AN EFFECTIVE DEFINITION OF A CONNECTED, LOCALLY CONNECTED AND PUNCTIFORM PLANE SET

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Let $S$ be a plane and choose in $S$ a system of rectangular axis. Let $I$ be the interval $\alpha \leq x \leq \beta; y = 0$, and $Q_n$ the rectangle $\alpha \leq x \leq \beta; 0 \leq y \leq 1$. For every $x$, the set $L(x)$ will be the line $x =$ constant and $0 \leq y \leq 1$. In the rectangle $Q_n$ we construct a set $A_n$ in the following way.

Let $C_n \subset$ be the Cantor discontinuum in $I$. We can divide $C_n$ in the set $C_{1,n}$ of the triadic rational points of $C_n$, and the set $C_{2,n}$ of the triadic irrational points of $C_n$.

Now $A_n$ is the set of all points $(x, y)$ of $Q_n$ so that: $x \subset C_{1,n}$ and $y$ rational or $x \subset C_{2,n}$ and $y$ irrational.

The set $A_n \subset Q_n$ has the following property: every continuum $K \subset Q_n$, which intersects every line $L(x)$ with $x \subset C_n$, also intersects $A_n$. To prove this 1) we suppose $K \cap A_n = 0$. Then for $x \subset C_{2,n}$, every point of $K \cap L(x)$ has a rational ordinate.

So $K \cap \bigcup_{x \subset C_{2,n}} L(x)$ is the sum of denumerably many sets $E_1, E_2, \ldots$ so that all points of $E_i (i = 1, 2, \ldots)$ have the same ordinate. It can easily be shown, that if $K \cap A_n = 0$, then every $E_i$ is compact.

Now let $F_i$ be the projection of $E_i$ on the $X$-axis. Then $F_i \subset C_{2,n}$ and $F_i$ is compact. From our supposition that $K$ intersects every line $L(x)$ with $x \subset C_n$, we see that $C_{2,n} = \bigcup_{i=1}^{\infty} F_i$. So $C = C_{1,n} \cup F_i$.

Now every $F_i$ is compact and $F_i \cap C_{1,n} = 0$. Moreover $C_{1,n}$ is everywhere dense in $C_n$. So $F_i$ is nowhere dense in $C_n$. Thus $C_{1,n} \cup \bigcup_{i=1}^{\infty} F_i$ is of first category, which is impossible. So $K \cap A_n \neq 0$.

If $A_n^* = A_n \setminus (L(x) + L(\beta))$, we see immediately, that also $K \cap A_n^* \neq 0$.

If we introduce the sets $B_n = A_n \setminus A_n^*$ and $B_n^* = B_n \setminus (L(x) + L(\beta))$, we can construct the desired set $A \subset Q_n$ in the following way.

Let $Q_0$ be the square $0 \leq x \leq 1; 0 \leq y \leq 1$. We construct the set $A_0 \subset Q_0$.

$Q_0 \setminus A_0$ contains denumerably many components. Every component is the interior of a rectangle $Q$. These rectangles will be enumerated in some way $Q_1, Q_2, \ldots$.

In $Q_1$ we construct the set $B_1$. In $Q_i$ $(i = 2, 3 \ldots)$ we construct the sets $A_i$.

Now the sets $P_i$ $(i = 0, 1, 2, \ldots)$ are defined as:

$$P_0 = A_0, \quad P_1 = B_1, \quad P_j = A_j \quad (j = 2, 3, \ldots).$$

Let for some $l$ the sets $P_{n_1 \ldots n_l}$ be constructed, so that $Q_{n_1 \ldots n_l} \setminus P_{n_1 \ldots n_l}$ contains again denumerably many components $R_i$. Every $R_i$ is the interior of a rectangle $Q_i$ which we call $Q_{n_1 \ldots n_l} \setminus P_{n_1 \ldots n_l}$ $(i = 1, 2, \ldots)$. Then we define the sets $P_{n_1 \ldots n_l}$ as follows:

$$P_{n_1 \ldots n_l} = B'_{n_1 \ldots n_l} \subset Q_{n_1 \ldots n_l},$$

$$P_{n_1 \ldots n_l} = A_{n_1 \ldots n_l} \subset Q_{n_1 \ldots n_l} \quad (i \geq 2).$$

Then $Q_{n_1 \ldots n_l} \setminus P_{n_1 \ldots n_l}$ $(i = 1, 2, \ldots)$ has again denumerably many components, and every component is the interior of a rectangle $Q_i$.

Let $I_1$ be the interval $0 \leq x \leq 1; \ y = 0$, and let

$$D = I_1 \setminus \bigcup_{l=1}^{\infty} \bigcup_{(n_1 \ldots n_l)} C_{n_1 \ldots n_l}.$$ 

Finally we take the set $R$ of all points of $Q_0$ with $x \subset D$ and $y$ rational.

The desired set $A$ is now defined as:

$$A = \bigcup_{l=1}^{\infty} \bigcup_{(n_1 \ldots n_l)} D_{n_1 \ldots n_l} \cup R.$$ 

To show that $A$ has the required properties, we prove that $A$ and $Q_0 - A$ are both punctiform. Let $K$ be an arbitrary continuum which is contained in $Q_0$. If $K$ is contained in a line parallel with the $Y$-axis, then it follows immediately from the construction that $K \cap A \neq 0$ and $K \cap (Q_0 \setminus A) \neq 0$.

If $K$ is not parallel with the $Y$-axis, then the projection $K_1$ of $K$ on the $X$-axis is a closed interval $\alpha \leq x \leq \beta$.

Now we can certainly find integers $n_1, n_2, \ldots n_l$, so that there exists a rectangle $Q_{n_1 \ldots n_l}$ for every point $p$ of which the condition $x \leq x_p \leq \beta$ is fulfilled. The continuum $K$ has then points in common with every line $L(x)$ with $\alpha \leq x \leq \beta$.

$$A \supset P_{n_1 \ldots n_l} = B'_{n_1 \ldots n_l} \subset C_{n_1 \ldots n_l}. \quad A'_{n_1 \ldots n_l} \subset B'_{n_1 \ldots n_l} \setminus B_{n_1 \ldots n_l} \subset Q_0 \setminus A.$$ 

From $K \cap A'_{n_1 \ldots n_l} \neq 0$, we conclude that $K \cap (Q_0 \setminus A) \neq 0$.

On the other hand $P_{n_1 \ldots n_l} = A_{n_1 \ldots n_l}$, and therefore $K \cap P_{n_1 \ldots n_l} \neq 0$. So $K \cap A \neq 0$.

We see that $K$ intersects with $A$ and with $Q_0 \setminus A$. Therefore $A$ and $Q_0 \setminus A$ are both punctiform.

Now if $A$ is not connected, then $A = A_1 \cup A_2$ with $A_1 \cap A_2 = 0$ and $A_1 \neq 0$ and $A_2 \neq 0$. Choose a point $a_1 \subset A_1$ and $a_2 \subset A_2$. 

According to Knaster and Kuratowski \(^2\) there exists a continuum \(L\), which separates the plane between \(a_1\) and \(a_2\) and which does not intersect \(A\). So \(L \cap A = 0\) and hence \(L' \cap Q_0 C Q_0 \setminus A\), which is impossible since \(Q_0 \setminus A\) is punctiform and \(L \cap Q_0\) has components of diameter \(> 0\). So \(A\) must be connected. In the same way one can show that \(A\) is locally connected. So \(A\) has the required properties.