The Great Leap to the Infinitely Small. Johann Bernoulli: Mathematician and Philosopher

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Summary
Johann Bernoulli is known mainly from his mathematical achievements concerning the differential and integral calculus, in which the concept of infinitely small plays a crucial part. This paper describes Bernoulli's struggle with this concept, especially the discrepancies that occur between his mathematical and philosophical interpretations. We concentrate on Bernoulli's Groningen period (1695–1705), in which the discussion with Leibniz about this question leads to a controversy between the two scholars.

Contents
1. Through the enemy lines to Groningen ........................................ 433
2. In Laudem Matheseos .............................................................. 437
3. Differentials as infinitely small quantities .................................... 440
4. Integration, the 'converse' of differentiation ................................ 443
5. The farthest frontier of the finite ................................................. 443
6. A leap that nature does not make: Leibniz's criticism .................... 446
7. Conclusions ............................................................................. 448

1. Through the enemy lines to Groningen

While the Nine Years' War was raging in Europe at the end of the seventeenth century, and the Church was losing its oppressively powerful position, scholars were racking their brains about the existence of infinitely small quantities. In 1697, the Treaty of Ryswyck brought this war to an end. A democratic trend began to emerge in politics. Every effort was made to legitimize the struggle for a new kind of society on a rational-humanist basis. In the intellectual climate that was created as a result, religion and science became increasingly opposed to one another, while at the same time the desire to reconcile the two grew. This desire was partly inspired by the wish to create a world-view that complied with the demands of the new era with its fundamental changes in the field of science. The theoretical expression of all kinds of new discoveries and inventions was thoroughly pondered in order to establish a philosophical meaning. The hypothesis of the actual existence of the infinitely small as the possible fundamental for all forms of existence rocked the Church, with divine omnipotence as its cornerstone, on its foundations. For, if the universe, down to the smallest detail, is subject to mathematical laws then the margin for a divinity to work

miracles disappears, and revelation has lost its lustre.\textsuperscript{2,3} Among others, it was Johann Bernoulli (1667–1748) who, during his time in Groningen (1695–1705) in particular, played an active part in this discussion.

In 1695 this already renowned mathematician, with his family in tow, took the plunge and crossed the enemy lines between Basel and little Groningen.\textsuperscript{4} For ten years Groningen would be in the front line of important mathematical and philosophical developments. During his journey, Bernoulli experienced the violent effects of the war, and these affected him deeply. In the opening words of his inaugural speech\textsuperscript{5} he referred to this as follows:

Both ancient and modern times teach us that one of the innumerable disasters that the ravages of war bring is the destruction of fine arts and sciences and the coarse barbarism that is the result of this. Robbery, arson, murder, robbery with murder, destruction, sacrilege, and numerous other such events, they are all nothing, because only external goods are affected which over the course of time can be replaced. If the study of science is lost, however, you too will acknowledge the extent of the loss when you know that mankind as a result will gradually descend to terrible aberrations, will be surrounded by the darkness of eternal ignorance and, so to speak, will become the equal of horses and foolish asses. ... Nevertheless, our time is blessed and is truly a golden age, that in the midst of the burning torches of war and, as it were, against the will of Mars, has seen the gifts of the Muse develop so far that all science and knowledge appear to have almost reached their apex.\textsuperscript{6}

Bernoulli balanced the arts and sciences against the irrational frenzy of acts of war. For ten years in Groningen he would be the champion of the ideal of man as a rational being, with a strong human side. We can say that he, with a divine respect for reason, developed some ideas of the Enlightenment. This is also shown by the following lines from Bernoulli's lecture:

... that an academic without mathematics is the same as an earth without a sun, or a body without a soul. For just as the sun lights up our whole universe, ... and as the soul gives movement to an otherwise motionless body, so does our nourishing mathematics extend its certainty and clarity to the other sciences

\textsuperscript{2} Chr. Wolff, Metaphysik oder Vernünftige Gedanken von Gott, der Welt und der Seele des Menschen, auch alten Dingen überhaupt (Halle, 1720).
\textsuperscript{3} A. G. Baumgarten, Metaphysik (Halle, 1739).
\textsuperscript{5} Joh. Bernoulli, Oration Inauguralis in Laudem Matheosae (Basel University Library Lla750B9, 1695); in a translation by H. A. Krop. The original text is in the Archive of the Bernoulli-Edition Basel (Leiter: Dr Fritz Nagel).
\textsuperscript{6} Oration inauguralis (note 5): 'Inter innumerus quas martis furor secum trahit calamitates, praecipuam esse interitum bonarum artium et scientiarum, atque crassam quae indit barbarism tam prisco quam moderna tempora nos docuerunt. Rapinias, incendia, caedes, latrocinia, vastationes, sacrilega et si quae sexcenta alia, haec omnia nihil sunt, percunt enim tantum bona externa quae cum tempore recuperari possunt: verum litterarum cultus amissus, jactura quanta sit ipse perpendit, ubi noveris, homines exinde sensim debabi ad errores horrendos, involvi tenerbris perpetuer ignorantiae et ut verbo dici similes fieri equis et multis quibus nullus est intelligentia. ... Foelix nihilominus tamen est et vere aureum seculum nostrum quod etiam inter ardentias bellorum faces et quasi invito Marte ita largitor sua expanderunt dona Musae, ut omnes scientiae et disciplinae fastigium fere suum attigisse videantur.'
and, as it were, lights the way for these to come to unquestionable and definite truths. ... Well, then... we notice that as we approach the true origin, that God himself was the most eminent mathematician and the first teacher to teach mathematics to men. ... Thus reason, as it were, receives the fruitful seeds of mathematics into its innermost places. It begins by reverently discovering the ideas of the numbers and the figures. In my opinion these are pure and
intelligible ideas which are not subject to any change without earthly and material interference, and which according to the healthy view of the Platonists are eternally in God and, just like Him, are not created. It is by means of these ideas, as if they were a staircase, that our souls rise up from the temporal to the eternal, from one point to infinity, from nothing to everything, and from themselves to their Creator.\footnote{Oratio Inauguralis (note 5): \textit{quod Academia sine Mathesi idem sit quod mundus sine sole, vel quod corpus sine anima; quaedammodum enim sol totum nostrum universum illustrat; \ldots et quaedammodum anima corpore aliquo inerti motum tribuit; sic alma nostra Matheis certitudinem suam et claritatem reliquis disciplinis feroeratur illisque quasi praecursus viam sternit quia ad inconcussas et solidas vertitates pervenitur; \ldots reprimis, si ad genuinae eorum origine ascendamus, quod Deus ipse fuerit Princeps et primus Professor qui ess homines docuat.\ldots Ratio recapt seu in sinum facundia matheoseos semina; inceptum cum admiracione detegere ideas numerorum et figurarum; istas inquam, ideas puras et intelligibilis quae nulli admixtioni terrestri et materiali nullique mutatione sunt oboeixae, et quae juxta sanam Platoniceorum mentem in Deo sunt ab aeterno et inercount ut ille ipse; istas ideas per quas tangam pesculum mens nostra cogitationes suas elevat a temporo ad aeternitatem, a puncto ad infinitum, a nihilo ad omne et a se ipsa ad Creatorum suum.}\footnote{In October 1684, Leibniz published his famous article in which he formulated the principles of differential calculus. G. W. Leibniz, \textit{\textit{Nova Methodus pro Maxima et Minimis, itemque Tangentibus, quae nec Fractas, nec Irrationales Quantitates Moratur, & Singular pro Ills Calculi Genus}. Acta Eruditorum, 3 (1684), 467–73. (English trans. in: Fauvel, Gray (1987) 426–32.)\footnote{See J. A. van Maanen (ed.), \textit{Een Complexe Grootheid; Leven en Werk van Johann Bernoulli, 1667–1748} (Utrecht, 1995), 49–69; also G. Sierksma, \textit{Johann Bernoulli (1667–1748). His Ten Turbulent Years in Groningen}. The Mathematical Intelligencer, 4/4 (1992), 22–31. Czech trans. in \textit{Pokroky Matematiky Fyziky & Astronomie}, 1/39 (1994), 14–26. Incidentally, the \textit{philosophia experimentalis}, with which Bernoulli carried out physics experiments in the chancel of the University Church, played an important role in this; see for this also Sierksma (note 4).}}

One of the most important scientific discoveries of the time was the invention of differential and integral calculus by Isaac Newton (1643–1727) in England and Gottfried Wilhelm Leibniz (1646–1716) in mainland Europe. Johann and his brother Jacob, twelve years older, were the first to understand and apply Leibniz's \textit{nova methodus}.\footnote{See Van Maanen (note 9), 56.} The method presupposes the existence of infinitely small quantities. The interpretation of precisely this concept led to vehement controversies, not only mathematical and philosophical, but also theological.

In his Groningen period in particular, Johann Bernoulli was involved in vehement controversies.\footnote{C. I. Gerhardt, \textit{G. W. Leibniz Mathematische Schriften}, Vol. III/2 Briefwechsel zwischen Leibniz, Jacob Bernoulli, Johann Bernoulli und Nicolaus Bernoulli (Hildesheim 1962), letter dated 5 July 1698, 500–1 (p. 504): \textit{Cavero tamen mihi, ne talia tangam apud Theologos quosdam hujus Civitatis, omnium libere philosophantium osores; haud dubie me ad rogum ablegarent, si tantas haereses a me audirent.}} Because of his too expressive and materialistic representation of man and of things he was accused of being a godless Cartesian and an atheistic Spinozist. Johann refuted the accusation by the Groningen theologians that he was denying the \textit{\textit{divine resurrection of the flesh}} by maintaining that the body is continually renewing itself, by working out for them that the body contained very thin threads, which he called \textit{stamina}, which were unchangeable and which would arise on the \textit{\textit{day of judgement}} covered in earthly matter.\footnote{\textit{\textit{Incidentia, I shall be wary of bringing my ideas before the attention of certain theologists in this land, the enemies of every free philosophy; they would undoubtedly condemn me to the stake if they knew about such heresies.}}} The denunciation eventually went so far that he was condemned from the pulpit as \textit{\textit{a corrupter of youth}}. He complained about the situation in a letter to Leibniz and planned to guard his tongue for the time being to avoid worse:

\begin{center}
Incidentally, I shall be wary of bringing my ideas before the attention of certain theologists in this land, the enemies of every free philosophy; they would undoubtedly condemn me to the stake if they knew about such heresies.\footnote{\textit{\textit{Incidentia, I shall be wary of bringing my ideas before the attention of certain theologists in this land, the enemies of every free philosophy; they would undoubtedly condemn me to the stake if they knew about such heresies.}}}\footnote{\textit{\textit{Incidentia, I shall be wary of bringing my ideas before the attention of certain theologists in this land, the enemies of every free philosophy; they would undoubtedly condemn me to the stake if they knew about such heresies.}}}\footnote{\textit{\textit{Incidentia, I shall be wary of bringing my ideas before the attention of certain theologists in this land, the enemies of every free philosophy; they would undoubtedly condemn me to the stake if they knew about such heresies.}}}
\end{center}
A later letter shows that although it never went that far, he certainly felt threatened:

"Because I want to leave the city within the hour I cannot at this time respond to the
content of your letter as I would wish."12

Despite the theological heresies, the Groningen years were very productive for the
scientific development of Bernoulli on both the mathematical and the philosophical
fronts.

2. In Laudem Matheseos

On 28 November 1695 Bernoulli gave his inaugural lecture In Laudem Matheseos
at the University of Groningen. In this lecture, some quotations from which appear
above, he praised the infinite possibilities and divine power of mathematics. This
lecture is a good example of how religious beliefs and scientific insights were
entangled at that time: 'in praise of mathematics' became the equivalent of 'in praise
of God'. Bernoulli's message was clear: whoever opposed mathematics and its
practitioners was opposing God, the greatest mathematician of all. According to him,
the opposition of the theologists not only endangered the development and use of
mathematics, but above and beyond all it impeded insight into God’s omnipotence
and comprehension of His greatest works of creation. We could almost say that he
was daring to draw a parallel between the destructive effects of acts of war on the arts
and sciences and the frustrating influence of theological shortsightedness on the
experience of faith.

It is as if he sensed the conflicts to come, because in this lecture he appears to be
covering himself against the coming theological allegations which would pester the
life out of him in Groningen. The last sentences of the lecture read as follows:

O Being of Beings, have mercy on me. Dispel all the slanderers who persecute
and condemn the Divine mathematics as if it were a hotbed of godlessness and
I don't know what other kind of irreligiosity. O intolerable dark forces, who
fight against a science that knows no equal, it would be better to practise it and
recognize that it has been the only thing since antiquity that has converted the
heathen to knowledge of the true God, to the Being of Beings. Learn, I say,
from Plato, that God is continually using geometry; He is the best geometrician
and an inimitable architect, who created heaven and earth out of nothing.13

In Bernoulli's time it was usual to contemplate the philosophical meaning that a
scientific discovery or mathematical method could have. Was mathematics only
concerned with arithmetical skills with practically applicable consequences, or did the
concepts of such a method stretch further, and did they eventually suggest general
forms or systematic structures for the world itself, which could be described by
metaphysics? The fact that Bernoulli tested mathematical concepts for their
philosophical content makes him into a true philosopher, as the rest of this narrative
will demonstrate.

In the second part of his lecture, Bernoulli paid attention to the concepts infinitely
large and infinitely small, two key concepts in the formulation of differential and
integral calculus at the time. This is what he says about the infinitely large:

12 Ibid., letter from 1698, 538–40 (p. 539): ‘Hac ipsa hora extra urbem abiturus, nunc ad literarum
Tuarum contenta proelie, prout vellem, respondere non possum’.

13 Oratio Inauguralis (note 5): ‘O! Ens Entiam miserere mei! Apague igitur illos Calumniatores qui
exagitam divinarum Mathesin et explodiant, ac s fomentum esset atheismi et nescio quaram impietatem;
O intolerabiles tenebrores qui ita invehunin in scientiam hanc quae sui parum non habet, discite potius
et agnoscite hanc unicum olim fuisse quae Ethnicos ad veri Dei agnationem, ad Ens entium converterent;
Discite inquam a Platone Deus semper geometrizare; Hic ille est Geometra sumnus et Architectus
inimitabilis, qui creavit coelum et terram ex nihil.’
It reaches to the poles of the first, the second, yea even the third heaven, past the Divine Fire, further and further in infinite series, later to form yet another new series, the first term of which is nothing less than unity, the second the last of all those which have gone before. Let it suppose there to be just as many terms in this series as in the previous ones. Let it in the same way duplicate and suppose that as many new series will be able to be formed as there are numbers, until exhausted reason is forced to give up. And let reason finally, after many attempts, acknowledge that all centuries are expressed by the truly terrible number at which reason is silent and that appears to approach eternity, disappears into it and is nothing in comparison with eternity, which absorbs and devours all things, as if it were only one year, one month, one day, one single hour, one single minute, and this moment that is now past, that disappears and perishes and, I think, passes and, even though it now exists, will never return and is only that which disappears and perishes in eternity.\footnote{Oratio Inauguralis (note 5): ‘Extendat illa cyphras ab uno terrae polo ad alterum, extendat usque ad polos caeli primi, vel caeli secondi, imo ad polos tertii: trajecit Empyreum, continuat insuper quamdu et quam procul velut immensam seriem progressionis suae: ut postea novam progressionem formet, cujus primus terminus sit sola unitas, secundus autem ultimus omnium illorum, quos percurrerit: Supponat in hac nova tot terminos quot in priori. Reiteret vol supponat eodem modo tot novas progressiones quot potuerit concipere numeros, donec fatigata succumbere cogatur: Et agnoscat tandem post tot conatus omnia secula expressa per numerum revera terribilem in quo subsistit, et qui (ut videtur) infinito proxime accedit evanescere tamen et annihili respectu aeternitatis; quae ea omnia absorberet et ingurgitat, ac si esset unicus annus, unicus mensis, unica dies, unica hora, unicum minutum aut hucce momentum quod nunc praeferabatur, quod eclipsatur et perit dum venit, quod praeterit nuncum reditum cum praesens puto, et demum quod evasit et fugit in aeternum.’}

The consideration of the concept of infinitely large brought Bernoulli too the question of the reality of the immeasurable, the infinite and the eternal, as derivatives of the measurable, the finite and the temporal. Bernoulli appears to have asked himself: ‘Can the space of this immeasurable perhaps be perceived as eternity, and can God be localized in it or is He the same thing as a kind of absolute time or absolute space?’ He says the following about the infinitely small:

Our speech ends with a no less remarkable wonder placed before our eyes by mathematics. Let our spirit closely examine very small objects, namely the invisible beings which a microscope reveals in their tens of thousands. Each of them has in its smallness parts which are incomparably smaller again. After all, they too have hearts and cardiac valves, veins and arteries with their many branches, which themselves branch, separate and divide into even smaller ones. The spirit, I say, examines the blood in this blood, the droplets in these humours, the vapours in these droplets, and the steam in these vapours. It divides these particles even further, until its imaginative powers are exhausted. Is this tiny substance the final object of our analysis? It perhaps believes that it has reached the absolute smallest.\footnote{Oratio Inauguralis (note 5): ‘Desunt nostra oratio ad aliud portentum non minus stupendum quod Mathesis nobis ob oculos posuit, consideret mens nostra attingit objecta exilissima, animalcula ista invisibilia, quorum microscopium multa multi simul detegit; quodvis horum in summa sua parvitati gaudent partibus incomparabiliter minoribus; habet enim cor suum et cordis valvulas, arterias et venas, et ramos illa insectos, et alios huc, qui iterum separatur et dividitur in alios minores; consideret inquiram sanguinem in his ramis, humores in hoc sanguine, guttulas in his humoribus, vapores in his guttulis, et spiritus fumantes in his vaporibus: Dividat usque et usque hucce particulas donec vim conceptionem exhaustat, sitque tandem corpusculum illud ad quod pervenit objectum nostri discursus; Credet farsan se ultimam omnino attigisse parvitatem.’}
Here, too, we see how Bernoulli’s thoughts result in a representation of the smallest ‘building blocks’ of life. Can these building blocks still be understood as material elements? By steadily dissolving visible material he appears to penetrate to an immaterial principle of life, namely ‘the steaming air in these vapours’. The soul, the anima or the life force was at that time often perceived as a current of air or a breath.

What is important is that Bernoulli as a mathematician was constantly concerned with such philosophical investigations. He took this research very seriously and persevered with it logically:

Geometry, however, opens up new abysses and clearly shows that this unimaginable particle can continue to be divided up infinitely, even if our imaginations completely seize up. Consequently this demonstrates that the measurements and relationships of this tiny world are just as refined, remarkable and complete in their incredible tininess as the world in which we breathe in its astonishing largeness. In the same way other particles are formed from this new particle, that again are themselves built up from new ones and this continues endlessly, which is more than sufficient proof that the omnipotence of God in the smallest of things is inexhaustible and infinite.

The last two quotations demonstrate that Bernoulli considered the infinitely large and the infinitely small to be connected with each other and that the unimaginable space of infinity could be ‘accessed’ by means of the intellect or the soul. In his eyes the world clearly formed a coherent whole, made up of an infinite number of forms of constantly varying quantities, ranged on a scale of joining, gradual changes. We shall return to this later.

In his inaugural lecture Bernoulli was voicing the generally accepted view of the time, that mathematics should stand model for the other sciences, including philosophy. Only that which could be understood as clearly and unequivocally as mathematical reasoning was considered to be true knowledge and thus incontrovertible. It is possible to state even more strongly that with his belief in the universal eloquence of mathematics, the mathesis universalis, he thought that he could penetrate the essence of nature. The lecture In Laudem Matheseos is, therefore, in the same tradition, begun in the Renaissance, as the work of, among others, Galileo, who formulated natural laws in mathematical formulae, Descartes, who derived the knowledge criterion claire et distincte from the evidential value of mathematics, and Spinoza, who remodelled his principal philosophical work on a geometrical basis and gave it the title Ethica, Ordine Geometrico demonstrata. On the other hand it was Leibniz who perceived that mathematics could not possibly be the basis of mechanics, let alone that of metaphysics:

16 Bernoulli was not only taken seriously as a mathematician, but also as a physician; see Sierksma (note 4), 72.

17 Oration Inauguralis (note 5): ‘Verum Geometriae nomen ipse aperit abyssos; et evidentissime ostendit hanc ipsam particulam, quae vim ejus conceptio subterfugit, adhuecum in infinitum divisibim esse, etiam imaginario expaesacat hoc. Et consequenter delineat in illa omnes dimensiones et proportiones Mundi contracti tam elegantis, tam mirabilibus et tam perficient in extrema sua exiguitate, quam mundus est in quo respiramus in stupenda sua magnitudine. Ipse eodem modo et alias delineat in novo hoc parvulo, in quibus iterum iterumque novos extrahit, continuatique hoc in infinitum: quod sane satis superque probat; Omnipotentiam Dei in singulis minimis inexactum et infinitum esse.’

18 In order to create the atmosphere of the Italian ‘rinascimento’, Fleckenstein calls on Leonardo da Vinci: ‘Whosoever spurns the great certainty of mathematics exposes his soul to confusion and will never ever be able to silence sophistical rhetoric’; see J. O. Fleckenstein, Gottfried Wilhelm Leibniz, Barock und Universalismus (Thun, München, 1958), 14.
Eventually the mechanics theory gained the upper hand and forced me to concern myself with mathematics. I first became familiar with its deepest secrets by associating with Mr Huygens in Paris. However, when I was looking for the last bases of mechanical observations, and entirely so when investigating the laws of movement, I discovered to my surprise that it was impossible to find them in mathematics, and that one had to turn back to metaphysics. This led me to the entelechies, and from matter back to form.\footnote{C. G. Gerhardt, ‘Leibniz an Remond, 10 Januar 1714’, in Leibniz, die Philosophischen Schriften, \textit{III}, ed. C. G. Gerhardt (Berlin, 1887), 606.}

Leibniz also stated that philosophy, just like metaphysics, needed a specific, still to be developed methodology and way of thinking of its own:

> It nevertheless appears to me, that as far as these questions are concerned, more so even than in mathematics, light and certainty are needed, because mathematical affairs contain their checks and proofs within themselves, which is the strongest reason for their success, whereas we in metaphysics manage without this convenience. This is why a particular method for making statements is necessary, a thread in the labyrinth, as it were, in order to bring the questions to a solution…\footnote{See G. W. Leibniz, ‘De Primae Philosophiae Emendatione, et de Notione Substantiae’, in \textit{G. W. Leibniz, Philosophische Schriften}, \textit{I}, ed. and trans. H. H. Holz (Frankfurt am Main, 1986), 194–200 (p. 197); see also W. Sierksma, \textit{Zur Ontologie des menschlichen Verstandes: das Verhältnis von Leibniz und Locke und der Seinsstatus des Denkens} (Köln, 1993), 20 ff.}

Leibniz was corresponding with Leibniz about mathematical problems.\footnote{No fewer than 283 letters have survived from the correspondence between Bernoulli and Leibniz.} In Groningen Bernoulli wrote for the first time about philosophical matters, too, particularly those concerning the interpretation of the concepts from differential and integral calculus. Before going into the philosophical side of calculus, we shall first discuss some aspects of it.

### 3. Differentials as infinitely small quantities

Once his brother Jacob had been appointed to the chair of mathematics in Basel, at the end of 1690, Johann began to travel. He remained in Paris for a year (until November 1692), about which he wrote in 1735:

> In this world-famous city I earned no mean reputation with many famous and learned members, but especially with the philosophers and mathematicians, particularly with the generally respected and excellent Marquis De l'Hôpital. … He was not ashamed to take lessons from me and to subject himself to my teachings on further mathematics, both in differential as well as integral calculus. … As a result of these lessons from me he subsequently compiled and published his book under the title \textit{Analyse des Infiniments Petits pour l'Intelligence des Lignes Courbes}, which to my pleasure he also acknowledges in his foreword.\footnote{See ‘Die Selbstbiographie von Johannes Bernoulli I’, in \textit{Gedenkbuch der Familie Bernoulli zum 300. Jahrestag ihrer Aufnahme in das Basler Bürgerrecht 1622–1922} (Basel, 1922), 81–103. The full title of De l'Hôpital's book is \textit{G. de l'Hôpital, Analyse des Infiniments Petits, pour l'Intelligence des Lignes Courbes} (Paris, 1696).}"

What De l'Hôpital actually wrote in his foreword was: ‘Incidentally, I owe a debt of gratitude to Messrs. Bernoulli for their many lucid ideas, especially to the younger of the two who is now the professor in Groningen.’ In fact, Johann was not really very...
SOLUTION D'UN PROBLEME

Concernant le calcul intégral, avec quelques diverses par rapport à ce calcul.

Par M. BERNOUlli Professeur à Groningue.

Le tout écrit d'une de ses Lettres écrites de Groningue
le 5. Août 1702. *

PROBLEME.

On la différentielle \( \frac{pdx}{q} \), dont \( p \) & \( q \) expriment des quantités rationnelles composées comme l'on voudra d'une seule variable \( x \) & de constantes; on demande l'intégrale, ou la somme algébrique, ou du moins qu'elle la réduise à la quadrature de l'hypelobe, ou du cercle; l'un ou l'autre étant toujours possible.

SOLUT. Soit divisée \( p \) par \( q \), jusqu'à ce qu'enfin la plus grande division de \( x \) dans le reste, soit moindre que dans \( q \); fasse \( y \), \( a \) moins que la plus grande dimension de \( x \) dans \( p \) ne soit dé- même que dans \( q \); auquel cas il n'y aura point de division à faire. Prenez ensuite l'intégrale du quotient de cette division; ce qui est toujours possible, puisque ce quotient (quant aux \( x \)) sera toujours entier & rationnel. Mais pour l'intégrale du reste (ce qui est proprement le point de la difficulté) voici comme on la trouve. Soit ce reste appelé \( r \), & supposons que \( rdx \) : \( q = adx : (x+h) + bdx : (x+z) + cdx : (x+n) + \&c. \) C'est à dire \( rdx : q \) égale à autant de différentielles logarithmiques que la plus grande dimen-

Figure 2. From Bernoulli's Opera Omnia. (See Bernoulli (note 24). The original Opera Omnia was published in four volumes by Gabriel Cramer in 1742, while Bernoulli was still alive.)

impressed with this token of appreciation. He considered the whole work to be plagiarism. Although Analyse des Infiniments Petits remained the standard work on differential calculus for decades, the quarrel about whether De l'Hôpital had really plagiarized was only decided in Johann Bernoulli's favour in 1923. In that year manuscripts were found in the university library of Basel which showed that Johann was indeed the spiritual father of the Analyse. 23

The manuscripts begin with three postulates, of which the first two are also in the Analyse:

Postulate 1.
A quantity which is reduced or increased by an infinitely smaller quantity is neither reduced nor increased.

Postulate 2.
Every curved line consists of an infinite number of straight ones, which are themselves infinitely small.

Postulate 3.
A figure which is delimited by two ordinates, the difference between the abscissas and an infinitely small part of an arbitrary curve, may be regarded as a parallelogram.

23 See J. A. van Maanen (note 5), 34 and 41. The notes which Nikolaus I Bernoulli made of the lessons from his uncle also confirm this view; see Johannis (I) Bernoulli, 'Lectiones de Calculo Differentialium', in Die Differentialrechnung, Ostwolds Klassiker Nr. 211, ed. P. Schafheitlin (Leipzig, 1924).
The first two postulates were used to support differential calculus and the last two integral calculus. De l’Hôpital’s *Analyse* deals only with differential calculus and contains postulates one and two. The manuscript found could indicate that Bernoulli was planning to write a more comprehensive work which would also have included integral calculus. In fact, he did later publish his *Lectiones mathematicae de Methodo Integralum*, in which postulates 2 and 3 were used. Incidentally, the postulates are not really all that clear. What exactly is to be understood by *infinitely small* quantities and do they actually exist? After the postulates come the arithmetical rules for differentials. The term *differential* was also left undefined. The proofs enable us to conclude that the differential \( d(f(x)) \) of the ‘quantité’ \( f(x) \) at the point \( x \) is the same as the infinitely small difference between its value in \( x + dx \) and its value in \( x \), with \( dx \) being the differential of \( x \). In other words, \( df(x) = f(x + dx) - f(x) \). Determining for example the differential of \( x^2 \) now went as follows:

\[
d(x^2) = (x + dx)^2 - x^2 = 2x \, dx + (dx)^2.
\]

Because \( dx \) is regarded as an infinitely small quantity, \((dx)^2\) was cancelled out against \( 2x \, dx \), so that:

\[
d(x^2) = 2x \, dx,
\]

which is still regarded as the correct expression. On the other hand, however, because according to the first postulate \( x + dx \) is the same as \( x \), then \( d(x^2) = 0 \), and that was not the intention. There appeared to be times in the calculations when the postulates could not be used. Leibniz and the two Bernoullis hardly made any effort at all to explain how the postulates ought be used. For Bernard Nieuwentiijt (1645–1718), however, doctor in and mayor of Purmerend, these uncertainties were the reason for strongly opposing and rejecting the *nova methodus* (the new differential and integral calculus). He published two articles, one in 1694 and the other in 1696, in which he strongly attacked the fact that, in the *nova methodus*, ‘things which are infinitely small but are not equal to zero are left out without mercy’.

Nieuwentiijt’s criticism is not entirely justified: the infinitely small quantities were not left out ‘without mercy’, although the systematics employed was not yet explicit. The new method worked perfectly, and famous, until then insoluble, problems were solved with it. Leibniz was aware from the start of the fact that a thorough foundation for the method was lacking. At the end of his *Nova Methodus pro Maximis et Minimis* he writes: ‘Other very learned gentlemen have searched along numerous roundabout ways for that which someone who is familiar with the new method can produce in three lines.’

Obviously Leibniz here takes working with infinitely small quantities for granted and is above all emphasizing the effectiveness of the method. Nevertheless he did not

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24 See Johannis Bernoulli, *Opera Omnia, tam antea sparsim edita, quam hactenus inedita* (Hildesheim, 1968).

25 See B. Nieuwentiijt, *Considerationes circa Analyseos ad Quantitatis Infine parvas Applicatae Principia, et Calculi Differentialibus usum in Resolvendis Problematibus Geometricis* (Amsterdam, 1694); also B. Nieuwentiijt, *Analysis Infiitorum seu Curvilineorum Proprietatis ex Polygonum Natura Deducita* (Amsterdam, 1695). The Irish philosopher and theologian George Berkeley (1685–1753) relativized the criticism of Newton’s fluxion arithmetic and said that whoever could digest all of those infinitely small quantities and instantaneous velocities of instantaneous velocities ‘need not be squeamish about any point in divinity’; see E. J. Dijkstra, *Clu’s Stiftkind*, Samengesteld en van einde in einde en Commentaar voorzien door K. van Berkel, (Amsterdam, 1990), 161–2.

26 *The Nonstandard Analysis* now provides a logical conceptual apparatus although it is not quite built up in the spirit of Leibniz’s work. The three postulates can be found there *mutatis mutandis* as theses; see A. Robinson, *Non-standard Analysis* (Princeton, 1996).

27 Leibniz, quoted in Van Maanen (note 9), 26.
ignore the fundamental discussions. He could do not less. He more than anyone else was searching for explanations for the relationship between things. He thought that he was on to this relationship with the concept monad. He regarded the monad as the shaping force which is a part of every type of being and which determines its uniqueness.28

4. Integration, the ‘converse’ of differentiation

The second and third postulates from the previous section were intended to be the basis of integral calculus. To calculate, for example, the area of a region enclosed by the x axis between x = 1 and x = 3 and the curve f(x) = x³, a quantity was sought with a differential which was the same as x³dx. This turned out to be \( \frac{1}{2}x^4 \), except for one constant. After all, if the first postulate was applied then d(\( \frac{1}{2}x^4 \)) = \( \frac{1}{2}(x + dx)^3 - \frac{1}{2}x^4 \) = x³dx. Mathematicians who did not wish to make use of the nova methodus divided the x axis into a number of equal segments between x = 1 and x = 3 and then determined the sum of the rectangular ‘rods’ from the segments to the curve. The more segments there were, the smaller the rods, and the closer they got to the value of the required area. However, this method only gave an approximate value, whereas the nova methodus gave the exact value. This fact was shown by taking an infinite division of the interval between x = 1 and x = 3, instead of a finite division. On the basis of the second postulate, the upper part of a rod, formed by a part of the curve, may be understood as a straight piece of line, and the whole rod, according to the third postulate, as a rectangle. The area of such a rod with a width dx and a length x³ was then equal to x³dx. The sum of all segments with a width dx was then given by \( \int x³dx \), with as boundaries x = 1 and x = 3.

In other words, the finite value of the area determined by integral calculus was at the same time equal to a ‘sum’ of an infinitely large number of infinitely small parts. Incidentally, the integration sign \( \int \) was thought up by Leibniz and accepted by Bernoulli in 1694 on condition that Leibniz replaced his term Summa by the term thought up by Johann. Integration. And so it came to pass. The same thing was true for integration as for differentiation: it worked! In a time when science and religion were still closely bound together, it is not surprising that the discovery of the fact that the combination of an infinitely large number of infinitely small quantities resulted in a finite quantity caused great commotion: mathematics was seated on the throne of the Creator, it created something out of nothing, it had crossed the border between heaven and earth.

5. The farthest frontier of the finite

It is important to realize that the interpretation of the new mathematical method took place in an atmosphere of conflict between (natural)philosophical, usually Cartesian, and theological viewpoints. There was anything but clarity about the question of whether, and if so how far, philosophy and theology ruled out or encompassed each other. Theologians and Cartesian were just as likely to fight each other as to form coalitions. The reason behind Bernoulli’s attempts to formulate a conceptual interpretation of the principles behind differential and integral calculus, and to enter into correspondence with Leibniz about philosophical questions, could very well be

28 Within the framework of this article we cannot go further into a possible basis for calculus in the terms of Leibniz’s monad theory. As a pointer for further research we would like to point out that the concept of the monad could be worked out as a repraesentatio totius mundi, with the theory of the petites perceptions appearing to be particularly significant in connection with the concept of the infinitely small; see W. Sierksma, Ontologie des Menschlichen Verstandes: das Verhältnis von Leibniz und Locke und der Seinstatus des Denkens (Dialectica Minora, Band 6, Köln: Dinter, 993).
because of his conflicts (from 1698 on) with Groningen theologians and students. In
order to be able to keep his head above water spiritually, Bernoulli was forced to take
philosophical position and turned to Leibniz for support, who in his turn thought
that he had found an ally in Bernoulli for his polemic against Cartesianism in The
Netherlands. Both scholars asked themselves how far the nova methodus could be a
model for the unity and cohesion of the world as a whole. The concepts continuity and
infinity were regarded by them as key concepts in this. The interpretation of the
infinitely small in connection with these two key concepts led Bernoulli to a
representation of the world in which:

...a world can exist in the smallest particle, with everything arranged in
accordance with the large world, and that turned the other way around our
world is nothing but a particle in a different, infinitely large world. According
to this representation, in my view, it is possible to pass through the world from
top to bottom without every coming across a border, so that our sort of
magnitude only occurs as an unusual result of an infinite number of gradations.29

By accepting the principle of continuity, the so-called lex continuitatis, the unity
and the cohesion of the whole cosmos followed as a matter of course.30 Argumentation
which assumed the principle of continuity was therefore not allowed to conflict with
the multitude and variety of beings in the world. In order not to conflict with this
variety, Bernoulli introduced different levels of infinity. With the conclusion that
beings possibly existed as varying levels of infinity, from the very smallest to the very
largest, the course was simultaneously set for the problem of the limitations of infinity
and the transition from the finite to the infinite. Bernoulli attempted to enlist
Leibniz’s support for his ideas by proving mathematically that there was no
contradiction between the continuity principle and the existence of outer limits for the
finite and the infinite. This limit could then theoretically be passed, not by man it is
true, but by nature itself. According to this theory, the finite would then border on
the infinite and, just like every finite thing, be defined on two sides by borders, and
therefore one can speak of separate things; thus the infinite should certainly have a
border on one side at least, that is, where it borders on the finite. When this frontier
was crossed, the infinite, too, should then actually be able to be entered as a spatial
world. The space of infinity was then no longer beyond our world, but rather its
extension and in a certain sense directly belonged to it. Bernoulli then asked himself
whether when crossing this frontier nature itself did not convert from the material
into the immaterial:

How far must I still proceed before I come to a simple, unique and individual
unit, off which I can immediately say that it is one substance, and not a
multitude of substances? To that end, indeed, matter would not only have to be
divided into the infinitely small, but also into the smallest parts, that is to say,
into points or non quanta, which do not exist as such.31

29 Gerhardt (note 11) letter dated July 5 1698, 500–16 (p. 504): ‘in minimo pulvisculo posse existere Mundum, in quo omnia proportionata sunt huic magno, et contra nostrum mundum nihil alium esse, quam pulvisculum alius infinites majoris; atque hanc conceptum continuari posse ascendendo et descendendo sine fine: unde nostrum genus quantitatum unicum tantum ex infinitis gradibus effecer.’
30 The way that the continuity principle can lead to mathematical discrepancies is described in Van
Maanen (note 9), 44.
31 Gerhardt (note 11), letter from 1698, 538–40 (p. 540): ‘quouquaque ergo progrescidendum, ut perveniam
ad simplicem unitatem singularem et individuum, ut possum dicere hanc esse substantiam, non substantias?
Sane materia non modo dividenda erit in partes infinite exiguis, sed in minimas, id est, in puncta seu non
quanta, quae non danur.’
Bernoulli appears to be saying here that the non quanta do not exist in the space of finite existence, but could be situated on the other side of the border. If nature also existed outside the limitations of finiteness, then Bernoulli found it hard to believe that in this qualitatively different space of infinity only the same things would be met with as in the space of finiteness. After all, did not the transition from the finite to the infinite mark the transition from the material-quantitative and measurable to the immaterial-qualitative and substantial? Was it not possible for God and the souls to be situated in this substantial space?

According to the continuity principle, the transition from the finite to the infinite cannot be an abrupt one; nature does not make leaps. According to Bernoulli, we are dealing with a marginal area with gradual and liquid transitions. For such transitions, the existence of infinitely small quantities was definitely constitutive. In order to prove that the world of the infinitely small and infinitely large bordered on the world of the finite, and that in addition the infinitely small existed, Bernoulli used the following ‘mathematical’ reasoning:

Imagine that a certain quantity is divided into parts, which decrease in the following geometrical series: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ As long as the number of terms is finite, I admit that the individual terms also have to be finite. If, however, the sum of the parts is actually present, then without doubt the most infinite part, and all of those following it, must also be infinitely small.

Bernoulli appears to be saying here that if the number of terms in the above series is infinitely large, then the series with the finite terms is, in a continual process, turning into a series with infinitely small terms. In fact Bernoulli is trying to prove that the material world borders on the immaterial world. Seeing that God could possibly be situated in that infinite world, it was for Bernoulli a conditio sine qua non that that world existed and that the infinitely small and the infinitely large also existed. In the following quotation Bernoulli is emphasizing the existence of infinitely small particles as independent objects and not only in connection with finite material:

If a finite body exists, consisting of an infinite number of parts, then that means, at least that is what I have always taken it to mean, that the least of these parts must be in an unparalleled or infinitely small relationship to the whole. Incidentally, an actual splitting apart is not necessary for this at all. It goes without saying that it is enough that such a particle exists not only in itself but simultaneously with the whole: in the same way as a mathematical line co-exists with the area, or the area with the body, or as any arbitrary differential at all co-exists with its integral.

In a later letter he repeated this reasoning using different words:

Further, a body that traces a line by its movement must actually exist at every separate point that I can imagine on that line. Therefore also at two points which are very close together, which means that this substance has indeed

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32 Ibid. (note 11), letter dated 16–26 August 1698, 528–33 (p. 529): ‘Conceive aliquam magnitudinem determinatam dividit in partes geometricas hac progressionem descendentes $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ Quamdiu numerus terminorum finitus est, factio singulas terminos fore atiam finitos; sed si omnes termini acti existant, crut sana infinitissimus omninoque sequentes infinite parvae magnitudines.’

33 Ibid. (note 11), letter dated 23 July 1698, 516–18 (p. 518): ‘nam si corpus finitum habet partes numero infinitas, credidi semper et etiam credo, minimum istarum partium debere habere ad totum rationem inassignabilem seu infinite parvum. Nec opus est actuali divisione, sufficit talem particulum in toto coexistere, quemadmodum linea mathematica coexistit cum superficie vel superficiibus cum corpore, vel quodlibet differentiale cum suo integrali, . . .’
crossed exactly that tiny interval, or exactly that infinitely minute particle. Because even if such an infinitely small particle did not itself exist independently, it would still have to co-exist with the whole.  

Thus according to Bernoulli the existence of the infinitely small follows on from the fact that in the total summation of all the gaps in the above series, not only are all of the finite terms included, but also the infinitely small ones which follow on from them. If the latter were not included, then the summation would not be the ‘total’. He wrote:

> It appears to me to be a contradiction to claim that all the elements in the series \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \) exist with the exception of the infinitesimal elements; for if the infinitesimals do not exist, then only the finite elements are available. That would mean—which is in conflict with the assumption—that not all elements exist.

Bernoulli with ‘all the elements in the series \( \frac{1}{2}, \frac{1}{3}, \ldots \)’ therefore meant all the finite terms plus all the infinitely small ones. Leibniz, who did not agree with Bernoulli’s argumentation, repeatedly called for the argumentation to be cast in the form of a conclusive reasoning. Bernoulli’s answer was:

> If there are ten elements, then the tenth must necessarily exist, if there are a hundred then the same is necessarily true for the hundredth, if there are a thousand then the same is necessarily true for the thousandth, if there are a numerically infinite number of elements available, then the most infinitesimal element must also exist.

What is noticeable is that Bernoulli effortlessly springs from the finite to the infinite. The next section will discuss Leibniz’s criticism of Bernoulli’s reasoning.

6. A leap that nature does not make: Leibniz’s criticism

Leibniz admired Bernoulli’s attempt to deploy the *lex continuitatis* against the Cartesian dualism of *res extensa* and *res cogitans*. He considered the fact that Bernoulli also had an eye for the variety of beings, with his many ‘levels of infinity’, as one of his greatest intellectual achievements. Thus he wrote appreciatively: ‘I understand... that you... have written deep and meaningful things about the various levels of infinite bodies.

At the same time he warned Bernoulli to be particularly careful about the making of generalizations and the drawing of conclusions:

> You employ a very striking example. Let us in fact assume that a line does indeed consist of these line segments, which may be expressed by \( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \)

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34 Ibid. (note 11), letter dated 16–26 August 1698, 528–33 (p. 529): ‘Praeterea corpus, quod motu suo describit lineam, existit utique actu in singulis punctis, quae in illa linea concipere possunt, ergo etiam in duobus, quae ego concepto infinite sibi vicina, adeoque actu intervallum illud seu particulam infinite exiguam emeasum est. Tandem licet talis particula infinite parva non existere separatam, coexistit tamen cum toto.’

35 Ibid. (note 11), letter dated 9 November 1698, 545–50 (pp. 545–6): ‘Videtur mihi contradicto, dicere omnes terminos hujus progressionis 1/2, 1/4, 1/8, 1/16 etc. existere, infinitesimos autem reversa non esse terminos: si enim infinitesimi non existunt, tunc finitum tantum sunt termini, ergo non omnes existunt, contra hypothesin.’

36 Ibid. (note 11), letter dated 7 January 1699, 561–3 (p. 563): ‘Si decem sunt termini, existit utique decimus; si centum sunt termini, existit utique centesimus; si mille sunt termini, existit utique millenium; ergo si numero infiniti sunt termini, existit infinitesimus.’

37 Ibid. (note 11), letter from 7 June 1698, 497–500 (p. 499): ‘Agnoeo ex ejus responsione, Te quaedam ad ipsum scrupuisse profunda et ingeniosa de corporibus variis infinitis.’
... etc., and that all the elements of this series do actually exist. In my opinion, however, nothing more can be concluded from this than that in fact any arbitrary finite and determinable fraction, however small, can exist. The same is true for the movement. Even if this extends over the total number of points it does not follow that any two of them are infinitely close to each other, and even less that two 'adjoining' points exist. I certainly understand a point to be not an element of the line, but rather a limitation or negation of its further progress, that is, a border on the line.\(^{38}\)

In this quotation Leibniz is pointing out, among other things, the limitations of a mathematical argumentation. In his opinion, mathematics was only relevant for the quantitative side of reality, and thus for the world in so far as it consisted of measurable, composite and finite beings. Leibniz demonstrated that Bernoulli's 'mathematical' argumentation did not lead to the existence of infinitely small and infinitely large elements.

According to Leibniz, a world perceived as a collection of finiteness consists only of things which are infinitely divisible, and each division results in something that is finite and again divisible, whereas Bernoulli believed that eventually only an infinitely small particle would remain that, because of its indivisibility, had to exist in the world of infinitenesses. On the other hand, Leibniz did not deny the infinite multitude:

I will concede the existence of the infinite multitude, but this multitude is neither a number nor a coherent whole. It means nothing more than that there are more parts, which could be referred to by any number at all, just as there is a multitude... of all numbers; this multitude, however, is itself neither a number nor a coherent whole.\(^{39}\)

According to Leibniz, Bernoulli's reasoning did not lead to something from another world:

Because even if I recognize that no material particle exists that is not simultaneously actually divided, we nevertheless do not as a result arrive at indivisible elements or smallest things, not even at infinitely small particles, but only at continually smaller ones, albeit actually ordinary quantities, just as when adding up we arrive at ever larger ones.\(^{40}\)

\(^{38}\) Ibid. (note 11), letter from 29 July 1698, 534–8 (p. 536): 'Ut etiam exemplo sano ad rem accommodato. Ponominus in linea actu datur, 1/2, 1/4, 1/8, 1/16, 1/32 etc. omnesque seriei hujus terminos actu existere; hine inferes dari et infinitesimum, sed ego nihil aliud hinc puto sequi, quam actu dari quamvis fractionem finitam assignabiliim cupiscunque parvitatis. Similiter in motu, etsi per omnia puncta transeatur, non tamen sequitur duo puncta dari sibi infinita vicina, et multo minus dari sibi proxima. Et revera puncta concipio, non ut elementa lineae, sed ut limites seu negationes progressus ulterioris, sive ut lineae terminos'. Earlier on pages 535–6 in the same letter he says: 'Just as there is no numerical element, that is to say, no smallest part of a unit and no minimum among the numbers, so too is there no smallest line, that is to say, no linear element; after all, the line allows itself, just like the unit, to be continually divided up into parts or fractions'. Latin: 'Quemadmodum autem non datur Elementum Numericum seu minima pars unitatis, vel minimum in Numeris, ita nec datur linea minima, seu elementum lineale; linea enim, ut Unitas, secari potest in partes vel fractiones'.

\(^{39}\) Ibid. (note 11), letter dated 21 February 1699, 574–6 (p. 575): 'Concedo multitudinem infinitum, sed haec multitudine non facit numerum seu unum totum; nec alium significat, quam pluris esse terminos, quam numero designari possint, prorsus quemadmodum datur multudo suo complexus omnium numerorum; sed haec multitudine non est numerus, nec unum totum.'

\(^{40}\) Ibid. (note 11), letter dated 29 July 1698, 521–7 (p. 524): 'Etsi enim concedam, nullam esse portionem materiae, quae non acta si secta, non tamen ideo devenitur ad elementum insecessibile, aut ad minimas portiones, imo nec ad infinitas parvas, sed tantum ad minores perpetuo, et tamen ordinariae; similiter ut ad majores perpetuo in augendo accedunt.'
The exchange of letters with Leibniz had thus reached a turning point. Whereas both scholars had originally agreed with each other, they now did not understand each other any more. Leibniz, finally, summarized his criticism as follows:

You reason: ‘if ten parts exist, then the tenth one exists. Therefore if an infinite number of parts exist, then the most infinite part exists.’ Here again it is possible to object that the conclusion of the finite for the infinite has no compelling power in this instance, and that if one says that an infinite number of parts exist, that this does not mean that a certain number is being referred to, but only that one wants to say that more exist than any finite number would be able to express.\(^4^1\)

According to Leibniz, Bernoulli was erroneously making the great leap to the infinitely small from the space of the finite beings. His ‘mathematical’ series did not take him into a different world. Further, in his opinion Bernoulli was not capable of indicating the links or of tracing the transition between the finite and the infinite.

At the heart of the matter was the fact that Bernoulli was still locked into the tradition of classic metaphysics. For he treated the infinite as if it was an empirical concept that could be known by observation and experimentation, by analogy with the finite. In order to be able to make sensible statements about the infinite it is necessary to have a specific philosophical thought mode. Leibniz had pointed out to Bernoulli that such a mode was missing. Mathematics offered no relief, either, because this only referred to the ‘detailed’ material. This was the point that caused the discussions between Leibniz and Bernoulli to grind to a halt.

7. Conclusions

Bernoulli looked to Leibniz for support in his conflict with the Groningen theologians. Leibniz praised him for his progressiveness in trying to place all forms of life including God and souls into a continuum. He did, however, point out that such a continuum of being is fundamentally different from a numerical series. According to Bernoulli’s theory it was intrinsic to the geometrical structure and in agreement with the infinite divisibility of the material that nature itself could cross the boundaries of the finite and therefore could change into intangible infinity. At the point where the finiteness of a simple being changed into the infiniteness of the very smallest thing, and where this smallest of things could never be the same as ‘nothing’, was it not there that its being, soul or pure form was situated? And could not God, too, as the Absolute Form, be situated on the other side of the farthest limits of our finite world, but which still bordered on it? By equating infinity with the place where God and the souls perhaps dwelt, and by making this place a part of the world itself, yea even to consider that it was accessible via mathematical knowledge, Bernoulli attempted to reconcile belief and science. He wanted to supply science, including the new developments, with a foundation without coming into conflict with the ruling theological viewpoints and his own religious convictions. He was not able to avoid conflicts. Bernoulli was eventually charged with heresy and the Groningen theologians were not convinced.

\(^{41}\) Ibid. (note 11), letter dated 13–23 January 1699, 564–6 (p. 566): ‘Pene oblitus eram questionis, utrum extant infinitesima! Dubitari potest an sequatur: Positis terminis decem, datur decimus: ergo positis terminis infinitis, datur infinitissimus. Dicet enim fortasse aliquis, argumentum de finito ad infinitum hic non valere: et cum dixerit dari infinitum, non dixerit dari eorum numeros terminatum, sed dari plura quovis numero terminato.’
Bernoulli’s ontological-geometrical proof of the existence of God remained trapped in the logical contradiction of every ontological proof of the existence of God. The transition from the finite to the infinite turned out to be a leap that he could not prove nature was capable of making. The transition from ‘matter with form’ to ‘form without matter’ remained discontinuous. Thus there was once again a dualism which was exactly what Bernoulli and Leibniz were trying to fight. Although Bernoulli made the Cartesian gap between the material and the ideal much smaller by making them border on each other, this did not result in them no longer being divided from each other. The border turned out be a hurdle that had to be taken by a leap. Thus in the final analysis Leibniz’s reproach to Bernoulli boils down to the fact that he turned out to be in conflict with this own starting point: the *lex continuitatis*. Bernoulli, when it came to philosophy, did not find the support he was looking for from Leibniz.

In 1705 the Bernoulli family returned to Basel. For ten years Groningen was in the forefront of philosophical struggles and mathematical developments, which have kept the *nova methodus* and its philosophical implications alive down to the present day.