Bounded real and positive real balanced truncation using $\Sigma$-normalised coprime factors

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December 2, 2009

Abstract

In this paper we will extend the method of balanced truncation using normalised right coprime factors of the system transfer matrix by Meyer [8] to balanced truncation with preservation of half line dissipativity. Special cases are preservation of positive realness and bounded realness. We consider a half line dissipative input-output system, with quadratic supply rate given by a nonsingular symmetric matrix $\Sigma$ with the property that its positive signature is equal to the number of input components of the system. The transfer matrix of such system allows a $\Sigma$-normalised right coprime factorisation. We associate with such factorisation two Lyapunov equations, one of which is a nonstandard one, involving the indefinite matrix $\Sigma$. Balancing will be based on making the unique solutions of these two Lyapunov equations equal and diagonal. The diagonal elements will be called the Hankel $\Sigma$-singular values, because their squares are the nonzero eigenvalues of the composition of the 'graph' Hankel operator, multiplication by $\Sigma$, and the adjoint graph Hankel operator. This method of balanced truncation will be shown to preserve stability, minimality, and half line dissipativeness. We will characterise the ‘classical’ positive real and bounded real characteristic values in terms of the new Hankel $\Sigma$-singular values. Finally, we will derive one-step error bounds for the special case of balanced truncation of bounded real systems.

Keywords: Model reduction, balanced truncation, bounded real balancing, positive real balancing, half line dissipativity, $\Sigma$-normalised coprime factorisation.

1 Introduction

This paper deals with model reduction by balanced truncation for linear finite-dimensional systems. Balanced truncation is one of the most prominent methods of model reduction. It is straightforward and simple, has a nice and convincing physical interpretation, preserves stability, controllability and observability, and, last but not least, comes with simple and effective $\mathcal{H}_\infty$ error bounds, see [14] and [5]. In [8] and [12], the method of balanced truncation was extended to unstable systems using normalised coprime factorisation of the system transfer matrix.

Starting with the seminal paper [4] by Desai and Pal on stochastic model reduction, there has been also an interest in balanced truncation methods that preserve typical structural properties of the original system. [4] introduces a balanced truncation method to approximate a given positive real transfer matrix by a reduced order positive real transfer matrix. In [6] this problem was revisited, and it was shown that also stability and minimality are preserved under this balanced truncation method. In [20], and later in [3], $\mathcal{H}_\infty$ error bounds for balanced reduction of strictly positive real transfer matrices were found. The closely related problem of balanced truncation of bounded real transfer matrices, including $\mathcal{H}_\infty$ error bounds, was studied extensively in [13]. For a nice overview, we refer to [1].

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More recently, a new method of positive realness preserving model reduction was introduced in [18] and [2], which is not based on balancing. In these papers it was shown that by interpolating a given positive real transfer matrix at a subset of the spectral zeros, a reduced order positive real transfer matrix is obtained. In [18], a method to perform this interpolation using eigenspace computations for the Hamiltonian matrix was derived. In [9], a behavioral generalization of this method was established. A drawback of these methods is that, up to now, no a priori error bounds are known.

In the present paper we revisit the problem of positive realness and bounded realness preserving model reduction by balanced truncation. We consider the properties of positive realness and bounded realness as special cases of half line dissipativity (see [26]) with respect to a given quadratic supply rate given by a nonsingular real symmetric matrix \( \Sigma \). The matrix \( \Sigma \) has the property that its positive signature equals the number of inputs of the system. The transfer matrix of such half line dissipative system allows a rational coprime factorisation which is \( \Sigma \)-normalised. We then apply so called \( \Sigma \)-balancing and balanced truncation to the system defined by these coprime factors. This leads to a set of invariants that we will call the Hankel \( \Sigma \)-singular values, and whose squares are the nonzero eigenvalues of the composition \( H^* \Sigma H \), where \( H \) is the Hankel operator corresponding to the \( \Sigma \)-normalised factors of \( G \). Balanced truncation based on \( \Sigma \)-balancing turns out to preserve half line dissipativity, stability, and minimality, and yields a \( \Sigma \)-normalised coprime factorisation of the transfer matrix of the reduced order system. The method of balanced truncation that we propose here can be considered as an extension of Meyer’s [8] method of balanced truncation using normalised right coprime factors, now incorporating preservation of bounded realness or positive realness, or more general, half line dissipativity.

A comparison with ‘classical’ positive real and bounded real balancing will show that the so called positive real characteristic values and bounded real characteristic values can be expressed in terms of our Hankel \( \Sigma \)-singular values. Finally, we establish \( \mathcal{H}_\infty \) error bounds for the error between the original and reduced order coprime factors.

**Notation and background material.** We denote by \( \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \) the space of all measurable functions \( w \) from \( \mathbb{R} \) to \( \mathbb{R}^n \) such that \( \int_{-\infty}^{\infty} \| w(t) \|^2 \text{d}t < \infty \) for all \( a, b \in \mathbb{R} \). \( \mathcal{L}_2(\mathbb{R}, \mathbb{R}^n) \) denotes the ambient space of all measurable functions \( w \) from \( \mathbb{R} \) to \( \mathbb{R}^n \) such that \( \int_{-\infty}^{\infty} \| w(t) \|^2 \text{d}t < \infty \). The \( L_2 \)-norm of \( w \) is \( \| w \|_2 := (\int_{-\infty}^{\infty} \| w(t) \|^2 \text{d}t)^{1/2} \). We denote by \( \mathbb{R}_- \) the set of negative real numbers, and by \( \mathbb{R}_+ \) the complementary set of nonnegative real numbers. \( \mathcal{L}_2(\mathbb{R}_-, \mathbb{R}^n) \) (\( \mathcal{L}_2(\mathbb{R}_+, \mathbb{R}^n) \)) denotes the space of all measurable functions \( w \) from \( \mathbb{R}_- \) (\( \mathbb{R}_+ \)) to \( \mathbb{R}^n \) such that \( \int_{-\infty}^{0} \| w(t) \|^2 \text{d}t < \infty \) (\( \int_{0}^{\infty} \| w(t) \|^2 \text{d}t < \infty \)). When the dimension of the codomain is clear from the context, we denote these spaces by \( \mathcal{L}_2(\mathbb{R}) \), \( \mathcal{L}_2(\mathbb{R}_-) \) and \( \mathcal{L}_2(\mathbb{R}_+) \). The inner product on \( \mathcal{L}_2(\mathbb{R}_-) \) is denoted by \( \langle w_1, w_2 \rangle_{\mathcal{L}_2(\mathbb{R}_-)} := \int_{-\infty}^{0} w_1^\top(t) w_2(t) \text{d}t \). The inner product on \( \mathcal{L}_2(\mathbb{R}_+) \) is denoted by \( \langle w_1, w_2 \rangle_{\mathcal{L}_2(\mathbb{R}_+)} := \int_{0}^{\infty} w_1^\top(t) w_2(t) \text{d}t \). \( C^{-} (C^{+}) \) is the subset of \( C \) of all \( \lambda \) such that \( \text{Re}(\lambda) < 0 \) (\( \text{Re}(\lambda) > 0 \)). For a given square matrix \( A \), \( \sigma(A) \) denotes the set of eigenvalues of \( A \). For a given nonsingular, symmetric matrix \( \Sigma \) we denote by \( \pi(\Sigma) \) (the positive signature of \( \Sigma \)) the number of positive eigenvalues of \( \Sigma \). For a given real rational matrix \( G(s) \), we denote \( G^{-}(s) := G^\top(-s) \).

## 2 Bounded real, positive real and half line dissipative systems

Consider the input-output system represented by

\[
\dot{x} = Ax + Bu, \quad y = Cx + Du,
\]  

(1)

with \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{m \times n} \) and \( D \in \mathbb{R}^{m \times n} \). We assume that the system is internally stable, i.e. \( \sigma(A) \subseteq \mathbb{C}^{-} \), and \((A, B)\) and \((C, A)\) are a controllable and observable pair, respectively. The equations (1) represent the external behavior \( \mathcal{B}_{\text{ext}} := \{ (u, y) \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^m) \mid \exists x \in \mathcal{L}^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \text{ such that } (1) \text{ holds } \} \).

For \( x_0 \in \mathbb{R}^n \), let \( \mathcal{B}_{\text{ext}}(x_0) \) be the subset of \( \mathcal{B}_{\text{ext}} \) consisting of all \((u, y)\) such that the corresponding (unique) state trajectory \( x \) satisfies \( x(0) = x_0 \). Denote by \( G(s) := C(sI - A)^{-1}B + D \) the transfer matrix from \( u \) to \( y \). Important properties in circuits, systems and control are bounded realness and positive realness of the transfer matrix. \( G \) is called bounded real if \( I_n - G^-(i\omega)G(i\omega) \geq 0 \) for all \( \omega \in \mathbb{R} \). It is called positive real if \( m = p \) and \( G(i\omega) + G^-(i\omega) \geq 0 \) for all \( \omega \in \mathbb{R} \). These properties are in fact transfer matrix...
characterizations of dissipativeness properties of the system (1). Bounded realness is associated with the supply rate \( \|u\|^2 - \|y\|^2 \). The system (1) is called half line dissipative with respect to the supply rate \( \|u\|^2 - \|y\|^2 \) if \( \int_0^\infty (\|u\|^2 - \|y\|^2) dt \geq 0 \) for all \((u, y) \in \mathcal{B}_{\text{ext}} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^m)\). This property is also called contractiveness. It is well known that this property is equivalent to the condition that \( G \) is bounded real.

On the other hand, positive realness is associated with the supply rate \( u \). The system (1) is called half line dissipative with respect to the supply rate \( u \) if \( \int_0^\infty u^\top y dt \geq 0 \) for all \((u, y) \in \mathcal{B}_{\text{ext}} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^m)\). This property is also called passivity. It is well known that (1) is passive if and only if \( G \) is positive real.

The above shows that the properties of bounded realness and positive realness can be studied simultaneously in the general framework of half line dissipativity with respect to a given supply rate

\[
s(u, y) = \begin{pmatrix} u \\ y \end{pmatrix} \Sigma \begin{pmatrix} u \\ y \end{pmatrix},
\]

where \( \Sigma \) is an arbitrary nonsingular real symmetric matrix with the property that its number of positive eigenvalues \( \pi(\Sigma) \) is equal to \( m \), the number of inputs. Indeed, both for \( \begin{pmatrix} I_m & 0 \\ 0 & -I_p \end{pmatrix} \) and \( \frac{1}{2} \begin{pmatrix} 0 & I_q \\ I_q & 0 \end{pmatrix} \)

this condition holds. In this paper we will study the system (1) together with supply rate (2), with \( \Sigma \) satisfying this signature condition. We will assume throughout that the system is half line dissipative with respect to the supply rate \( s(u, y) \), i.e. \( \int_0^\infty s(u, y) dt \geq 0 \) for all \((u, y) \in \mathcal{B}_{\text{ext}} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^m)\). It is understood that the bounded real case and the positive real case are special cases.

Partition \( \Sigma \) compatible with the input output partition as

\[
\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^\top & \Sigma_{22} \end{pmatrix}.
\]

Then, since it is stable, the system (1) is half line dissipative with respect to the supply rate \( s(u, y) \) if and only if the frequency domain inequality

\[
\Sigma_{11} + G\!\!\!\!'^\top(i\omega)\Sigma_{12} + \Sigma_{12}G(i\omega) + G\!\!\!\!'^\top(i\omega)\Sigma_{22}G(i\omega) \geq 0 \quad \text{for all } \omega \in \mathbb{R}
\]

holds (see [25], Theorem 6.4). For convenience, assume that

\[
\det \left( \Sigma_{11} + G\!\!\!\!'^\top(i\omega)\Sigma_{12} + \Sigma_{12}G(i\omega) + G\!\!\!\!'^\top(i\omega)\Sigma_{22}G(i\omega) \right) \neq 0 \quad \text{for all } \omega \in \mathbb{R}.
\]

For the bounded real case this assumption requires that \( I_m - G\!\!\!\!'^\topG \) has no zeros on the imaginary axis, for the positive real case it requires the same for \( G + G\!\!\!\!'^\top \). We also assume that the following regularity condition holds:

\[
R := \Sigma_{11} + D\!\!\!\!'^\top \Sigma_{12} + \Sigma_{12}D + D\!\!\!\!'^\top \Sigma_{22}D > 0.
\]

In the bounded real case and positive real case this requires \( I_m - D\!\!\!\!'^\top D > 0 \) and \( D + D\!\!\!\!'^\top > 0 \), respectively. It is well known that under condition (5), the system (1) is half line dissipative with respect to the supply rate \( s(u, y) \) if and only if the algebraic Riccati equation

\[
A\!\!\!\!'^\top P + PA - C\!\!\!\!'^\top \Sigma_{22}C + \left( PB - C\!\!\!\!'^\top \left( \Sigma_{12} + \Sigma_{22}D \right) \right) R^{-1} \left( B\!\!\!\!'^\top P - \left( \Sigma_{12} + D\!\!\!\!'^\top \Sigma_{22} \right) C \right) = 0.
\]

has at least one real symmetric solution \( P \geq 0 \), see [22] or [19]. If this is the case, it has a smallest and a largest real symmetric solution, \( P_- \) and \( P_+ \). Due to the conditions \( \pi(\Sigma) = m \) and (4), these satisfy \( 0 < P_- < P_+ \). Furthermore, the eigenvalues of \( A + BR^{-1} \left( B\!\!\!\!'^\top P - \left( \Sigma_{12} + D\!\!\!\!'^\top \Sigma_{22} \right) C \right) \) are contained in \( \mathbb{C}^- \). The smallest real symmetric solution \( P_- \) yields the available storage for the dissipative system: for all \( x_0 \in \mathbb{R}^n \) we have

\[
x_0\!\!\!\!'^\top P_- x_0 = V_{av}(x_0) := \sup \{ -\int_0^\infty s(u, y) dt \mid (u, y) \in \mathcal{B}_{\text{ext}}(x_0) \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^m) \},
\]

and the largest real symmetric solution \( P_+ \) yields the required supply

\[
x_0\!\!\!\!'^\top P_+ x_0 = V_{req}(x_0) := \inf \{ \int_{-\infty}^0 s(u, y) dt \mid (u, y) \in \mathcal{B}_{\text{ext}}(x_0) \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^m) \}.
\]

For background material on dissipative systems, storage functions and their relation with the algebraic Riccati equation, we refer to [23], [19], or more recent [25, 26].
Remark 2.1: Clearly, half line dissipativity only implies $\Sigma_{11} + D^T\Sigma_{12}^T + \Sigma_{12}D + D^T\Sigma_{22}D \geq 0$. In case that the strict inequality (5) does not hold, the algebraic Riccati equation (6) is not defined. However, there do exist real symmetric solutions of a corresponding linear matrix inequality. The development in this paper will however highly depend on the use of the algebraic Riccati equation, and it is therefore a subject of future research to investigate whether its role in this paper can be replaced by this linear matrix inequality.

3 Lyapunov balancing of $\Sigma$-normalised coprime factors

Using the smallest solution $P_-$ of (6) we can obtain a minimum phase spectral factorisation of $\Sigma_{11} + G^\ast\Sigma_{12} + \Sigma_{12}G + G^\ast\Sigma_{22}G$. Define $K := R^{-1}(B^TP_+ - (\Sigma_{12} + D^T\Sigma_{22})C)$. Define $F(s) := K(sI - A)^{-1}B - R\Sigma$. Then $\Sigma_{11} + G^\ast\Sigma_{12} + \Sigma_{12}G + G^\ast\Sigma_{22}G = F^\ast F$. Moreover, $F$ has a stable inverse $F^{-1}(s) = -K(sI - A - BK)^{-1}BR^{-1} - R^{-1}$. Define now $M := -F^{-1}$ and $N := -GF^{-1}$. Then we obviously have $G = NM^{-1}$ and

$$(M^- N^-) \Sigma \begin{pmatrix} M \\ N \end{pmatrix} = I_n. \quad (7)$$

Thus we have obtained a right coprime factorisation of the transfer matrix $G$ over the ring of stable proper rational matrices (see [11]). Since it satisfies property (7), we call the factorisation $\Sigma$-normalised. The rational matrix $\text{col}(M, N)$ is the transfer matrix from $v$ to $\text{col}(u, y)$ of the system

$$\dot{x} = (A + BK)x + BR^{-1}\Sigma v, \quad \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} K \\ C + DK \end{pmatrix}x + \begin{pmatrix} R^{-1} \\ DR^{-1} \end{pmatrix}v. \quad (8)$$

The representation (8) is an alternative state space representation of the external behavior $\mathfrak{B}_{\text{ext}}$ of (1), called a driving variable representation. The variable $v$ is called a driving variable. This variable should not be interpreted as input of our original system (which is still $u$), but as a variable that 'generates' the input-output trajectories in $\mathfrak{B}_{\text{ext}}$ (see [17], [24]). Associated with (8), consider the following pair of Lyapunov equations

$$\begin{align*}
(A + BK)^\top W_O + W_O(A + BK) - \begin{pmatrix} K \\ C + DK \end{pmatrix}^\top \Sigma \begin{pmatrix} K \\ C + DK \end{pmatrix} &= 0, \\
(A + BK)W_C + W_C(A + BK)^\top + BR^{-1}B^\top &= 0.
\end{align*} \quad (9, 10)$$

Since $A + BK$ is stable, unique real symmetric solutions $W_O$ and $W_C$ exist. Obviously, $W_C > 0$ is the controllability Gramian of (8). We want to give an interpretation for $W_O$. By comparing (9) and (10) we see that, in fact, $W_O = P_+$, the smallest real symmetric solution of the algebraic Riccati equation. Thus, $W_O > 0$ and for each state $x_0$, $x_0^\top W_Ox_0$ is equal to the available storage in $x_0$. Clearly, by suitable state space transformation, $W_O$ and $W_C$ can be brought to the same real diagonal form, say $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$. The squares $\lambda_i^2$ are the eigenvalues of $W_OW_C$. In analogy with classical balancing, these eigenvalues turn out to admit a characterization in terms of the Hankel operator from $v$ to $(u, y)$ of (8). Let $H : L_2(\mathbb{R}^-, \mathbb{R}^m) \to L_2(\mathbb{R}^+, \mathbb{R}^m \times \mathbb{R}^p)$ denote this Hankel operator, given by

$$H(v_-)(t) := \int_{-\infty}^0 \begin{pmatrix} K \\ C + DK \end{pmatrix} e^{(A + BK)(t-s)}BR^{-1}\Sigma v_-(s)ds.$$

Let $H^* : L_2(\mathbb{R}^+, \mathbb{R}^m \times \mathbb{R}^p) \to L_2(\mathbb{R}^-, \mathbb{R}^m)$ denote the adjoint of $H$. Let $\Sigma$ be the map in $L_2(\mathbb{R}^+ \times \mathbb{R}^m \times \mathbb{R}^p)$ defined by pointwise multiplication with the matrix $\Sigma$. Then the following can be proven using standard arguments involving Hankel operators and Lyapunov equations:

**Proposition 3.1:** The operator $H^*\Sigma H$ is nonnegative, i.e. $<(H^*\Sigma H)(v_-), v_- >_{L_2(\mathbb{R}^-)} \geq 0$ for all $v_- \in L_2(\mathbb{R}^-, \mathbb{R}^m)$, and it has an $n$-dimensional image. Its non-zero eigenvalues are $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$, the eigenvalues of $W_OW_C$. 


Remark 3.2 : In the context of classical Lyapunov balancing, the square roots of the eigenvalues of $W_0 W_C$ are the nonzero singular values of the Hankel operator. It can be shown that in our context, the $\lambda_i$’s can also be given an interpretation of singular values. Indeed, the operator $H^* \Sigma$ can be shown to be equal to the adjoint $H^\Sigma$ of $H$, where the adjoint $H^\Sigma$ is understood to be taken with respect to the indefinite inner product $< w_1, \Sigma w_2 >_{\Sigma_2(\mathbb{R}^+)}$ on $\Sigma_2(\mathbb{R}^+, \mathbb{R}^n \times \mathbb{R}^m)$ (instead of the standard inner product $< w_1, w_2 >_{\Sigma_2(\mathbb{R}^+)}$, and the standard inner product $< v_1, v_2 >_{\Sigma_2(\mathbb{R}^-)}$ on $\Sigma_2(\mathbb{R}^-, \mathbb{R}^m)$. By Theorem 3.1, the composition $H^* \Sigma H$ is nonnegative, and its nonzero eigenvalues are $\lambda_i^2$. Thus their square roots $\lambda_i$’s are singular values of $H$ in an indefinite inner product sense.

Any state space transformation that transforms $W_0$ and $W_C$ to $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$ will be called a $\Sigma$-balancing transformation, and the state representations (1) and (8) will be said to be in $\Sigma$-balanced coordinates. The $\lambda_i$’s will be called the Hankel $\Sigma$-singular values. Suppose the state space transformation transforms $A$ to $\hat{A}$, $B$ to $\hat{B}$, and $C$ to $\hat{C}$. Since $W_0 = P_-$, the transformation will transform $K$ to $\hat{K} := R^{-1} (\hat{B}^T \Lambda - (\Sigma_{12} + D^T \Sigma_{22}) \hat{C})$. Thus, in $\Sigma$-balanced coordinates a driving variable representation of $\mathfrak{F}_{\text{ext}}$ is given by

$$\begin{align*}
\dot{x} &= (\hat{A} + \hat{B} \hat{K}) x + \hat{B} R^{-\frac{1}{2}} v, \\
\begin{bmatrix} u \\ y \end{bmatrix} &= \begin{bmatrix} \hat{K} \\ \hat{C} + D \hat{K} \end{bmatrix} x + \begin{bmatrix} R^{-\frac{1}{2}} \\ DR^{-\frac{1}{2}} \end{bmatrix} v, \tag{11}
\end{align*}$$

where the Lyapunov equations take the form

$$\begin{align*}
(\hat{A} + \hat{B} \hat{K})^T \Lambda + \Lambda (\hat{A} + \hat{B} \hat{K}) - \left( \begin{array}{c} \hat{K} \\ \hat{C} + D \hat{K} \end{array} \right)^T \Sigma \left( \begin{array}{c} \hat{K} \\ \hat{C} + D \hat{K} \end{array} \right) &= 0, \\
(\hat{A} + \hat{B} \hat{K}) \Lambda + \Lambda (\hat{A} + \hat{B} \hat{K})^T + BR^{-1} B^T &= 0. \tag{12, 13}
\end{align*}$$

Remark 3.3 : The actual computation leading to a $\Sigma$-balanced representation $(\hat{A}, \hat{B}, \hat{C})$ requires the following steps:

1. Compute the smallest (the stabilizing) real symmetric solution $P_- > 0$ of the algebraic Riccati equation (6). Compute $K := R^{-1} (B^T P_- - (\Sigma_{12} + D^T \Sigma_{22}) C)$.
2. Compute the unique solution $W_C > 0$ of the Lyapunov equation (10).
3. Compute a nonsingular matrix $S \in \mathbb{R}^{n \times n}$ that simultaneously diagonalizes $P_-$ and $W_C$, i.e. $SP_-=S^{-1}QS^{-1}=\Lambda=\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ (we refer to [1] for a treatment of numerical issues in the computation of such $S$).
4. Set $\hat{A} = SAS^{-1}$, $\hat{B} = SB$ and $\hat{C} = CS^{-1}$.

Thus, our method requires solving one Riccati equation and one Lyapunov equation. Classical methods of bounded real and positive real balancing ([3, 4, 6, 13, 20]) require the computation of both the minimal and the maximal real symmetric solution of the algebraic Riccati equation (6). In general this is done by introducing a second Riccati equation (the dual of the original one), and solving both the original and the dual one. Thus, in effect two Riccati equations need to be solved in classical bounded real and positive real balancing.

### 4 Half line dissipativity preserving model reduction by $\Sigma$-balanced truncation

In this section we discuss model reduction by balanced truncation based on the concept of $\Sigma$-balancing introduced in the previous section. Recall that for each $x_0 \in \mathbb{R}^n$ the quantity $x_0^T A x_0$ is equal to the
available storage 'in' $x_0$, and $x_0^T \Lambda^{-1} x_0$ is equal to the minimal amount of driving variable energy $\|v\|_2^2(\mathbb{R}^-)$ over the past to reach $x_0$. Thus $\Sigma$-balanced truncation favours those states that require little energy to be reached in the past, and that hold a large amount of internal storage in the sense of the given supply rate (depending on $\Sigma$).

Starting point is the $\Sigma$-normalised right coprime factorisation $G = NM^{-1}$. The driving variable system (11) is a $\Sigma$-balanced state space realisation of the transfer matrix $\text{col}(M, N)$. In this section, we will switch from the notation using 'bars': \[ \bar{A}, \bar{B}, \bar{K}, \text{etc.}, \] used at the end of the previous section, back to $A, B, K, \text{etc.}$ Assume that $\lambda_1 > \lambda_2 > \ldots > \lambda_N$ are the \textit{distinct} Hankel $\Sigma$-singular values, where $\lambda_i$ appears $n_i$ times. Then $\Lambda = \text{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \ldots, \lambda_N I_{n_N})$, where $I_i$ is the $n_i \times n_i$ identity matrix. Partition $\Lambda = \text{diag}(A_1, A_2)$, with $A_1 = \text{diag}(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \ldots, \lambda_r I_r)$ and $A_2 = \text{diag}(\lambda_{r+1} I_{r+1}, \lambda_{r+2} I_{r+2}, \ldots, \lambda_N I_N)$. Partition conformably with $\Lambda$:

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}.
\]

The truncated driving variable system is then given by

\[
\dot{x}_1 = (A_{11} + B_1 K_1)x_1 + B_1 R^{-\frac{1}{2}} v, \\
\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} K_1 \\ C_1 + DK_1 \end{pmatrix} x_1 + \begin{pmatrix} R^{-\frac{1}{2}} \\ DR^{-\frac{1}{2}} \end{pmatrix} v,
\]

where $K_1 := R^{-1} (B_1^T \Lambda_1 (\Sigma_{12} + D^T \Sigma_{22}) C_1)$. Let $\text{col}(M_r, N_r)$ be the transfer matrix from $v$ to $\text{col}(u, y)$ of (14), i.e.

\[
M_r(s) = K_1 s I - A_{11} - B_1 K_1)^{-1} B_1 R^{-\frac{1}{2}} + R^{-\frac{1}{2}}
\]

\[
N_r(s) = (C_1 + DK_1) s I - A_{11} - B_1 K_1)^{-1} B_1 R^{-\frac{1}{2}} + DK_1 R^{-\frac{1}{2}}.
\]

Then obviously $N_r(s)M_r^{-1}(s) = G_r(s) = C_1 (s I - A_{11})^{-1} B_1 + D$, the transfer matrix of the truncation

\[
\dot{x}_1 = A_{11} x_1 + B_1 u, \quad y = C_1 x + Du
\]

of the original input-output representation (1). The truncated input-output representation (17) represents the external behavior $\mathfrak{B}_{\text{ext}} := \{(u, y) \in \mathcal{L}_2^\infty(\mathbb{R}, \mathbb{R}^m \times \mathbb{R}^q) \mid \exists x \in \mathcal{L}_2^\infty(\mathbb{R}, \mathbb{R}^n) \text{ such that (17) holds}\}$. Note that (14) is a driving variable representation of $\mathfrak{B}_{\text{ext}}$. Of course, $\Lambda_1$ satisfies the reduced order Lyapunov equations

\[
(A_{11} + B_1 K_1)^T \Lambda_1 + \Lambda_1 (A_{11} + B_1 K_1) - \begin{pmatrix} K_1 \\ C_1 + DK_1 \end{pmatrix}^T \Sigma \begin{pmatrix} K_1 \\ C_1 + DK_1 \end{pmatrix} = 0,
\]

\[
(A_{11} + B_1 K_1)^T \Lambda_1 + \Lambda_1 (A_{11} + B_1 K_1)^T + B_1 R^{-1} B_1^T = 0.
\]

\textbf{Theorem 4.1} : Consider the system $\mathfrak{B}_{\text{ext}}$ represented by (1). Assume $(A, B)$ is controllable and $(C, A)$ is observable. Let $\Sigma$ be a nonsingular symmetric matrix such that $\pi(\Sigma) = m$, the number of inputs. Assume that (1) is half line dissipative with respect to the supply rate $s(u, y)$ given by (2). Assume that (4) and (5) holds. Finally, assume that $\Sigma_{22} \leq 0$. Then the system $\mathfrak{B}_{\text{ext}}$ represented by (17) obtained after $\Sigma$-balanced truncation satisfies the following properties:

1. $\sigma(A_{11}) \subseteq \mathbb{C}^-$,
2. $(A_{11}, B_1)$ is controllable, $(C_1, A_{11})$ is observable,
3. $\mathfrak{B}_{\text{ext}}$ is half line dissipative with respect to the supply rate $s(u, y)$,
4. $\sigma(A_{11} + B_1 K_1) \subseteq \mathbb{C}^-$,
5. the factorisation $G_r = N_r M_r^{-1}$ is $\Sigma$-normalised.
**Proof :** 1.) The proof of this is entirely based on the original stability proof by Pernebo and Silverman [14] in the context of classical Lyapunov balancing. Clearly, after balancing, the full order Lyapunov equation (12) can be rewritten as the full order Riccati equation (6)

\[ A^\top \Lambda + \Lambda A - C^\top \Sigma_{22} C + \left( \Lambda B - C^\top (\Sigma_{12} + \Sigma_{22} D) \right) R^{-1} \left( B^\top \Lambda - (\Sigma_{12} + D^\top \Sigma_{22}) C \right) = 0. \]  

(20)

Pre- and postmultiplying the second Lyapunov equation (13) by \( \Lambda^{-1} \) and adding it to (20), we find that also \( \Lambda + \Lambda^{-1} \) is a (diagonal) solution to (20). Using the assumption \( \Sigma_{22} \leq 0 \), factorise

\[-C^\top \Sigma_{22} C + \left( \Lambda B - C^\top (\Sigma_{12} + \Sigma_{22} D) \right) R^{-1} \left( B^\top \Lambda - (\Sigma_{12} + D^\top \Sigma_{22}) C \right) = K^\top K, \]

yielding to the Lyapunov equation

\[ A^\top \Lambda + \Lambda A + K^\top K = 0. \]  

(21)

Put \( N := (\Lambda + \Lambda^{-1})^{-1} \). Then, with \( G := NK^\top \), we have the second Lyapunov equation

\[ AN + NA^\top + GG^\top = 0. \]  

(22)

Now, we have \( \Lambda = \text{diag}(\Lambda_1, \Lambda_2) \) and \( N = \text{diag}((\Lambda_1 + \Lambda_1^{-1})^{-1}, (\Lambda_2 + \Lambda_2^{-1})^{-1}) \). Although these diagonal solutions are not equal, we do have that the diagonal elements of the products \( \Lambda_1(\Lambda_1 + \Lambda_1^{-1})^{-1} \) and \( \Lambda_2(\Lambda_2 + \Lambda_2^{-1})^{-1} \) form disjoint sets. Using this, the original proof of Pernebo and Silverman in [14] can be adapted to show \( \sigma(A_{11}) \subset \mathbb{C}^- \).

2.) The reduced order Lyapunov equation (18) can be rewritten as the Riccati equation

\[ A_{11}^\top A_1 + \Lambda_1 A_1 A_1 - C_{11}^\top \Sigma_{22} C_1 + \left( \Lambda_1 B_1 - C_{11}^\top (\Sigma_{12} + \Sigma_{22} D) \right) R^{-1} \left( B_1^\top \Lambda_1 - (\Sigma_{12} + D^\top \Sigma_{22}) C_1 \right) = 0. \]  

(23)

By pre- and postmultiplying the reduced order Lyapunov equation (19) by \( \Lambda_1^{-1} \), and adding the resulting equation to (23), we find that also \( \Lambda_1 + \Lambda_1^{-1} \) is a solution to the Riccati equation (23). Now, by reordering terms, (23) can be rewritten as

\[ \tilde{A}_{11}^\top A_1 + \Lambda_1 \tilde{A}_{11} + \tilde{Q}_{11} + \Lambda_1 \tilde{B}_1 \tilde{B}_1^\top \Lambda_1 = 0, \]  

(24)

where

\[
\begin{align*}
\tilde{A}_{11} &:= A_{11} + B_1 R^{-1}(\Sigma_{12} + D^\top \Sigma_{22}) C_1, \quad B_1 = B_1 R^{-\frac{1}{2}}, \\
\tilde{Q}_{11} &:= -C_{11}^\top \Sigma_{22} C_1 + C_{11}^\top (\Sigma_{12} + D^\top \Sigma_{22}) R^{-1}(\Sigma_{12} + D^\top \Sigma_{22}) C_1.
\end{align*}
\]

We now first prove that, in addition to \( A_{11} \), also \( \tilde{A}_{11} \) is stable. Let \( \mu \in \mathbb{C}, 0 \neq v \in \mathbb{C}^n \) such that \( A_{11} v = \mu v \). Using (24) we obtain

\[ 2\text{Re}(\mu) v^\top \Lambda_1 v = -v^\top \tilde{Q}_{11} v - \| \tilde{B}_1^\top \Lambda_1 v \|^2. \]

By the assumption \( \Sigma_{22} \leq 0 \) we have \( \tilde{Q}_{11} \geq 0 \). This implies \( \text{Re}(\mu) \leq 0 \). Assume \( \text{Re}(\mu) = 0 \). Then we have \( \tilde{B}_1 A_{11} v = 0 \) and \( v^\top \tilde{Q}_{11} v = 0 \). Again using \( \Sigma_{22} \leq 0 \), this implies that \( (\Sigma_{12} + D^\top \Sigma_{22}) C_1 v = 0 \). Therefore, \( A_{11} v = \mu v \), which contradicts stability of \( A_{11} \).

Also \( \Lambda_1 + \Lambda_1^{-1} \) satisfies (24):

\[
\tilde{A}_{11}^\top (\Lambda_1 + \Lambda_1^{-1}) + (\Lambda_1 + \Lambda_1^{-1}) \tilde{A}_{11} + \tilde{Q}_{11} + (\Lambda_1 + \Lambda_1^{-1}) \tilde{B}_1 \tilde{B}_1^\top (\Lambda_1 + \Lambda_1^{-1}) = 0.
\]

(25)

By subtracting (24) from (25) we get

\[
\tilde{A}_{11}^\top A_1 - \Lambda_1^{-1} \tilde{A}_{11} + (\Lambda_1 + \Lambda_1^{-1}) \tilde{B}_1 \tilde{B}_1^\top (\Lambda_1 + \Lambda_1^{-1}) - \Lambda_1 \tilde{B}_1 \tilde{B}_1^\top A_{11} = 0.
\]

(26)

We will show that \( (\tilde{A}_{11}, \tilde{B}_1) \) is controllable. Assume it is not. Then there exists a vector \( 0 \neq v \in \mathbb{C}^n \) and \( \mu \in \mathbb{C} \) such that \( v^\top \tilde{A}_{11} = \mu v^* \) and \( v^\top \tilde{B}_1 = 0 \). Pre- and postmultiply (26) by \( v^\top \Lambda_1 \) and \( \Lambda_1 v \) respectively. Then we obtain

\[ 2\text{Re}(\mu) v^\top \Lambda_1 v = v^\top (\Lambda_1^2 + I) \tilde{B}_1 \tilde{B}_1^\top (\Lambda_1^2 + I) v - v^\top \Lambda_1^2 \tilde{B}_1 \tilde{B}_1^\top \Lambda_1^2 v = 0. \]

(27)

Since \( \text{Re}(\mu) < 0 \) by stability of \( \tilde{A}_{11} \), and \( v^\top \Lambda_1 v > 0 \), this yields a contradiction. Thus \( (\tilde{A}_{11}, \tilde{B}_1) \) is controllable. From this we conclude that \( (A_{11}, B_1) \) is controllable as well.
We now prove observability. From \((24)\) and \((25)\) we obtain
\[
\begin{align*}
\Lambda_1^{-1}\bar{A}_{11} + \bar{A}_{11}\Lambda_1^{-1} + \bar{B}_1\bar{B}_1^\top + \Lambda_1^{-2}\bar{Q}_{11}\Lambda_1^{-1} &= 0, \quad (28) \\
(\Lambda_1 + \Lambda_1^{-1})^{-1}\bar{A}_{11} + \bar{A}_{11}(\Lambda_1 + \Lambda_1^{-1})^{-1} + \bar{B}_1\bar{B}_1^\top + (\Lambda_1 + \Lambda_1^{-1})^{-1}\bar{Q}_{11}(\Lambda_1 + \Lambda_1^{-1})^{-1} &= 0. \quad (29)
\end{align*}
\]
Note that since \(\Lambda_1\) is a diagonal matrix, \((\Lambda_1 + \Lambda_1^{-1})^{-1}\) is also diagonal. The diagonal elements are then the Hankel \(\Sigma\)-singular values, the nonnegative numbers whose squares are the nonzero eigenvalues of the solutions of the Lyapunov equations \((9)\) and \((10)\) equal and diagonal. The diagonal elements are thus the Hankel \(\Sigma\)-singular values of this transfer matrix. In \([8]\), the Hankel singular values of this transfer matrix are called the graph Hankel singular values.

We will show that \((C_1, \bar{A}_{11})\) is observable. Assume it is not. Then there exists a vector \(0 \neq v \in \mathbb{C}^n\) and \(\mu \in \mathbb{C}\) such that \(\bar{A}_{11}v = \mu v\) and \(C_1v = 0\). Pre- and postmultiply \((30)\) by \(v^*(\Lambda_1^2 + I)A_1\) and \(A_1^2v\), respectively. Using that \((\Lambda_1 + \Lambda_1^{-1})^{-1}(\Lambda_1^2 + I)A_1 = \Lambda_1^2\) and \(\bar{Q}_{11}v = 0\) we then obtain
\[
2\text{Re}(\mu)v^*(\Lambda_1^2 + I)v = v^*\Lambda_1^2\bar{Q}_{11}\Lambda_1^2v - v^*(\Lambda_1^2 + I)\bar{Q}_{11}(\Lambda_1^2 + I)v = 0.
\]
Again, this contradicts \(\text{Re}(\mu) < 0\). We conclude that \((C_1, \bar{A}_{11})\) is observable. This yields observability of \((C_1, A_{11})\) as well.

3.) The fact that the system represented by \((17)\) is half line dissipative follows from the fact that the Riccati equation \((23)\) has a non-negative solution.

4.) This follows immediately from the Lyapunov equation \((19)\) together with controllability of \((A_{11}, B_1)\).

5.) Since \(A_1\) satisfies the reduced order Riccati equation \((23)\) we have \(\Sigma_{11} + G_r^\top \Sigma_1^r + \Sigma_2G_r + G_r^\top \Sigma_2^r G_r = F_1^\top F_1\), where \(F_1(s) := K_1(sI - A_{11})^{-1}B_1 - R_1\). Clearly, \(-F_1^{-1}(s) = M_r(s)\), which implies
\[
(M_r^\top \ N_r^\top) \Sigma (M_r \ N_r) = I_n.
\]  
\[\square\]

Remark 4.2 : Note that both for the bounded real case \((\Sigma_{22} = -I_p)\) and the positive real case \((\Sigma_{22} = 0)\) the above theorem applies. Thus, the balanced truncation method proposed here preserves stability, minimality and bounded realness (positive realness).

Remark 4.3 : In the paper \([8]\) by Meyer, balanced truncation is based on right coprime factorisation of the system transfer matrix \(G\) as \(G = NM^{-1}\), with \(\text{col}(M, N)\) normalised in the sense that \(M^\top M + N^\top N = I\). Balanced truncation is then applied to the system corresponding to the transfer matrix \(\text{col}(M, N)\), called the graph operator of the system. In \([8]\), the Hankel singular values of this transfer matrix are called the graph Hankel singular values. We note that \([8]\) does not address the problem of preservation of half line dissipativity. The method presented in our paper can be considered as an extension of the work of Meyer to balanced truncation with preservation of half line dissipativity. It turns out that this requires \(\Sigma\)-balanced factorisation, instead of ordinary normalised factorisation.

5 Comparison with classical bounded real and positive real balancing

In section 3 we have proposed to choose a balancing state state transformation that makes the unique solutions of the Lyapunov equations \((9)\) and \((10)\) equal and diagonal. The diagonal elements are then the Hankel \(\Sigma\)-singular values, the nonnegative numbers whose squares are the nonzero eigenvalues of the nonnegative operator \(H^\top \Sigma H\). In this section we show that this balancing transformation also diagonalizes both the smallest and the largest real symmetric solutions of the algebraic Riccati equation \((6)\). As before, let \(P_+\) and \(P_\ast\) denote these extremal solutions. Recall that \(W_O\) and \(W_C\) are the solutions of \((9)\) and \((10)\) The following now holds:
Related results can be found in [7].

Lemma 5.1 : 
\[ W_O = P_-, \quad W_C = (P_+ - P_-)^{-1}. \]

**Proof:** The fact that \( W_O = P_- \) was already noted in section (3). Note that the Riccati equation (6) can be rewritten as
\[
\bar{A}^T P + P \bar{A} + \bar{Q} + P \bar{B} \bar{B}^T P = 0,
\]
where
\[
\bar{A} := A + BR^{-1}(\Sigma_{12} + D^T \Sigma_{22})C, \quad \bar{B} = BR^{-\frac{1}{2}},
\]
\[
\bar{Q} := -C^T \Sigma_{22} C + C^T (\Sigma_{12} + D^T \Sigma_{22}) R^{-1}(\Sigma_{12} + D^T \Sigma_{22}) C.
\]
Using that \( P_- \) and \( P_+ \) are solutions of (33), this yields
\[
(P_+ - P_-)(A + BR^{-1}(\Sigma_{12} + D^T \Sigma_{22})C) + (P_+ - P_-)B \bar{B}^T (P_+ - P_-) = P_+ \bar{A} + P_+ \bar{B} \bar{B}^T P_- - P_- \bar{A} + P_- \bar{B} \bar{B}^T P_+ + \bar{A}^T P_+ + P_- \bar{B} \bar{B}^T P_+ - P_+ \bar{B} \bar{B}^T P_- - P_- \bar{B} \bar{B}^T P_+ = 0,
\]
Noting that \( A + BK = \bar{A} + \bar{B} \bar{B}^T P_- \) we then get
\[
(A + BK)(P_+ - P_-)^{-1} + (P_+ - P_-)^{-1} (A + BK)^T + BR^{-1}B^T = 0,
\]
so \((P_+ - P_-)^{-1} = W_C\), the unique solution of the Lyapunov equation (10). □

Obviously, the above implies that \( P_+ = W_C + W_C^{-1} \). Since our balancing transformation results in \( W_O = W_C = \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \), it therefore also diagonalizes both \( P_- \) and \( P_+ \) to \( P_- = \Lambda \) and \( P_+ = \Lambda + \Lambda^{-1} \).

In classical bounded real and positive real balancing (see [3], [6], [13], [16]), the balancing transformation is chosen such that \( P_+^{-1} = P_- \) and are diagonal. In the case of bounded real balancing, this yields \( P_+^{-1} = P_- = \text{diag}(\beta_1, \beta_2, \ldots, \beta_n) \), where the \( \beta_i \) are called the bounded real characteristic values of the system (1). In the case of positive real balancing we get \( P_+^{-1} = P_- = \text{diag}(\pi_1, \pi_2, \ldots, \pi_n) \), and the \( \pi_i \) are called the positive real characteristic values of (1). All this can be generalized to general half line dissipative systems (1), with supply rates \( s(u, y) \) given by (2), with \( \Sigma \) nonsingular and satisfying the signature condition \( \pi(\Sigma) = \frac{m}{n} \). In that case we obtain \( P_+^{-1} = P_- = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \), with \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0 \). The \( \sigma_i \) are the characteristic values of (1) (see [10]). Obviously, the \( \sigma_i^2 \) are the eigenvalues of the product \( P_- P_+^{-1} \). Since \( 0 < P_- < P_+ \), we have \( \sigma_i < 1 \). By lemma (5.1), \( W_OW_C = P_- (P_+ - P_-)^{-1} = P_- P_+^{-1} (I - P_- P_+^{-1})^{-1} \). This implies the following relation between the \( \lambda_i \)'s and \( \sigma_i \)'s:

\[
\lambda_i^2 = \frac{\sigma_i^2}{1 + \lambda_i^2}, \quad \sigma_i = \frac{\lambda_i^2}{1 + \lambda_i^2}.
\]

Thus we obtain the following intrinsic Hankel-operator characterization of the bounded real and positive real characteristic values:

**Corollary 5.2 :** Let \( \beta_i \) and \( \pi_i \) be the bounded real and positive real characteristic values of the system (1), respectively. Let \( \text{col}(H_u, H_y) \) the Hankel operator from \( v \) to \( \text{col}(u, y) \) in (8). Then we have

1. \( \beta_i^2 = \frac{\lambda_i^2}{1 + \lambda_i^2} \), where the \( \lambda_i^2 \) are the nonzero eigenvalues of \( H_u^* H_u - H_y^* H_y \),
2. \( \pi_i^2 = \frac{\lambda_i^2}{1 + \lambda_i^2} \), where the \( \lambda_i^2 \) are the nonzero eigenvalues of \( H_y^* H_u + H_u^* H_y \).

Related results can be found in [7].
In this section we study a priori error bounds for \(\Sigma\)-balanced truncation. Again consider the system

\[
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &= Cx + Du
\end{align*}
\]

Interpretations of classical bounded real and positive real balanced truncation in terms of available storage and required supply can be found in the literature, see for example [13] and [16]. Basically, the idea is that those states are neglected that require a relatively large amount of supply to reach in the past, but contribute little to the supply that can be extracted in the future. In the remainder of this section we will give a physical interpretation of \(\Sigma\)-balanced reduction in terms of dissipation of supply.

Assume that our system (1) is half line dissipative with respect to the supply rate \(s(u, y)\). Since half line dissipativity implies dissipativity (see [26]), we then have

\[
\int_{-\infty}^{\infty} s(u, y) dt \geq 0
\]

for all \((u, y) \in \mathcal{B}_{\text{ext}} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^a \times \mathbb{R}^p)\). The left hand side of this inequality is equal to the total supply that is dissipated if the system is taken through the trajectory \((u, y)\). The total supply that is dissipated depends on the 'gap' \(W_C^{-1} = P_+ - P_-\) between the largest and smallest real symmetric solutions of the Riccati equation. This is made precise as follows:

**Proposition 5.4 :** For all \(x_0 \in \mathbb{R}^n\) we have

\[
\inf \{ \int_{-\infty}^{\infty} s(u, y) dt \mid (u, y) \in \mathcal{B}_{\text{ext}}(x_0) \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^a \times \mathbb{R}^p) \} = x_0^\top W_C^{-1} x_0. \tag{34}
\]

**Proof :** Let \(x_0 \in \mathbb{R}^n\) and \((u, y) \in \mathcal{B}_{\text{ext}}(x_0) \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^a \times \mathbb{R}^p)\). Then we have:

\[
\int_{-\infty}^{\infty} s(u, y) dt = \int_{-\infty}^{0} \int_{-\infty}^{0} s(u, y) dt - \int_{0}^{\infty} s(u, y) dt \geq \int_{0}^{\infty} s(u, y) dt \geq x_0^\top P_+ x_0 - x_0^\top P_- x_0.
\]

Now let \(\epsilon > 0\). There exists \((u_1, y_1) \in \mathcal{B}_{\text{ext}}(x_0) \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^a \times \mathbb{R}^p)\) such that \(\int_{0}^{\infty} s(u_1, y_1) dt \leq x_0^\top P_+ x_0 + \epsilon/2\), and \((u_2, y_2) \in \mathcal{B}_{\text{ext}}(x_0) \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^a \times \mathbb{R}^p)\) such that \(\int_{0}^{\infty} s(u_2, y_2) dt \geq x_0^\top P_- x_0 - \epsilon/2\). For the concatenation \((u, y)\) of \((u_1, y_1)\) and \((u_2, y_2)\) at \(t = 0\) (which is again in \(\mathcal{B}_{\text{ext}}(x_0)\)) we then have \(\int_{0}^{\infty} s(u, y) dt \leq x_0^\top P_+ x_0 - x_0^\top P_- x_0 + \epsilon\). This proves the claim of the proposition. \(\square\)

Now let \(1 > \lambda_1 \geq \lambda_2 \geq \ldots \lambda_n > 0\) be the Hankel \(\Sigma\)-singular values of the half line dissipative system (1), and assume the system is in \(\Sigma\)-balanced coordinates. Then we have \(W_C^{-1} = \Lambda^{-1}\). Therefore, in \(\Sigma\)-balanced coordinates \(x_0 = (\xi_1, \xi_2, \ldots, \xi_n)\) we have

\[
\inf \{ \int_{-\infty}^{\infty} s(u, y) dt \mid (u, y) \in \mathcal{B}_{\text{ext}}(x_0) \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^a \times \mathbb{R}^p) \} = \sum_{i=1}^{n} \xi_i^2 \frac{1}{\lambda_i}.
\]

If \(x_0 = e_i\), the \(i\)th standard basis vector of \(\mathbb{R}^n\), then for \((u, y) \in w \in \mathcal{B}_{\text{ext}}(e_i)\) the total dissipated supply is at least equal to \(1/\lambda_i\). Thus, since \(0 < \frac{1}{\lambda_1} \leq \frac{1}{\lambda_2} \leq \ldots \leq \frac{1}{\lambda_n}\), a nice physical interpretation of \(\Sigma\)-balanced truncation is that the reduction procedure 'removes' states that correspond to trajectories along which a relatively large amount of supply is dissipated.

6 Error bounds

In this section we study a priori error bounds for \(\Sigma\)-balanced truncation. Again consider the system \(\mathcal{B}_{\text{ext}}\) represented by (1), together with the supply rare \(s(u, y)\) given by (2), with \(\Sigma\) nonsingular, satisfying
\( \pi(\Sigma) = \mathbb{m} \) and \( \Sigma_{22} \leq 0 \). Assume the system is half line dissipative with respect to this supply rate. Let \( \text{col}(M,N) \) be the transfer matrix from \( v \) to \( \text{col}(u,y) \) of the driving variable representation (8) of \( \mathfrak{B}_{\text{ext}} \), corresponding to the \( \Sigma \)-normalised factorisation \( G = NM^{-1} \). Let the distinct Hankel \( \Sigma \)-singular values be \( \lambda_1 > \lambda_2 > \ldots > \lambda_N > 0 \), where \( \lambda_i \) appears \( n_i \) times, so \( \Lambda = \text{diag}(\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_N I_N) \), with \( I_i \) the \( n_i \times n_i \) identity matrix.

Suppose now that we do a one-step balanced truncation corresponding to the smallest Hankel singular value \( \lambda_N \), i.e. partition \( \Lambda = \text{blockdiag}(\Lambda_1, \Lambda_2) \), with \( \Lambda_2 = \lambda_N I_N \). Let (14) be the truncated driving variable system, and let \( \text{col}(\tilde{M},\tilde{N}) \) be the transfer matrix from \( v \) to \( \text{col}(u,y) \) of (14) (in other words, \( \tilde{M} : = M_{N-1} \) and \( \tilde{N} : = N_{N-1} \)). We will study the error between the original system \( \mathfrak{B}_{\text{ext}} \) and its balanced truncation \( \mathfrak{B}_{\text{ext}} \) in terms of the difference

\[
E := \begin{pmatrix} M - \tilde{M} \\ N - \tilde{N} \end{pmatrix}.
\]

The following theorem holds:

**Theorem 6.1**: The rational matrix \( E \) is stable. For all \( \omega \in \mathbb{R} \) we have

\[
0 \leq -E^\top (i\omega) \Sigma E(i\omega) \leq 4\lambda_N^2.
\]

**Proof**: Denote \( I_N \) by \( I \). Consider the driving variable representation (8) and, compatible with the partition \( \Lambda = \text{blockdiag}(\Lambda_1, \Lambda_2) \), partition

\[
A + BK = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{11} & \tilde{A}_{12} \end{pmatrix}, \quad BR^{-2} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}, \quad \begin{pmatrix} K \\ C + DK \end{pmatrix} = \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix}.
\]

By straightforward calculation it can be shown that the difference \( E \) is equal to \( E(s) = C(s)(sI - A(s))^{-1}B(s) \), where

\[
A(s) := \tilde{A}_{22} + \tilde{A}_{21}(sI - \tilde{A}_{11})^{-1}\tilde{A}_{12},
\]

\[
B(s) := \tilde{B}_2 + \tilde{A}_{21}(sI - \tilde{A}_{11})^{-1}\tilde{B}_1,
\]

\[
C(s) := \tilde{C}_2 + \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{A}_{12}.
\]

\( \Lambda \) is partitioned as \( \Lambda = \text{blockdiag}(\Lambda_1, \Lambda_2) \), with \( \Lambda_2 = \lambda_N I_N \). Using the fact that \( \Lambda \) satisfies both full order Lyapunov equations (9) and (10) it follows by straightforward calculation that

\[
A(-s)^\top \Lambda_2 + \Lambda_2 A(s) - C(-s)^\top \Sigma C(s) = 0,
\]

\[
A(s)\Lambda_2 + \Lambda_2 A(-s)^\top + B(s)B(-s)^\top = 0.
\]

Thus we obtain

\[
\lambda_N A(-s)^\top + \lambda_N A(s) - C(-s)^\top \Sigma C(s) = 0
\]

\[
\Leftrightarrow \lambda_N [-sI - A(-s)^\top] + \lambda_N [sI - A(s)] = -C(-s)^\top \Sigma C(s)
\]

\[
\Leftrightarrow \lambda_N [-sI - A(-s)^\top]^{-1} + \lambda_N [sI - A(s)]^{-1} =
\]

\[
\Leftrightarrow \lambda_N B(-s)^\top [-sI - A(-s)^\top]^{-1}B(s) + \lambda_N B(-s)^\top [sI - A(s)]^{-1}B(s) =
\]

\[
\Leftrightarrow \lambda_N R(-s)^\top + \lambda_N R(s) = -E(-s)^\top \Sigma E(s),
\]

where \( R(s) := B(-s)^\top [sI - A(s)]^{-1}B(s) \) and \( E(s) = C(s)(sI - A(s))^{-1}B(s) \) as above. Similarly, from the Lyapunov equation (38) we get

\[
\lambda_N R(-s)^\top + \lambda_N R(s) = R(-s)^\top R(s).
\]
Now, from (39) and (40)

\[
R^T(-s)R(s) = -E(-s)^T \Sigma E(s) = \lambda_N R(-s)^T + \lambda_N R(s) \\
= 2\lambda_N R(-s)^T + 2\lambda_N R(s) - R(-s)^T R(s) \\
= 4\lambda_N^2 - \left[R(-s)^T - 2\lambda_N I\right] [R(s) - 2\lambda_N I]
\]

Now let \( s = i\omega \) to obtain \( 0 \leq -E(-i\omega)^T \Sigma E(i\omega) \leq 4\lambda_N^2 \) for all \( \omega \in \mathbb{R} \). \( \square \)

Of course, the question arises what the inequality (36) means physically. It was shown in [19] that a system in driving variable representation \( \dot{x} = A + Bv, w = Cx + Dv \) with \( G(s) := D + C(sI - A)^{-1}B \) is dissipative with respect to the supply rate \( w^T \Sigma w \) if and only if \( G^-(i\omega) \Sigma G(i\omega) \geq 0 \) for all \( \omega \). By Theorem 6.1 for the transfer matrix \( E \) of the error system we have \( 0 \leq E^-(i\omega) (-\Sigma) E(i\omega) \leq 4\lambda_N^2 \). Thus the error system is always dissipative with respect to the supply rate \( -s(u, y) \), however for \( \lambda_N \) close to 0, it is close to being lossless.

We now turn to the question in what sense (36) can be interpreted as an error bound. Since

\[
(M^- \ N^-) \Sigma \begin{pmatrix} M \ \\
N \end{pmatrix} = I_n, \quad (\hat{M}^- \ \hat{N}^-) \Sigma \begin{pmatrix} \hat{M} \ \\
\hat{N} \end{pmatrix} = I_n,
\]

it is easily seen that (36) is equivalent with: for all \( \omega \in \mathbb{R} \)

\[
0 \leq W^-(i\omega) + W(i\omega) \leq 4\lambda_N^2 I, \quad \text{(41)}
\]

where

\[
W := (M^- \ N^-) \Sigma \begin{pmatrix} M - \hat{M} \ \\
N - \hat{N} \end{pmatrix}.
\]

The inequality (41) can be interpreted as an estimate of the Hermitian part of the weighted error \( W \).

For the special case of positive real balanced truncation it seems hard to derive a more relevant error bound from the inequality (36). We will now study the special case of bounded real balanced truncation. In that case, (6.1) is equivalent to: for all \( \omega \in \mathbb{R} \) we have

\[
[M(i\omega) - \hat{M}(i\omega)]^T [M(i\omega) - \hat{M}(i\omega)] \leq [N(i\omega) - \hat{N}(i\omega)]^T [N(i\omega) - \hat{N}(i\omega)]
\leq 4\lambda_N^2 I + [M(i\omega) - \hat{M}(i\omega)]^T [M(i\omega) - \hat{M}(i\omega)].
\]

This yields the following inequalities for the \( \mathcal{H}_\infty \)-norms of the differences between the coprime factors:

\[
\|M - \hat{M}\|_\infty \leq \|N - \hat{N}\|_\infty \leq 2\lambda_N + \|M - \hat{M}\|_\infty. \quad \text{(42)}
\]

Now recall that \( G = NM^{-1} \) is a \( \Sigma \)-normalized factorisation of the transfer matrix \( G \) of (1), in other words, \( M^-M - N^-N = I \). This implies that

\[
I - G^- G = M^{*-1} M^{-1},
\]

i.e., \( M^{-1} \) is a minimum phase spectral factor of \( I - G^- G \). From Theorem 4.1, (5), we also have that \( \tilde{M}^{-1} \) is a minimum phase spectral factor of \( I - \tilde{G}^- \tilde{G} \), where \( \tilde{G} \) is the transfer matrix of the one-step truncated system (17).

In the following, we will apply the following well-known error bound on the norm of the difference between the transfer matrices \( G \) and \( G_r \), and their minimum phase spectral factors, obtained after truncating the \( N - r \) smallest bounded real characteristic values \( \beta_{r+1}, \ldots, \beta_N \) (see [13]):

\[
\| \begin{pmatrix} G - G_r \\
M^{-1} - M_r^{-1} \end{pmatrix} \|_\infty \leq 2 \sum_{i=r+1}^{N} \beta_i. \quad \text{(43)}
\]

By applying this, we obtain the following error bound for the one-step error \( E \) given by (35):
Theorem 6.2 : Let $\hat{M}$ and $\hat{N}$ be obtained by one-step $\Sigma$-balanced truncation of $M$ and $N$ with $s(u,y) = \|u\|^2 - \|y\|^2$. Then the following bound on the relative error between $M$ and $\hat{M}$ holds:

$$\|\hat{M}^{-1}(M - \hat{M})\|_\infty \leq 2\frac{\lambda_N}{\sqrt{1 + \lambda_N^2}} \|M\|_\infty. \tag{44}$$

In particular, this yields

$$\|\left(\begin{array}{c} M - \hat{M} \\ N - \hat{N} \end{array}\right)\|_\infty \leq 2\lambda_N \left(1 + \frac{2}{\sqrt{1 + \lambda_N^2}} \|M\|_\infty \|\hat{M}\|_\infty\right). \tag{45}$$

Proof : The estimate (44) follows from $\hat{M}^{-1}(M - \hat{M}) = (\hat{M}^{-1} - M^{-1})M$, from the estimate (43), and the relation between the $\beta_i$ and the $\lambda_i$ in Corollary 5.2. The estimate (45) is obtained as follows:

$$\|\left(\begin{array}{c} M - \hat{M} \\ N - \hat{N} \end{array}\right)\|_\infty \leq \|M - \hat{M}\|_\infty + \|N - \hat{N}\|_\infty \leq 2\lambda_N + 2\|M - \hat{M}\|_\infty,$$

(see (42)). Then, finally $\|M - \hat{M}\|_\infty \leq 2\frac{\lambda_N}{\sqrt{1 + \lambda_N^2}} \|M\|_\infty \|\hat{M}\|_\infty$. \hfill $\square$

Remark 6.3 : In [8], for the normalised right coprime factors $M, N$ and $M_r, N_r$ the error bound

$$\|\left(\begin{array}{c} M - \hat{M} \\ N - \hat{N} \end{array}\right)\|_\infty \leq 2 \sum_{i=r+1}^N \gamma_i \tag{46}$$

was derived, where the $\gamma_i$ are the graph Hankel singular values, see also Remark 5.3. Due to the fact that the factors are normalised (in the sense that $M^{-1}M + N^{-1}N = I$, etc.), the left hand side of (46) is an upper bound for the gap between the original system and its balanced truncation. In the context of our paper, with $\Sigma$-normalisation instead of normalisation, this gap interpretation no longer holds.

7 Example

In this section we apply the method of $\Sigma$-balanced truncation to the bounded real system with transfer function

$$G(s) = \frac{(s + 1)(s + 2)}{(s + 3)(s + 4)(s + 5)} \tag{47}$$

(see also [13]). For this example, the transfer matrix of the $\Sigma$-normalised driving variable representation (8) is computed as

$$\begin{pmatrix} G_u \\ G_y \end{pmatrix} := \begin{pmatrix} \frac{s^2 + 12s^2 + 47s + 60}{s^3 + 11.95s^2 + 46.37s + 99.97} \\ \frac{s^2 + 3.99s + 2.002}{s^3 + 11.95s^2 + 46.37s + 99.97} \end{pmatrix}. \tag{48}$$

After solving the Lyapunov equations (9) and (10), the Hankel $\Sigma$-singular values of this transfer matrix can be computed to be

$$\lambda_1 = 0.0522, \lambda_2 = 0.0361, \lambda_3 = 0.0006.$$ 

Using the solutions to the Lyapunov equations, a $\Sigma$-balancing transformation can be computed bringing the driving variable representation (8) to $\Sigma$-balanced coordinates. The transfer matrices of the second-order and first-order truncated system are, respectively

$$\begin{pmatrix} G^2_u \\ G^2_y \end{pmatrix} := \begin{pmatrix} \frac{s^2 + 9.747s + 26.06}{s^2 + 9.694s + 26.05} \\ \frac{0.9971s + 0.8362}{s^2 + 9.694s + 26.05} \end{pmatrix},$$

$$\begin{pmatrix} G^1_u \\ G^1_y \end{pmatrix} := \begin{pmatrix} \frac{1 + 9.657}{1 + 9.605} \\ \frac{1.004}{1 + 9.605} \end{pmatrix}.$$ 

The Bode plots Fig. 1. and Fig. 2. show that the second order truncations $G^2_u$ and $G^2_y$ of $G_u$ and $G_y$ respectively, are very close, while the first order truncations differ a lot.
Figure 1: $\Sigma$-balanced truncation; solid line $G_u(s)$; crossed line $G^2_u(s)$; dashed line $G^3_u(s)$.

8 Conclusions

In this paper we have extended Meyer’s method [8] of balanced truncation using normalised right coprime factors of the system transfer matrix to balanced truncation with preservation of half line dissipativity. Two important special cases are preservation of positive realness and bounded realness. We have considered half line dissipative input-output systems, with quadratic supply rates given by nonsingular symmetric matrices $\Sigma$ with positive signature equal to the number of input components of the system. We have applied balancing to a $\Sigma$-normalised coprime factorisation of the transfer matrix. We have associated with such factorisation two Lyapunov equations, one of which is a nonstandard one, involving the matrix $\Sigma$. Balancing has then been based on making the unique solutions of these two Lyapunov equations equal and diagonal. The diagonal elements have been called the Hankel $\Sigma$-singular values because their squares are the nonzero eigenvalues of the composition of the ‘graph’ Hankel operator, multiplication by $\Sigma$, and the adjoint graph Hankel operator. We have shown that this notion of balanced truncation preserves stability, minimality, and half line dissipativeness. It turns out that our balancing transformation also diagonalises the extremal solutions of the Riccati equation associated with our dissipative system. Using this, we have given an interpretation of the ‘classical’ positive real and bounded real characteristic values in terms of the new Hankel $\Sigma$-singular values. Finally, we have studied the issue of a priori error bounds, and have derived one-step error bounds for the special case of bounded real systems.

As a subject for future research we mention the possible extension of the material in this paper to descriptor systems, by representing its external behavior in the form of a driving variable representation with $\Sigma$-normalised transfer matrix. This will provide an alternative to the methods of positive real and bounded real balancing for descriptor systems as developed in [16] and, more recently, in [15].

References

Figure 2: Σ-balanced truncation; solid line $G_y(s)$; crossed line $G^2_y(s)$; dashed line $G^1_y(s)$.


