b-i}.
\]
\[
\frac{a!}{(b+c+d)!} d! (a-c-1)! (b-c)! (a-c+d)!
\]

and (6.91) follows.

(6.92), (6.93) The l.h.s. of (6.92) is
\[
\sum_{k=n}^{m+n} \binom{N}{k} \binom{k}{m} \binom{m}{k-n}
\]
\[
\sum_{h=0}^{m} \left( \binom{N}{n+h} \binom{n+h}{m} \right) \left( \binom{m}{h} \right)
\]

and (6.92) follows from (6.88) with \( n \)

replaced by \( m \) and then \( d = n, \ y = m, \ x = N \).

For \( m \leq n \), the l.h.s. of (6.93) is
\[
\sum_{h=0}^{m} \left( \binom{N}{h+n} \binom{m}{h} \right) \left( \binom{m}{h+n} \right)
\]

the same as above. And similarly for \( n \leq m \).

For a direct proof write (\( \ast \)) as
\[
\sum_{h=0}^{N-n} \left( \binom{N}{h+n} \binom{m}{h} \right) \left( \binom{m}{h+n} \right)
\]

and cancel and rearrange factorials.

Interpretation: Let \( A \) and \( B \) be independent random subsets of \( \{1, 2, \ldots, N\} \) with \( |A| = m, |B| = n \). Put \( X = |A \cup B|, Y = |A \cap B| \).
Then $X + Y = m + n$, $m \leq X \leq m + n$, $0 \leq Y \leq m \Delta n$. We have

$$\binom{N}{m} \binom{N}{n} \mathbb{P}(A \cup B = \{i_1, \ldots, i_k\}) = \frac{k!}{(m-j)! (n-j)!},$$

where $j = m+n-k$. This follows since the favorable outcomes are found by dividing $\{i_1, \ldots, i_k\}$ into the three subsets $A - B$, $A \cap B$, and $B - A$ with $|A - B| = m-j$, $|A \cap B| = j$, $|B - A| = n-j$. Then

$$\binom{N}{m} \binom{N}{n} \mathbb{P}(X = k) = \binom{N}{k} \binom{k}{m} \binom{n}{k-m} = \binom{N}{k} \binom{k}{m} \binom{m}{k-n},$$

as is seen by writing the binomial coefficients as quotients of factorials, noting that $k = m+n-j$. The identity (6.92) states that

$$\sum_{k=m}^{m+n} \mathbb{P}(X = k) = 1.$$

Also

$$\mathbb{P}(Y = j) = \mathbb{P}(X = m+n-j).$$
\[
\binom{n}{m+n-j}\binom{n}{j}\binom{n}{m-j}^{-1}\binom{n}{j}^{-1} = \\
\binom{n}{j}\binom{N-n}{m-j}\binom{N}{m} = \binom{m}{j}\binom{N-m}{n-j}\binom{N}{n}^{-1},
\]

\((m+n-N)\quad 0 \leq j \leq m \land n\). So \(Y\) has a hypergeometric distribution, see D(26a).

We have, for \(n \leq m\),

\[
\sum_{j=0}^{n} \binom{N}{j}\binom{m+n-j}{n-j}\binom{N}{n} = \binom{N}{m}\binom{N}{n}.
\]

Putting \(n-j = h\) we obtain (6.93) for \(n \leq m\).

For the combinatorial interpretation, see Andrews (1975) and Hurt et al. (1982).

(6.95), (6.96) Proofs by operator identity, see I18. In (C(189)) put \(\lambda^{-2} = z\):

\[
\sum_{2k \leq n} \binom{n}{2k}\binom{-1/2}{k} z^k = \\
(-1)^n \sum_{j=0}^{n} \binom{n-j}{j}\binom{-1/2}{n-j} (z+1)^j 2^{n-2j}.
\]

Replace \(z\) by \(-E\) and apply the operator identity to the function \(\binom{x}{M}\) to
(6.97) Proof by operator identity, see pp. 13-18. With \( D(x) \), canceling and rearranging factorials, we may write (3.565) as

\[
\sum_{2k \leq n} (-1)^k \binom{n}{k} \frac{(2n-2k)}{n} (1+2z)^{n-2k} = 2^n \sum_{h=0}^{\infty} \binom{n}{h} \frac{(n+h)}{h} z^h.
\]

Replacing \( z \) by \(-\frac{1}{2}E \) and applying the resulting operator identity to the function \( \binom{x}{m} \) we obtain (6.97).

(6.98) Proof by operator identity, see pp. 13-18. With (2.189):

With Vandermonde's convolution and by canceling and rearranging factorials, the L.H.S. is equal to

\[ \sum_{n} (\mathcal{a})(1-x)^{n} \sum_{k=0}^{\infty} \binom{x}{x-k} \]

\[ \leq \sum_{j=0}^{\infty} \binom{x}{x-j} \leq \prod_{k=0}^{\infty} \binom{n-k}{x-k} \]

\[ \sum_{l=0}^{\infty} \left( \binom{i}{i} \right) \left( \binom{n}{i} \right) \]
\[ \binom{n}{m} \sum_{h=0}^{n-m} (-1)^h \binom{n-m}{h} (u + ma + ha) = \]

\[ \Delta^n \left( \binom{\nu+x}{\frac{\nu+x}{\varepsilon}} \right) \bigg|_{x=0} = \]

\[ \Delta^n \left( \binom{a + b \cdot x}{\frac{c + d \cdot x}{n - x}} \right) \bigg|_{x=0} = \]
(6.128) Define

\[ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \]

With \( G(2\mathbb{R}) \), \( \text{So}(2^a) \): \( A(n, m, \varepsilon, u, v) = 0 \) when \( n > m + \varepsilon \)

\[
\sum_{i=[n-m]}^{n} (-1)^i (i) (m-i) (\varepsilon-n+i) =
\]
\[ \sum_{j=[n-m]}^{n \land \ell} (-1)^j \binom{n-j}{j} \binom{n}{m-n+j} (v-j) \quad \text{Also} \]

\[ (4) \quad A(n, m, \xi, u, \nu) = (-1)^n \sum_{i=0}^{n} \binom{n}{i} \Delta^i \left( \frac{\nu+n-x}{\xi}\right) \bigg| E \Delta^{n-i} \left( \frac{u+i}{m}\right) \bigg|_0 = \]

\[ \sum_{i=[n-m]}^{n \land \nu} (-1)^{n-i} \binom{n}{i} \binom{\nu+n-i}{\xi-i} \binom{u+i}{m-n+i} \]

\[ \sum_{j=[n-\nu]}^{m \land n} (-1)^j \binom{n}{j} \binom{\nu+j}{\xi-n+j} \binom{u+n-j}{m-j} \]

From (4) and \( D(24) \) and the third member of (3),

\[ \sum_{j=[n-\nu]}^{m \land n} (-1)^j \binom{n}{j} \binom{\nu+j}{\xi-n+j} \binom{u+n-j}{m-j} \]

\[ = (-1)^{m+n-t-n} A(n, m, \xi, m-u-n-1, \xi-\nu-n-1) \]
From (5) and (3) \[ m \geq n \]

\[ (-1)^{-e} \binom{m}{e-n+m} \times e = \cdots \]

\[ = 0, \quad n-e > m \]

(9) \[ A(n, m, \epsilon, u, \bar{\epsilon}-n-1) = (-1)^{m+n} A(n, m, \epsilon, m-u-n-1, 0) \]

\[ = (-1)^{n-\epsilon} \binom{n}{\epsilon} \binom{m-u-n-1}{\epsilon-n+m} \times e \]

\[ = 0, \quad n-\epsilon > m \]

From (1) \[ j=0 \]
From (3) and (6.10) we have

(12) \[ A(n, n, n, u, u) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (u-i) (i) \binom{u}{i} = \]

\[ (-1)^s \binom{2s}{s} \binom{u+s}{2s}, \quad n = 2s, \quad s \in \mathbb{N}_0, \quad u = 0, \quad n \text{ odd}. \]

(13) \[ \sum_{i=1}^{n} \frac{(-1)^i}{i} \binom{n}{i} (i) (m-i) (s-n+i) = \]

\[ \begin{cases} n \leq r, \quad 0 \leq m, \quad s \leq n \end{cases} \]
\[
\sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \frac{u+k}{m} \binom{v+n-k}{r} = 
\]

\[
(1+x)(1+y) \left(1 - \frac{1}{1+y}\right) = \left(1+y\right)^2 \\
\text{Then with (1) on p. 112 (proof of (6.38), (6.39))}
\]
\[
\sum_{j=0}^{\infty} \left( m + \epsilon - u + \nu - n - 1 \right)_j j
\]

\[
\sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} A(n, \alpha, \beta, u - \epsilon, v - m) x^{\alpha+j} y^{\beta+j} = 
\]

\[(\text{where } u > v + \ldots) \quad \sum_{i=\lfloor n-\epsilon \rfloor}^{\infty} (-1)^i \frac{n! u! v!}{i! (n-i)! (m-i)! (u-m+i)! (v-\epsilon+n-i)!} \]

Way is (5).
For this to make sense we should have 
$m + r > n$. But see (20).

When we take $u \in \mathbb{N}$, $v \in \mathbb{N}$ we should 
have $u + v + n > m + r$. In fact we have 

\[(19) \ A(n, m, r, u, v) = 0 \text{ when } u \in \mathbb{N}_0, \ v \in \mathbb{N}_0, \ u + v + n < m + r,
\]

since the nonzero terms in (3) have 
$m + r \leq u + v + n$.

vanish, i.e.

\[(21) \ i = m - u = v + n - r.\]

This term is present only if 
$m - u \leq m \wedge n$, $m - u \geq 0$, $m - u \geq n - r$, i.e.
(23) \[ A(m, n, \varepsilon, u, \nu) = (-1)^{m-u} \frac{n!}{(m-u)! (\varepsilon-u)!} = (-1)^{m-u} \binom{n}{m-u} = (-1)^{m-u} \binom{n}{\varepsilon-u}. \]
\[ \sum_{m \leq n} \binom{m}{i} \binom{n-i}{\xi-i} \binom{u+v+n-2-m}{\xi-m} \]

(29) \( H(n, m, \varepsilon, u, \varepsilon-n) = \binom{m}{\varepsilon} \), \( \varepsilon \geq n \),
as also is seen from (1).
\[
\begin{align*}
\mathfrak{z} &= \mathbb{Z}^{m - n - \varepsilon} \setminus \{0\}, \\
-x &\in \mathbb{N}, -\varepsilon &\in \mathbb{N}^+, \\
\sum_{n-m}^{n} J_n &\left( u + x \right) + E^{i} \Delta^{n - i} / \left( w + x \right) \\
\sum_{n-m}^{n} (-1)^{i} \left( \begin{array}{c} n \\ i \end{array} \right) \left( \begin{array}{c} u \\ m - i \end{array} \right) \left( \begin{array}{c} w + i \\ x - n + i \end{array} \right) \\
\sum_{n-m}^{n} (-1)^{i} \left( \begin{array}{c} n \\ i \end{array} \right) \left( \begin{array}{c} u \\ m - i \end{array} \right) \left( \begin{array}{c} w + i \\ x - n + i \end{array} \right) \\
\sum_{n-m}^{n} (-1)^{i} \mathfrak{A} \left( n, m, \varepsilon, u, x - n - w - 1 \right)
\end{align*}
\]
The proof by partial symmetry, \( \text{Eqn. (11) \& (12)} \), now follows by replacing \( z \) by \( \frac{1}{2} \Delta \) and therefore \( 1 + 2z \) by \( E \). Applying the resulting operator identity to the function \( \phi_n \) we obtain (6.131).

\[
\binom{n + \nu}{\nu} \geq (-1)^k \binom{n}{2} \binom{2n - 2k}{n + \nu} =
\]
Proof by operator identity, see pp. 178 - 178. In (x) in the proof of (6.131) replace $z$ by $-\frac{1}{2} E^{-A}$, so $1 + z$ by $E^{-1}$ and apply the resulting operator identity to the function $(\hat{M})$.

However, this is not the type of proof by operator generated by the $-\frac{1}{2} E^{-A}$, we now appeal to the isomorphism between formal generating functions $\sum_{i=0}^{\infty} a_i z^i$ and shift-invariant operators.
\[ \sum_{i=0}^{\infty} a_i \Delta^i, \text{ see pp. C10-12}. \]
(6.150) By canceling and rearranging factorials we see that the l.h.s. is equal to
\[
\binom{n}{m} \sum_{k=m}^{n} (-1)^{k-m} \binom{n-m}{k-m} \frac{u+k \alpha + M \epsilon}{u+k \alpha + M} \binom{M}{M} =
\]
\[
\binom{n}{m} \sum_{h=0}^{n-m} (-1)^{h} \binom{n-m}{h} \frac{u+\alpha M + h \alpha}{u+\alpha M + h \alpha + M} \binom{M}{M} =
\]
\[
(-1)^{n-m} \binom{n}{m} \Delta^{n-m} \frac{u+\alpha x}{u+\alpha x + M} \binom{M}{M},
\]
\(x=m\)

For \(\alpha=1\) apply \(G(y_0)\). For \(\alpha=-1\) put \(u-m-n=\) \(u-m-n+h\) and apply \(G(2h)\) and \(G(y_0)\). Cf. (6.121)

(6.151) Proof by operator identity, see pp. T3-T8

By canceling and rearranging factorials we may write (3.5.65) as
\[
\sum_{2k \leq n} (-1)^{k} \binom{n}{k} \binom{2n-2k}{n} (1+2z)^{n-2k} =
\]
\[
2^n \sum_{h=0}^{n} \binom{n}{h} \binom{n+h}{n} z^h.
\]

Replacing \(z\) by \(\frac{\Lambda}{\Delta}\) and applying the resulting operator identity to the function \(x(x+M)\) we obtain (6.151).

Cf. (6.97).
(6.166) Both sides are polynomials in $x$ and $y$. So it is sufficient to prove (6.166) for $x = n \in \mathbb{N}$, $y = \epsilon \in \mathbb{N}$. The l.h.s. then

$$\sum_{k=0}^{m} \binom{n}{k} \binom{\epsilon}{k} \binom{n+\epsilon+m-k}{n+\epsilon} \cdot$$

vanish since then $0 \leq n+\epsilon+m-k \leq n+\epsilon$. (Note $k \leq n$). With (6.68) the last sum is equal to
(6.170) Since both sides are polynomials in $x$ and $y$, it is sufficient to prove (6.170) for $x = p \in N$ and $y = q \in N$. For $p + q \leq n$, both sides vanish. For $p + q \geq n$ the L.H.S. is equal to

$$n \binom{p}{q} \binom{p + q + m + 1}{p + q + 1}$$

in the sum we have $i \leq j \leq n \leq p + q$. So with (6.60) and Vandermonde's convolution $D_{12}'$ the above sum is equal

$$\sum_{i=0}^{n} \binom{p + q + m}{p + q - i} \binom{p}{i} \binom{p + q - i}{n - i} = \binom{p + q + m}{p + q - i} \sum_{i=0}^{n} \binom{p}{i} \binom{m + n}{n - i} =$$

$$\binom{p + q + m}{m + n} \binom{p + m + n}{n}.$$
(6.171) Since both sides are polynomials in \( u \) and \( v \) it is sufficient to prove (6.171) for \( u \), and \( v \), (large) positive integers.

\[
\sum_{h=0}^{\min(u,v)} \binom{n-h}{i} \binom{m+h}{i} = \sum_{l=0}^{\min(u,v)} \binom{u+v-m}{u+v-i} \sum_{h=0}^{\min(n-u,v)} \binom{h}{i} \binom{m+h}{i}.
\]

By (6.87) with \( y = u \), \( x = v \), \( d = m \) and \( c = i \) this is equal to

\[
\sum_{l=0}^{\min(u,v)} \binom{u+v-m}{u+v-l} \binom{n}{v} \binom{n}{u+v-i} = \sum_{l=0}^{\min(u,v)} \binom{u+v-m}{u+v-l} \frac{\binom{n}{v}}{\binom{n}{u+v-i}}.
\]

\[
\binom{u+v-m}{u+v-l} \sum_{l=0}^{m+n-1} \binom{v}{i} \binom{n}{m+n-i} = \sum_{l=0}^{m+n-1} \binom{v}{i} \binom{n}{m+n-i}.
\]

\[
\binom{n}{v} \binom{n}{u+v-i} = \frac{n!}{v!(n-v)!} \frac{n!}{(u+v-i)!(n-u-v+i)!} = \binom{n}{v} \binom{n}{u+v-i}.
\]
\[ \sum_{2j \leq n} \binom{a}{j} \binom{a-j}{n-2j} (uv)^j (u+v)^{n-j} \]

(6.170) by cancelling corresponding factorials one sees that the first and second member are equal and also equal to

\[ \binom{x}{n} \sum_{k=0}^{\infty} \binom{n}{k} \frac{1}{(m-k)!} \]

\[ \sum_{h=0}^{\infty} \binom{2-n}{h} \binom{n-2+n}{h} \]
From (6.170) with \( x = z+r-n, \, n = \xi, \, m = n - r \) this is equal to

\[
\sum_{n=0}^{\infty} \frac{x+1}{x+a} \binom{x}{n} \sum_{k=0}^{n} \binom{n}{k} (k+a-1). 
\]

From (6.171) with \( u = y, \, \nu = z+r-n, \, m = z - n \) this is equal to

factorials the L.h.s. is equal to

\[
\frac{x+1}{x+a} \binom{x}{n} \sum_{k=0}^{n} \binom{n}{k} (k+a-1). 
\]

By (3.33) with \( \xi = a-1 \) and \( x \) replaced by \( x+a \) the sum over \( k \) is equal to

\[
\binom{x+a+n}{a-1+n}. 
\]

The identity (6.182) then follows by rearranging and canceling factorials.
For a proof of (6.183) by operator...
(6.184) With Vandermonde's convolution

\[ F(a, b, m, n) = \]

\[ \sum_{j=0}^{\text{min}(a, n)} \binom{a+n}{j} \frac{\Gamma(m-j)}{\Gamma(m)} \sum_{k=0}^{\text{min}(b, n-j)} \binom{b+n-j}{k} \frac{\Gamma(n-j-k)}{\Gamma(n-j)} = \]

\[ \sum_{k=0}^{\text{min}(b, n)} \binom{b+m}{k} \frac{\Gamma(a+n)}{\Gamma(a)} \sum_{j=0}^{\text{min}(a, n-k)} \binom{a+n-k}{j} \frac{\Gamma(m-k-j)}{\Gamma(m-k)} = \]
Here we replace \( t \) by \(-\Delta/4\). Then apply the resulting operator identity to the function \((x)\).


(6.196) Proof by operator identity, see pp. 13–18.

\[
\sum_{k=0}^{\infty} (-1)^k \binom{\frac{n}{2}}{k} \binom{\frac{n}{2}}{k} z^k = \\
\sum_{2k \leq n} \frac{(n)!(2k)!}{(2k)(k)!} z^k (1-xz)^{n-2k},
\]

replace \( z \) by \(-\Delta/2\) and apply the resulting operator identity to the function \((x)\).
and (6.198) follows with $B(14)$.

A direct proof by operator identity (see pp. 13 - 18). In (3.566):

\[
\sum_{k=0}^{n} \binom{2n-2k+1}{n-k} \binom{2n-2k}{k} \frac{2^k}{k} (1-4t)^k =
\]

\[
y^n \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \binom{2k}{k} t^k,
\]
\[-1\] \[\sum_{h=0}^{m} (-1)^h \binom{2h+1}{h+1} \binom{m-h}{h+1} (h+1)(t-\frac{1}{2})^h\]

Replace \( t \) by \(-\frac{\Delta}{4}\) and apply the resulting operator identity to the function \( \chi \).

(6.204) Proof by operator identity, see pp. 13-18. In (3.417) \( z(\lambda) \) put \( \lambda - \kappa = z \).

Replace \( z \) by \(-\frac{\Delta}{4}\) and apply the resulting operator identity to the function \( \chi \).
\[(n) \sum_{j=0}^{m} j^M \cdot \sum_{i=0}^{\infty} (-1)^i (\frac{a-M}{i}) (\frac{a+b-M-i}{n-M-i}) = \]

\[
\left( \frac{(2\alpha-1)!}{\alpha!(\alpha-1)!} \right) \sum_{k=0}^{n} \frac{\binom{k+\alpha-1}{\alpha-1} (2n)}{(n-k)} = \sum_{h=0}^{n} (-1)^h \binom{n}{h} (\frac{a}{h}) (\frac{b}{n-h}) \]

(6.216) By canceling and rearranging factorials and then D(24) the l.h.s. is seen to be equal to
\[
\frac{(2a-1)!}{a! (a-1)!} \left( -\frac{1}{a} \right)^2 (\pi a^{-1/2}) = \]

\[
= \left( 1 - \frac{a}{a+1} \right) (a+1)! (a+2)! - (a) (n) .
\]

(6.217), (6.218) Proofs by operator identity
(see pp. 13–18). With B(2\( \gamma \)) and (3.89)

\[
\sum_{j=0}^{\infty} l^j J(j) \left( n-j \right) l^{n-j} J =
\]
Taking the operator identity to the function gives (6.217).
Replacing $z$ by $E$ and applying the resulting operator identity to the function $f(x)$ gives (6.219). Similarly, we obtain (6.220) by replacing $z$ by $E^{-}$. 

(6.221) The second equality is $G(22)$. 

With $G(28)$ and then (3.452) we have

\[
\sum_{j=0}^{m-h} \binom{m-h}{j} \binom{m-h-j}{v+j} 
\]

This sum is over

\[ A = \{ (i, j, h) \in \mathbb{N}_0 : u+j \leq m, v \leq h \leq m-j \} . \]

So our sum is equal to
\[
\sum_{i+j \leq m} \frac{m! (-2)^{m-i-j}}{i! j! (m-i-j)!} \sum_{k=0}^{m-i-j} \binom{m-i-j}{k} (u+i)(v+j)(w+i+j+k) = \\
(\binom{2\pi i}{m}) \sum_{n=0}^{\infty} \sum_{j \leq n} \binom{n+x+1}{n-x-j} \binom{x+j}{x} (1+yz)^j = \\
\sum_{j=0}^{\infty} \binom{x+j}{j} w^j (1+yz)^j \sum_{m=0}^{\infty} \binom{m+x+j+2z+1}{m} w^m = \\
\sum_{j=0}^{\infty} (x+i)^j w^j (1+i)^j \\
(1-w) \left(1-w (1+yz)(1-w)\right)^j = \\
\left(1-2w-yzw^2\right)^{-x-1} = \\
\sum_{h=0}^{\infty} \binom{x+h}{h} 2^h w^h (1+2zw)^h =
\]
\[ \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \binom{r+h}{h} \binom{h}{k} z^{h+k} w^{h+k} z^k = \]

\[ \sum_{n=0}^{\infty} w^n z^n \sum_{k=0}^{n} \binom{n-k}{k} \frac{(r+h-k)}{(n-k)} z^k, \quad \text{so} \]

\( \sum_{2j \leq n} \binom{n+2j+1}{n-2j} \binom{r+j}{j} (1+yz)^j = \)

\[ \sum_{2m \leq n} \binom{r+n-m}{n-m} \binom{n-m}{m} z^m. \]

The L.H.S. in (\( \star \)) is

\[ \sum_{2j \leq n} \binom{n+2j+1}{n-2j} \binom{r+j}{j} \sum_{m=0}^{j} \binom{j}{m} y^m z^m = \]

\[ \sum_{m=0}^{\left\lfloor n/2 \right\rfloor} y^m z^m \sum_{j=m}^{\left\lfloor n/2 \right\rfloor} \binom{n+2j+1}{n-2j} \binom{r+j}{j} \binom{j}{m} \]

Equating coefficients of \( z^m \) in both sides of (\( \star \)) gives (6.222).

A different proof. By canceling and rearranging factorials we see that the L.H.S. of (6.222) is equal to

\[ \binom{r+m}{r} \sum_{2m \leq 2j \leq n} \binom{n+2j+1}{n-2j} \binom{r+j}{j-m} = \]
Now in (3.361):

take \( N = n - 2m \), \( x = \frac{n + 2}{2} \), so that \( x + \frac{1}{2} N = \frac{n + 2}{2} + m \). Then the L.H.S. of (6.222) is seen to be equal to

\[
2^{n-m} \binom{n-m}{m} \binom{\frac{n + 2}{2} + n - m}{n-m}.
\]
Application of this identity to the function \( \chi_M \) proves (6,228).

However, we may not apply the isomorphism of pp. 13-18 since that is about polynomials whereas here we have an infinite series. Instead, we apply formal power series and series of powers of a delta operator, see pp. C10-11. This isomorphism and (3,490) gives...
for shift-invariant operators (Theorem C.4). From \( C(27) \) with \( R = \Delta \), \( q_n(x) = \binom{x}{n} \), \( T = E^{-m^{-1/2}} \) we have:

\[
E^{-m^{-1/2}} = \sum_{k=0}^{\infty} a_k \Delta^k,
\]

\[
a_k = E^{-m^{-1/2}} \left( \frac{x}{k} \right) \bigg|_{x=0} = \binom{m^{-1/2}}{k},
\]

and \((*)\) follows.

(6.232) Canceling and rearranging factorials shows that the l.h.s. is equal to:

\[
\binom{n}{r} \sum_{i = \lfloor r \rfloor}^{n} \binom{n-i}{i} \binom{n-r}{n-i} 2^{n-2i} =
\]

\[
\binom{n}{r} \sum_{i = 0}^{n} \binom{n-i}{i} \binom{n-r}{n-i} 2^{n-2i} =
\]

\[
\binom{n}{r} \sum_{j=0}^{n} \binom{j}{n-j} \binom{n-r}{j} 2^{2j-n} =
\]

\[
\binom{n}{r} \binom{2n-2r}{n} \text{ by } (3.397)
\]

(6.234), (6.235) Proofs by operator identity, see pp. 18-28. From (14) in the discussion of (3.434), Part I, p. 259,
\[
\sum_{k=0}^{n} (-1)^k \binom{n+1}{k}(2k)^t k^t = \\
\tag{\star}
(-1)^n \sum_{k=0}^{n} \binom{2k}{k} \left( \frac{-1}{n+1} \right)^k \left( t + \frac{1}{4} \right)^k
\]

Replacing \( t \) by \( E/y \) and applying the resulting operator identity to the function \( \binom{x}{y} \) proves (6.234).

Similarly, replacing \( t \) by \( E/y \) and therefore \( t + \frac{1}{4} \) by \( -\frac{1}{4} E^{-1} \Delta \) leads to (6.235).


\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k}(2k)^t k^t = \\
\sum_{2k \leq n} \binom{n}{2k} \binom{2k}{k} t^{2k} (1-2t)^{n-2k}
\]

Replace \( t \) by \( E/2 \) and apply the resulting operator identity to the function \( \binom{x}{y} \). Cf. (6.95), (6.96)
(6.140) Proof by operator identity, cf. pp. 13 - 18. In (*) in the proof of (6.204), p. 65, replace \( z \) by \( E/y \) and apply the resulting operator identity to the function \( (x^*) \), cf. (6.266).
(6.253) With \( D(24) \) and (6.166) the L.h.s. is equal to
\[
\sum_{k=0}^{n} \frac{(-1)^{n-k}}{ (n)_k (n-k)_(-y-1) } (-y-1)^k
\]

(6.256) By canceling and rearranging factorials and applying \( D(24) \) and \( D(26) \)
\[
(n) \sum_{k=0}^{n} (-1)^k (k)_k (k)_n =
\]
\[ (-1) (\hat{n}) \left( -\frac{n}{n} \right) = (\hat{n}). \]

With the inverse pair \( a_{nk}, b_{nk} \) on \( p. \text{IR11} \) of Part I, we may write (6.256) as

\[
\sum_{k=0}^{n} a_{nk} (x)(x+k) = (x)^2.
\]

So
\[
\sum_{k=0}^{n} b_{nk} (x) (x+n) = \frac{(x)^2}{(x+n)}.
\]

So (6.256) and (6.168) are companions in the sense of Part I, \( p. \text{IR7} \), for the above inverse pair.
The sum here may be extended to \((h, s, s) \in \mathbb{N}^3\) since the added terms vanish because

\[
\begin{align*}
&h \geq [b-a], \quad h \leq a+b. \\
&\text{This is the only restriction. So the coefficient is}
\end{align*}
\]

\[
\sum_{h=\lfloor b-a \rfloor}^{\lfloor a+b \rfloor} (-1)^h \frac{1}{(h)(a-b+h)} (a+b-h).
\]

To find the coefficient of \(x^{\alpha} y^{\beta}\), we have to take

\[
h \geq [a_1 - a_2]_+ \quad \text{and} \quad h \leq a_1 + a_2,
\]


\[
h \geq \lfloor b-a \rfloor.
\]
Since the l.h.s. of (x) has only even powers of \(x\) and \(y\), we obtain (6.259)

Note that in order to determine the coefficient of \(x^{2a}y^{2b+1}\) and \(x^{2a+1}y^{2b}\) we should take \(r+s=2a\), \(2h-3r+s=2b+1\) and \(r+s=2a+1\), \(2h-3r+s=2b\), respectively, both cases implying \(2h+2s=2a+2b+1\), so that an empty sum results.

(6.260) With (6.13) the l.h.s. is equal to

\[
(-4)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{-1/2}{n-k} (-1/2)^n \binom{-1/2}{k}.
\]

With (6.10) this is zero for \(n\) odd and for \(n=2m\) it is equal to

\[
\frac{2m}{m} (-\frac{1}{2} \binom{m}{m}) = (2m)^2.
\]
(6.266) In (*) in the proof of (6.204) replace $z$ by $E^{-1} \Delta / y$, and therefore $(1-yz)$ by $E^{-1}$ and apply the resulting operator.

(6.267) The first equality follows from $D(24)$ and $B(13)$. Write (6.92) with $M = n$ as

$$y^n(-1)^{\frac{n}{2}} \frac{1}{2h+1} (-1)^{\frac{n}{2}} \left( \frac{1}{h} \right) (h^{\frac{1}{2}}) (n-h).$$

With (6.10) where $x = -\frac{1}{2}$ this is equal to zero when $n$ is odd and to

$$y^{2m} \left( \begin{pmatrix} -\frac{1}{2} \end{pmatrix} \right)^2 = \left( \begin{pmatrix} 2m \end{pmatrix} \right)^2$$

when $n = 2m$. 
\[ (-1)^h \sum_{h=0}^{n} \binom{n}{h} \binom{-n}{h} (-\frac{1}{2} n^{h}) = \sum_{n=1}^{\infty} \frac{(-1)^h (n+1) \cdot \ldots \cdot (2k)}{(2k)!} n^{k} \]

So from (6.267) we obtain

\[ m=0 \binom{m}{r} \]
\( (6.29b) \) From \( D(\varphi) \) and Vandermonde's convolution \( D(26) \)

\[
\sum_{n=0}^{\infty} \binom{n}{k} \binom{n-k}{k} = (-1)^k \binom{n+2k}{2k} = \left(\frac{m+\varepsilon+n}{n}\right).
\]

With \( T(\varphi) \) and \( R(\eta) \) this may be written as
(6.272), (6.278). We start with the identity \((\star \star)\) in Remark 2 to the proof of (6.267):

\[
\sum_{k=0}^{\infty} \left( \begin{array}{c} 2k \\ k \end{array} \right) (-z)^{k} (1-yz)^{k} = \sum_{j=0}^{\infty} \left( \begin{array}{c} 2j \\ j \end{array} \right) y^{2j}.
\]

\((\star \star)\)

The proofs by operator identity as on pp. 73-18 suggest replacing \(z\) by \(-\Delta/4\) and \(E^{-\Delta/4}\), respectively. Then \(1-yz\) is replaced by \(I+\Delta = E\) and \(I-E^{-1}(E-I) = E^{-1}\). We then obtain, formally, the operator identities:

\[
\sum_{k=0}^{\infty} \left( \begin{array}{c} 2k \\ k \end{array} \right) \Delta^{k} y^{-2k} = \sum_{j=0}^{\infty} \left( \begin{array}{c} 2j \\ j \end{array} \right) \Delta^{2j} y^{-2j} \quad \text{and} \quad \sum_{k=0}^{\infty} \left( \begin{array}{c} 2k \\ k \end{array} \right) \Delta^{k} E^{k+1} (-4)^{k} = \sum_{j=0}^{\infty} \left( \begin{array}{c} 2j \\ j \end{array} \right) E^{-2j} \Delta^{2j} y^{-2j}.
\]

Applying these identities to the function \((\star \star)\) we obtain (6.277) and (6.278).

However, the proofs suggested on pp. 73-18 assume polynomials in \(z\), not series. We now appeal to the isomorphism \(\Phi\)
between formal power series and shift invariant operators defined in Chapter C, pp. C10-C12. It maps complex functions analytic in a neighborhood of \( z=0 \). Note that both sides of (**) are such functions. This restriction still is linear.

\[
f_1(z) = \sum_{k=0}^{\infty} \binom{w}{k} (2^k)^w (-z)^k (1-4z)^{-2k-1}
\]
Since \( f(z) \) starts with \( z^{M+1} \), its image \( \Phi f \) starts with \( \Delta^{M+1} \) and it annihilates \( (x) \). So
\[
(\Phi f)(x) = (\Phi f)(x).
\]
By the linearity and isomorphy of \( \Phi_a \)
\[
\Phi_a f = \sum_{k=0}^{M} \binom{2k}{k} \Phi_a (-z)^k (1-yz)^{2k-1} = \sum_{k=0}^{M} \binom{2k}{k} \Phi_a (-z)^k \Phi_a (1-yz)^{2k-1}
\]
Here \( \Phi_a (-z) = \Delta/4 \)
\[
\Phi_a (1-yz)^{-1} = \Phi_a \left( \sum_{k=0}^{\infty} \frac{4^k z^k}{k!} \right)
\]
\[
\sum_{k=0}^{\infty} (-1)^k \Delta^k = E^{-1}.
\]
The last step applies Chapter C, Theorem, with \( Q = \Delta \):
\[
E^{-1} = \sum_{k=0}^{\infty} a_k \Delta^k, \quad a_k = E_q (0),
\]
where the basic sequence \( q_n \) of \( \Delta \) is
\[
q_n(x) = \binom{x}{n} \text{ by C(21), so}
\]
\[
q_k = \left. E^{-1}(x) \right|_{x=0} = \left( -1 \right)^k.
\]
\[ \text{and (6.277) follows.} \]

\[ \sum_{j=0}^{x} (\begin{array}{c} n-j \\ j \end{array}) \binom{n-2j}{n-j} (-y \mu)^j (1+\mu)^{n-2j} = \]

\[ \sum_{j=0}^{x} (n-j)^{j} \binom{2n-2j}{n-j} (-y)^j \sum_{j=0}^{x} \binom{n-2j}{n-j} \mu^{i+j} \]

\[ \sum_{j=0}^{x} \binom{n-j}{j} (n-j)^{(n-j)-(x-j)} \]

When \( x \leq n/2 \) the sum over \( j \) is for \( j \leq x = x \wedge (n/2) \). When \( x \geq n/2 \) the sum

\[ \text{is} \]

\[ \text{constant} \]
(7.3) The first equality is $I(13)$. With $D(14)$, by rearranging and canceling factorials and with $D(26)$ (Vandermonde's convolution) the l.h.s. is equal to

$$
\sum_{k=0}^{n} \binom{n}{k} \binom{\nu}{k} (-w-1+k)^{-1} =
$$

$$(\nu-w-1)^{-1} \sum_{k=0}^{n} \binom{\nu}{k} (-w-k)^{-1} =$$

$$(\nu-w-1)^{-1} \binom{\nu+w-1}{n} = \binom{\nu}{n} \binom{\nu-w}{n},$$

again with $D(24)$.

(7.4) With $D(14)$ and by rearranging and canceling factorials we see that the l.h.s. is equal to

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{\nu}{k} (\nu-n+k)^{-1} =
$$

$$(\nu)^{-1} \sum_{k=0}^{n} (-1)^k \binom{\nu}{k} (\nu-k)^{-1} =$$

and (7.4) follows with (3.67).

In (7.3) of Gould (1972e) this sum is given as

$$
2^n (\nu-n)! \pi^{1/2} / (\nu - \frac{1}{2} n)! (-\frac{1}{\kappa} n - \frac{1}{\kappa})! .
$$
This is the same as our (7.4) when one defines \(1/\xi! = 0\) for \(-\xi \in \mathbb{N}\). For \(n = 2m\) apply \(B(2m)\) first and then \(B(1)\). Note that \(\pi^{\frac{1}{2}} = (-\frac{1}{2})!\).

The l.h.s. of (7.4) is \(F(-n, -v; v+1-n; -1)\), which is in the form of Kummer's theorem (18), however with \(-\frac{1}{2} \in \mathbb{N}\). So to apply (18) a limiting process is needed.

(7.5) Induction on \(n\). Adding the induction contribution and the term \(k = n+1\) is done by writing these terms as products and quotients of factorials, see D(14).

(7.6) The first equality is (13). The second one follows by canceling and rearranging factorials. The last one follows by D(24).

(7.8) By canceling and rearranging factorials the l.h.s. is equal to

\[
\sum_{r=0}^{n} \frac{(2i)!(2n-2i)}{(2i)!(2n-2i)}.
\]
(7.9) This identity is a consequence of the orthogonality of the Legendre polynomials. We need the relations \( S(11) \) and \( S(12) \) in Chapter 5 of Part I:

\[ (1) = S(1) \int_{-1}^{1} p_n(x) \, p_m(x) \, dx = 0 \quad \text{when} \quad f \quad \text{is a polynomial of degree} \quad < n. \]

\[ (2) = S(1) \int_{-1}^{1} x^n \, p_m(x) \, dx = 2^{n+1} n! \, n! / (2n+1)! \]

We also need the integral

\[ (3) \int_{-1}^{1} (x+1)^r (x-1)^s \, dx = (-1)^s \frac{2^{r+s+1} \int_{0}^{1} t^r (1-t)^s \, dt}{\Gamma (r+1) \Gamma (s+1)} \]

\[ = (-1)^s 2^{r+s+1} \frac{\Gamma (r+1) \Gamma (s+1)}{\Gamma (r+s+x)} = \frac{(-1)^s 2^{r+s+1} \, r! \, s! \, (r+s+1)!}{\Gamma (r+s+x)} \quad \text{Res} \quad r > -1 \]

With \( S(3) \):

\[ p_n(x) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} (x+1)^k (x-1)^{n-k} \]

and (3) we have

\[ (4) \int_{-1}^{1} (x+1)^k (x-1)^q \, p_n(x) \, dx = \]
\[ \sum_{k=0}^{\infty} \binom{n}{k} \int_{-1}^{1} (x+1)^{p+k} (x-1)^{q+n-k} \, dx = 2^{p+q+1} \sum_{k=0}^{\infty} (-1)^{q+n-k} \binom{n}{k} x^{p+k} (p+k)! (q+n-k)! \frac{(p+q+n+1)!}{(p+q+n+1)!} \]
\[(7.10)\quad S(p, q) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{p+q+n}{p+k} \right)^n = \sum_{h=0}^{n} \binom{n}{h} \left( \frac{p+q+n}{p+h} \right)^n\]

These identities are inspired by Problem 810 in Nieuw Arch. (n), 9, 1991, 217-230. We follow the solution by A. A. Toney.
\[ n \]

\[ 
\begin{align*}
\mu! (i-t)^{w-k-q} / (u-k)! &= (-1)^{k} (i-t)^{w-u-q} D^{k} (1-u)^{k}.
\end{align*}
\]

So, with Leibniz's formula

\[ 
\int_{0}^{1} dt \ t^{z} \ (1-t)^{y} \ \sum_{k=0}^{\infty} \binom{k}{l} D^{l} (1-t)^{y} t^{z} =
\]

\[ 
\sum_{i=0}^{M} \binom{M}{i} D^{i} t^{d-v+M} D^{M-i} (1-t)^{w-u-q}.
\]

Here we want to integrate by parts in \( z \) and \( y \) times. All terms at 0 and 1 vanish when

\[ (3) \quad d > 0, \ \text{Re} w \to d + M. \]

Absolutely.

From (4)

\[ S' = (w+1)(-1)^{M}(M!)^{-1} \int_{0}^{1} dt \ t^{w} (1-t)^{d}. \]

\[ \sum_{i=0}^{M} \binom{M}{i} D^{i} t^{d-v+M} D^{M-i} (1-t)^{w-u-q} = \]
\[(w+1) \int_0^1 dt \ t^w \ (1-t)^u \]

\[\sum_{i=0}^{M} \frac{(-1)^i}{i! (M-i)!} \binom{q-v+M}{i} t^{w-u-q-M+i} \]

\[= (w+1) \sum_{i=0}^{M} (-1)^i \binom{q-v+M}{i} \binom{w-u-q}{M-i} \]

\[\int_0^1 dt \ t^{q-M-i} (1-t)^{w-q-M+i} \]

\[= (w!)^{-1} \sum_{i=0}^{M} (-1)^i \binom{q-v+M}{i} \binom{w-u-q}{M-i} \]

\[= \begin{cases} \sum_{i=0}^{M} (-1)^i \binom{q-v+M}{i} \binom{w-u-q}{M-i} (q+M-i)^{-1} \\ \sum_{k=0}^{M} (-1)^k \binom{w-u-q}{k} (q-M-k) (q+k) \end{cases} \]

(5) \[S(M, u, v, w, q) = (-1)^M S(M, w-u-q, q-v+M, w, q) \]

**SPECIAL CASES**

\[v = q+M+1, \quad w = u+q+M+1 = u+v \]

The conditions (3) now become
\[(w+1)(-1)^{u} \int_{0}^{1} dt \ t^{\nu}(1-t)^{u} D^{w} t^{\nu-1}(1-t)^{m+1} \]
\[= \]
\[(w+1) \int_{0}^{1} dt \ t^{\nu}(1-t)^{u} (t^{m-1} - 1) = \]
\[(w+1) \frac{(\nu-M-1)! u!}{(\nu-M+u)!} - (w+1) \frac{\nu! u!}{(u+v+1)!} = \]
\[(u+q+m+2) \frac{q! u!}{(q+u+1)!} - (q+m+1)! u! \]
\[(u+q+m+1)! \]

This special case is the problem 810 mentioned above.

\[b) \ \nu = q + M - \tau, \ w = u + q + s, \ \tau, s \in \mathbb{N}. \]

\[(a) \ q > \nu, \ \text{and} \ \ u > m - s. \]

From (4), since now \( t^{q-\nu+m} / (1-t)^{w-u-q} \)
For \( r+s = M \) we see from (4)

\[(10) \quad S(M, u, q+M-r, u+q+s, q) = \]

\[(-1)^M (w+1) \int_0^1 dt \ t^v (1-t)^u (-1)^s = \]

\[(-1)^r (w+1) v! u! / (u+v+1)! = \]

\[(-1)^r v! u! / (u+v)! = (-1)^r \left( \frac{u+v}{v} \right) = (-1)^r \left( \frac{u+q+s}{q} \right) \]

\[\text{if } v = q-1, \quad w = u+q-1 \]

The conditions (3) now become

\[(11) \quad q > 0, \quad \text{Re} u > M+1.\]

Then from (4)

\[(12) \quad S(M, u, q-1, u+q-1, q) = \]

\[(-1)^M (w+1)(M!) \int_0^1 dt \ t^v (1-t)^u D^M t^{M+1}(1-t)^v = \]

\[(-1)^M (w+1) \int_0^1 dt \ t^v (1-t)^u \left\{ (1-t)^{-M-1} - 1 \right\} dt = \]

\[(-1)^M (w+1) \int_0^1 dt \left\{ t^{q-1} (1-t)^{u-M-1} - t^{q-1} (1-t)^u \right\} \]
\((-1)^{\frac{m}{n+1}} \left\{ \frac{(q-1)! (u-M-1)!}{(q+u-M-1)!} - \frac{(q-1)! u!}{(q+u)!} \right\} = \right\}

\((-1)^{\frac{m}{u}} (q+u-M-1)^{-1} + (-1)^{M+1} \left( \frac{q+u-1}{q-1} \right)^{-1} \right)\)
replace $z$ by $E$, so $z-1$ by $E-I = \Delta$, and apply the resulting operator identity to the function $(x)^{n+1}$. See $(*)$ and the discussion on p. 15.

(7.14) As (7.9) this identity is a consequence of the orthogonality of the Legendre polynomials. We need the relations $S(11)$ and $S(12)$ in Chapter 5 of Part I:

(1) $S(11) \int f(x) P_n(x) \, dx = 0$ when $f$ is a polynomial of degree $\leq n$.

(2) $S(12) \int x^n P_n(x) \, dx = \frac{2^{n+1} n! n!}{(2n+1)!}$

We also need the integral

(3) $\int (x+1)^r (x-1)^s \, dx = (-1)^r \frac{2^{r+s+1}}{r! s!} \int (1-t)^{r+s+1} \, dt$

From $S(5)$

(4) $P_n(x) = \sum_{h=0}^{n} \binom{n}{h} \binom{n+h}{h} 2^{-h} (x-1)^h$.

So

(5) $\int (x+1)^r (x-1)^s P_n(x) \, dx =$
\[
\sum_{h=0}^{n} \left( \binom{n}{h} \binom{n+h}{n} \right) 2^{-h} \int_{-1}^{1} (x+1)^{p} (x-1)^{q+h} \, dx = n.
\]

So from (1) when \( p+q < n \)
\[
\sum_{h=0}^{n} \binom{n}{h} \binom{n+h}{n} (n+h)^{-1} = 0.
\]
\[
\lim_{x \to a} f(x) = \frac{1}{(a-x)^2} \quad (a-x) \\
\text{From } G(x) \text{ and } G(x) \text{ we see that} \\
\text{is also equal to} \\
\text{at } \lim_{x \to a} f(x) \\
\]
(7.20) Write (7.19) as \[ \sum_{j=0}^{n} (-1)^j \binom{n}{j} x_j = x_n, \]

Since the lower triangular matrix \[ a_{nj} = (-1)^j \binom{n}{j} \] is its own inverse (see which is (7.20). So (7.19) and (7.20) are each other's companions in the sense of Part I, p. 157.
(7.22) - (7.23). The first equality in (7.22) and (7.23) is (\text{Iq}).

\[ \int_0^1 (1-\tau)^u \, d\tau = (1+1)^u, \quad u = \int \Gamma\left(\frac{a}{2} + \Gamma - t \right) \frac{\Gamma\left(\frac{b}{2} - \frac{a}{2} + t \right)}{\Gamma\left(\frac{b}{2} - \frac{a}{2} + t \right)} \, dt. \]
\[
\sum_{j=0}^{2\nu-1} (-1)^j \binom{2\nu-1}{j} \int_0^1 u^\frac{1}{2} a + \frac{1}{2} j - 1 \left(1 - u\right)^{-\frac{1}{2} b - \frac{1}{2} a - \nu} \, du,
\]

where we should have

(3) \( \Re a > 0, \Re \left(\frac{1}{2} b - \frac{1}{2} a - \nu\right) > -1. \)

So

(4) \[ _2F_1\left(a, \frac{b}{2}; \frac{1}{2} a + \frac{1}{2} b + \nu; \frac{1}{2}\right) =
\]

\[
2^{a-1} \Gamma\left(\frac{1}{2} a + \frac{1}{2} b + \nu\right) \left[\frac{\Gamma(a) \Gamma\left(\frac{1}{2} b - \frac{1}{2} a + \nu\right)}{\Gamma(a) \Gamma\left(\frac{1}{2} b - \frac{1}{2} a + \nu + 1\right)}\right]^{-1} \sum_{j=0}^{2\nu-1} (-1)^j \binom{2\nu-1}{j} \Gamma\left(\frac{1}{2} a + \frac{1}{2} j - \nu\right) \Gamma\left(\frac{1}{2} b + \frac{1}{2} j - \nu + 1\right) \left[\frac{\Gamma\left(\frac{1}{2} b - \frac{1}{2} a + \nu\right)}{\Gamma\left(\frac{1}{2} b - \frac{1}{2} a + \nu + 1\right)}\right].
\]

The conditions (1), (2), (3) may be combined to \( \Re (b - a) > 2\nu - 2, \) since \( \nu \in \mathbb{N}, \) and \( \Re a > 0. \)
For \( x = \frac{1}{2} \) there is only the term \( j = 0 \) in the r.h.s. of \((4)\). We then find

\[
(5) F_1 (a, b ; \frac{1}{2} a + \frac{1}{2} b + \frac{1}{2} ; \frac{1}{2}) = \\
2^{a-1} \frac{\Gamma (\frac{1}{2} a) \Gamma (\frac{1}{2} a + \frac{1}{2} b + \frac{1}{2})}{\Gamma (a) \Gamma (\frac{1}{2} b + \frac{1}{2})}.
\]

This is \((7.21)\), as is seen by the duplication formula of the gamma function, see Part I, p. B3; Rainville (1960), § 19.

\[(7.23) \text{This is (7.22) with } x = 1. \text{ The r.h.s. of (4) now becomes}
\]

\[
2^{a-1} \frac{\Gamma (\frac{1}{2} a) \Gamma (\frac{1}{2} a + \frac{1}{2} b + 1)}{\Gamma (\frac{1}{2} b - \frac{1}{2} a)}
\]

\[
= \Gamma (\frac{1}{2} b) - \Gamma (\frac{1}{2} b + \frac{1}{2})
\]

\[(7.24) \text{When}
\]

\[
\{ \Gamma (n) \} \xrightarrow{n \to 0} 1
\]
\[
\begin{align*}
\frac{\Gamma(a, b; \frac{1}{2}a+\frac{1}{2}b - x; \frac{1}{2})}{2^a \Gamma\left(a, \frac{1}{2}a - \frac{1}{2}b + x; \frac{1}{2}a + \frac{1}{2}b - x; -1\right)} &= \frac{\Gamma\left(\frac{1}{2}a - \frac{1}{2}b + x; a; \frac{1}{2}a + \frac{1}{2}b - x; -1\right)}{2^a \Gamma\left(\frac{1}{2}a + \frac{1}{2}b - x; a; \frac{1}{2}a + \frac{1}{2}b - x; -1\right)}. \\
\text{With Pochhammer's theorem, when} \quad (2) \quad \text{Re}a > 0, \text{ Re}\left(\frac{1}{2}b - \frac{1}{2}a - x\right) > 0, \\
\text{the last expression is equal to} \quad 2^a \Gamma\left(\frac{1}{2}a + \frac{1}{2}b - x\right) \left[\Gamma(a) \Gamma\left(\frac{1}{2}b - \frac{1}{2}a - x\right)\right]^{-1} \\
(3) \quad \int_0^1 t^{a-1} (1-t)^\frac{1}{2}b-\frac{1}{2}a-\frac{1}{2} (1+t)^\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2} \, dt = \\
\int_0^1 t^{a-1} (1-t^2)^\frac{1}{2}b-\frac{1}{2}a-\frac{1}{2} (1+t)^\frac{1}{2}b-\frac{1}{2}a+\frac{1}{2} \, dt \cdot \\
2^a \Gamma\left(\frac{1}{2}a + \frac{1}{2}b - x\right) \left[\Gamma(a) \Gamma\left(\frac{1}{2}b - \frac{1}{2}a - x\right)\right]^{-1} + \\
\sum_{i=0}^{2n+1} (2i+1) \int_0^1 t^{a+i-1} (1-t^2)^\frac{1}{2}b-\frac{1}{2}a-\frac{1}{2} \, dt = 
\end{align*}
\]
\( F(a, b; \frac{1}{2}a + \frac{1}{2}b; \frac{1}{2}) = \)
(7.26) Write (3.202) with \( a = -1 \) as

\[
\left( \begin{array}{c} b - v_m \\ n \end{array} \right) = v \left( \begin{array}{c} v+n \\ n \end{array} \right) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{b+k}{m-k} \frac{1}{v+k}, \quad m \leq n.
\]

With (22), (28), (38), (42) this may be written as

\[
\left( \begin{array}{c} b - v_m \\ n \end{array} \right) = (-1)^n v \left( \begin{array}{c} v+n \\ n \end{array} \right) \Delta^n \left( \begin{array}{c} b+x \\ m \end{array} \right) \frac{1}{v+x} \bigg|_{x=0} =
\]

\[
(-1)^n v \left( \begin{array}{c} v+n \end{array} \right) \sum_{k=0}^{n} \binom{n}{k} \Delta^k \left( \begin{array}{c} b+x \end{array} \right) \Delta \left( \begin{array}{c} b+x+k \end{array} \right) \left( \frac{v+n}{v+k} \right)^{-1} \bigg|_{x=0}
\]

which is (7.26). We also may write

\[
\left( \begin{array}{c} b - v_m \\ n \end{array} \right) = (-1)^n v \left( \begin{array}{c} v+n \\ n \end{array} \right) \sum_{k=n-m}^{n} \binom{n}{k} \Delta^k \Delta \left( \begin{array}{c} b+k \end{array} \right) \left( \frac{v+n}{v+k} \right)^{-1} \bigg|_{x=0}
\]

\[
\left( \begin{array}{c} v+n \\ n \end{array} \right) \sum_{k=n-m}^{n} (-1)^{n-k} \binom{n}{k} \binom{b+k}{m-n+k} \binom{v+k}{k}^{-1}
\]

This is (7.61).
(7.27) Put $m = n + r$ where $0 \leq r \leq n$ since $n \leq m \leq 2n$. The l.h.s. of (7.27), by canceling and rearranging factorials, is equal to

$$\frac{n!}{m!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(m-k)!}{m-2k+1} =$$

With (3.10a) where $b = n + r$, $a = 1$ and where $m$ is replaced by $r$, this is equal to

$$(n+r)^{-1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(n-r-k)!}{n+r-1} =$$

(7.28) For $m = n$ this is (3.10a). When $m > n$ put $m = n + r$, where $1 \leq r \leq n$ since $n \leq m \leq 2n$. By canceling and rearranging

$$(n+r)^{-1} \sum_{k=1}^{n} (-1)^k \frac{k}{k} \binom{n}{k} (n+r-k) =$$
By (3.204) with \( a = 1 \), \( b = n + \epsilon \) and \( m \) replaced by \( \epsilon \), noting that \( 1 \leq \epsilon \leq n \), we see that the last expression is equal to

\[
\left( \frac{n+\epsilon}{n} \right)^{\frac{\epsilon}{n}} \sum_{j=0}^{\epsilon-1} \frac{1}{n^j} - \left( \frac{n+\epsilon}{n} \right)^{\frac{\epsilon}{n}} \sum_{j=1}^{\epsilon} \frac{1}{j} = 
\sum_{i=n+1}^{n+\epsilon} \frac{1}{i} - \sum_{j=1}^{\epsilon} \frac{1}{j} = 
\sum_{i=n+1}^{m-1} \frac{1}{i} - \sum_{j=1}^{\epsilon} \frac{1}{j}
\]
\[
\sum_{k=0}^{n} \frac{(-1)^k k^n (n-k)^n}{(k+x)!} x \int_{0}^{1} (-1)^{n-k} (\frac{1}{n}) (2n-k) \int_{0}^{1} (1-t)^k t^{x-1} dt =
\]

(7.41) The first identity is

\[
\sum_{k=0}^{n} \frac{(2n+1)}{(n-k)} \frac{(\gamma)}{(k+1)} \frac{(\gamma + k + 1)}{k}^{-1}
\]

for \( y = 2 \in \{0, 1, 2, \ldots, n\} \).

Canceling and rearranging factorials we may write (1) as
(1) \[ \sum_{k=0}^{n} \frac{(2n+1)}{(n-k)!} \frac{(2y+1)!}{(y-k)! (y+k+1)!} = \]

\[ \frac{y^n (2y+1)! (y+n+\frac{1}{2})!}{y! (y+\frac{1}{2})! (y+n+1)!} \]

With the duplication formula of the gamma function, in terms of the factorial function:

\[ y!(y+\frac{1}{2})! = 2^{-2y} (\frac{1}{2})! (2y+1)! \]

we may write (2) as

(3) \[ \sum_{k=0}^{n} \frac{(2n+1)}{(n-k)!} \frac{(2y+1)!}{(y-k)! (y+k+1)!} = \]

\[ \frac{y^{n+y} (y+n+\frac{1}{2})! / (\frac{1}{2})! (y+n+1)!}{(2y+2n+1)! / (y+n)! (y+n+1)!} \]

Again with the duplication formula, now applied to \( (y+n+\frac{1}{2})! \) we see that (3) is equivalent with

(4) \[ \sum_{k=0}^{n} \frac{(2n+1)}{(n-k)!} \frac{(2y+1)!}{(y-k)! (y+k+1)!} = \]

\[ (2y+2n+1)! / (y+n)! (y+n+1)! \]

So the identities (1) - (4) are equivalent.
\[ \sum_{k=0}^{n} (2n+1)^{2k} + \sum_{h=0}^{n} (2n+1)^{2h} = 2 \sum_{h=0}^{n} (2n+1)^{2h} \]

\[ U(n, \xi) = \sum_{k=0}^{n} \binom{2n+1}{n-k} \binom{2n+1}{\xi-k} \]. Then for

... So (1) holds for \( n = m+1 \) by the above remarks.

The second identity follows from the above computations, e.g. from (4).

These relations are descendants of
(7.46) By rearranging and canceling factorials we see that the l.h.s. is equal to

\[
\frac{1}{y-x} \left\{ (Y) - (X) \right\}
\]
\[ \Delta^n \left( \frac{2n-x}{n} \right) \left( \frac{x}{n} \right) \bigg|_{x=0} \]

With G(28), G(46), and G(52) this last expression is equal to

\[ \sum_{k=0}^{n} \binom{n}{k} \Delta^k \left( \frac{2n-x}{n} \right) E^k \Delta^{n-k} \left( \frac{x}{n} \right)^{-1} \bigg|_{x=0} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left( \frac{2n-k}{n-k} \right) \left( \frac{x+n}{x+n-k} \right)^{-1} \]

\[ (-1)^n \sum_{k=0}^{n} \binom{n}{k} \left( \frac{2n-k}{n} \right) \left( \frac{x+n}{x+n-k} \right)^{-1} \]

Application of G(28) the other way does not give a useful identity.

(7.49) As (7.9) and (7.14) this identity is a consequence of the orthogonality of the Legendre polynomials. We need the identities

\[ (1) = S(11) \int f(x) P_n(x) \, dx = 0, \text{ for a polynomial of degree } \leq n \]

\[ (2) = S(12) \int x^n P_n(x) \, dx = 2^{n+1} n! n! / (2n+1)! \]

and the integral, for \( \tau \) even,
\[
\int_{-1}^{1} x^n (x^2 - 1)^5 \, dx = 2 \int_{0}^{1} x^n (x^2 - 1)^5 \, dx = \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{n}{2} + 1\right)
\]

So with (3)
\[
\int_{-1}^{1} \prod_{i \neq j} b_i (\partial_{x_i}) x^j \, dx
\]
The coefficient of $x^{p+q}$ in $x^k (x-1)^{2k}$ is 1.

$$\sum_{2k \leq n} (-1)^k \binom{n}{2k} \frac{(2k)!}{k!} \frac{1}{(n-2k)!} \frac{1}{(n+k)!} \left( \frac{1}{2} \right)^{n+1} \left( 1 + \frac{n+1}{2} + \frac{k}{2} \right)^{-1}$$

$$= 2^{n+1} n! n! / (2n+1)!$$

(7.50) By (7.17) and then canceling and rearranging factorials the l.h.s. is

$$\sum_{2k \leq n} (-1)^k \binom{n}{2k} \frac{n!}{k! (n-2k)!} \left( \frac{1}{2} \right)^{n+k} \left( \frac{m+n-k}{m+1} \right)! \left( \frac{\frac{1}{2} m + \frac{1}{2} n}{m+1} \right)! \left( \frac{\frac{1}{2} m + \frac{1}{2} n - k}{m+1} \right)!$$

$$= \left( \frac{m+n-2k+1}{m+1} \right)!$$

$$= \frac{n! m!}{(n+m+1)!} \sum_{2k \leq n} (-1)^k \left( \frac{\frac{k}{2} m + \frac{k}{2} n}{m} \right) \binom{m+n-2k}{m}$$

Let $m+n$ be even, $m+n = 2\epsilon$. Then this is equal to

$$\frac{m! n!}{(n+m+1)!} \sum_{2k \leq n} (-1)^k \binom{k}{m} \binom{2\epsilon-2k}{m}$$

Here we may extend the sum to $0 \leq k \leq m$ since for $m/2 < k \leq \epsilon$ we have $2\epsilon-2k > 0$ and $2\epsilon-2k = m+n-2k < m$. So with $\Delta(m)$ the l.h.s. of (7.50) is equal to

$$\frac{m! n!}{(n+m+1)!} \Delta(2\epsilon) \left( \frac{2\epsilon}{m} \right)$$

$$\sum_{k=0}^{m} \Delta(2\epsilon) \left( \frac{2\epsilon}{m} \right)$$
For $n > m$, this is zero since then $r > m$.
For $n \leq m$, so $r \leq m$, this is equal to

$$\frac{m! n!}{(n + m + 1)!} \left( \frac{r}{m - r} \right)^{2n - m} = \text{see } (3.155)$$

$$\frac{m! n!}{(n + m + 1)!} \left( \frac{\frac{1}{2} m + \frac{1}{2} n}{\frac{1}{2} m - \frac{1}{2} n} \right)^{2n}.$$  

From (7.49) with $q = 0$, we obtain (7.50) with $m = n$.

(7.52) - (7.55) The first equalities in (7.52) and (7.53) follow from $B(13)$ and $T(9)$. In (7.54) and (7.55) they follow from $B(14)$ and $T(9)$.

We start from the differential equation satisfied by the hypergeometric function $F(a, b; c; z)$:

(1) $z(1-z)F''(z) + (c-z(a+b+1))F'(z) - abF(z) = 0$,

and derive from (1) a differential equation for

(2) $G(x) = \frac{1}{2} F(a, b; c; \sin^2 x)$.

We have, with $E$ for $F$,

$$G'(x) = F'(\sin^2 x) 2 \sin x \cos x = F(\sin^2 x) \sin 2x.$$
\[ G''(x) = F''(\sin^2 x) \sin^2 2x + 2F'(\sin^2 x) \cos 2x \]

From (1) for \( x \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right) \)

\[
F''(\sin^2 x) = \frac{\left(\alpha + \beta + 1\right) \sin^2 x - \alpha^2}{\sin^2 x \cos^2 x} F'(\sin^2 x) + ab F(\sin^2 x)
\]

\[ 4 \left(\alpha + \beta + 1\right) \sin^2 x - 4c + 2 \cos 2x \]

\[ \sin^2 2x \]

so

(3) \[ G''(x) = 4abG(x) + \]

\[ 4 \left(\alpha + \beta + 1\right) \sin^2 x - 4c + 2 \cos 2x \]

\[ \sin^2 2x \]

(\(-\frac{\pi}{2}, 0\) \cup \(0, \frac{\pi}{2}\))

In (7.52) we have \( a = \frac{u}{2}, b = -\frac{u}{2}, \)

\( c = \frac{u^2}{4} \). So the differential equation

(3) becomes \[ G''(x) = -\frac{u^2}{4} G(x) \]

Since \( G(0) = 1, G'(0) = 0 \) we have \( G(x) = \cos u x \), i.e. (7.52) holds.

Note that this differential equation also holds for \( x = 0 \), by continuity or from the l.h.s. of (7.52).
In (7.53) we have \( a = \frac{u + 1}{2} \), \( b = -\frac{u + 1}{2} \), \( c = 1/2 \).

So the differential equation (3) becomes

\[
G''(x) = \frac{2 \sin x}{\cos x} G'(x) + (1 - u^2) G(x).
\]

This equation also holds for \( x = 0 \), by continuity or from the L.H.S. of (7.53).

The r.h.s. \( \cos u x / \cos x \) of (7.53) satisfies the same differential equation. Both sides have the same value 1 for \( x = 0 \) and the same derivative 0 for \( x = 0 \). So they are equal and (7.53) is proved.

Write (7.54) as

\[
\sum_{k=0}^{\infty} \left( -\frac{1}{2k+1} \right) \left( \frac{u}{2k+1} - \frac{1}{2} \right) \frac{1}{2k+1} \left( \frac{1}{2k+1} \right)^{-1} y (\sin x) = \frac{1}{4} \sin u x.
\]

From (7.53) we see that both sides here have the same derivative w.r. to \( x \). Both sides vanish for \( x = 0 \). So they are equal, i.e. (7.54) holds.

For (7.55) we apply Euler's formula \( I(16) \) to the second member and then we find with (7.54)
\[
F \left( \frac{u+1}{2}, -\frac{u+1}{2}; \frac{3}{2}; \sin^2 x \right) =
\]

\[
(1 - \sin^2 x)^{-\frac{1}{2}} F \left( -\frac{u+1}{2}, \frac{u+1}{2}; \frac{3}{2}; \sin^2 x \right) =
\]

\[
\frac{1}{\cos x} F \left( \frac{u+1}{2}, -\frac{u+1}{2}; \frac{3}{2}; \sin^2 x \right) =
\]

\[
\frac{1}{\cot x} \frac{\sin u x}{u \sin x} = \frac{2 \sin u x}{u \sin 2x}
\]

Applying Euler's formula (16) to (7.52) we obtain (7.53), a shorter proof than the above one.
(7.64) With $B(13)$ the l.h.s. is equal to
\[ \sum_{k=0}^{n} \binom{n}{k} \left( \frac{-1/2}{k} \right) \left( \frac{x+k}{k} \right) \circ \]

With (7.64) and (7.38) this is equal to
\[ \frac{(-n, 1/2; x+1; 1)}{x} = (x-\frac{1}{2} + n) \left( \binom{x+n}{n} \right)^{-1} \]
the last equality follows with $B(16)$.

(7.66) By canceling and rearranging factorials we have
\[ \sum_{2k \leq n} \binom{n}{2k} \binom{2k}{k} \binom{k+x}{k} \binom{n+x}{n} 2^{n-2k} = \]
\[ \sum_{2k \leq n} \binom{n-k}{k} \binom{n+x}{n-k} 2^{n-2k} = \]
\[ \sum_{\frac{n}{2} \leq h \leq n} \binom{n+x}{h} \binom{h}{n-h} 2^{2h-n} = \]
\[ \sum_{h=0}^{n} \binom{n+x}{h} \binom{h}{n-h} 2^{2h-n} \]

Now (3.397).
\[(7.67) - (7.69) \text{. We have by } B(13), B(14),\]
\[B(24) \text{ and canceling factorials}
\[
\sum_{k=0}^{n} \binom{2n}{n+k} \frac{(2n+2k+1)^{-1}}{(n+k)} y^k =
\]
\[(-1)^n \sum_{k=0}^{n} \binom{2n}{n+k} \frac{(-1/2)}{(n+k)} (-3/2)^{-1} y^k =
\]
\[\mu^n \sum_{k=0}^{n} \binom{2n}{n+k} (k-1/2)^{-1} \left(\sqrt{n+k+1/2}\right)^{-1} y^k
\]

With the beta integral the last expression is equal to
For $y=1$ the expression (***) is equal to

$$n \quad -n \quad \cdots \quad -1$$

$$\left(\frac{2n}{2n-1}\right)^{1/2^n - 1/2^n}$$
\[ \sum_{k=0}^{n} \frac{(-1)^k (k - \frac{1}{2})!}{k! (n-k)! (n+k+\frac{1}{2})!} = \int_{0}^{\infty} \frac{\sum_{k=0}^{n} (-1)^k (n-k)! (n+k+\frac{1}{2})!}{n! (n+1)!} \, dx. \]

With (4.46) and $B(12)$ this is equal to

\[ \int_{0}^{\infty} \frac{2^{2n} (2n)! (n! (4n+1)!)}{n! (n+1)!} \, dx, \]

which is (7.69).

(7.70) With $B(13)$, $D(24)$ and then (7.3) the l.h.s. is equal to

\[ x^{-1} \sum_{k=0}^{n} \frac{(-1)^k (n-k)! (-x)(-\frac{1}{2})}{k!} \left( x \right)^{-k} \]

\[ = x^{-1} \sum_{k=0}^{n} (-1)^k \frac{(n-k)! (-x)(-\frac{1}{2})}{k!} \left( x \right)^{-k} \]

\[ = (-1)^n x^{-1} (x^{-\frac{1}{2}}) \left( \frac{1}{n} \right)^{-\frac{1}{2}} \]

With $B(13)$ and $B(11)$ this is equal to

\[ (-1)^n x^{-1} \left( \frac{x^{-\frac{1}{2}}}{n} \right)^{\frac{1}{2}} = \]

\[ (-1)^n 4^n x^{-1} \frac{(x^{-\frac{1}{2}})!}{(x^{-\frac{1}{2}}-n)!} \frac{n!}{(2n)!} = \]
\[-\frac{(-1)^n x^{-1} (2x)!(x-n)!}{(2x-2n)! x! (2n)!} = (-1)^n x^{-1} \left(\frac{2x}{2n}\right) \left(\frac{x}{n}\right)^{-1}\]

(7.71) With $B(n)$, $D(2y)$ and then (7.3)

the L.H.S. is equal to

\[\sum_{k=0}^{n} (-1)^k \left(\frac{n}{k}\right) \left(-\frac{x}{k}\right) \left(-\frac{3}{2}\right)^{n-k} = \left(\frac{x-1/2}{n}\right) \left(-\frac{3}{2}\right)^{-1}\]

The second equality follows with $B(n)$. 

\[\frac{1}{n} \left(n \lln 2 \right) = \left(n \lln 1 \right) = \left(n \lln 2 \right)^{-1}\]
(7.81) By canceling and rearranging factorials and by $D(24)$ the l.h.s. is equal to

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2n-k}{n-k} (x+k)^{-1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \binom{n-k}{k} (x+k)^{-1}.$$ Now (7.40).

(7.82) The first equality follows with $B(29)$ that the l.h.s. is equal to

$$\text{E}_{\frac{1}{2}, \frac{1}{2}} \left( \frac{1}{2}, \frac{3}{2} ; \frac{3}{4} \right)$$

follows from I(2).

The last equality is proved by M.R. Spiegel, Monthly 59, 1962, 894-896. For $p = \frac{1}{4}$ see Problem 550, Nieuw Arch. (3), 18, 1980, 215-216.
Table 8, hypergeometric functions
and Fibonacci

(8.42) The first equality is \( I(b) \).
The second one follows with \( B(30) \).
Applying \( B(16) \) to the third member
and canceling and rearranging factorials
we obtain the third equality. The last
one is proved by bisection of the
binomial series, see G(9). See Dieder (3.8).

Special cases:

For \( u = (n-1)/2 \) we obtain

\[
\text{For } n \neq 0 \quad \text{with } z = \sqrt{uy}, \text{ this becomes, by (8.10), (8.11),}
\]

\[
(8.42.2) \quad _2 F_1 \left( \frac{1-n}{2}, 1-\frac{n}{2}; 2; z^2 \right) = \frac{1}{2n} \left\{ (1+z)^n - (1-z)^n \right\}, \quad n \neq 0
\]
(finitely many terms when \( n \geq 1 \))

From (8.42.2) with \( y = 1 \) or directly from
(8.7), (8.8), of Dieder (3.1)

\[
(8.42.3) \quad _2 F_1 \left( \frac{1-n}{2}, 1-\frac{n}{2}; \frac{3}{2}; 5 \right) = \frac{2^{n-1}}{n} F^{(5)}_{n-1}, \quad n \geq 1.
\]
Examples of special values of \( y \): From (8.42.2) and (8.31)
\[
(8.42.4) \quad {}_2F_1 \left( \frac{1-n}{2}, 1 - \frac{n}{2}; \frac{3}{2}; \frac{5}{9} \right) = \frac{2^{n-1}}{n} \frac{1}{3^{n-1}} {}_2F_{2n-1},
\]
\( n \neq 0 \).

Taking \( z = \sqrt{1+5y} = 5^{-1/2} \), \( y = -1/5 \) in (8.42.2)
we find \( (n \neq 0) \)
\[
(8.42.5) \quad {}_2F_1 \left( \frac{1-n}{2}, 1 - \frac{n}{2}; \frac{3}{2}; \frac{1}{5} \right) = \frac{2^{n-1}}{n} \Phi_{2n-1} \left( -\frac{1}{5} \right),
\]
and then with (8.24) and (8.23) or directly
with (8.42.1) and Binet's formula
\[
(8.42.5) \quad {}_2F_1 \left( \frac{1-m}{2}, 1 - m; \frac{3}{2}; \frac{1}{5} \right) = \frac{4^{m-1}}{m} \frac{1}{5^{1-m}} {}_2F_{2m-1},
\]
\( (m \neq 0) \),
\[
(8.42.6) \quad {}_2F_1 \left( -m, \frac{1-m}{2}; \frac{3}{2}; \frac{1}{5} \right) = \frac{4^m}{2m+1} \frac{1}{5^{-m}} \frac{L_{2m+1}}{2m+1}.
\]

\( \frac{\Gamma(2m+1)}{2m+1} \)
\[
For \ z = 5^{-1/2} \) in (8.42.7)
\[
\frac{5^{1/2}}{4^n} \left[ \frac{5^n}{4^{2n}} \left( 1 - \sqrt{5} \right)^{2n} - \frac{5^n}{4^{2n}} \left( 1 + \sqrt{5} \right)^{2n} \right] = \\
\frac{1}{\sqrt{5}} \frac{n+1}{2^n} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{2n} - \left( \frac{1 - \sqrt{5}}{2} \right)^{2n} \right] = \\
\frac{1}{n} 5^{n+1} \frac{F_{2n-1}}{2} \quad n \neq 0 \\
\text{cf. Dilcher (4,13).}
\]

\[(8.43)\] The first equality is 1(6). The second one follows with B(29). The third one follows by applying B(16) and then canceling and rearranging factorials. The last one is bisection of the binomial series, see G(8).

For \( u = \frac{n}{2} \), cf. Dilcher (10.15), (10.16),

\[(8.43.1) \quad F\left( -\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2}; 2 \right) = \frac{1}{2} \left( 1 + z \right)^n + \left( 1 - z \right)^n \]

and for \( z = \sqrt{1+4y} \) with (8.12)

\[(8.43.2) \quad F\left( -\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2}; 1+4y \right) = 2^{n-1} \frac{\Delta_n (y)}{\Delta_n (1)} \]

Here with \( y = 1 \), or \( z = \sqrt{5} \), or from Binet's formula (8.4),

\[(8.43.3) \quad F\left( -\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2}; 5 \right) = 2^{n-1} L_n \]
For \( n = 2m + 1 \) and \( n = 2m \) this becomes, by \((8.26)\) and \((8.25)\) respectively

\[
(8.43.5) \quad \frac{\Gamma \left( -m - \frac{1}{2}, -m \frac{3}{2} \right)}{\Gamma \left( -m - \frac{1}{2} \right)} = 2^{n-1} \left( -m \right)^{\frac{n-1}{2} - m},
\]

\[
(8.43.7) \quad \frac{\Gamma \left( n + \frac{1}{2}, n + 1 \frac{1}{2} \right)}{\Gamma \left( n + \frac{1}{2} \right)} = \frac{\Gamma \left( n + 1 \frac{1}{2} \right)}{\Gamma \left( n + \frac{1}{2} \right)} z^2.
\]
These identities are the same as those in (8.42). Put \( u = y - 1/2 \).

A direct proof: The first equality is (8.16).

With \( B(17) \) and \( B(30) \) and then canceling and rearranging factorials we prove the second equality.

The last one is Bisection of the binomial series, cf. G(9).

For \( u = n - 1 \) the identity (8.44) becomes

\[
\left(8.44.1\right) \quad F\left(1-n, \frac{1}{2} - n, \frac{3}{2}; z^2\right) = \frac{1}{2n} \left\{ \left(1+z^2\right)^{2n} - \left(1-z^2\right)^{2n} \right\}, \quad n \geq 1
\]

(Or \( |z| < 1 \) when \( n < 0 \)).

For \( z = \sqrt{1+y} \) this becomes with (8.10), (8.11)

\[
\left(8.44.2\right) \quad F\left(1-n, \frac{1}{2} - n, \frac{3}{2}; 1+y\right) = \frac{y^{n-1}}{2n} \left\{ \left(\frac{1+\sqrt{1+y}}{2}\right)^{2n} - \left(\frac{1-\sqrt{1+y}}{2}\right)^{2n} \right\}
\]

in particular with \( y = 1 \) or directly with Binet's formula (8.7), (8.8)

\[
\left(8.44.3\right) \quad F\left(1-n, \frac{1}{2} - n, \frac{3}{2}; 5\right) = \frac{y^{n-1}}{n} F_{2n-1}, \quad n \geq 1
\]

For \( z = (1+y)^{1/2} \) in (8.44.1)
\[ n \left( \frac{1+4y}{2} \right)^n \left[ \left( \frac{1}{2} \right) - \left( \frac{1}{2} \right)^m \right] = \frac{4^{n-1} (1+4y)^{n-1} \Phi_{2n-1}(y)}{n}, \quad n \geq 1. \]

In particular, for \( y = 1 \), or directly with Binet's formula (8.7), (8.8)

\[(8.44.5) \quad \frac{\phi_{2} (1-n, \frac{1}{2}, -n; \frac{3}{2}; \frac{1}{2})}{\phi_{2} (1, \frac{1}{2})} = \frac{1}{n} \left( \frac{4}{5} \right)^{n-1} \Phi_{2n-1}, \quad n \geq 1. \quad \text{cf. Dilcher (4.11)}
\]

\[ n \neq 0. \text{ See (8.16).} \]

\[(8.45) \text{ The same as (8.43). Put } v = u + \frac{1}{2}. \]

Direct proof. The first equality follows with \( \text{T}(6) \). The second one is shown by applying \( B(17) \) and \( B(19) \) and canceling and regrouping factorials. The last one is a consequence of the binomial series, see G(8).

Special cases may be proved in the same way as in (8.12) and (8.11).
(8.46.1) The first equality is $I(3)$. The second one follows from $B(30)$ and then canceling and rearranging factorials. The last one follows from $\Phi(132)$ by writing

$$(-yz)^k = (-yz)^{n-k} \cdot \left( \frac{1}{-yz} \right)^{n-k}.$$ 

With (8.24) we have

$$\left( \frac{F}{n} \right)_{21} \left( 1-n, 1+n; \frac{3}{2}; \frac{-1}{4} \right) =$$

$$\frac{1}{n} \left( -5 \right)^{n-1} \Phi_{2n-1} \left( -1/5 \right) = \frac{1}{n} (-1)^n \frac{F_{4n-1}}{F_{2n-1}}, \quad n \geq 1,$$

see Dilcher (4.45) where a factor $-1$ is missing.

From (8.28)

$$\left( \frac{F}{n} \right)_{21} \left( 1-n, 1+n; \frac{3}{2}; -5/4 \right) =$$

$$\frac{1}{n} \cdot 5^{n-1} \Phi_{2n-1} (1/5) = \frac{1}{3n} \frac{F_{4n-1}}{F_{2n-1}}, \quad n \geq 1.$$

Cf. Dilcher (4.40).

(8.46.3) For $z = -1/4$ we have the special value

$$2 \frac{F}{n} \left( 1-n, 1+n; \frac{3}{2}; -1/4 \right) = \frac{1}{n} \frac{F_{4n-1}}{F_{2n-1}}, \quad n \geq 1.$$ 

It could be derived directly by $F(67)$

See Dilcher (4.48).
\[ n \geq 1, \text{ cf. Dilcher (3.6),} \]
and by complex conjugation, or with (8.19),

\[(8.47) \text{ The first equality follows from } I(3). \]
The second one follows by \( B(29) \) and canceling and rearranging factorials.

Write the third member as

\[
\frac{1}{n} (-yz)^{n-1} \sum_{k=0}^{n} \frac{n!}{(n-k)! (2k)!} \left( \frac{1}{2} \right)^{n-1-k} (-yz)^{n-1-k}.
\]
From (5.21):
\[
\Phi(2m)(x) = \sum_{h=0}^{m} \binom{m+h}{2h} x^{m-h},
\]

So,
\[
\Phi(2m)(x) = \sum_{h=0}^{m} \binom{m+h}{2h} x^{m-h}.
\]
From (8.13) and (8.14):
\[
\Phi_n(x) = \frac{1}{n+1} \left( \frac{n+1}{n} \right)^{-1} \left( \frac{n+1}{n} \right) x - 2 \Phi_n(x).
\]
\[
\frac{(-1)^n}{n} \left\{ (2n+1) L_{2n} - 2 L_{2n+1} \right\} =
\]

Eq. 8.47.1
\[
\sum_{j=0}^{\infty} \binom{2j+1}{j} \frac{1}{(2+4s)^{j+1}} = \frac{1}{\sqrt{\pi s}} \frac{1}{2+4s} \quad \text{with} \quad (s < 0.5).
\]
\[ (8.47.2) \quad \frac{\Gamma\left(1 - n, n + 1 - \frac{1}{2}; -\frac{5}{4}\right)}{2} = \]

\[
\frac{5^n}{q_n} \left\{ \frac{(2n+1)}{2n} \Phi_{2n} \left( \frac{1}{5} \right) - (2n+1) \Phi_{2n-1} \left( \frac{1}{5} \right) \right\} =
\]

\[
\frac{5^n}{q_n} \left\{ \frac{4n}{2} \Phi_{2n} \left( \frac{1}{5} \right) - (2n+1) \Phi_{2n-1} \left( \frac{1}{5} \right) \right\} =
\]

\[
\frac{1}{q_n} \left\{ \frac{(2n+1)}{2n} \frac{4n}{3} \Phi_{2n} \left( \frac{1}{5} \right) - \frac{2}{3} \Phi_{2n+2} \left( \frac{1}{5} \right) \right\} =
\]

\[
\frac{1}{q_n} \left\{ \frac{(2n+1)}{3} \frac{4n}{3} \Phi_{2n+2} \left( \frac{1}{5} \right) - \frac{5}{3} (2n+1) \Phi_{2n-1} \left( \frac{1}{5} \right) \right\}, \quad n \geq 1
\]

(8.48) The first equality is \(I(10)\). The second one follows from \(B(30)\) and by canceling and rearranging factorials. The last equality is proved by \(Q(124)\).

Special cases:

With (8.26) (cf. Dilcher (4.4))

\[ (8.48.1) \quad \frac{\Gamma\left(-n, n+1 ; \frac{3}{2}; -\frac{5}{4}\right)}{\Gamma\left(2n+1, 2n+1; \frac{3}{2}; -\frac{5}{4}\right)} =
\]

\[
\frac{(-5)^n}{2n+1} \frac{\Phi_{2n+1} \left( -\frac{1}{5} \right) - \frac{1}{2} \Phi_{2n} \left( -\frac{1}{5} \right)}{2n+1} \quad n \geq 0.
\]

And with (8.30), cf. Dilcher (4.39),
\[(8.48.2) \quad \frac{5^n}{2n+1} \Gamma(-n, n+1; \frac{3}{2}; -\frac{1}{4}) = \frac{1}{2n+1} F_{\frac{1}{4}, \frac{1}{4}}^{\frac{3}{2}, \frac{3}{2}}(n \geq 0),\]

With (8.14)

\[(8.48.3) \quad \frac{5^n}{2n+1} \Gamma(-n, n+1; \frac{3}{2}; -\frac{1}{4}) = \frac{(-1)^n}{2n+1} \cos \left(\frac{(2n+1)\pi}{3}\right) (n \geq 0),\]

(8.49) The first equality is \(I(10).\) The second one follows from \(B(29)\) by canceling and rearranging factorials. The last one follows from \(\phi(21).\)

Cf. Dilcher (4.14) From (8.50)

(8.50) The first equality is \(I(10).\) The second one follows from \(B(29)\) by canceling and rearranging factorials. The last one follows from \(\phi(21).\)

From (8.35), (8.33), (8.35), (8.34) we obtain, with

\[y_1 = 2 + \sqrt{5}, \quad y_2 = 2 - \sqrt{5},\]

the special cases (Cf. Dilcher, (6.3)).
\[ (8.50.2) \quad F\left( -2m-1, 2m+1; \frac{1}{2}; \frac{1}{4} (2 + \sqrt{5}) \right) = \frac{1}{2} \left( -\gamma_1 \right)^{2m+1} \frac{\Gamma_{2m+2} \left( -\frac{1}{\gamma_1} \right) + \frac{1}{2} \sqrt{5} F_{2m}}{m \geq 1}, \]

\[ (8.50.3) \quad F\left( -2m, 2m; \frac{1}{2}; \frac{1}{4} (2 - \sqrt{5}) \right) = \frac{1}{2} \left( -\gamma_2 \right)^{2m} \frac{\Gamma_{2m} \left( -\frac{1}{\gamma_2} \right)}{m \geq 1}, \]

\[ (8.50.4) \quad F\left( -2m-1, 2m+1; \frac{1}{2}; \frac{1}{4} (2 - \sqrt{5}) \right) = \]

\[ (8.51) \quad \text{The first equality is } I(13). \text{ The second one follows from } B(17) \text{ by canceling and rearranging factorials. The last one follows from } \Phi(124). \]

\[ (8.50.1) \quad F\left( -n, -\frac{1}{2} - n; -2n \right) = \]

\[ A_{\frac{1}{2}+1} (-\frac{1}{5}) = 5^{-n} F_{\frac{3}{2}}, \quad \text{cf. Dilcher (4.14)}, \]

\[ n \geq 0, \]
\[(8.51.2) \quad F_{11}(-n, -\frac{1}{z}, -2n, -\frac{1}{z}, -1) = \Lambda_{2n+1}(-\frac{1}{z}) = 5^{-n} F_{4n+1}, \text{ of Dilcher (4.36) for } n > 0.\]

We cannot apply (8.33) - (8.35) since the index of \(\Lambda\) in (8.51) is odd. We therefore apply \(\Phi(44)\) giving

\[(*) \quad F_{11}(-n, -\frac{1}{z}, -2n, -\frac{1}{z}, -1) = \Phi_{2n+1}(-\frac{1}{z}) - \frac{1}{y^2} \Phi_{2n-1}(-\frac{1}{z}).\]

We use the notation of (8.33) - (8.37):

\[\gamma_1 = 2 + \sqrt{5}, \quad \gamma_2 = 2 - \sqrt{5}, \quad \gamma_1 \gamma_2 = -1.\]

First we want to take \(-\frac{1}{z} = -\frac{1}{\gamma_1} = \gamma_2\), \(z = -\gamma_2 \gamma_1 = -4(\gamma_2 + \sqrt{5})\).

Then from (*) with \(n = 2m\), by (8.36), (8.37)

\[(8.51.3) \quad F_{11}(-2m, -2m; -\frac{1}{\gamma_2}, -4m; -\gamma_2) = \]

\[\Phi_{2m+1}(-\frac{1}{\gamma_1}) - \frac{1}{y_1} \Phi_{2m-1}(-\frac{1}{\gamma_1}) = \]

\[\gamma_1^{-2m} L_{2m+1} - \frac{1}{y_1} (-y_1)^{-2m+1} (-5^{1/2}) F_{2m-1} = \]

\[\gamma_2^{-2m} \left( L_{2m+1} - 5^{1/2} F_{2m-1} \right).\]
And from (2) with \( n = 2m+1 \) by (8.36), (8.37)

\[
(8.51.4) \quad \Phi \left( \frac{-2m-1}{2}, -2m-\frac{3}{2}; -4m-2; -4\gamma_1 \right) =
\]

\[
\Phi \left( \frac{-1}{\gamma_1} \right) = \frac{1}{\gamma_1} \Phi \left( \frac{-1}{\gamma_1} \right) =
\]

\[
-5^{1/2} \left( -\gamma_1 \right)^{-2m-1} \frac{\gamma_1}{2m+1} \sum_{s=0}^{m} \frac{1}{\gamma_1^{s+1}} \gamma_1^{-2m+1} L_{2m+1} =
\]

\[
\frac{\gamma_1^{2m+1}}{2} \left( -5^{1/2} \frac{\gamma_1}{2m+1} + L_{2m+1} \right).
\]

Then we take \( \frac{1}{4} z = -\frac{1}{\gamma_2} = \gamma_1 \)

\[
z = -\gamma_1 = -4(2+\sqrt{5})
\]

Then from (2) with \( n = 2m \) by (8.36), (8.38),

\[
(8.51.5) \quad \Phi \left( -2m, -2m-\frac{1}{2}; -4m; -4\gamma_1 \right) =
\]

\[
\Phi \left( \frac{-1}{\gamma_2} \right) = \frac{1}{\gamma_2} \Phi \left( \frac{-1}{\gamma_2} \right) =
\]

\[
\frac{\gamma_1^{-2m+1}}{2} \sum_{s=0}^{m} \frac{1}{\gamma_2^{s+1}} \gamma_1^{2m} L_{2m+1} =
\]

\[
\frac{\gamma_2^{2m}}{2} \left( -5^{1/2} \gamma_1^{2m} L_{2m+1} + 5^{1/2} \gamma_1^{2m} \gamma_1 \frac{\gamma_2}{2m-1} \right).
\]

and for \( n = 2m+1 \) with (8.38), (8.36)
\( (8.51.6) \quad F_{2m} \left( -2m-1, -2m-\frac{3}{2}; -ym-2; -\gamma z \right) = \)
\[
\Phi_{ym+3} \left( -\frac{1}{\gamma z} \right) - \frac{1}{\gamma z} \Phi_y {ym+1} \left( -\frac{1}{\gamma z} \right) = \\
5^{-\frac{1}{2}} (-\gamma z)^{-2m-1} \frac{\Gamma}{2m+1} \frac{1}{\gamma z} \gamma^{-2m} L_{2m+1} =
\]

\( (8.52) \) The first equality is \( I(q) \). The second one follows with \( B(16) \) by canceling and rearranging factorials. The last one follows from \( \Phi(68) \).

For \( z = y/5 \) and \( z = -y/5 \) we obtain the special cases

\( (8.52.1) \quad F_{1-n, \frac{1}{2}-n; 1-2n; y/5} = \)
The first equality is \( T(3) \). The second one follows from \( B(19) \) by canceling and rearranging factorials. The third equality follows from \( C(132) \) by bisection of series, see \( G(\theta) \).

For \( a = m \) the third member in \( (8.53) \)
is equal, by \( \Phi(123) \), to

\[
\frac{1}{2} (4z^2)^m \Lambda_{\frac{1}{2}m} \left( \frac{1}{4z^2} \right).
\]

That this is equal to the fourth member in \( (8.53) \) may be seen by \( \Phi(18), (17) \). See also \( (8.50) \).

For \( a = n + \frac{1}{2} \), \( z = \frac{1}{2} \) in \( (8.53) \), we obtain

\[
(8.53.2) \quad \frac{\Gamma(-m, m; \frac{1}{2}; -z^2)}{\Gamma(-m, m; \frac{1}{2}; -z^2)} = \frac{1}{2} (4z^2)^m \Lambda_{\frac{1}{2}m} \left( \frac{1}{4z^2} \right).
\]

That this is equal to \( \frac{1}{2} F_{2n} \sqrt{5} \), cf. Ditcher (4:19).

The first equality is \( T(4) \). The second one follows with \( B(30) \) by canceling and rearranging factorials. Writing the third member as...
\[
\frac{1}{\zeta(2\alpha-1)} \sum_{k=0}^{\infty} \frac{a^{-\frac{1}{2}}}{a^{-\frac{1}{2}} + (2k+1)^{\frac{1}{2}}} \left( \frac{a^{-\frac{1}{2}} + (2k+1)^{\frac{1}{2}}}{2k+1} \right)^{2k+1} (2z)^{2k+1} \]

we prove the last equality from (6.132) by binomial expansion, see (6.19).

Remark For \( a = 1/2 \) we obtain from the first equality in (8.54)

\[
(8.54.1) \quad \frac{F \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2 \right)}{2} = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \frac{(-1)^k}{k} z^{2k},
\]

and from the last equality

\[
(8.54.2) \quad \frac{F \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2 \right)}{2} = \frac{1}{2Z} \log \frac{\sqrt{1+z^2} + z}{\sqrt{1+z^2} - z} = \frac{1}{2} \log \left( z + \sqrt{1+z^2} \right)
\]

For \( a = m \geq 1 \) the third member in (8.54) becomes, with \( \Psi (124) \)

\[
\frac{(4z^2)^{m-1}}{2^{m-1}} \sum_{k=0}^{m-1} \frac{2(m-1)+1}{m-1+k} \left( \frac{1}{2k+1} \right) (4z^2)^{m-1-k} (4z^2)^{m-1} \]

\[
\frac{(4z^2)^{m-1}}{2^{m-1}} A_{2m-1} (1/4z^2)
\]

So
(8.54.3) \( F_\frac{1}{2} (m, 1-m; \frac{3}{2}; -z^2) = \frac{(yz^2)^{m-1} \Gamma(1/2)}{2m-1} \cdot \) 

That this is the same as the last member in (8.54) follows with \( \Phi(18), (27) \) 

For \( a = m + \frac{1}{2} \), we obtain, with \( \Phi(18), \Phi(26) \) 

\[
(8.54.4) \quad F_\frac{1}{2} (m + \frac{1}{2}, 1-m; \frac{3}{2}; -z^2) = \frac{1}{4mz^2} \left[ (z + \sqrt{1+z^2})^{2m} - (z - \sqrt{1+z^2})^{2m} \right] = \]
\[
\frac{z^{2m-2}}{4m^2} \left[ \left( 1 + \frac{\sqrt{1+z^2}}{yz^2} \right)^{2m} - \left( 1 - \frac{\sqrt{1+z^2}}{yz^2} \right)^{2m} \right] = \]
\[
\frac{(2z)^{2m-2}}{m} \left( 1+ z^2 \right)^{-\frac{1}{2}} \Phi_\frac{1}{2m-1} \left( \frac{1}{yz^2} \right), \quad m \geq 1, \quad |z| < 1 \]

In particular for \( z = \frac{1}{2} \), more direct from \( (8.54) \) and \( (8.7), (8.8) \), cf. Disterher (4.20) 

\[
(8.54.5) \quad F_\frac{1}{2} (m + \frac{1}{2}, 1-m; \frac{3}{2}; -\frac{1}{4}) = \frac{\sqrt{5}}{2m} F_\frac{1}{2m-1}, \quad m \geq 1. \]

From (8.54.4) and (8.54), third member 

\[
(8.54.6) \quad \sum_{k=0}^{\infty} \frac{1}{m^{\frac{1}{2}+k}} \left( \frac{m^{\frac{1}{2}+k}}{2k+1} \right) (2z)^{2k} = \]
\[
\frac{(2z)^{2m-3}}{m-1} (1+z^2)^{-\frac{1}{2}} \Phi_\frac{1}{2m-1} \left( \frac{1}{yz^2} \right), \quad m \geq 1, \quad |z| < 1. \]
(8.55) The first equality is \( I(9) \). The second one follows with \( B(19) \) by rearranging and canceling factorials. The last one is proved by bisection of \( C(134a) \), see \( G(8) \).

For \( \alpha = \beta \) the third member of (8.55) is equal to

\[
(yz^2)^{m-1} \sum_{k=0}^{m-1} \frac{(m-1+k)!}{2^k k!} \left( \frac{1}{y^2} \right)^k = (yz^2)^{m-1} \Phi_{2m-2} \left( \frac{1}{yz^2} \right), \text{ see } G(121).
\]

(8.55) \[ F_l \left( m, 1-m; \frac{1}{2}; -z^2 \right) = (yz^2)^{m-1} \Phi_{2m-2} \left( \frac{1}{yz^2} \right), \text{ } m \geq 1. \]

That this is the same as the last member of (8.55) follows with \( \Phi \) (18), (26).

For \( \alpha = m+\frac{1}{2} \) we have from (8.85), (8.40), (8.12)

(8.55.2) \[ F_l \left( m+\frac{1}{2}, 1-m; \frac{1}{2}; -z^2 \right) = \]

\[
\frac{1}{2} \left( 1+z^2 \right)^{-1/2} \left[ (z+\sqrt{1+z^2})^{2m} + (z-\sqrt{1+z^2})^{2m} \right] =
\]

\[
2^{m-1} z^{2m} (1+z^2)^{-1/2} \left[ \left( \frac{1}{2} + \frac{1}{2} \sqrt{1+z^2} \right)^{2m} + \left( \frac{1}{2} - \frac{1}{2} \sqrt{1+z^2} \right)^{2m} \right]
\]

\[
= 2^{m-1} z^{2m} (1+z^2)^{-1/2} \frac{1}{2m} \Phi_{2m} \left( \frac{1}{yz^2} \right), |z| < 1.
\]
From (8.55.2) and the third member of (8.55)

\[
\sum_{k=0}^{\infty} \left( \frac{m+k-\frac{1}{2}}{2} \right) (2z)^{2k} = \nabla
\]

\[
2^{2m-1} z^{m} (1+z)^{-\frac{1}{2}} I_{2m} \left( \frac{1}{yz^2} \right), \quad |z| < 1.
\]

(8.56) The first equality is \( I(q) \). The second one follows from (8.16) by canceling and rearranging factorials. The last one follows from \( C(18,6) \).

Remark: When \( a = m \in N \), we have to be careful. By (8.68) we have

\[
\sum_{k=0}^{m} \binom{2m-k}{k} \left( -\frac{1}{y} z \right)^{k} = \Phi_{2m} \left( -\frac{1}{y} z \right).
\]

But when we take \( a=m \) in the third member of (5.56) the terms with \( k > 2m \) do not vanish.
(10.2) The first equality is \( G(12) \). Since 
\[ x^3 \] is a polynomial of degree 2n.

In particular, 
\[ \triangle^n \binom{x}{n} = \frac{(2n)!}{(n!)^2} \]
\[(10.3) \quad \sum_{k=0}^{n} \left( \binom{n}{k} \binom{2n-k}{n} \right) k^2 \Delta_k f(n, x) \bigg|_{x=0} = \]

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(n, j) =
\]

\[
\sum_{j=0}^{n} \frac{1}{j!} f(n, j) \sum_{j=0}^{n} (-1)^{k-j} \binom{2n-k}{n-k} \frac{(2n-k)!}{(n-k)! (n-k)! (k-j)!} =
\]

\[
\sum_{j=0}^{n} \frac{1}{j!} f(n, j) \sum_{h=0}^{n-j} (-1)^{h} \binom{2n-j-h}{n-j-h} \frac{(2n-j-h)!}{(n-j-h)! (n-j-h)! h!} =
\]

\[
\sum_{j=0}^{n} \binom{n}{j} f(n, j) \sum_{h=0}^{n-j} (-1)^{h} \binom{2n-j-h}{n-j-h} \binom{n-j}{h} =
\]

\[
\sum_{j=0}^{n} \binom{n}{j} f(n, j) \Delta_{n-j} \left( \frac{x}{n} \right) \bigg|_{x=n} =
\]

\[
\sum_{j=0}^{n} \binom{n}{j} f(n, j)
\]

For \( f(n, j) = \binom{n}{j} \) this general summation formula reduces to (10.3).

(10.4) The l.h.s. is equal to

\[
\sum_{k=0}^{3n+1} \left( \binom{3n+1}{k} \frac{x+3n-k}{n} \right)^3 + (-1)^{3n} \frac{(x-1)^3}{n}.
\]
Here the first term is the \((3n+1)\)th difference of a polynomial of degree \(3n\). So it vanishes. For \(x=0\), the r.h.s. of (10.4) reduces to \((-1)^n(-1)^{3n}=1\). The sum may be restricted to \(0 \leq k \leq 2n\).

\[(10.5) \text{ We have } S'(m, \xi) = \sum_{j=0}^{m+r} \left( \begin{array}{c} m \end{array} \right)^2 \sum_{k=0}^{m+r} (-1)^k \left( \frac{\xi}{k} \right)^2 \left( \frac{m+r}{k} \right)^{-1} = \sum_{j=0}^{m} \left( \begin{array}{c} m \end{array} \right)^2 \sum_{h=0}^{m+r-j} (-1)^{j+h} \left( \frac{\xi}{h} \right)^0 \left( \frac{m+r}{j+h} \right)^{-1} = \sum_{j=0}^{m} (-1)^j \left( \begin{array}{c} m \end{array} \right)^2 \sum_{h=0}^{r} (-1)^h \left( \frac{\xi}{h} \right)^0 \left( \frac{m+r}{j+h} \right)^{-1} \]

From (7.9) with \(n=\xi\), \(p=j\), \(q=m-j\) we see that the sum over \(h\) vanishes for \(p+q<n\), i.e., \(m<\xi\) and is equal to \((-1)^j (2\xi)^{-1}\) for \(m=\xi\). So

\[S'(m, \xi) = 0, \quad m<\xi, \]

\[S'(\xi, \xi) = \sum_{j=0}^{m} \left( \begin{array}{c} m \end{array} \right)^2 \left( \frac{\xi}{\xi} \right)^{-1} = \left( \frac{\xi}{\xi} \right)^{-1} \left( \frac{m}{m} \right) = 1. \]

By (3.1)

The assertion for \(m>\xi\) follows.
since $S(m, r) = S(r, m)$ as is seen by writing
\[ S(m, r) = \sum_{k=0}^{m+r} \binom{k}{k} (m+r)^{-1} \sum_{i+j=k} (m)^2 \binom{m}{i} \binom{m}{j}. \]

\[ (10.6) \sum_{k=0}^{m+r} \binom{k}{k} \sum_{j=0}^{k} \binom{m}{j} (r)^2 (k-j)^2 = \]
\[ \sum_{j=0}^{m} \binom{m}{j} \sum_{k=j}^{m+r} (-1)^{k-j} (k-j)^2 \]
\[ \sum_{j=0}^{m} \binom{m}{j} \sum_{h=0}^{r} (-1)^{j+h} (r)^2 (h)^2 = \]
\[ \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} \cdot \sum_{h=0}^{r} (-1) (h)^2, \]
and (10.6) follows with (3.7).

\[ (10.7) \sum_{j=0}^{n} \binom{c-a-b}{n-j} \binom{a-c}{j} \binom{b-c}{j} (-c)^{-1} \]
\[ = \binom{-a}{n} \binom{-b}{n} (-c)^{-1}. \]
This identity is a consequence of
Euler's formula I (16), see Slater (1960), (15.12). With (15.9) and the binomial series, this may be written as

\[
\sum_{n=0}^{\infty} (-1)^n \binom{-a}{n} \binom{-b}{n} \binom{-c}{n} z^n = \sum_{h=0}^{\infty} (-1)^h \binom{a-a-b}{h} z^h.
\]

and (10.7) follows with the convolution (or product) property of generating functions.

(10.8) The L.h.s. is equal to

\[
(\gamma) \sum_{k=0}^{n} \binom{n}{k} (\frac{x}{k}) (\frac{y-x}{k}) (\frac{x+k}{k}) (-1)^{x+y+n} = \]

\[
(\gamma) \sum_{k=0}^{n} (-n+k-1)! (\frac{-x+k-1}{k}) (-y+2+k-1)! (x+y+n-1) \binom{x+y+n}{k} \]

\[
\times (\frac{k+1+k-1}{k}) (-x-y-n+k-1) \binom{-x-y-n+k-1}{k} =
\]

\[
\binom{n}{k} F(-n, -x, -y+x; \frac{k+1}{k} -x-y-n; z).
\]

We now apply Saalschütz's formula,
see G.K.P. (1988), (5.97), p.214:

\[ 3F_2 \left( -n, a, b; c, a+b-c-n+1 \right) = \]

\[ (c-a)^{(n)}(c-b)^{(n)}/c^{(n)}(c-a-b)^{(n)} = \]

\[ \binom{c-a+n-1}{n} \binom{c-b+n-1}{n} \]

\[ \binom{c+n-1}{n}^{-1} \binom{c-a-b+n-1}{n}^{-1} \]

For \( a = -\gamma, b = -\gamma + 2, c = \gamma + 1 \) this gives

\[ \binom{\gamma}{n} 3F_2 \left( -n, -\gamma, -\gamma + 2; \gamma + 1, -\gamma - n+1 \right) = \]

\[ \binom{\gamma}{n} \binom{x+\gamma+n}{n} \binom{\gamma+n}{n}^{-1} \binom{x+\gamma+n}{n}^{-1} \]

\[ \binom{x+\gamma+n}{n} \binom{\gamma+n}{n+2} \binom{x+\gamma+n}{n}^{-1} \]

\[ \sum_{n=0}^{\infty} w^n \sum_{k=0}^{n} \binom{x}{k} \binom{\gamma}{k} \binom{\gamma}{n-k} \left( \frac{z}{x+y+z} \right)^{-1} = \]
\[
\sum_{k=0}^{\infty} \binom{x}{k} \binom{y}{k} \binom{x+y+z}{k}^{-1} \sum_{n=0}^{\infty} w^n \binom{z}{n-k} = 
\]
\[
(1+w)^z \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \binom{x}{k} \binom{y}{k} \binom{x+y+z}{k}^{-1} (-w)^k 
\]
\[
(1+w)^z \ {}_2F_1 \left( -x, -y; -x-y-z; -w \right),
\]
see I(g). We apply Euler's formula I(16), see Slater (1966), (1.5.12):
\[
(1+w)^{c-a-b} \ {}_2F_1 \left( c-a, c-b; c; -w \right) = \ {}_2F_1 \left( c-a, c-b; c; -w \right).
\]
For \( a= -x-z \), \( b= -y-z \), \( c= -x-y-z \) this gives
\[
(1+w)^z \ {}_2F_1 \left( -x, -y; -x-y-z; -w \right) = 
\]
\[
(1+w)^z \ {}_2F_1 \left( -y, -x; -x-y-z; -w \right) = 
\]
\[
\ {}_2F_1 \left( -x-z, -y-z; -x-y-z; -w \right) = 
\]
\[
\sum_{n=0}^{\infty} \binom{x+z}{n} \binom{y+z}{n} \binom{x+y+z}{n}^{-1} w^n,
\]
see I(g). Our identity now follows.

Changing the order of summation is justified since the double sum is absolutely convergent: From B(45)
\[ \sum_{n=0}^{\infty} \frac{1}{(x+y+z)^{n+k}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(z)^k} \left( \frac{1}{(n-k)} \right)^{-1} \]

\[ \sum_{k=0}^{\infty} \frac{1}{(x+y+z)^{n+k}} \sum_{m=0}^{\infty} \frac{1}{(z)^m} \left( \frac{1}{(m)} \right)^{-1} \]

\[ \sum_{k=0}^{\infty} \frac{1}{(x+y+z)^{n+k}} \sum_{m=0}^{\infty} \frac{1}{(z)^m} \left( \frac{1}{(m)} \right)^{-1} \]

which converges for small \( w \) since the binomial coefficients behave as powers of \( k \) by \( B(\gamma) \).

\[ \sum_{k=0}^{\infty} \frac{1}{(x+y+z)^{n+k}} \sum_{m=0}^{\infty} \frac{1}{(z)^m} \left( \frac{1}{(m)} \right)^{-1} \]

(10.10) By canceling and rearranging factorials the l.h.s. is equal to

\[ \binom{n+y}{y} \sum_{k=0}^{n} \binom{y}{n-k} \binom{y}{n+k} = \]

\[ \binom{n+y}{y} \sum_{h=0}^{n} \binom{y}{h} \binom{y}{2n-h}, \]

and (10.10) follows with (3.40).

(10.11) By canceling and rearranging factorials the l.h.s. is equal to
\[
\frac{(n+y)^{-2} \sum_{k=0}^{n} (-1)^k \binom{n+y}{n-k} \binom{n+y}{n+k}}{(n+y)^{-2} \sum_{h=0}^{n} (-1)^{n-h} \binom{n+y}{h} \binom{n+y}{2n-h})} =
\]

From (3.67) we have:

\[
(-1)^n \binom{x}{n} = \sum_{k=0}^{2n} (-1)^k \binom{x}{k} \binom{x}{2n-k} =
\]

\[
\sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{x}{2n-k} + \sum_{k=n}^{2n} (-1)^k \binom{x}{k} \binom{x}{2n-k} + (-1)^{n+1} \binom{x}{n}
\]

\[
\sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{x}{2n-k} + \sum_{h=0}^{n} \binom{x}{h} \binom{h}{2n-h} + (-1)^{n+1} \binom{x}{n}
\]

2 \sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{x}{2n-k} + (-1)^{n+1} \binom{x}{n}.

So,

\[
\sum_{k=0}^{n} (-1)^k \binom{x}{k} \binom{x}{2n-k} = \frac{1}{2} (-1)^n \binom{x}{n} + \frac{1}{2} (-1)^n \binom{x}{n}. \]

In (\star) this gives (10.11).

(10.12) By canceling and rearranging factorials we see that the L.H.S. is equal to
\[
\left( \frac{n+y}{n} \right)^{-1} \left( \frac{n+y}{n+1} \right)^{-1} \sum_{k=0}^{n} \binom{n+y}{n-k} \left( \frac{n+y}{n+1+k} \right) \]

\[
\left( \frac{n+y}{n} \right)^{-1} \left( \frac{n+y}{n+1} \right)^{-1} \sum_{h=0}^{n} \binom{n+y}{h} \left( \frac{n+y}{2n+1-h} \right),
\]

and (10.12) now follows with (3.41).

The identity (12.5) in Gould (1972c) with \( n \) replaced by \( n+1 \) evaluates the above sum as

\[
\frac{1}{2} \left( -1 \right)^{n+1} \left( \frac{-y+1/2}{n} \right) \left( \frac{-y+1}{n} \right)^{-1} \left( \frac{-1/2}{n} \right)^{-1}
\]

That this is the same as our result may be derived with B(13) and B(16).

(10.13) When \( a+b = c+d \) we have

\[
\Delta \left( \frac{(a+x)!(b+x)!}{(c+x)!(d+x)!} \right) = \left( \frac{(a+x)!(b+x)!}{(a+x)!(d+x)!} \right) \left( \frac{(a+x+1)(b+x+1)}{(c+x+1)(d+x+1)} - 1 \right) = \]

\[
\left( \frac{ab-cd}{a+b} \right) \left( \frac{(a+x)!(b+x)!}{(c+x+1)!(d+x+1)!} \right)
\]

So

\[
\left( \frac{ab-cd}{a+b} \right) \sum_{k=0}^{n} \frac{(a+k)!(b+k)!}{(c+k+1)!(d+k+1)!} = \]

\[
\frac{(a+n+1)!(b+n+1)!}{(c+n+1)!(d+n+1)!} - a!b!/c!d!.
\]
Putting \((a+k)! = a! k! \left(\frac{a+k}{k}\right)\) etc.,
the identity (10.13) follows.
Brouwer, A., Automatic summation using.


Ebbenhorst Tengbergen, C. van, De identiteiten, ... , Nieuw Arch. (2) 18, 1934 (or 1936), 1-7.

Evans, R., M.E.H. Ismail and D. Stanton, Coefficients in expansions of certain

H.W. Gould, The operator \((a^x A)^n\) and Stirling numbers of the second kind,

23, 1965, 66-69 "

Gould, H.W., A variant of Pascal's
Janardan, K. V. and D. J. Schaeffer,
A generalization of the Markov-Pólya distribution, its extensions


Numbers with Applications. Wiley-Interscience 2001


Mathai, A.M. and R.K. Saxera, Generalized Hypergeometric Functions with Applications in Statistics and,


Slater, L.J., Generalized Hypergeometric Functions, Camb. Univ. Press 1966

