Using mixture of Gammas for Bayesian analysis in an M/G/1 queue with optional second service

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M/G/1 queuing system with optional second service
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- Markov process
- General distribution
- Main Queue
- Service
- Departure
  - $1 - p$
  - $p$
- Failed Queue
- Re-service
- $N1$: number of customer
- $N2$: number of customer
M/G/1 queuing system with optional second service

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- \( \text{Service} \)
- \( \text{Departure} \)
- \( 1 - p \)
- \( p \)
- \( \text{Failed Queue} \)
- \( \text{Re-service} \)

- \( B_1(.) \)
  - \( \text{mean} = \mu_1 \)
  - \( \text{Variance} = \delta_1 \)

- \( B_2(.) \)
  - \( \text{mean} = \mu_2 \)
  - \( \text{Variance} = \delta_2 \)
M/G/1 queuing system with optional second service

- $s_1 = \{s_{11}, \ldots, s_{2n_s}\}$
- $B_1(.)$
  - $\text{mean} = \mu_1$
  - $\text{Variance} = \delta_1$
- $x = \{x_1, \ldots, x_{n_p}\}$
- $p$
- $B_2(.)$
  - $\text{mean} = \mu_2$
  - $\text{Variance} = \delta_2$

- $s_2 = \{s_{21}, \ldots, s_{2n_s}\}$
- Failed Queue
- Re-service
Some performance measures in this queuing system

Probability of idle period

\[ Pr(\text{idle period}) = \frac{1 - \lambda \mu_1}{1 + p^2 \lambda \mu_2} \]

Mean number of customers in MQ

\[ E(N1) = \rho_1 + \frac{\lambda^2 \delta_1 + \rho_1^2}{2(1 - \rho_1)} + \frac{p[\lambda^2 \delta_2 + \rho_2^2 + p \rho_2^2(\frac{\lambda^2 \delta_1 + \rho_1(2 - \rho_1)}{1 - \rho_1})]}{2(p \rho_2 + (1 - \rho_1)\psi(1 - p + pB_2^*(\lambda)))} \]

Mean number of customers in FQ

\[ E(N2) = \frac{p(2(1 - \rho_1)^2 + \lambda^2 \delta_1 + \rho_1(1 - \rho_1))}{2(p \rho_2 + (1 - \rho_1)\psi(1 - p + pB_2^*(\lambda)))} \]

\[ \psi(u) = uB_1^*[\lambda(1 - \psi(u))] \]
Main idea

Problems

- Estimation of performance measures and parameters of the system.
- Approximation of general distributions of service and re-service time densities ($B_1(.)$ and $B_2(.)$).

Bayesian solutions

1. Using mixture of Gamma distributions with unknown number of components to approximate $B_1(.)$ and $B_2(.)$.
2. Determine the Birth-Death MCMC algorithm to simulate from posterior distributions.
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2. Determine the Birth-Death MCMC algorithm to simulate from posterior distributions.
Mixture of Gamma distributions

We approximate general distribution of service and re-service times with mixture of Gamma distributions:

\[ B(s|\theta) = \sum_{i=1}^{k} \pi_i g(s|\alpha_i, \beta_i) \]

in which

\[ \theta = (k, \pi, \alpha, \beta) \]
Prior and Conditional distributions

Prior distributions

\[ P(K = k) \propto \frac{\gamma^k}{k!}, \quad k = 1, \ldots, k_{\text{max}} \]

\[ \pi|k \sim D(\phi_1, \ldots, \phi_k) \]

\[ \alpha_i|k \sim \Gamma(\nu, \nu), \quad i = 1, \ldots, k \]

\[ \beta_i|k \sim \Gamma(\eta, \tau), \quad i = 1, \ldots, k \]

Conditional distributions

\[ \pi|... \sim D(\phi_1 + n_1, \ldots, \phi_k + n_k) \]

\[ \beta_i|... \sim \Gamma(\eta + n_i\alpha_i, \tau + \sum_{j:z_j=i} s_j), \quad i = 1, \ldots, k \]
Improvement over previous schemes

\[ f(\alpha_i|...) \propto \left( \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \right)^{n_i} \left( \prod_{j:z_j=i} s_j \right)^{\alpha_i} \alpha_i^{\nu-1} e^{-\nu \alpha_i} \]

Where we introduce Metropolis-Hastings. From shape of the target distribution, we propose \( \Gamma(\nu, \nu) \). So, acceptance probability, is

\[ Pr(\alpha_r, \alpha^*) = \min \left\{ 1, \left( \frac{\Gamma(\alpha_r)}{\Gamma(\alpha^*)} \right)^{n_r} \left( \prod_{j:z_j=r} \beta_r s_j \right)^{\alpha^*-\alpha_r} \right\} \]

Remark

Acceptance probability for our model is around 25 percent which is optimal rate. With compare with 5 percent (previous work) is quite suitable.
Improvement over previous schemes

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Birth-Death MCMC algorithm

The algorithm is based on a birth-death process. In this approach, mixture size, $k$, changes so that births and deaths of the mixture components occur in continuous time.
Birth-Death MCMC algorithm

Starting with initial values, $k^{(0)}$, $\pi^{(0)}$, $\mu^{(0)}$, $\nu^{(0)}$

Birth - Death process:

$$k^{(1)} = \begin{cases} 
  k^{(0)} + 1 & \text{if it is the time for birth} \\
  k^{(0)} - 1 & \text{if it is the time for death} \\
  k^{(0)} & \text{if it is not time for birth or death}
\end{cases}$$

MCMC steps:

Gibbs sampling for $z$, $\pi$ and $\beta$

Metropolis – Hastings sampling for $\alpha$
Predictive densities

Given BD-MCMC output, for service parameters, \((k_1, \pi_1, \alpha_1, \beta_1)\), and re-service parameters, \((k_2, \pi_2, \alpha_2, \beta_2)\), the quantities of interest can be consistently estimated by the sample path averages.

Predictive density of service and re-service time distributions is

\[
\hat{B}(s) \approx \frac{1}{N} \sum_{j=1}^{N} \sum_{i=1}^{k^{(j)}} \pi^{(j)}_i g(s|\alpha^{(j)}_i, \beta^{(j)}_i)
\]
We consider samples of 1000 service data from mixture of two truncated Normal distributions on interval \((-\infty, 0)\):

\[
B_1(s) = 0.4 \, TN_{(0,\infty)}(1.4, 2.3) + 0.6 \, TN_{(0,\infty)}(0.2, 0.3),
\]

For re-service, 1000 data simulated from single Log-Normal distribution:

\[
B_2(s) = LN(1, 0.5).
\]

We ran 200,000 iterations of BD-MCMC algorithm with 100,000 iterations as burn-in.
Simulations

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Predictive densities

(left) service time data and (right) re-service time data.
Cumulative occupancy fractions

(left) service data, $k_1$, (right) re-service data, $k_2$, for a complete run including burn-in.
$$\rho = 0.952$$, real value of traffic intensity. Considering values of SD, all true values lie inside their 95 percent credible intervals.

| Performance measures | $E[\text{busy period}]$ | $P(\text{idle period})$ | $E(X_n)$ | $E(Y_n)$ | $P(\rho < 1 | \text{data})$ | $E(\rho | \text{data})$ |
|----------------------|-------------------------|--------------------------|----------|----------|-----------------------------|--------------------------|
| True value           | 5.7974                  | 0.381                    | 1.661    | 0.227    |                             | 0.983                    |
| Estimates            | 5.8445                  | 0.3793                   | 1.5863   | 0.2297   | 0.973                       | 0.972                    |
| SD                   | 0.0793                  | 0.0032                   | 0.0597   | 0.0024   | 0.0075                      | 0.0075                   |
Conclusions

- automatic general distribution complexity selection \((k)\)
- very efficient sampler for \(\alpha\)
- output of the performance measure of the queue

Thanks for your attention
Conclusions

- automatic general distribution complexity selection (k)
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- output of the performance measure of the queue

Thanks for your attention

The **birth rate** occur at a constant rate, $\gamma$. The **death rate** of every mixture component is a likelihood ratio with and without this component:

\[ \Delta_j = \frac{n_s}{\pi_j g(s_r | \alpha_j, \beta_j)} \cdot \frac{B(s_r)}{(1 - \pi_j) B(s_r)} \], \quad j = 1, \ldots, k

The total death rate is $\Delta = \sum_{j=1}^{k} \Delta_j$. The **birth and death processes** are independent Poisson processes. Thus, the **time of birth/death event** is exponentially distributed with mean $1/(\Delta + \gamma)$. 
Birth-Death MCMC algorithm

Starting with initial values $k^{(0)}$, $\pi^{(0)}$, $\alpha^{(0)}$, $\beta^{(0)}$, iterate the following steps:

Run the birth-death process for a fixed time $t_0$

1. Compute the death rates for each component, $\Delta_j$ and $\Delta = \sum_{j=1}^{k} \Delta_j$
2. Simulate the time to the next jump from an $\text{Exp}(\Delta + \gamma)$
3. If the run time is less than $t_0$ continue
4. Simulate the type of jump:
   \[ Pr(\text{birth}) = \frac{\gamma}{\gamma + \Delta}, \quad Pr(\text{death}) = \frac{\Delta}{\gamma + \Delta} \]
5. Adjust the mixture components
Birth-Death MCMC algorithm

MCMC steps conditional on \( k \) fixed

1. Update the latent variables from

\[
\mathbf{Z}^{(i+1)} \sim \mathbf{Z} | \mathbf{s}, k^{(i+1)}, \pi^{(i)}, \alpha^{(i)}, \beta^{(i)}
\]

2. Update the weights from

\[
\pi^{(i+1)} \sim \pi | \mathbf{s}, k^{(i+1)}, \mathbf{Z}^{(i+1)}
\]

3. for \( r = 1, \ldots, k^{(i+1)} \)
   - Update the \( \beta_r \) from \( \beta_r^{(i+1)} \sim \beta_r | \mathbf{s}, k^{(i+1)}, \mathbf{Z}^{(i+1)} \)
   - Update \( \alpha_r \) using a Metropolis-Hastings

4. Set \( i = i + 1 \) and go to step 1.
Reminder

Full conditional distributions:

\[ P(Z_j = i|...) \propto \pi_i g(s_j|\alpha_i, \beta_i), \quad i = 1, \ldots, k \]

\[ \pi|... \sim D(\phi_1 + n_1, \ldots, \phi_k + n_k) \]

\[ \beta_i|... \sim \Gamma(\eta + n_i \alpha_i, \tau + \sum_{j:z_j=i} s_j), \quad i = 1, \ldots, k \]

\[ f(\alpha_i|...) \propto \left( \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \right)^{n_i} \left( \prod_{j:z_j=i} s_j \right)^{\alpha_i} \alpha_i^{\nu-1} e^{-\nu \alpha_i} \]

where \( n_i = \# \{z_j = i\} \) for \( i = 1, \ldots, k \).
Birth-Death MCMC algorithm

Steps 1, 2 and 3.1 are Gibbs sampling. The only complicated is step 3.2, where we introduce a Metropolis-Hastings. From the shape of the target distribution, we propose a $\Gamma(\nu, \nu)$. With this proposal distribution the acceptance probability, is

$$Pr(\alpha_r, \alpha^*) = \min \left\{ 1, \left( \frac{\Gamma(\alpha_r)}{\Gamma(\alpha^*)} \right)^{n_r} \left( \prod_{j: z_j = r} \beta_r s_j \right)^{\alpha^* - \alpha_r} \right\}$$

Remark

The acceptance probability for our model is around 25 percent which is optimal rate. With compare with 5 percent (previous work) is quite suitable.
Number of component trace

(left) a trace of $k_1$ for 100,000 iterations after 100,000 burn-in iterations and (right) for the $k_2$. 
posterior distribution of $k$

(left) the estimation of posterior distribution of $k_1$, and (light) for $k_2$. 
Mean number of customers in MQ and FQ:

\[ E(X) = \rho_1 + \frac{\lambda^2 \delta_1 + \rho_1^2}{2(1 - \rho_1)} + \frac{p[\lambda^2 \delta_2 + \rho_2^2 + p \rho_2^2 \left( \frac{\lambda^2 \delta_1 + \rho_1 (2 - \rho_1)}{(1 - \rho_1)} \right)]}{2(p \rho_2 + (1 - \rho_1) \psi(1 - p + p B_2^*(\lambda)))} \]

\[ E(Y) = \frac{p(2(1 - \rho_1)^2 + \lambda^2 \delta_1 + \rho_1 (1 - \rho_1))}{2(p \rho_2 + (1 - \rho_1) \psi(1 - p + p B_2^*(\lambda)))} \]

where

\[ \psi(u) = u B_1^*[\lambda(1 - \psi(u))], \]

\( B_1^*(.) \) and \( B_2^*(.) \) are the Laplace Stieltjes Transform of service and re-service times density.