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A stability theorem for networks containing synchronous generators

recently appeared in SCL, authored by:

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Background on the stability of power networks

The stability of power networks is a highly relevant topic due to the penetration of distributed generators and the dramatic changes taking place or being planned for utility grids, such as the need for the autonomous operation of microgrids. Finding sufficient conditions for a power network to be stable is extremely difficult, because the system is nonlinear and of high dimension (it was called the most complex machine engineered by mankind in Kundur (1994). Every type of analysis is based on an approximate model of the system. An individual synchronous generator (SG) can be modeled under various assumptions (with 0, 1 or 2 damper windings, with constant field current or with field current dynamics, etc). The complexity of the SG model ranges from order 9, see for instance the nice survey Stegink, De Persis and Van der Schaft (2019) on SG modeling, down to order 2 (phase angle and angular velocity), see Chiang (2011), **DB**= Dörfler and Bullo (SICON, 2012), Sauer and Pai (1997), Schiffer, Efimov and Ortega (2019).

Modelling power networks

The prime movers can be modeled as complex dynamical systems, or as static systems where the generated power depends on the grid frequency (via the droop curve), or simply as constant power sources. The transmission lines and the loads can also be modeled as dynamical systems, such as in Gross, Arghir and Dörfler (2016), or a quasi-static approximation can be adopted, where the network is represented by its admittance matrix, with the dependence on frequency neglected, such as in Chiang (2011), Casagrande, Astolfi, Ortega and Langarica (2012), Colombino, Gross, Brouillon and Dörfler (2018), **DB**. Clearly, the more complex the model of the power network, the more difficult it is to prove stability results for it. It is an inescapable conclusion that non-local stability results can be derived only for very simple models that neglect many aspects of the true power system.

A challenge (orthogonal to the main story line)

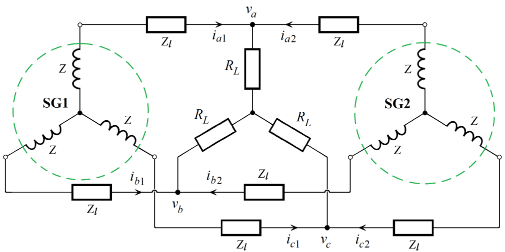


Figure: A system with two identical SGs with identical prime movers (not shown). The rotor windings (not shown) have equal and constant currents, the line impedances $Z_l = R_l + sL_l$.

This is a system of order 7, with the last state variable, the rotor angle difference δ, being curled up to be an element of the unit circle $\mathbb{T}$. It is not difficult to find the 2 equilibrium points and to find conditions for local stability around the equilibrium point with $\delta = 0$ (the other equilibrium has $\delta = \pi$).

A challenge (continued)

Open question: For what set of parameters is this system almost globally asymptotically stable? This means that for all initial states except those in a set of measure zero, the state trajectory converges to the stable equilibrium point.

The work of Andrieu, Jayawardhana and Praly (2016) on transverse exponential stability is helpful to find a stable 3 dimensional manifold in the state space.

A similar question has been analyzed for a single SG connected to an infinite bus (i.e., symmetric sinusoidal AC voltage source) in Barabanov, Schiffer, Ortega and Efimov (2017) and in Natarajan and Weiss (2018), with different sets of sufficient conditions on the parameters.

The NK and NRPS models

When studying the synchronization or the stability of a power network containing SGs (or inverters emulating SGs) using a second order model for the SGs (called the swing equation) and a quasi-static model of the transmission lines and loads, one is lead to two famous models of coupled oscillators, called the nonuniform Kuramoto (NK) model and the network-reduced power system (NRPS) model, see **DB**, Dörfler and Bullo (2014) for more background on these models. The NK model is actually the inertia-less (singlar perturbation) limit version of the NRPS model, and under certain assumptions the angle trajectories of the NRPS model are very close to those of the corresponding NK model, which in turn is easier to analyze. **DB** contains synchronization and stability results for these two models, that have had a large impact on subsequent research. We have noticed that these results can be strengthened in several aspects, and the stronger form of the results is better suited for studying the stability of power networks.

Our aim

Our aim in this research is to give the stronger statement of the main results from **DB** and to indicate how the proofs in **DB** can be adjusted to prove the stronger results. Given this aim, it is natural that we will closely follow the notation and terminology of **DB**. After presenting the NK and the NRPS models, we will briefly indicate how these models can been derived from physically motivated equations, using approximations.

Notation. We denote by $\mathbb{T}$ the unit circle (torus), in which angles are represented by numbers in $(-\pi, \pi]$ and addition (subtraction) is performed modulo 2π. If $\theta_1, \theta_2 \in \mathbb{T}$, then the *geodesic distance* between them is

$$|\theta_1 - \theta_2|_g = \min\{|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|\}.$$

Note that if $\theta : [0, \infty) \to \mathbb{T}$ is differentiable at a point t, then its derivative $\dot{\theta}(t)$ is a real number.

More notation

In the sequel, we work with a fixed natural number n whose meaning is the number of SGs in the power network. For any $\gamma \in (0, \pi]$ we define

$$\Delta(\gamma) = \left\{ \theta \in \mathbb{T}^n \middle| \, \exists \theta_0 \in \mathbb{T} \; \middle| \; |\theta_j - \theta_0|_g < \frac{\gamma}{2}, \; 1 \le j \le n \right\}.$$

In other words, $\theta \in \Delta(\gamma)$ iff there is an open arc of length γ in $\mathbb{T}$ that contains all the angles θ_j (and θ_0 is the midpoint of this arc). For $\gamma \in (0, \pi]$ we define $\overline{\Delta}(\gamma)$ as the closure of $\Delta(\gamma)$ in $\mathbb{T}^n$. For any $\gamma \in (0, \pi]$ we introduce the set $\Delta_{\mathrm{grnd}}(\gamma) \subset \mathbb{R}^{n-1}$ as follows:

$$\Delta_{\mathrm{grnd}}(\gamma) = \left\{ \delta \in \mathbb{R}^{n-1} \;\middle|\; \begin{array}{c} |\delta_j| < \gamma, \\ |\delta_j - \delta_k| < \gamma, \end{array} \quad 1 \le j, k \le n-1 \right\}.$$

More notation and parameters

Define the mapping $\mathrm{grnd} : \mathbb{T}^n \to \mathbb{R}^{n-1}$ as follows: $\delta = \mathrm{grnd}(\theta)$ if

$$\delta_j = \theta_j - \theta_n, \quad 1 \leq j \leq n-1. \tag{1}$$

Notice that the image of $\Delta(\gamma)$ through the mapping grnd is exactly $\Delta_{\mathrm{grnd}}(\gamma)$:

$$\mathrm{grnd}(\Delta(\gamma)) = \Delta_{\mathrm{grnd}}(\gamma), \quad \gamma \in (0, \pi].$$

Sometimes we also need the notation $\overline{\Delta}_{\mathrm{grnd}}(\gamma)$ for the closure of $\Delta_{\mathrm{grnd}}(\gamma)$ in $\mathbb{R}^{n-1}$.

Parameters of the models. The matrix $[a_{jk}] \in \mathbb{R}^{n \times n}$ is such that

$$a_{jk} = a_{kj} > 0 \quad \text{for} \quad 1 \leq j, k \leq n, \ j \neq k$$

and $a_{jj} = 0$ for $j \in \{1, \ldots n\}$.

More parameters of the models

Consider also an array of phase shifts $\varphi_{jk} \in \mathbb{T}$ that satisfies

$$\varphi_{jk} = \varphi_{kj} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \text{for} \quad 1 \leq j, k \leq n,\ j \neq k\,. \tag{2}$$

The diagonal values φ_{jj} are not relevant. We also introduce three sets of n real numbers: $M_j > 0$, $D_j > 0$ and $p_j \in \mathbb{R}$, $j \in \{1, 2, ... n\}$, where the intuitive meaning of M_j and D_j is the moment of inertia and the damping constant (due to viscous friction and frequency droop) of the SG with label j. The intuitive meaning of p_j is less straightforward, as it is a combination of different power terms, including the setpoint power of the generator j, as we shall see later. In **DB** the notation ω_j is used instead of p_j, which we have not followed because it might lead to confusion with the angular velocity ω_j of the generator j.

The NK model

This is a collection of n first order differential equations on $\mathbb{T}^n$: for $1 \leq j \leq n$,

$$D_j \dot{\theta}_j = p_j - \sum_{k=1}^{n} a_{jk} \sin(\theta_j - \theta_k - \varphi_{jk}) . \qquad (3)$$

The state of this system is $\theta = (\theta_1, \ldots \theta_n)$, which is in the state space $\mathbb{T}^n$. The intuitive meaning of θ_j is the rotor angle of the SG with label j. We will denote $\omega_j = \dot{\theta}_j$. The vector $\omega = (\omega_1, \ldots \omega_n)$ is obviously a function of θ:

$$\omega_j = h_j^0(\theta) = \frac{1}{D_j} \left(p_j - \sum_{k=1}^{n} a_{jk} \sin(\theta_j - \theta_k - \varphi_{jk}) \right) . \qquad (4)$$

The NRPS model

This has been derived in Sauer and Pai (1997) (see also Chapter 6 in Chiang (2011)) and it consists of a collection of n second order differential equations on $\mathbb{T}$: for $1 \leq j \leq n$,

$$M_j \ddot{\theta}_j + D_j \dot{\theta}_j = p_j - \sum_{k=1}^{n} a_{jk} \sin(\theta_j - \theta_k - \varphi_{jk}). \tag{5}$$

Denoting again $\omega_j = \dot{\theta}_j$, the state of the above system is

$$(\theta, \omega) = (\theta_1, ...\theta_n, \omega_1, ... \omega_n)$$

and this evolves in the state space $\mathbb{T}^n \times \mathbb{R}^n$.
Both models can be formulated in the generic form $\dot{x} = F(x)$ where F is globally Lipschitz, the state space of the NK model is compact, and the trajectories of the NRPS model are easily shown to be bounded. It follows that both models have unique and bounded solutions defined for all $t \geq 0$ (for any initial state).

The small parameter ε

Our main result concerns not one NRPS model but a family of such models, parametrized by a variable $\varepsilon > 0$. We will assume that there are constants $m_j > 0$ (for $1 \leq j \leq n$) such that the inertia parameters M_j are given by

$$M_j = \varepsilon m_j, \quad 1 \leq j \leq n. \tag{6}$$

Using these ε-dependent parameters in (5), we obtain a system denoted by NRPS(ε). Our results for this family of systems are of an asymptotic nature, i.e., our estimates are about the behavior of certain error estimates when $\varepsilon \to 0$. Unfortunately, our results about the family NRPS(ε) say *nothing* about the solutions of NRPS(ε) for one specific choice of ε. A crucial observation for our discussion is that the singlar perturbation reduced system of the NRPS model, when $\varepsilon \to 0$, is the NK model. Thus, we can apply Tikhonov's theorem, as presented in Khalil (2002).

Derivation of the NRPS model

Consider a power system with n synchronous machines (or virtual synchronous machines) that are connected through a linear transmission network. The network is assumed to be balanced and the internal synchronous voltages of the generators are assumed to be sinusoidal functions of their respective rotor angles θ_j ($1 \leq j \leq n$). Moreover, the internal synchronous voltages are arranged in positive sequence (meaning with equal amplitudes and phase shifts of $2\pi/3$ from phase to phase), and their frequencies $\omega_j = \dot{\theta}_j$ are assumed to be in a narrow range around the nominal frequency ω_{nom}.

The system is also assumed to be quasi-static, meaning that the time derivatives of the Park transformed voltages and currents (with respect to the reference angle $\omega_{nom} t$) are negligible when compared to the same quantities multiplied with ω_{nom}. Then the currents and voltages can be accurately and conveniently modeled using time-varying phasors. For instance, the time-varying phasor of a three-phase voltage v is defined as

$$\overline{v} = v_d + \mathbf{i} v_q , \qquad \mathbf{i} = \sqrt{-1},$$

where v_d, v_q are the d, q components of the Park transformed v.

Derivation of the NRPS model - continued 1

The loads and the output filters of the inverters are assumed to be linear, and are modeled as part of the transmission network. For our derivation, it does not matter if the synchronous machines in the network are real or virtual. All the machines are assumed to have two magnetic poles on their rotors. An illustrative example:

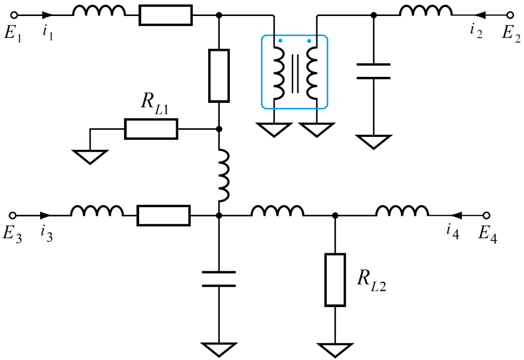


Figure: A transmission network that connects four generators with internal synchronous voltages $E_1, \ldots E_4$, one phase shown.

Derivation of the NRPS model - continued 2

The angular acceleration of the rotor of machine j is governed by the approximate swing equation

$$J_j \omega_{nom} \dot{\omega}_j = P_{m,j} - P_{e,j}\,, \tag{7}$$

where $1 \leq j \leq n$, J_j is the moment of inertia of everything rotating with the rotor, $P_{m,j}$ is the mechanical power accelerating the rotor (typically from a prime mover), and $P_{e,j}$ is the electromagnetic power decelerating the rotor (due to the power flowing to the grid). The mechanical power is assumed to be regulated by a droop controller according to

$$P_{m,j} = P_{set,j} - D_j \left(\omega_j - \omega_{nom}\right)\,, \tag{8}$$

where $P_{set,j}$ is the reference power and D_j is the damping coefficient (or droop constant). Each machine has a (three phase) “synchronous internal voltage” source (the voltage induced in the stator by the moving rotor field) described by a time-varying phasor E_j. This voltage source is connected to the grid connection point of the generator via the synchronous impedance of the stator.

Derivation of the NRPS model - continued 3

In the case of a virtual synchronous machine, this stator impedance is replaced with the output filter of the inverter combined with virtual circuit elements, see for instance Natarajan and Weiss (2017). The synchronous impedance is modeled as part of the transmission network. We use the polar decomposition of E_j:

$$E_j = |E_j| e^{i(\theta_j - \omega_{nom} t)},$$

where $|E_j|$ is assumed to be constant. Furthermore, for each machine the electromagnetic power $P_{e,j}$ is given by

$$P_{e,j} = \mathrm{Re}\left[E_j \overline{I_j}\right], \tag{9}$$

where I_j is a phasor describing the stator output current and $\overline{I_j}$ is its complex conjugate. Since the transmission network is assumed to be quasi-static, this current may be expressed as

$$I_j = \sum_{k=1}^{n} Y_{jk} E_k, \tag{10}$$

where Y_{jk} are the elements of the network's Kron-reduced admittance matrix computed at the nominal frequency ω_{nom}. Usually $Y_{jk} = Y_{kj}$.

Derivation of the NRPS model - continued 4

Each of these elements is given by

$$Y_{jk} = |Y_{jk}| e^{i\left(\varphi_{jk} + \pi/2\right)}, \tag{11}$$

where $|Y_{jk}|$ and φ_{jk} depend on the parameters of the network. It follows from (9), (10) and (11) that

$$P_{e,j} = \sum_{k=1}^{n} |E_j| \cdot |E_k| \cdot |Y_{jk}| \cdot \sin(\theta_j - \theta_k - \varphi_{jk}) .$$

Substituting this into (7) and using $\dot{\theta}_j = \omega_j$ leads to the NRPS model (5), where the constants for $1 \leq j, k \leq n$ are

$$a_{jj} = 0, \quad a_{jk} = |E_j| \cdot |E_k| \cdot |Y_{jk}| \ \text{ for } \ j \neq k,$$
$$M_j = J_j \omega_{nom}, \quad p_j = P_{set,j} + D_j \omega_{nom} + |E_j|^2 |Y_{jj}| \sin \varphi_{jj} .$$

The NK model is obtained from (5) by setting $M_j = 0$. Note that many practical transmission networks are *predominantly inductive*, which is defined by $-\frac{\pi}{2} < \varphi_{jk} < \frac{\pi}{2}$. This name is motivated by the fact that for a transmission network consisting of inductors only, $\varphi_{jk} = 0$.

The main result

Theorem. With the above notation and assumptions, define

$$\Gamma_{min} = n \min_{j \neq k} \left\{ \frac{a_{jk}}{D_j} \cos \varphi_{jk} \right\} , \quad \varphi_{max} = \max_{j \neq k} \left\{ |\varphi_{jk}| \right\} ,$$

$$\Gamma_{crit} = \frac{1}{\cos \varphi_{max}} \left(\max \left| \frac{p_j}{D_j} - \frac{p_k}{D_k} \right| + 2 \max_{1 \leq j \leq n} \sum_{k=1}^{n} \frac{a_{jk}}{D_j} \sin \varphi_{jk} \right) .$$

Assume that $\Gamma_{min} > \Gamma_{crit}$. Define $\gamma_{min} \in [0, \frac{\pi}{2} - \varphi_{max})$ as the unique solution of

$$\sin \gamma_{min} = \frac{\Gamma_{crit}}{\Gamma_{min}} \cos \varphi_{max} ,$$

and set $\gamma_{max} = \pi - \gamma_{min}$.
Then **for the NK model,** the following holds:
(1) The set $\overline{\Delta}(\gamma)$ is positively invariant for every $\gamma \in [\gamma_{min}, \gamma_{max}]$ and every state trajectory starting in $\Delta(\gamma_{max})$ reaches $\overline{\Delta}(\gamma_{min})$ (and remains there).

The main result - continued a

(2) There exists a unique vector $\delta^* = (\delta_1^*, \dots \delta_{n-1}^*) \in \overline{\Delta}_{\mathrm{grnd}}(\gamma_{min})$ such that for any initial state $\theta(0) \in \Delta(\gamma_{max})$ we have

$$\lim_{t \to \infty} [\theta_j(t) - \theta_n(t)] = \delta_j^*, \quad 1 \leq j \leq n-1. \tag{12}$$

Denote $\omega^* = h_j^0(\delta_1^*, \delta_2^*, \dots \delta_{n-1}^*, 0)$. Then ω^* is independent of $j \in \{1, 2, \dots n-1\}$ and for any initial state $\theta(0) \in \Delta(\gamma_{max})$ we have

$$\min_{1 \leq j \leq n} \omega_j(0) \leq \omega^* \leq \max_{1 \leq j \leq n} \omega_j(0) \tag{13}$$

and

$$\lim_{t \to \infty} \omega_j(t) = \omega^*, \quad 1 \leq j \leq n, \tag{14}$$

The convergence in (12) and (14) is at an exponential rate.

The main result - continued b

For the family of models NRPS(ε), the following holds:
(3) For every $\gamma_0 \in (\gamma_{min}, \gamma_{max})$ and every $\omega_{max} > 0$, there exist constants $\varepsilon_* > 0$ and $m > 0$ such that if $0 < \varepsilon < \varepsilon_*$, then for any initial state $(\theta(0), \omega(0))$ satisfying

$$\theta(0) \in \overline{\Delta}(\gamma_0), \quad \|\omega(0)\| \leq \omega_{max}, \tag{15}$$

the following holds: Denote, for $1 \leq j \leq n$,

$$\delta_j(t) = \theta_j(t) - \theta_n(t), \quad \overline{\delta}_j(t) = \overline{\theta}_j(t) - \overline{\theta}_n(t),$$

where (θ, ω) is the state of the NRPS(ε) model and $\overline{\theta}$ is the solution of the NK model starting from the same initial angles $\theta(0)$ as the NRPS(ε) model. Then

$$\|\delta(t) - \overline{\delta}(t)\| \leq m\varepsilon \qquad \forall\, t \geq 0.$$

The main result - continued c

Moreover, for every $t_b > 0$ there exists $\varepsilon^* \in (0, \varepsilon_*]$ and $\tilde{m} > 0$ such that if $\varepsilon < \varepsilon^*$, denoting $\overline{\omega} = \dot{\overline{\theta}}$,

$$\|\omega(t) - \overline{\omega}(t)\| \leq \tilde{m}\varepsilon \qquad \forall\, t \geq t_b\,.$$

(4) For every $\gamma_0 \in (\gamma_{min}, \gamma_{max})$ and every $\omega_{max} > 0$, there exists a constant $\varepsilon^{**} > 0$ such that if $0 < \varepsilon < \varepsilon^{**}$, then for any initial state $(\theta(0), \omega(0))$ satisfying (15),

$$\delta_j(t) \to \delta_j^*, \;\; 1 \leq j \leq n-1\,, \quad \omega_j(t) \to \omega^*, \;\; 1 \leq j \leq n,$$

at an exponential rate. Here, δ_j^* and ω^* are as in point (2).

For the remainder of this presentation, we compare our theorem with Theorem 2.1 in **DB**, which is a bit complicated. We discuss the differences one by one:

1. The stability of the NK model

In the original state variables θ, the NK model is not stable, as the angles θ_j keep rotating in $\mathbb{T}$. The stability of the NK model should be discussed using the vector of "grounded angles" δ from (1) as the state of the model, so that the state space changes to $\mathbb{R}^{n-1}$. Then the model becomes:

$$\dot{\delta}_j = h_j(\delta) - h_n(\delta), \tag{16}$$

where $h : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is defined by

$$h_j(\delta) = h_j^0(\delta_1, \delta_2, \dots \delta_{n-1}, 0), \quad 1 \leq j \leq n,$$

and h^0 is the function from (4). Thus, the condition $\theta(0) \in \Delta(\gamma_{max})$ is equivalent to $\delta(0) \in \Delta_{\mathrm{grnd}}(\gamma_{max})$. The model (16) is called the *grounded Kuramoto model* in **DB** and we shall also call it the *grounded NK model*.

The stability of the NK model - continued

It follows from Lemma 3.1 and Theorem 4.3 in **DB** that every state tajectory δ of (16) starting from an initial state in $\Delta_{\mathrm{grnd}}(\gamma_{max})$ converges to an isolated exponentially stable equilibrium point located in $\overline{\Delta}_{\mathrm{grnd}}(\gamma_{min})$. Moreover, all the frequencies $\omega_j = h_j(\delta)$ converge exponentially to the same limit $\dot{\theta}_\infty$. This latter property is called *exponential frequency synchronization* in **DB**. However, **DB** leaves open the possibility that the limit of δ and also $\dot{\theta}_\infty$ may depend on $\delta(0)$.

We have improved this by stating the existence of an exponentially stable equilibrium point δ^* whose region of attraction contains all initial states in $\Delta_{\mathrm{grnd}}(\gamma_{max})$ (this follows from (12)). As a simple consequence, we also have a unique limit ω^* for the frequencies ω_j, see (14).

2. The position of quantifiers in points (3) and (4)

Point (3) of Theorem 2.1 in **DB** seems to say the following: first we choose parameters for the NRPS model, from a reasonable set of parameters (they must satisfy certain inequalities). Then we choose an initial state, from a reasonable set of states, so that a state trajectory $(\dot{\theta}, \theta)$ has been determined. For this trajectory there exists a constant $\varepsilon^* > 0$ such that if $\varepsilon = \max\{M_j\} / \min\{D_j\}$ satisfies $\varepsilon < \varepsilon^*$, then certain estimates hold involving an asymptotic estimate term $\mathcal{O}(\varepsilon)$. There is a problem of logic here: if the parameters have already been fixed, then ε is already fixed, and so we cannot make asymptotic statements about what happens when ε becomes very small. Point (4) of Theorem 2.1 in **DB** suffers from a similar problem: it states that certain errors tend to zero if ε is less than a bound that is even smaller than in point (3). But again, ε is already fixed, so this may possibly lead to statements (3) and (4) being empty.

The differences - continued

To remedy this problem, we must decide which parameters are fixed and which can vary as functions of ε. Then, the quantifiers "there exist $\varepsilon^* > 0$ and $m > 0$ such that for every" have to be positioned much earlier in the statements, after the fixed parameters have been chosen. The estimates involving ε should hold for any $\varepsilon \in (0, \varepsilon^*)$ and for any choice of initial state (from a reasonable set of initial states). This is precisely what we have done in our theorem above.

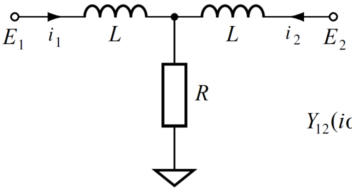
3. The dependence on ε. The way in which we introduce the parameter ε is rather different from **DB**, where it is set as $\varepsilon = \max\{M_j\}/\min\{D_j\}$. Our reasoning is as follows: In singular perturbations theory, see for instance Chapter 11 in Khalil (2002), setting $\varepsilon = 0$ and expressing the fast state variables as functions of the slow ones leads to the so-called reduced model, and in our case this is exactly (16).

The differences - continued

The reduced model cannot depend on ε, because we have set $\varepsilon = 0$ to obtain it. Our reduced model clearly depends on p_j/D_j and on a_{jk}/D_j, so these cannot depend on ε. The concluson is that only M_j can depend on ε, and this leads to the way in which we have defined the dependence of the parameters on ε in (6).

4. The generality of the model, the range of the angles φ_{jk}. If we model a power network using the NRPS model, then the meaning of φ_{jk} is that $\varphi_{jk} + \pi/2$ is the angle of the entry Y_{jk} in the admittance matrix of the network. In **DB** it is assumed that $\varphi_{jk} \in [0, \pi/2)$, see the text after the NRPS model (2.3) in **DB.** The case of a purely inductive network corresponds to $\varphi_{jk} = 0$. However, a network containing inductors and resistors (such as loads) can easily lead to $\varphi_{jk} \in (-\pi/2, 0)$. A very simple network with this property is shown in Fig. 1 (where $R > 0$ and $L > 0$), but here we omit the (simple) discussion of how to derive the equations, as it is off topic.

The generality of the model - continued



$$Y_{12}(i\omega) = \frac{R}{L^2\omega^2 - 2iRL\omega}$$

Figure: A very simple network that connects two generators producing the voltages E_1 and E_2, and contains one load. The figure shows one of three identical phases. (This is the same system as in the first figure.) For this network, $\varphi_{12} \in (-\pi/2, 0)$.

A network containing inductors, resistors, capacitors and transformers can have its angles φ_{jk} anywhere in $\mathbb{T}$. If the dominant network elements at the nominal grid frequency are inductors, transformers and resistors (i.e., if the capacitors play a small role only), then there is a high chance to have (2) satisfied, and we call such networks *predominantly inductive*.

The differences - continued

5. Exponential stability of the grounded NRPS(ε) model for sufficiently small ε. As in **DB**, it is convenient to reformulate the NRPS(ε) model using the vector of grounded angles δ from (1) in place of θ. The resulting model, with state space $\mathbb{R}^{n-1} \times \mathbb{R}^{n}$, is called the *grounded* $\mathrm{NRPS}(\varepsilon)$ *model* (see also Step 3 of the proof). Then point (4) of the theorem becomes a (local) exponential stability result for this model. To reach this conclusion, the main result of **DB** requires ε and φ_{max} to be sufficiently small, since the analysis is based on asymptotic and continuity arguments. Here, we again require ε to be sufficiently small, but no further assumptions are made on φ_{max}, since the proof is based on a converse Lyapunov theorem and singular-perturbation-inspired Lyapunov arguments in Khalil (2002). We note that these methods in principle also allow to quantify how small ε should be as a function of Lyapunov decay rates. However, the bounds are implicit and conservative and not worth reporting here.

Conclusions

Remark. Formally, point (3) of the theorem follows from point (4), given that we do not offer any estimates for the constants $\varepsilon_*, \varepsilon^*, \varepsilon^{**}$ and $m, \tilde{m}$. In reality, point (3) has its own legitimacy because we expect ε^{**} to be smaller than ε^* (see the end of our proof), so that there may be a range of values of ε for which the conclusions of point (3) hold, but not the conclusions of point (4). We use point (3) to prove point (4).

Conclusion 1. Under suitable conditions, inverter based power grids, where the inverters act as synchronverters with small inertia, have trajectories that are close to stable trajectories that correspond to the inertia-less case. The main assumptions for this to be true are that the network is "predominantly inductive" (in the sense of (2)) and $\Gamma_{min} > \Gamma_{crit}$. This is true for initial states in a large set, if the inertia constants are small enough. Moreover, if the inertia constants are even smaller, then the power grid model is locally exponentially stable, with a large region of attraction of the stable equilibrium point.

Conclusions - continued

Conclusion 2. It is not clear if designing inverter-based power grids to have very small inertia, so that the above mentioned results hold, is a good idea. The drawback of such power grids would be relatively large frequency variations triggered by load changes. Low inertia levels are hardly compatible with the legacy system requiring tight frequency regulation, but they may be conceivable in microgrids and future power transmission systems, as they have certain benefits, see Tayeb et al (2019). This paper might be regarded as proving the benefits of low inertia. Another avenue to explore is to achieve a similar stabilizing effect without reducing the inertia, by adding “virtual friction” to the power grid, in the sense of Blau and Weiss (2018). To prove this, the main result of this paper could turn out to be an important tool.

QUESTIONS?