To Count or to Think, That is the Question.

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1. HANDSHAKES

My wife and I were invited to a party recently, a party attended by four other couples, making a total of ten people. As the guests arrived, some of them shook hands with some of the ones already there, but, of course, people did not shake their own hands and spouses did not shake hands with one another.

It quickly became visible, as each couple arrived, that there was a pattern in the previous acquaintanceships of the people involved: the husband of the just arriving couple always knew all the men already there, and the wife knew all the women, but none of the men was previously acquainted with any of the women, and, of course, vice versa.

In all fairness, however, let me warn you that these facts by themselves do not imply anything about who shook hands with whom. Some of the guests didn’t hold with handshaking and some did, some were polite and some were not – it could perfectly well be that old friends did, or else did not, shake hands, and that newly made acquaintances did, or else did not, shake hands with the people they were being introduced to. There is, however, something else that you should be told and that will greatly enrich your information.

To wit: after everyone arrived, I went around and asked all of my nine companions (including my own wife, but, of course, not including myself) how many hands they shook. The total number of possible answers was obviously 9 – right? – since people did not shake their own hands or their spouses’, that left eight others to shake hands with, and that means that the number of hands that any one person could possibly shake is either 0, 1, or 0, and so on, down to 1.
and even to zero: the numbers between 0 and 8 inclusive are the nine possible answers. Here is the most curious fact about my numerical investigation: I did in fact receive all possible answers. There was (exactly) one person who shook no hands at all, exactly one who shook just one hand, exactly one who shook two, and so on up the line — exactly one who shook exactly eight hands.

Here is the puzzle: was the person who shook four hands a man or a woman?

I mention this problem not because it is what I want mainly to say (but I will parenthetically whisper that it is a legitimate, bona fide problem, and my mathematician colleagues could have solved it, and so can you), but because it illustrates, sort of, one of the basic mathematical principles that I do want to discuss. Here is another question, that leads to another basic mathematical principle: is there a number with the property that when it is multiplied by itself five times the result is the same as if we had just added 2 to it? Anyone who didn’t manage to remain completely innocent of high-school algebra will recognize the question as a nonsymbolic way of asking whether the equation $x^5 = x + 2$ has any solutions.

I am not demanding that you answer either puzzle right here and now, but I do ask that you look at them with me. Most of the population of the world does NOT consist of mathematicians, and consequently knows very little about what mathematicians do and the little they know is usually inaccurate. The most common notion is that mathematicians ask and answer questions about numbers, such as these: “how much does it weigh?” or “how many are there?” or “how long is it?”. The handshake problem I just posed is not like that: the desired answer is not 6, or 4.359, or $\sqrt{19}$, or $\frac{5}{2}$, but “man” or “woman”.

I would like to maintain that the puzzle is much more typical of the kind that mathematicians are usually concerned with than a numerical question would have been. Most mathematicians do not think about numbers, especially not nowadays. We have learned to live with computers, and we can and do leave the number work to them; what mathematicians are concerned with are designs, patterns, abstract ideas, and the logical connections among them.

Yes, that’s right usually what a mathematician wants to know is not a number, nor a fact, nor yet a theorem, and not an example and not even a proof — most often what we want is a method, what we want is understanding, what we want is insight into an idea. Abstract ideas are what we try to juggle — abstract ideas such as symmetry, continuity, order, chance, size, and connectedness — that’s the stuff that is our daily bread and butter.

The question in the title above is not really well posed — it’s a little like asking whether we should do without our hearts or our lungs, or like asking whether in running the left foot or the right is more important. The truth is that most of the time we should, we must, do both — both count and think. Most of us, however, prefer one or the other, or are much better at one than at the other.

You have probably guessed by now that I tend toward the “think” answer rather than the “count” one — I like the conceptual more than the computational. As long as I am making personal confessions, let me go on and tell you that I like words more than numbers, and I always did. Why then, you might
well ask, did I become a mathematician? The truth is that I don’t know.

2. **Tennis Players**

To make another point about the count versus think question, let me put before you another puzzle.

Imagine a society of 1025 tennis players. The mathematically minded ones among you, if you haven’t already heard about this famous problem, have been immediately alerted by the number. It is known to anyone who ever kept on doubling something, anything, that 1024 is \(2^{10}\), the product of ten factors all equal to 2. All cognoscenti know, therefore, that the presence in the statement of a problem of a number such as \(1 + 2^{10}\) is bound to be a strong hint to its solution; the chances are, and this can be guessed even before the statement of the problem is complete, that the solution will depend on doubling — or halving — something ten times. The more knowledgeable cognoscenti will also admit the possibility that the number is not a hint but a trap. Imagine then that the tennis players are about to conduct a gigantic tournament in the following manner. They draw lots to pair off as far as they can, the odd man sits out the first round, and the paired players play their matches. In the second round only the winners of the first round participate, and the whilom odd man. The procedure is the same for the second round as for the first — pair off and play at random, with the new odd man (if any) waiting it out. The rules demand that this procedure be continued, over and over again, until the champion of the society is selected. The champion, in this sense, didn’t exactly beat everyone else, but he can say, of each of his fellow players, that he beat some one, who beat some one, . . . , who beat that player. The question is: how many matches were played altogether, in all the rounds of the whole tournament?

There are several ways of attacking the problem, and even the most naive ones works. According to it, the first round has 512 matches (since 1025 is odd and 512 is a half of 1024), the second round has 256 (since the 512 winners in the first round, together with the odd man of that round, make 513, which is odd again, and 256 is a half of 512), etc. The “etcetera” yields, after 512 and 256, the numbers 128, 64, 32, 16, 8, 4, 2, 1, and 1 (the very last round, consisting of only one match, is the only one where there is no odd man), and all that is necessary is to add them up. That’s a simple job that pencil and paper can accomplish in a few seconds; the answer (and hence the solution of the problem) is 1024.

A mathematical student might proceed a little differently. He would quickly recognize, as advertised, that the problem has to do with repeated halvings, so that the numbers to be added up are the successive powers of 2, from the ninth down to the first, — no, from the ninth down to the zeroth! — together with the last 1 caused by the obviously malicious attempt of the problem-setter to confuse the problem-solver by using 1025 instead of 1024. The student would then proudly exhibit his knowledge of the formula for the sum of a geometric progression, he would therefore know (without carrying out the addition) that the sum of 512, 256, . . . , 8, 4, 2, and 1 is 1023, and he would then add the odd
1 to get the same total of 1024.

The trouble with the student's solution is that it's much too special. If the number of tennis players had been 1000 instead of 1025, the student would be no better off than the naive layman. The student's solution works, but it is as free of inspiration as the layman's. It is shorter but it is still, in the mathematician's contemptuous word, computational.

The problem has also an inspired solution, that requires no computation, no formulas, no numbers — just pure thought. Reason like this: each match has a winner and a loser. A loser cannot participate in any later rounds; every one in the society, except only the champion, loses exactly one match. There are, therefore, exactly as many matches as there are losers, and, consequently, the number of matches is exactly one less than the membership of the society. If the number of members is 1025, the answer is 1024. If the number had been 1000, the answer would be 999, and, obviously, the present pure thought method gives the answer, with no computation, for every possible number of players.

I offer the tennis player problem as a microcosmic example of an abstract and pretty piece of mathematics. The example is bad because, after all my warning that mathematicians are interested in other things than counting, it deals with counting; it's bad because it does not, cannot, exhibit any of the conceptual power and intellectual technique of non-trivial mathematics; and it's bad because it illustrates applied mathematics (that is mathematics as applied to a "real life" problem) more than it illustrates pure mathematics (that is, the distilled form of a question about the logical interrelations of concepts — concepts, not tennis players, and tournaments, and matches). For an example, for a parable, it does pretty well nevertheless; if your imagination is good enough mentally to reconstruct the ocean from a drop of water, then you can reconstruct abstract mathematics from the problem of the tennis players.

3. Geometry
Is the distinction between counting and thinking of importance to people who are not mathematicians? It may be. Let me suggest two extremes for your consideration. The lightning calculator is frequently not a mathematician — he sees instantly that

\[ 123 \times 456 = 56088 \]

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but he has no insight into why. He may be able to add up the number of matches played in the tennis tournament in a flash, but it is quite possible that he is completely unable to understand (and even more unable ever to invent) the abstraction involved in the one-to-one correspondence that the conceptual solution depends on. The other extreme might be represented by a typical victim of what used to be called "math anxiety". Many such victims are creative, intelligent, and even brilliant people — quite possibly people to whom abstract set theory and the subtleties of mathematical logic can be a source of esthetic pleasure — but they are people who are turned off, or frightened, or paralyzed by numerals, and symbols, and computations. They have
my sympathy – looking at a mathematical proof they feel, I think, the way I feel when I look under the hood of my car: worried, bewildered, and totally powerless.

I do not mean to suggest that counting and thinking are enemies and that each of us must swear allegiance to one camp or the other. The purpose of computing is insight – insight is what mathematicians are always after. The computer, I say, is less of a contribution to mathematics than the discovery of arabic numerals and of decimal notation – and that was mighty little. Mathematics – by which I mean all our familiar classical subjects such as algebra and geometry and number theory – will go on after the computer exactly as it went on before – no better, no worse. Archimedes, Diophantos, and Euclid didn’t need and didn’t use arabic numerals and decimal points.

The count or think duality is visible in elementary geometry too – let me give you a nice example, the 8 × 8 tiling problem. Partition each of the sides of a square into eight intervals of equal length, and consider the partition so induced of the given square into 64 small squares. Remove two diagonally opposite ones of those small squares and then tile, if you can, the remaining 62 of them with 31 dominoes each the size of two adjacent small squares. The question is of the yes or no kind – can it be done or can’t it? – and the answer turns out to be no. It’s easy enough to program the problem and run it through a computer, or, in other words, systematically to list all possible ways of laying down dominoes, one after another, as long as no obstacles are encountered; the conclusion is that no tiling is possible – they all encounter obstacles. (Try it with 4 × 4 instead of 8 × 8; there are only two cases.)

That’s the counting way to proceed; there is also a thinking way, based on a one-word hint: chessboard. Indeed: the diagonally opposite removed squares are of the same color on a chess board, black say; of the remaining 62 squares then 30 are black and 32 are white. Each domino covers one square of each of the two colors; 31 dominoes will necessarily cover 31 black squares and 31 white ones, and never 30 black ones and 32 white ones.

Isn’t that an insight that beats the systematic exhaustion of cases? Isn’t that more profound – isn’t that better in every way?

4. IS MATHEMATICS USEFUL?

As time goes on, is mathematics becoming more useful or less? Let us take a quick look at history: how did it all start? A good mathematician to start with is Archimedes, who lived approximately 2200 years ago. He is sometimes remembered as a very pure mathematician – he is the one who chided the Roman soldier with murder in his eye, the one who interrupted Archimedes’s study of geometry on the beach at Syracuse, by saying “do not disturb my circles”. He occupied himself with other amusements, such as this: throw a ball up, let it trace a parabolic arc, and calculate the area under that arc. Many professional football players, and university administrators, and elected politicians, are inclined to sneer at such occupations and say “so what?” – but it turns out that Archimedes was in fact really solving a differential equation, and that is generally regarded as one of the most practical and indispensable
activities in mathematics; for Archimedes it won naval battles.

Ptolemy was an astronomer and an applied mathematician. He said that the sun moves in circles around the earth and, when he ran into predictive troubles, he introduced a fudge factor – well, really, it moves in little circles around a big circle, and then again, and again. He is sometimes laughed at for this – it is fashionable to say that he missed the boat and Copernicus and Kepler were needed to set him straight – but in fact he worked things out to a remarkable degree of accuracy, and, for many purposes of navigation, what he knew was as good as the most sophisticated modern mathematics.

Mathematical games have fascinated mathematicians for a long time. Chess, for instance, is approximately 1500 years old. In the beginning of the 20th century, Émile Borel and others started seriously thinking about games such as chess, just for the fun of it. Mathematics connects with everything, including games. The biggest step forward in this direction came in 1928, and it was taken by John von Neumann. He proved the so-called "minimax theorem" – and thus he proved, in effect, that for most games that people play there is actually no practical point in playing them! Some 14 or 15 years later, this idle amusement became an 800 page book (Theory of games and economic behavior, in collaboration with Oskar Morgenstern) and has since become an essential part of economics and many other planning procedures, including the ones for wars and recessions.

In the 19th century an Irishman, George Boole, investigated "the laws of thought" – and in the course of his investigations he discovered a trivial but beautiful kind of algebra that has been the joy of freshman students ever since: plus is the same as minus, and it is almost impossible to make an arithmetical mistake. Logicians have enjoyed Boolean algebra – and then, to everybody’s astonishment, it became a vital part of everybody’s daily life – electrical circuit designs and computing machines are based on Boolean algebra.

I seem to be proving that mathematicians are useful – or were once – but I think I am being dishonest when I say that. I wouldn’t want a student to come to a course to learn how to be practical – or, in any event, to come for that reason only – mathematics touches all phases of human existence and the “practical” ones are only a part.

The very universality of mathematics may be one reason for its abstractness and its difficulty. Sure, mathematics touches the grocery store, and television design, and pure number theory, and is in a sense the highest, purest, truth and beauty in its way of thinking that affects everything else – but, at the same time, it is, in a sense, independent of all those things. It is the only field of human endeavor that enters everywhere, but it has nothing to do with human beings as such. Physiology and physics and history and linguistics and psychology and economics – all have to do with what people feel or see or do or say or think or eat – mathematics exists on its own abstract level and is therefore considered distasteful and difficult by many.

It is sometimes said that mathematics, to remain alive, has to be constantly revitalized by contact with reality – with balls rolling down inclined planes or genes forming intricate spiralled conspiracies. Is that true? Has chess ever been
useful? I think not — but it is still flourishing — there are millions (literally millions) of members of chess clubs and every now and then the whole civilized world spends days watching a young American play a young Russian. Despite all that, chess has not been revived in 1500 years. To point to mathematics being useful is as honest as to say that Beethoven’s Ninth Symphony was useful — after all the directors of several popular and money-making movies made essential use of it. (Has any of you ever seen the Clockwork Orange?) What do we mean by “useful” anyway? Is something useful if it prolongs life (e.g., penicillin) — or if it enriches the soul (e.g., a church service or a novel by Kafka) — or if it provides pleasure (e.g., a question and answer show on television) — or if it fills a demand (e.g., a football game) — or if it makes a profit? (In the latter case, by the way, an injection to prevent influenza is arguably not useful?)

We might try to make a classification of useful things. I suppose something is useful if it provides energy (the ability to do work), or if it enables us to make predictions, or if it facilitates a search (for water, oil, gold, food), or if it facilitates communication, or navigation, or the distribution of goods already produced. Isn’t it true, however, that everything that we normally say is good for something, is “useful”, is for that very reason not regarded as highly as the thing that it is good for? When we ask “what is this good for?”, and receive the answer, then we can turn right around and ask, what’s that good for, and get the answer, and proceed this way for a long time.

Is there a philosopher in the house? He would often have heard what I learned when I was a student of philosophy (before I changed over to become a student of really pure thought, mathematics) — namely, that everything that’s good for something is, ultimately, good for something that’s good for nothing.

5. Numbers, Phonemes, and Species
The main concept I would like to discuss is that of abstraction. Consider, for an illustration, the following question: what is there in common between the biologist’s concept of a species, the linguist’s concept of a phoneme, and the mathematician’s concept of a number?

I’ll turn to the three parts of the question in reverse order, but before doing that I’d like to describe the leitmotif of the whole discussion.

What is a black hole? I don’t know, but I get a vague idea every now and then from a casual article in a newspaper or a magazine. It seems to be something very very heavy — so heavy that nothing that once enters its domain of gravitational attraction can ever escape from it, not even light, and, as a result, it is something that we can never perceive with any of our senses — something we can never see, hear, smell, taste, or feel. It has a measurable influence on some of the world we can perceive, but it itself is an abstraction. I probably said that all wrong, but I feel that even my vague and erroneous notion of a black hole is worth knowing — when I learned it, my soul grew. I became richer. I saw a vista I had never dreamt of before, my imagination was stimulated in a new way.

All parts of human intellectual endeavor have their abstractions: the economist’s utility, the psychologist’s id, the chemist’s molecule, the biologist’s
species, the linguist’s phoneme, and, of course, the mathematician’s number — all these are abstractions, and each is a seminal part of the field it belongs to. When I asked, however, what species, phoneme, and number have in common, I wasn’t just leading up to the shallow answer that they are all abstractions. There is more to the question than that.

A dictionary might define a phoneme of a language as “a smallest unit of speech that distinguishes one word from another”. That’s too quick, too shallow, and too simplistic, but it is a beginning of a definition. An example will help to clarify the issue. If I consider the English word “bat” and then replace b by m, I get another English word, “mat”, which means something completely different; that’s why b and m belong to two different English phonemes.

Consider, on the other hand, the words “stone” and “tone”. Does everybody realize that the t sounds different in those words? In “tone” it is aspirated, and in “stone” it is not — by which the linguist means that if I hold a slip of paper two or three inches from my lips and then say “tone”, the paper will move, but if I say “stone” it will not. There are languages (I believe Hindi is one of them) in which the replacement of an unaspirated t by an aspirated one can change meaning (just as the replacement of b by m changes the meaning of “bat”). In English, however, although phoneticians and their machines can distinguish between the two t’s, there is no context in which the replacement of one by the other changes the meaning. If a person who is not a native speaker of English uses an unaspirated t where he shouldn’t, we feel that there is something slightly off, that he has a foreign accent in some sense, but we don’t have any trouble understanding him. As far as English is concerned, the two t’s are “isosemantic”. There is no such word — I just made it up — but everybody can probably guess what it would mean if it existed: it would mean that the replacement of one by the other preserves meaning.

What then is a phoneme? Or, better asked, what is the phoneme of a sound? Answer: the collection of all sounds isosemantic with it. Since b and m are not isosemantic, b does not belong to the phoneme of m, but that in “tone” does belong to the phoneme of the t in “stone”.

A similar analysis of the concept of species is possible, but I will not enter into it now. A dictionary might define a species as “a collection of organisms capable of interbreeding”, but before we could discuss the pertinent analogue of “isosemantic”, we would need to sort out the sexes, and that digression, while possibly interesting, would take too long.

The concept of number is nearer at hand, and, at least in mathematical circles, very well known. We all use words such as “five” every day, but do many people ask themselves what “five” is? And, by the way, if they don’t, shouldn’t they be ashamed of themselves? We wouldn’t use words such as “grandfather”, or “tax”, or “lawnmower” without being able to define them — without, to be specific, being able to tell a ten-year old child exactly what a grandfather, or a tax, or a lawnmower is, but our present challenge is to tell him exactly what a number is. I don’t mean what a number DOES, or how a number can be used — I mean what it is.

All right: what is “5”? We may not know that, but we know that if it is the
answer to “How many fingers are there on my right hand?”, then it’s also the answer to “How many players are there on a basketball team?” In other words, while we may not know what “number” is, we do know when two sets of objects (be they fingers, or whatever) are “equinumerous”. They are that just when we can establish a correspondence between them (for example by pointing to each basketball player on the team with a different finger) that is a one-to-one correspondence — each object in each set corresponds to a unique object in the other set.

What then is a number? Or, better asked, what is the number of objects in a set? Answer: the collection of all sets equinumerous with it.

6. ABSTRACTION AND ATTITUDE: EQUIVALENCE RELATIONS AND EXTENSIONALISM

That is an abstract definition, it is a frightening definition, it’s an ingenious definition. It is due to Bertrand Russell, and it leads me now to comment on two things: one, an abstraction, a basic mathematical concept, that includes the way species, phones, numbers, and many other concepts in many parts of life are best thought of, and two, an attitude, a philosophical stand, that some mathematicians embrace, and that contributes greatly to the clarity and precision of mathematics. The name of the concept is “equivalence relation”, and it is well known and standard; the name of the attitude is “extensionalism”, and, while the attitude is not uncommon, the name, so far as I know, is something that I’ve been using for some time in private, but no one else has ever heard of.

An equivalence relation is a relation that has three properties in common with the relations of being isosemantic and equinumerous, namely that it is reflexive, symmetric, and transitive. The replacement of an utterance by itself (which is, of course, no replacement at all) surely preserves meaning, and each set has a one-to-one correspondence with itself — that’s what “reflexive” means in the special cases of words and numbers. Officially and generally: a relation is reflexive in case every object in its realm does bear that relation to itself. So, for instance, fatherhood is NOT reflexive — no one can be his own father — and whether brotherhood among, let us say, human males is or is not reflexive is a small hairsplitting debate about how you want to use words. Am I my own brother?

To say that a relation is “symmetric” means that the roles of two objects in the relation can always be reversed. Example: if the initial sound in “pit” is isosemantic with the initial sound in “pendulum”, then, vice versa, the initial sound in “pendulum” is isosemantic with the initial sound in “pit”. Similarly: if a basketball team is equinumerous with the fingers on my right hand, then, vice versa, the fingers on my right hand are equinumerous with a basketball team. Here are a couple of non-examples: fatherhood is not symmetric (my father bears that relation to me, but I do not bear the same relation to him), and fondness is not symmetric — while it may often happen that someone I am fond of is fond of me, it is not guaranteed, and one single exception disproves the universality of the property.
“Transitivity” is just as easy a concept, but it takes a little longer to say. If three sounds are isosemantic in order, that is, the first and the second are isosemantic, and the second and the third are, then it follows that the first and the third also are. A well-known non-example is friendship: it’s not always true that if Tom is Dick’s friend and Dick is Harry’s, then Tom and Harry also bear the relation of friendship to each other.

So, that’s what an equivalence relation is: one that is reflexive, symmetric, and transitive. And any time we run across an equivalence relation, the objects to which it applies can be split up into what are called equivalence classes — and, using that language, I can now say that a phoneme is an equivalence class of the relation of being isosemantic, and a number is an equivalence class of being equinumerous.

That’s one of my main points, and when I learned it, I felt that I gained a thrilling insight — that’s what I mean by a thrill of abstraction. The notion of equivalence relation is one of the basic building blocks out of which all mathematical thought is constructed. It is simple, it is general, it is widely applicable, and it is 100% explicit and precise. And, what’s more, it has nothing to do with columns of numbers or triangles or electronic computers or whatever mathematics is sometimes thought to be about — it is abstract pure thought.

Now, about “extensionalism” — there I am not sure I can explain what I feel. In a short sentence what I am trying to say is that to a mathematician — well, in any event, to me — a concept IS its extension. Consider, for example, the number 5. What is it? Not “What does it do?”, “How can it be used?”, or “How can I tell it apart from others?”, but “What IS it?” Mathematicians usually ask such questions: their insistence on definitions and their insistence on complete precision in the definitions and complete consistency in their use is one of the distinguishing features of their art. The “extension” of a property (an old, established philosophical term) is the class of all objects that possess it. Thus, the extension of “blue” is the class of all blue things — the sky, the Danube, all blue books, all blue neckties, whatever — whatever — everything that happens to be blue. The extension of “5” is the class of all quintuples — a basketball team, the fingers of a hand, whatever. A cautious lexicographer might be willing to go this far with the mathematician: very well, he might say, 5 is the property that is common to all quintuples. The rigorous mathematician would consider that pussyfooting, however. Just what, pray, is a “property”? he would ask. And how dare we speak of “the” common property of the set of all quintuples — how do we know there is only one? No, sir!, he would say: all I really know about fiveness is that I am willing to assert it of the fingers on my right hand, and, extending from there, of any other set that I can put in one-to-one correspondence with those fingers. In others words, he would say, I know the extension of fiveness, and that’s all I know about it. The only courageous way to define 5, therefore, is to follow the principle that a concept IS its extension, and, as a religiously observant extensionalist, I therefore DEFINE 5 to be the equivalence class of equinumerousness to which the set of fingers on my right hand belongs.

There is something cold and forbidding, something impersonal and fright-
ening, about this definition — one might feel that while it is intellectually, logistically defensible, it somehow misses the essence of the concept being defined. It reminds me of the classical, and equally unsatisfying definition of a man as a “featherless biped”. When I first heard that I objected. Surely, I thought, there is more to humanity than that. What about soul, what about humor, what about art, culture, technology, war, friendship, motherhood — what about all these “essential” characteristics of humanity — doesn’t a cold-blooded definition such as “featherless biped” miss them all, and therefore miss the point? After many years of becoming used to the idea, I no longer feel that discomfort in the presence of an extensionalist definition. If it is indeed true that humanity is coextensive with the class of featherless bipeds (I repeat: If it is indeed true — I am not asserting that it is), then humanity is the class of featherless bipeds. And, similarly, since “fiveness” jolly well is coextensive with the class of quintuples, I happily embrace the definition according to which 5 IS that class.

7. DREAMERS AND NON-CONSTRUCTIVE PROOFS
It's about time I turned to the second of the two questions that I raised at the beginning, in order to describe a second, very different, basic mathematical belief and behavior. Is there, I asked, a number that when multiplied by itself 5 times gives the same result as adding 2 to it? There are those, both among dreamers and among very practical people, who would answer that question by yes only if they could explicitly produce a number with the property described, or, at the very least, if they could explicitly describe an algorithm, a procedure of calculation, that will produce such a number. Thus, for instance, if I change the number 2 in the problem to 240, and if I go on to observe that 3² = 243, which is the same as 3 + 240, then, I think, we would all agree that the changed question has been answered, and answered in the affirmative.

There is, however, another way to answer such questions, the way of non-constructive proofs, of which I’ll give a modest example. Imagine that I have an ultra-efficient but not especially intelligent computer, programmed to tell me instantaneously which is greater, x² or x + 2, whenever I ask it about any particular whole number x, but that knows about whole numbers only. All right, I say to the computer, let’s go: x = 0. It says: x + 2 is greater. I say: x = 1. It says: x + 2 is greater. I say: x = 2. It says x² is greater. I say: Hurray! — the game is over, and the answer is yes. That’s right, isn’t it? If I imagine myself moving along the line, scanning all the numbers from 0 on up, and if I know that somewhere (say when x = 1) x + 2 is the larger of x² and x + 2, and somewhat later (when x = 2) x² is the larger, then, by an intuitively obvious and rigorously provable property of continuity I can rest assured that somewhere in between x² and x + 2 will be exactly equal.

What do I know now that I didn’t know before? Do I know a number x such that x² = x + 2? No, I do not. All I know, but that I know for sure, is that although I am not (not yet) able to construct one, such a number does exist.

I have just given, as I promised, a modest example of a non-constructive existence proof. I call it a “modest” example, because, as a matter of fact, with
a little trouble it can be converted into a concrete algorithm that will produce a number of the kind that is wanted as accurately as desired: for the benefit of those of us who are just dying of curiosity, I'll put on record that rounded off to five decimals the answer is 1.26717.

Genuine non-constructive existence proofs, the kind that cannot be converted into a computational procedure, are sometimes a source of heated debates in the mathematical family. They are impressive demonstrations of human ingenuity and of the depth of mathematical thought. Sometimes, for instance, in order to prove that a certain set (such as a set of points on the number line) contains at least one object of a particular kind (such as a number \( x \) for which \( x^3 = x + 2 \)), a mathematician might use a "stochastic" method. That's a complicated concept whose detailed description would take us too far afield (and, besides, some of you might remember that it was discussed and explained by Professor Kalman, here in Groningen, a couple of years ago) — but, in qualitative terms it means something like this. Design a gambling game, a dice game, say, whose possible outcomes are the objects in the set under consideration. Using the properties that are demanded of the particular objects whose existence is in question, compute the probability that the gambling game will produce one of those objects. If that probability turns out to be a positive number (in other words, not 0), then we can be sure that the set of desired objects is not empty — objects like that must exist — even though the method of proof doesn't even yield a hope this side of heaven of ever concretely exhibiting one.

The stochastic method is a much fairer example of a non-constructive existence proof than the "modest" one based on continuity. Many non-constructive existence proofs use some notion of the "size" of a set (such as probability, or dimension, or even just cardinal number), and achieve their end by proving that the size of the set of objects not known to exist is large — large enough to guarantee that it is not zero!

8. SCHUBFACHPRINZIP
The very first question that I asked (remember? — the handshake question?) can be used to illustrate a third basic principle of thrilling, pure, abstract mathematical thought, the so-called Schubfachprinzip or pigeonhole principle, but I think I'll yield to my congenital tendency to mathematical sadism, and let that question stand as a puzzle for you — I'll use a different question to explain the Schubfachprinzip.

Suppose that a bunch of us are together in a room, 100 of us, say, and we form temporarily a small society of our own. In this closed society there are a certain number of acquaintanceships: some of us are acquainted with some others. I don't know which ones of us are acquainted with which others, but I'm sure of one thing: I'll bet that there are at least two of us that have the same number of acquaintances.

Believe it? Let's see if I can make it convincing. Suppose that someone asked us, each of us, myself included: "How many other people in this closed society are you acquainted with?" We could all tell him an answer, somewhere between 0 and 100. No, wait a minute. If there are exactly 100 of us, then nobody is
acquainted with 100 OTHER people; the largest number can be no larger than 99. As far as 0 is concerned, that's all right, there could well be some hermits among us, but it's not likely, and, in any event, I can easily settle that case. If there are two or more hermits, then I've already won my bet; any two hermits have the same number of acquaintances. If there is only one hermit, then let's ostracize him – let's not count him – let's go so far as to pretend that he isn't here. I must still prove that among the remaining 99 there are two of us with the same number of acquaintances, and I'll do so – but because 100 is easier to say than 99, let me assume that even if the possible hermit is not counted there are still 100 of us left.

So then, what possible numbers will each of the 100 of us give to the questioner? Answer: any number between 1 and 99 inclusive. What is it that I am betting? Answer: that two of us will give the questioner the same number. Indeed: how could we fail? There are only 99 numbers to tell him and there are 100 of us telling: there must be at least one repetition.

Isn't that pretty? I think it is, and, by the way, it is an application of the impressive sounding but childishly easy Schubfachprinzip. The principle says that if we have a number of pigeonholes with letters in them, and if there are more letters than pigeonholes, then at least one pigeonhole will end up with more than one letter in it. That childishly easy principle is still another basic building block of mathematics – it occurs over and over again, sometimes in very sophisticated contexts, and it is the backbone of all so-called finite or combinatorial mathematics.

Note that the three basic principles I have described so far are of three different kinds. "Equivalence relation" is a concept; "non-constructive existence proof" is a technique (and an attitude); and the Schubfachprinzip is a theorem, a fact (with, to be sure, many millions of applications and very different-sounding special cases). I could have, and for greater clarity I feel sure I should have, given other examples of the domains of application of the three basic principles already mentioned, and, by the same token, I could and should have given other principles, that arise in other problems. Anything like completeness in a discussion such as this is impossible in an hour or so – but perhaps I could do more justice to both the subject and the audience by at least mentioning what else could have been said.

Thus, for instance, is it obvious that the face of a clock is, in effect, a picture of an equivalence relation? (I have in mind the relation between two numbers that holds when one is obtained from the other by adding 12 to it, or, for that matter, any multiple of 12 – so that 13 o'clock is the "same" as 1 o'clock.) Or is it obvious that round-off downward (permitted by the United States Internal Revenue Service, or so I am informed, when we calculate our income tax) defines an equivalence relation? (In this sense a tax of $317.23 is equivalent to $317.00; more generally two possible calculated taxes are equivalent if ignoring the pennies, any number of them from 1 to 99, makes them equal.)

As for examples of non-constructive existence proofs: many of them depend on the famous (for some people infamous) law of excluded middle. Do we want to prove that a certain mathematical construct "exists"? Very well – let
us assume that there is no number or triangle, or whatever that satisfies the condition we are working with; let us proceed to reason from that assumption, and, if we're lucky, we shall presently arrive at a contradiction. Conclusion: non-existence is logically untenable, and therefore at least one instance of the object must indeed exist. This kind of non-constructive existence proof makes the people who don't believe in it angrier than most other kinds.

9. Do normal numbers exist?
Other examples of non-constructive proofs can occur in the theory of so-called "transcendental numbers" (there are, in the sense of Cantor's set theory, "more" transcendental numbers than non-transcendental ones, hence there must be at least one), and in the theory of "normal" numbers (the "length" of the set of normal numbers, or, in other words, the "probability" that a number be normal, is not zero, and hence there must be at least one such number).

The last thing I mentioned is sufficiently interesting that I am strongly tempted to go into a bit of technical detail. I promise it won't last long.

In this discussion the "numbers" I want to consider are proper fractions — the positive numbers that have no whole number part, such as

.500060000...
.333000000...
.333333333...
.142857142857...
.12345678901234567890...

When we look at the decimal form of such a number, we can ask: how often does the digit 8 occur in that form, in the long run average? The answer for the first three numbers is "never" — 8 just isn't in the act. The answer for the fourth number is "one sixth of the time". Isn't that clear? There is exactly one 8 in each successive group of six digits; among the first million digits the number of 8's is approximately one sixth, and if we replace "million" by more and more, the approximation to one sixth becomes more and more nearly perfect. For the last number in the list, the answer is "one tenth"; the reasoning is the same as before.

There are ten digits available to us, and we might consider that a number is "fair" if it treats each of them the same as all others — in other words, if each of the ten digits occurs in that number exactly one tenth of the time, in the long run average. In this sense only the fifth of my five sample numbers is fair.

There is a more sophisticated notion of fairness, however, according to which none of my sample numbers is fair. To illustrate what I mean, let me ask this question. Given a number (in decimal form, with no whole number part), how often do the digits 5 and 7 appear in it, next to each other, in that order (in the same long run sense as before)? The answer is "never" in all my examples,
except the fourth, and in that case it is "one sixth". To see what I mean, go along the digits, count all "blocks" of length two and keep track of what proportion of them are "57".

What should the answer be if we are to regard the number as fair, if it is fair not only to each individual digit, but fair to each conceivable pair as well? The answer depends on how many possible pairs there are – and the answer to that is 100. Clear? Sure it is: we could just count them,

00, 01, 02, ... , 09, 10, ... , 97, 98, 99.

One hundred it is, and, consequently, the only way a number can be “pair fair” is if it has each possible pair in it one hundredth of the time (in the long run average).

Can we write down a number that is fair to each digit and to each pair of digits as well? Sure we could, with paper, pencil, and some time – but as soon as the task was finished, I’d be ready with a new question that demands to be asked. The new question is about triples, such as 293. I would now refuse to call a number fair unless it treated fairly each digit (with frequency one in ten), each pair (with frequency one in a hundred), and each triple (with frequency – surely the answer is guessable – one in a thousand). And once the pattern is clear, I can continue it: in my infinite greed for justice I’ll demand an absolutely fair number, by which I mean one in which all blocks of all lengths occur with the “right” frequency (one in ten, or hundred, or thousand, or ten thousand, etc., for singles, doubles, triples, quadruples, etc.). The usual technical, mathematical name is not “absolutely fair” but “normal”, and now we’ve got a question, a bona fide, hard, juicy mathematical question. Can all these, infinitely many, conditions be satisfied simultaneously? In other words: do there exist any normal numbers?

That one I don’t think most people can do, not unless they are professional card-carrying dues-paying members of the mathematician’s union. But the mathematician who is not afraid of non-constructive existence proofs, and who has a small amount of training in modern probability theory, can sail right through it. All he needs to do is to consider the process of choosing a number at random, by, for instance, throwing an arrow randomly at the segment of the number line that lies between 0 and 1, compute the probability that the number he hits is normal, and observe that the answer is not 0. The computation is not trivial – that’s where some mathematical technique is really needed. The probability that it yields is not only different from 0, but it is as different as it could possibly be: it is equal to 1. In other words: it is almost certain that a randomly chosen number will be normal – which surely guarantees that normal numbers do exist.

The number of what I have called “basic mathematical principles” is surprisingly small. No one has ever listed them, and it would be a risky, controversial thing to do, but most mathematicians agree that mathematics is a unit – it all hangs together, with all subjects interwoven, and all concepts applicable everywhere – the number of bricks needed to build such a marvelously compact edifice cannot be very large.
That's one general comment; I'd like to make one more. I have been discussing the thrills of abstraction, and, in particular, the thrills of mathematics, which most people consider very abstract indeed. Would it be a contradiction if I now said that mathematics is an experimental science? I do think that mathematics is abstract, and I do think that mathematics is an experimental science, and I do not think that those two beliefs contradict one another.

To solve a mathematical problem is not a deductive act — it is a matter of guessing, of trial and error, of experiment. To solve the handshake problem, for instance, for five couples, we could do a lot worse than just plain guess. Guess, for instance, that the one who shook four hands was a woman, and then try to see what, if anything, is wrong with that guess. Another procedure, a more dignified one that more nearly deserves to be called an experiment, is to vary the conditions and try to solve some related but, we hope, easier problem. What, for instance, happens to the handshake question if we ask it for only four couples?, or three?, or two?, or even just one?

10. Abstractions are facts
That's the sort of way that a typical working mathematician proceeds — his attitude is not that of creation but of discovery. The answer is there somewhere, and we have no control over what it is — all we are trying to do is find it. The concepts, techniques, and theorems are abstract all right — but our learning about them proceeds in the same way as our learning about the boiling point of a chemical and the acceleration of a falling body. The abstractions are FACTS, facts outside of us, facts that we do not "invent" but that are there for us to find if we can.

Some of you will recognize, of course, that the position I have thereby "proved" is that of an unreconstructed die-hard Platonist, but you won't, I hope, hold that against me. My convictions (please do not call them prejudices) took a long time to grow and I would hate to have to give them up. I am convinced that mathematics is infinite in its extent and applications, yet a unity in its conceptual way of looking at things and describing them; the facts of mathematics are there waiting for us to guess at, experiment on, and finally stumble across; the concepts, the techniques, and the facts are abstract, and, in their very abstraction, one of the most thrilling phenomena of the universe.

11. Epilog
No machine can ever do everything. Machines can systematize and reproduce everything I have discovered till yesterday — but they cannot produce insights (such as establish a one-to-one correspondence between matches and losers in the tennis problem, or such as color the $8 \times 8$ squares like a chessboard) — they cannot predict what I'll think up tomorrow. I have the faith that we, humanity, will keep on getting our deep mystic insights and will keep on finding new and exciting solutions to "unsolvable" problems. Count, by all means, but don't ever stop thinking.

P.S. As for the handshaking problem: the person who shook four hands was my wife.