Fully Distributed Robust Synchronization of Networked Lur’e Systems with Incremental Nonlinearities

Fan Zhang\textsuperscript{a,c}, Harry L. Trentelman\textsuperscript{b}, Jacquelien M.A. Scherpen\textsuperscript{a}

\textsuperscript{a} Research Institute of Technology and Management, University of Groningen, 9747 AG Groningen, The Netherlands
\textsuperscript{b} Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, 9700 AV Groningen, The Netherlands
\textsuperscript{c} Research Center of Satellite Technology, Harbin Institute of Technology, 150080 Harbin, China

Abstract

This paper deals with robust synchronization problems for uncertain dynamical networks of diffusively interconnected identical Lur’e systems subject to incrementally passive nonlinearities and incrementally sector bounded nonlinearities, respectively, in a fully distributed fashion. Whereas in stabilization of one single Lur’e system the conditions of passivity and sector boundedness for the uncertain nonlinear function in the negative feedback loop are commonly assumed, in our context of networked Lur’e systems we adopt the stronger assumptions of incremental passivity and incremental sector boundedness. Throughout this paper the interconnection topologies among these Lur’e agents are assumed to be undirected and connected. We design robustly synchronizing protocols and subsequently implement these protocols in a fully distributed way by means of an adaptive control law that adjusts the coupling weights between neighboring agents. Both for the cases of incrementally passive as well as incrementally sector bounded nonlinearities we obtain sufficient conditions for the existence of fully distributed robustly synchronizing protocols. The state feedback matrices are computed by solving LMI’s in terms of the matrices defining the individual agent dynamics. Numerical simulation examples illustrate our theoretical results.

Key words: Lur’e system; incremental nonlinearity; uncertain network; robust synchronization; adaptive control.

1 Introduction

Synchronization of complex dynamical networks has attracted a lot of attention from multidisciplinary research communities over the last decade, see e.g. [31,1,28,23,29,13] to name just a few. This is due to the fact that collective behaviors of multiple interconnected dynamical systems are widespread in nature, technology and human society, and have potential applications in wide areas such as spatiotemporal planning, cooperative multitasking and formation control [26,18]. The essence of synchronization is the collective objective of the agents in a network to reach an agreement about certain variables of interest.

Synchronization problems for linear multi-agent networks have been extensively studied, see [17,22] and the references therein. A unified viewpoint for synchronization of linear multi-agent networks was presented in [13]. Internal model principle based necessary and sufficient conditions for linear output synchronization were given in [30]. Recent research interests have also been aimed at nonlinear multi-agent networks, possibly with model uncertainties, time delays, data dropouts and quantized interconnections etc. [10,4,28,2,7,6,21]. In [2], a passivity-based group coordination framework was proposed, especially applicable to nonlinear multi-agent networks. The research in these directions is motivated by practical applications such as assembling a network of mobile robots (e.g. smart sensors, unmanned aerial vehicles, satellites) to collaborate with each other to fulfill certain complex tasks.

In this paper, we consider nonlinear multi-agent networks in which the dynamics of the individual agents is described by a Lur’e system, i.e. a nonlinear system consisting of the negative feedback interconnection of a nominal linear system with an uncertain static nonlinear function.
fully distributed robustly synchronizing protocols, both for incrementally passive and for incrementally sector bounded nonlinearities.

The remainder of this paper is organized as follows. Section 2 introduces some preliminaries and the individual agent dynamics we will consider in this paper. Our main results are presented in Sections 3 and 4. In Section 3 we establish sufficient conditions for the existence of robustly synchronizing protocols and discuss how to compute these protocols. In Section 4 we explain how to modify these protocols in order to be able to implement them in a fully distributed way. Numerical simulation examples are given to illustrate these results in Section 5. The paper closes with some concluding remarks in Section 6.

2 Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. We denote $\mathbb{R}^+ := [0, \infty)$, $\mathbb{R}^{m_1 \times m_2}$ ($\mathbb{C}^{m_1 \times m_2}$) the space of $m_1$ by $m_2$ real (complex) matrices. Matrices, if not explicitly stated, are assumed to have compatible dimensions. The superscript $(\cdot)^T$ denotes the transpose of a real matrix, and the superscript $(\cdot)^*$ denotes the conjugate transpose of a complex matrix. We denote the block diagonal matrix with matrices $M_0, b = 1, \ldots, d$, on its diagonal by $\text{diag}(M_0, \ldots, M_d)$. The Kronecker product of matrices $M_1$ and $M_2$ is denoted by $M_1 \otimes M_2$. An important property of the Kronecker product is that $(M_1 \otimes M_2)(M_3 \otimes M_4) = (M_1 M_3) \otimes (M_2 M_4)$. We denote by $\mathbf{0}$ and $\mathbf{1}$ the zero and identity matrices, respectively, of compatible dimensions. By $\mathbf{0}_N$ and $\mathbf{1}_N$ we denote the column vector of dimension $N$ with all its elements equal to 0 and 1, respectively.

In this paper, the interconnection topology of a network of bidirectionally interconnected dynamical systems is represented by an undirected graph $\mathcal{G}$ that consists of a finite and nonempty node set $\mathcal{V} = \{1, 2, \ldots, N\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ with the property that $(i,j) \in \mathcal{E} \iff (j,i) \in \mathcal{E}$ for all $i, j = 1, 2, \ldots, N$. We assume that the graph $\mathcal{G}$ is simple, i.e. it does not contain any self-loop $(i,i)$ and there is at most one undirected edge between any two different nodes. An undirected path connecting nodes $i_0$ and $i_l$ is a sequence of undirected edges of the form $(i_{p-1}, i_p)$, $p = 1, \ldots, l$. The graph $\mathcal{G}$ is connected if there is an undirected path between any pair of distinct nodes. The adjacency matrix $A$ associated with the graph $\mathcal{G}$ is defined as $[A]_{ij} = a_{ij}$ if $(j,i) \in \mathcal{E}$ and $[A]_{ij} = 0$ otherwise, where $a_{ij} > 0$ is the edge weight of $(j,i)$. The degree of node $i$ is defined as $d_i = \sum_{j=1}^N a_{ij}$. $\mathcal{D} := \text{diag}(d_1, d_2, \ldots, d_N)$ is called the degree matrix of the graph $\mathcal{G}$. The Laplacian matrix $\mathcal{L}$ of the graph $\mathcal{G}$ is defined by $\mathcal{L} := \mathcal{D} - \mathcal{A}$. It is well known that $\mathcal{L} \mathbf{1}_N = \mathbf{0}_N$, i.e. $\mathbf{1}_N$ is an eigenvector associated with the eigenvalue 0.

Let $\mathcal{G}$ be an undirected graph with $N$ nodes, where $N \geq 2$. The graph $\mathcal{G}$ is connected if and only if its Laplacian eigenvalue 0 has geometric multiplicity one, see [17]. In this case, the eigenvalues of the Laplacian matrix $\mathcal{L}$ associated with $\mathcal{G}$ can be ordered as $\lambda_1 = 0 < \lambda_2 < \ldots < \lambda_N$.
\(\lambda_2 \leq \cdots \leq \lambda_N\). Furthermore, there exists an orthogonal matrix \(U = [u_1 \quad u_2 \quad \cdots \quad u_N]\), where \(U \in \mathbb{R}^{N \times (N-1)}\), such that \(U^T U \mathcal{L} = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)\). It is obvious that \(U^T U^2 = I_{N-1}\) and \(U^T U^2 = I_N - \frac{1}{\lambda} 1_N 1_N^T\). We denote \(\Lambda := \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)\), which is symmetric. The following lemma will play a crucial role in our main results.

**Lemma 1** For any two vectors \(a = [a_1^T, a_2^T, \cdots, a_N^T]^T\) and \(b = [b_1^T, b_2^T, \cdots, b_N^T]^T\), where \(a_i, b_i \in \mathbb{R}^n, i = 1, 2, \cdots, N\), we have

\[
a^T (U U^T \otimes I_n) b = \frac{1}{N} \sum_{1 \leq i < j \leq N} (a_i - a_j)^T (b_i - b_j).
\]

**Proof.**

\[
a^T (U U^T \otimes I_n) b = a^T \left( I_N - \frac{1}{N} 1_N 1_N^T \right) \otimes I_n b
\]

\[
= \sum_{i=1}^{N} a_i^T b_i - \frac{1}{N} \sum_{i=1}^{N} \left( \sum_{j=1}^{N} a_j^T \right) b_i
\]

\[
= \sum_{i=1}^{N} a_i^T b_i - \frac{1}{N} \sum_{j=1}^{N} a_j^T b_i = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} (a_i - a_j)^T b_i
\]

\[
= \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} (a_i - a_j)^T (b_i - b_j)
\]

\[
= \frac{1}{N} \sum_{1 \leq i < j \leq N} (a_i - a_j)^T (b_i - b_j).
\]

Below we give the definition of minimal left annihilator of a given matrix.

**Definition 1** ([11]) For a given matrix \(B \in \mathbb{C}^{n \times m}\) with rank \(r < \min(m, n)\), we denote by \(B^\perp \in \mathbb{C}^{(n-r) \times n}\) any matrix of full row rank such that \(B^\perp B = 0\). Any such matrix \(B^\perp\) is called a **minimal left annihilator** of \(B\).

Note that, for a given \(B\), a minimal left annihilator \(B^\perp\) exists if and only if \(B\) has linearly dependent rows. The set of all such matrices is given by \(B^\perp = T U^2\), where \(T\) is any nonsingular matrix and \(U^2\) is obtained from the singular value decomposition \(B = [u_1 \ u_2 \ \cdots \ u_m] \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}\).

Thus, for a given \(B\), \(B^\perp\) is not unique. Throughout this paper, \(B^\perp\) will denote any choice from this set of matrices.

In this paper, we consider a multi-agent network of \(N \geq 2\) nonlinear dynamical systems described by the following identical Lur’e systems (see Fig. 1),

\[
\begin{align*}
\dot{x}_i &= Ax_i + Bu_i + Ez_i, \\
y_i &= Cx_i, \\
z_i &= -\phi(y_i, t),
\end{align*}
\]

where \(x_i(t) \in \mathbb{R}^n, u_i(t) \in \mathbb{R}^m\) and \(y_i(t) \in \mathbb{R}^n\) are the state to be synchronized, the diffusive coupling input and the output of the \(i\)th agent, respectively. The equation \(z_i = -\phi(y_i, t)\) represents a time-varying, memoryless, nonlinear negative feedback loop. The function \(\phi(\cdot, t)\) from \(\mathbb{R}^n \times \mathbb{R}^n\) to \(\mathbb{R}^n\) is uncertain and can be any function from a set of functions to be specified later. \(A, B, C\) and \(E\) are known constant matrices of compatible dimensions. We assume that the number of control inputs \(m\) is strictly less than the state space dimension \(n\). In this case the rows of \(B\) are linearly dependent and thus \(B^\perp\) exists. We also assume that \((A, B)\) is stabilizable. The interconnection topology among these agents is represented by the connected undirected graph \(G\).

### 3 Robust Synchronization

In this section, the agents (1) in the network are assumed to be interconnected by means of the following distributed static protocol

\[
u_i = F \sum_{j=1}^{N} a_{ij}(x_i - x_j), \quad i = 1, 2, \cdots, N,
\]

where \(F \in \mathbb{R}^{m \times n}\) is a common feedback gain matrix to be determined later, and \(A = [a_{ij}]\) is the adjacency matrix of the graph \(G\).

**Definition 2** The network of agents (1) with the protocol (2) is robustly synchronized if \(x_i(t) - x_j(t) \to 0\) as \(t \to \infty, \forall i, j = 1, 2, \cdots, N\), for all initial conditions and all uncertainties \(\phi(\cdot, t)\).

By interconnecting (1) and (2) we get the Lur’e dynamical network

\[
\begin{align*}
\dot{x} &= (I_N \otimes A + \mathcal{L} \otimes BF)x - (I_N \otimes E)\Phi(y, t), \\
y &= (I_N \otimes C)x
\end{align*}
\]

where \(x = [x_1^T, x_2^T, \cdots, x_N^T]^T, \quad y = [y_1^T, y_2^T, \cdots, y_N^T]^T, \quad \Phi(y, t) = [\phi(y_1, t)^T, \phi(y_2, t)^T, \cdots, \phi(y_N, t)^T]^T, \quad I_N\) is the identity matrix of dimension \(N\), and \(\mathcal{L}\) is the Laplacian matrix of the graph \(G\).
In the next two subsections we will discuss robust synchronization of the Lur’e network (3). In the first subsection, the uncertainty in the feedback loop is modeled by assuming the uncertain nonlinear functions \( \phi_i(t) \) to be incrementally passive. In the second subsection, we model the uncertainty by assuming that the functions \( \phi_i(t) \) satisfy an incremental sector boundedness condition.

3.1 Incrementally Passive Nonlinearities

In this subsection we assume that the uncertain functions \( \phi_i(t) \) are incrementally passive. Incremental passivity for static systems of the form

\[
    z = \phi(y, t)
\]

with input \( y(t) \in \mathbb{R}^s \) and output \( z(t) \in \mathbb{R}^s \) is defined as follows.

**Definition 3** ([20]) The system (4) is called incrementally passive if the function \( \phi_i(t) \) satisfies

\[
    (y_1 - y_2)^T (\phi(y_1, t) - \phi(y_2, t)) \geq 0
\]

for all \( y_1, y_2 \in \mathbb{R}^s \) and \( t \in \mathbb{R}^+ \).

In general, incremental passivity is stronger than the property of passivity which is defined by \( y^T \Phi(y, t) \geq 0 \) for all \( y \in \mathbb{R}^s \) and \( t \in \mathbb{R}^+ \). Passivity implies incremental passivity for linear systems, and also for monotone increasing static nonlinearities [25].

**Lemma 2** Assume that the uncertain functions \( \phi_i(t) \) satisfy (5) for all \( y_1, y_2 \in \mathbb{R}^s \) and \( t \in \mathbb{R}^+ \). If there exist matrices \( P > 0 \) and \( F \) such that

\[
    (A + \lambda_i BF)^T P + P (A + \lambda_i BF) < 0
\]

for all \( i = 2, \cdots, N \), and

\[
    PE = C^T,
\]

then the network of Lur’e agents (1) with the protocol (2) is robustly synchronized, i.e. the Lur’e network (3) is synchronized for all incrementally passive \( \phi_i(t) \).

**Proof.** Let \( U \) be an orthogonal matrix such that \( U^T \Lambda U = \Lambda \) as defined in Section 2. All the other notation introduced in Section 2 will be also used without redefinitions or statements throughout this paper. Let \( \bar{x} = [U_1^T \otimes I_n] x \) and \( \bar{x} = [U_2^T \otimes I_n] x \), where \( \bar{x} = [\tilde{x}_1^T, \tilde{x}_2^T, \cdots, \tilde{x}_N^T]^T \) and \( \bar{x} = [\tilde{x}_{1}^T, \cdots, \tilde{x}_{N}^T]^T \). Then after the coordinate transformation the differential equation in (3) takes the form

\[
    \dot{\tilde{x}} = (I_{N-1} \otimes A + \bar{\Lambda} \otimes BF) \tilde{x} - (U_1 C \otimes E) \Phi(y, t),
\]

We know that \( x_i(t) - x_j(t) \to 0 \) as \( t \to \infty \), \( \forall i, j = 1, 2, \cdots, N \), if and only if \( \bar{x}(t) \to 0 \) as \( t \to \infty \), see Lemma 3.2 in [28]. Thus the robust synchronization of \( x \) is equivalent to the global asymptotical stability of \( \tilde{x} \).

By Lemma 1, we have

\[
    x^T (U_2 I_N \otimes C) \Phi(y, t)
\]

\[
    x^T (I_N \otimes C) (U_1 I_N \otimes I_n) \Phi(y, t)
\]

\[
    y^T (U_2 I_N \otimes I_n) \Phi(y, t)
\]

\[
    = \frac{1}{N} \sum_{i < j \leq N} (y_i - y_j)^T (\phi(y_i, t) - \phi(y_j, t)) \geq 0.
\]

Choose a quadratic Lyapunov function candidate

\[
    V_1(\tilde{x}) = \frac{1}{2} \tilde{x}^T (I_{N-1} \otimes P) \tilde{x},
\]

where \( P > 0 \) together with the feedback gain matrix \( F \) satisfies (6) and (7). Obviously, \( V_1(\tilde{x}) \) is positive definite and radially unbounded. The time derivative of \( V_1(\tilde{x}) \) along the trajectories of the system (8) is given by

\[
    \dot{V}_1(\tilde{x}) = \tilde{x}^T (I_{N-1} \otimes P) \tilde{x}
\]

\[
    = \tilde{x}^T (I_{N-1} \otimes P) [(I_{N-1} \otimes A + \bar{\Lambda} \otimes BF) \tilde{x} - (U_1 C \otimes E) \Phi(y, t)]
\]

\[
    = \tilde{x}^T (I_{N-1} \otimes PA + \bar{\Lambda} \otimes PBF) \tilde{x}
\]

\[
    - \tilde{x}^T (U_1 \otimes E) \Phi(y, t)
\]

\[
    = \sum_{i=2}^{N} \tilde{x}_i^T (PA + \lambda_i BF) \tilde{x}_i
\]

\[
    - \tilde{x}^T (U_2 \otimes I_n) (U_1 C \otimes C) \Phi(y, t)
\]

\[
    = \sum_{i=2}^{N} \tilde{x}_i^T (PA + \lambda_i BF) \tilde{x}_i - \tilde{x}^T (U_1 C \otimes C) \Phi(y, t)
\]

\[
    \leq \frac{1}{2} \sum_{i=2}^{N} \tilde{x}_i^T [(A + \lambda_i BF)^T P + P (A + \lambda_i BF)] \tilde{x}_i,
\]

which is negative definite. Thus the system (8) is globally asymptotically stable, i.e. the Lur’e network (3) is robustly synchronized. This completes the proof. \( \square \)

**Remark 1** It is well known that the nonzero eigenvalues of the graph Laplacian matrix in general play an important role in synchronization conditions, see e.g. [13], where it was shown that the protocol (2) synchronizes the linear network if and only if the feedback gain matrix \( F \) stabilizes \( A + \lambda_i BF \) for all \( i = 2, \cdots, N \). In our work, due to the strict passivity requirements (6) and (7), a common \( P > 0 \) is required as well as a common \( F \). Without the Lur’e-type nonlinearities, the Lur’e network (3) becomes a linear network, and the requirement of a common \( P \) and the condition (7) are removed.

By applying the Kalman-Yakubovich-Popov lemma, the existence of \( P > 0 \) and \( F \) such that (6) and (7) hold is equivalent to the existence of a common state feedback...
control law $u_i = Fx_i$ for the $N - 1$ systems
\begin{align*}
\dot{x}_i &= Ax_i + \lambda_i Bu_i + Ez_i, \\
y_i &= Cx_i,
\end{align*}
(9)
which renders all the resulting closed-loop systems strictly passive from $z_i$ to $y_i$ with the common storage function $\frac{1}{2}x_i^TPx_i$.

We will now study the problem of finding necessary and sufficient conditions under which such common $P > 0$ and $F$ exist for the set of systems (9). We first consider the case that we have a single system $\dot{x} = Ax + Bu + Ez, y = Cx$, and obtain necessary and sufficient conditions for the existence of a state feedback control law $u = Fx$ that renders this system strictly passive from $z$ to $y$.

Lemma 3 There exist matrices $P > 0$ and $F$ such that $(A + BF)^TP + P(A + BF) < 0, \ PE = CT$ if and only if there exists a matrix $Q > 0$ such that
\begin{equation}
B^T(QA^T + AQ)(B^T)^T < 0, \tag{10}
\end{equation}
\begin{equation}
E = QC^T. \tag{11}
\end{equation}
In this case, a suitable $P$ is given by $P = Q^{-1}$, and a suitable $F$ is given by $F = \mu B^T Q^{-1}$, where $\mu$ is any real number satisfying $QA^T + AQ + 2\mu BB^T < 0$.

Proof. For the ‘only if’ part, let $Q = P^{-1}$. Then we get (11) and $QA^T + AQ + QF^T B^T + BFQ < 0$ which yields (10). For the ‘if’ part, by Finsler’s lemma (see [11]), there exists a real $\mu$ such that $QA^T + AQ + 2\mu BB^T < 0$. Let $P = Q^{-1}$ and $F := \mu B^T Q^{-1}$. Then we get $PE = CT$ and
\begin{equation}
(A + BF)^TP + P(A + BF) = (A + \mu BB^T Q^{-1})^T Q^{-1} + Q^{-1} (A + \mu BB^T Q^{-1}) = A^T Q^{-1} + Q^{-1} A + 2\mu Q^{-1} BB^T Q^{-1} < 0.
\end{equation}
This completes the proof. \qed

Remark 2 Obviously, if $P > 0$ and $F$ satisfying the conditions in Lemma 3 exist, then the state feedback control law $u = Fx$ renders the system $\dot{x} = Ax + Bu + Ez, y = Cx$ strictly passive from $z$ to $y$. Equivalently, the resulting closed-loop system is robustly stabilized against passive uncertain nonlinearities in its negative feedback loop $z = -\phi(y, t)$, see also [3]. More on static output feedback passification can be found in [24].

Remark 3 If $\mu$ is a real number as in the formulation of Lemma 3, then by the fact that $BB^T$ is positive semidefinite, $QA^T + AQ - 2\mu BB^T < 0$ for any $\mu$ such that $\mu \geq -\mu$. Clearly, such $\mu$ can always be taken positive. In this case, $F := -\mu B^T Q^{-1}$ is also a suitable control law.

Next we will focus on conditions for the existence of a protocol (2) that robustly synchronizes the Lur’e network (3), i.e. on finding a common $P > 0$ and $F$ such that (6) and (7) in Lemma 2 hold. The following theorem establishes necessary and sufficient conditions for the existence of such common $P$ and $F$:

Theorem 1 There exist matrices $P > 0$ and $F$ such that (6) and (7) hold for all $i = 2, \ldots, N$ if and only if there exists a matrix $Q > 0$ such that (10) and (11) hold. In this case, a suitable $P$ is given by $P = Q^{-1}$, and a suitable $F$ is given by $F = -k B^T Q^{-1}$, where the positive real number $k$ satisfies
\begin{equation}
QA^T + AQ - 2k \lambda_2 BB^T < 0. \tag{12}
\end{equation}

Proof. The ‘only if’ part is obvious. For the ‘if’ part, there exists $k > 0$ such that (12) holds. Let $P = Q^{-1}$ and $F := -k B^T Q^{-1}$. Then we get $PE = CT$ and
\begin{equation}
(A + \lambda_i BF)^TP + P(A + \lambda_i BF) = (A - k \lambda_i BB^T Q^{-1})^T Q^{-1} + Q^{-1} (A - k \lambda_i BB^T Q^{-1}) = A^T Q^{-1} + Q^{-1} A - 2k \lambda_i Q^{-1} BB^T Q^{-1} < 0
\end{equation}
for all $i = 2, \ldots, N$. This completes the proof. \qed

Remark 4 We want to stress that our conditions in Theorem 1 are equivalent to the existence of $P > 0$ and $F$ for a single agent, see Lemma 3. From the computational point of view this is advantageous since it reduces the computation of a synchronization protocol for a possibly large network ($N$ large) to the computation of a state feedback controller that renders a single agent strictly passive.

### 3.2 Incrementally Sector Bounded Nonlinearities

In this subsection we consider uncertain feedback nonlinearities $\phi(\cdot, t)$ given by incrementally sector bounded functions within sector bounds $[S_1, S_2]$, where $S_1, S_2 \in \mathbb{R}^{n \times n}$ are real symmetric matrices with $0 \leq S_1 < S_2$. This property is defined as follows.

Definition 4 The system (4) is called incrementally sector bounded within sector bounds $[S_1, S_2]$ if the function $\phi(\cdot, t)$ satisfies
\begin{equation}
[z_1 - z_2 - S_1(y_1 - y_2)]^T [z_1 - z_2 - S_2(y_1 - y_2)] \leq 0 \tag{13}
\end{equation}
for all $y_1, y_2 \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$, where $z_1 = \phi(y_1, t)$ and $z_2 = \phi(y_2, t)$.

Incremental sector boundedness was investigated before, for example in [5]. Note that any function $\phi(\cdot, t)$ satisfying the incremental sector boundedness condition (13) also satisfies the ordinary sector boundedness condition, i.e. $\phi(y, t) - S_1 y)^T (\phi(y, t) - S_2 y) \leq 0$ for all $y \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$ for the SISO case, i.e. the case that $z, y$
are scalars and hence $S_1 = \alpha$, $S_2 = \beta$ are real numbers [34], the incremental sector boundedness condition with $0 \leq \alpha < \beta$ is equivalent to the slope-restrictedness condition that was used in [8,12]. In contrast to [8,12], the incremental sector boundedness condition allows us to explore Lur’e network synchronization via linear matrix inequalities immediately.

**Lemma 4** Assume that the uncertain functions $\phi(\cdot,t)$ satisfy (13) for all $y_i, y_j \in \mathbb{R}^a$ and $t \in \mathbb{R}^+$. If there exist matrices $P > 0$, $F$ and a positive real number $\tau$ such that the strict inequality

$$
\begin{bmatrix}
(A + \lambda_l BF)^T P & -PE \\
+P(A + \lambda_l BF) & -\tau C^T(S_1 + S_2) \bar{C} \\
-\tau C^T(S_1 + S_2) \bar{C} & -2\tau I_s
\end{bmatrix} < 0
$$

holds for all $i = 2, \ldots, N$, then the network of Lur’e agents (1) with the protocol (2) is robustly synchronized for all incrementally sector bounded $\phi(\cdot,t)$.

**Proof.** As in the proof of Lemma 2, choose the Lyapunov function candidate $V_1(\bar{x}) = \frac{1}{2} \bar{x}^T (I_{N-1} \otimes \bar{P}) \bar{x}$, where $P > 0$ together with $F$ and $\tau > 0$ satisfies (14). Then the time derivative of $V_1(\bar{x})$ along the trajectories of the system (8) is equal to

$$
\dot{V}_1(\bar{x}) = \frac{1}{2} \begin{bmatrix}
\bar{x}^T [I_{N-1} \otimes PA + \bar{A} \otimes PB F] \bar{x} \\
-(I_{N-1} \otimes PE) (\bar{U}_s^T \otimes I_s) \Phi(y,t)
\end{bmatrix}^T
$$

By taking the Schur complement, (14) is equivalent to the existence of a common state feedback control law $u_i = Fx_i$ for the
such that the resulting closed-loop systems are dissipative with respect to the supply rate $s(z_i, y_i) = \tau z_i^T z_i - y_i^T y_i$ with the common storage function $x_i^T P x_i$. In particular the common state feedback law $u_i = F x_i$ renders the $H_\infty$ gains from $z_i$ to $y_i$ less than or equal to $\sqrt{2\tau}$.

Using the same idea as in Lemma 3, we now establish necessary and sufficient conditions for the existence of $P > 0$, $F$ and $\tau > 0$ satisfying (14), thus obtaining a static protocol that robustly synchronizes the Lur’e network (3) against incrementally sector bounded nonlinearities:

**Theorem 2** There exist $P > 0$, $F$ and $\tau > 0$ such that (14) holds for all $i = 2, \ldots, N$ if and only if there exist a matrix $Q > 0$ and a positive real number $\rho$ such that the following LMI holds:

$$
\begin{bmatrix}
Q(A - \frac{1}{2} E(S_1 + S_2) C)^T & QC^T \\
\phantom{Q} + (A - \frac{1}{2} E(S_1 + S_2) C) Q + \frac{1}{2} \rho E E^T & C Q \\
\end{bmatrix} < 0,
$$

(15)

In this case, a suitable $P$ is given by $P = Q^{-1}$, a suitable $\tau$ is given by $\tau = \frac{1}{\rho}$, and a suitable $F$ is given by $F = -k B^T Q^{-1}$, where the positive real number $k$ is chosen to satisfy

$$
\begin{bmatrix}
Q(A - \frac{1}{2} E(S_1 + S_2) C)^T & QC^T \\
\phantom{Q} + (A - \frac{1}{2} E(S_1 + S_2) C) Q + \frac{1}{2} \rho E E^T - 2k \lambda_2 B B^T & C Q \\
\end{bmatrix} < 0.
$$

(16)

**Proof.** For the ‘only if’ part, by taking the Schur complement, (14) is also equivalent to

$$
\begin{bmatrix}
(A - \frac{1}{2} E(S_1 + S_2) C)^T P + P(A - \frac{1}{2} E(S_1 + S_2) C) + \lambda_i (F E B^T + BF E) \\
\phantom{A} + \frac{1}{2} \rho E E^T P \phantom{A} & C P \\
\end{bmatrix} < 0,
$$

(17)

Let $Q = P^{-1}$ and $\rho = \frac{1}{2}$. Then we get

$$
\begin{bmatrix}
Q(A - \frac{1}{2} E(S_1 + S_2) C)^T + (A - \frac{1}{2} E(S_1 + S_2) C) Q + \lambda_i (F E B^T + BF E) \\
\phantom{Q} + \frac{1}{2} \rho E E^T \phantom{A} & C Q \\
\end{bmatrix} < 0.
$$

Without loss of generality, we have

$$
B = \begin{bmatrix}
B & 0_{o \times n} \\
0_{s \times n} & I_{s \times s}
\end{bmatrix},
$$

(17)

By premultiplying with (17) and postmultiplying with the transpose of (17), (15) is obtained.

For the ‘if’ part, again by taking the Schur complement, (15) implies

$$
B^T \begin{bmatrix}
Q \left(A - \frac{1}{2} E(S_1 + S_2) C \right)^T + \left(A - \frac{1}{2} E(S_1 + S_2) C \right) Q + \frac{1}{2} \rho E E^T + \frac{1}{2} QC^T (S_2 - S_1)^2 C Q - 2k \lambda_2 B B^T & 0
\end{bmatrix}^T < 0.
$$

By Finsler’s lemma, it follows that there exists $k > 0$ such that

$$
\begin{bmatrix}
Q \left(A - \frac{1}{2} E(S_1 + S_2) C \right)^T + \left(A - \frac{1}{2} E(S_1 + S_2) C \right) Q + \frac{1}{2} \rho E E^T + \frac{1}{2} QC^T (S_2 - S_1)^2 C Q - 2k \lambda_2 B B^T & 0
\end{bmatrix}^T < 0,
$$

i.e. (16). Let $P = Q^{-1}$, $\tau = \frac{1}{\rho}$ and $F := -k B^T P$. Then we get

$$
\Theta_2 = \begin{bmatrix}
\left(A - \frac{1}{2} E(S_1 + S_2) C \right)^T & P \\
\phantom{A} + \left(A - \frac{1}{2} E(S_1 + S_2) C \right) & -2k \lambda_2 P B B^T P + \frac{1}{2} \tau C^T (S_2 - S_1)^2 C + \frac{1}{2} P E E^T P < 0.
\end{bmatrix}
$$

Thus $\Theta_1 \leq \Theta_2 < 0$ for all $i = 2, \ldots, N$. This completes the proof. \hspace{1cm} \Box

**Remark 5** Obviously, if we take sector bounds $[0, kI]$ with $k$ a positive real number, then by formally setting $k = +\infty$ we obtain incremental passivity. Thus the question arises whether we can obtain the conditions of Theorem 1 by letting $k \to +\infty$ in the condition of Theorem 2. In the case of sector bounds $[0, kI]$ the LMI (15) in
Theorem 2 becomes
\[
\begin{bmatrix}
B^+ & Q(A - \frac{1}{2}kEC)^T + (A - \frac{1}{2}kEC)Q \\
0 & +\frac{1}{2}\rho EE^T + \frac{1}{2}k^2QC^T CQ
\end{bmatrix}
\begin{bmatrix}
B^+ \\
0
\end{bmatrix}^T < 0
\]

or, equivalently,
\[
B^+ \left[ Q \left( A - \frac{1}{2}kEC \right)^T + \left( A - \frac{1}{2}kEC \right)Q \right]
+ \frac{1}{2}\rho EE^T + \frac{1}{2}k^2QC^T CQ \left( B^+ \right)^T < 0
\]

for some \( Q > 0 \) and \( \rho > 0 \). Note that, in fact, \( Q \) and \( \rho \) depend on \( k \), i.e., \( Q = Q(k) \) and \( \rho = \rho(k) \). Clearly, the strict inequality above is equivalent to
\[
B^+ \left[ QA^T + AQ + \frac{1}{\rho} \left( E - \frac{1}{\rho}kQC^T \right) \right]
\left( B^+ \right)^T < 0.
\]

From this we indeed immediately obtain
\[
B^+ \left( QA^T + AQ \right) \left( B^+ \right)^T < 0,
\]

which is exactly the first condition, (10), of Theorem 1. The question of course then arises whether also the second condition of Theorem 1, i.e., \( QC^T = E \) can be obtained. Here the idea would be to let \( k \) run off to infinity. If we would know that \( \frac{1}{\rho(k)} \to 1 \) and \( Q(k) \to \bar{Q} \) for \( k \to +\infty \) (for some limit \( \bar{Q} \)), then from the latter inequality we would indeed obtain \( QC^T = E \). However, it is unclear how \( Q(k) \) and \( \rho(k) \) depend on the sector bound \( k \), and what happens if \( k \) goes to infinity. Summarizing, the condition of Theorem 2 indeed immediately implies the first condition of Theorem 1, but it is unclear how the second condition of Theorem 1 follows from the condition of Theorem 2.

4 Fully Distributed Robust Synchronization

As shown in the previous section, in order to compute the feedback gain matrix \( F \) in the synchronization protocol (2), a positive real number \( k \) must be computed that satisfies (12) or (16), respectively. Computation of such \( k \) involves the second smallest eigenvalue \( \lambda_2 \) of the Laplacian matrix \( L \), which is called the algebraic connectivity of the undirected graph \( G \). However, knowledge of the exact value of \( \lambda_2 \) involves global information in the sense that each agent has to know \( L \) and consequently the entire interconnection topology \( G \) to compute it. Thus, in effect, the protocol (2) cannot be implemented by each agent in a fully distributed fashion.

In order to overcome this drawback, we adopt the following fully distributed synchronization protocol with an adaptive control law that adjusts the coupling weights in real time (see also [8, 14, 15]):

\[
u_i = F \sum_{j=1}^{N} a_{ij} c_{ij}(x_i - x_j),
\]

(18)

\[
\dot{c}_{ij} = a_{ij}(x_i - x_j)^T H(x_i - x_j), \quad i, j = 1, 2, \ldots, N,
\]

where \( c_{ij}(t) \) is a time-varying coupling weight between agents \( i \) and \( j \) with \( c_{ij}(0) = c_{jj}(0) \), \( H \in \mathbb{R}_{+}^{r \times r} \) is a coupling gain matrix to be determined later. By interconnecting the agents (1) with the adaptive protocol (18), we obtain that the network is represented by

\[
\dot{x}_i = Ax_i + BF \sum_{j=1}^{N} a_{ij} c_{ij}(x_i - x_j) - E\phi(Cx_i, t),
\]

(19)

\[
\dot{c}_{ij} = a_{ij}(x_i - x_j)^T H(x_i - x_j), \quad i, j = 1, 2, \ldots, N.
\]

We will now design \( F \) and \( H \) in (18) such that the Lur’

\[e = x - \bar{X}\]

\[e_T = [e_1^T, e_2^T, \ldots, e_N^T]^T
\]

Then we get

\[
\dot{e}_i = Ae_i + BF \sum_{j=1}^{N} a_{ij} c_{ij}(e_i - e_j) - E\phi(Cx_i, t)
\]

\[
+ \frac{1}{N} \sum_{j=1}^{N} E\phi(Cx_j, t),
\]

(20)

\[
\dot{c}_{ij} = a_{ij}(e_i - e_j)^T H(e_i - e_j), \quad i, j = 1, 2, \ldots, N.
\]

It is obvious that \( x_i - x_j = 0 \) for all \( i, j = 1, 2, \ldots, N \) if and only if \( e = 0 \). Therefore, the synchronization of \( x \) is
equivalent to the global asymptotical stability of \( e \).

Let \( e = [e_1, \cdots, e_N]^T \). Choose the Lyapunov function candidate \( V_2(e, c) = \sum_{i=1}^{N} e_i^T Q^{-1} e_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (c_{ij} - \bar{c})^2 \), where \( Q > 0 \) satisfies (10) and (11), and \( \bar{c} \) is a positive real number to be determined later. The time derivative of \( V_2(e, c) \) along the trajectories of (20) is given by

\[
V_2(e, c) = 2 \sum_{i=1}^{N} e_i^T Q^{-1} e_i + \sum_{i=1}^{N} \sum_{j=1}^{N} (c_{ij} - \bar{c}) \dot{c}_{ij} \\
= 2 \sum_{i=1}^{N} e_i^T Q^{-1} \left[ Ae_i + BF \sum_{j=1}^{N} a_{ij} c_{ij}(e_i - e_j) \right] \\
- E \phi(Cx_i, t) + \frac{1}{N} \sum_{j=1}^{N} E \phi(Cx_j, t) \\
+ \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (c_{ij} - \bar{c})(e_i - e_j)^T H(e_i - e_j).
\]

Since \( a_{ij} = a_{ji}, c_{ij}(0) = c_{ji}(0) \) and \( \dot{c}_{ij} = \dot{c}_{ji}, \forall i, j = 1, 2, \cdots, N, \) we know that \( c_{ij}(t) = c_{ji}(t), \forall t \geq 0, i, j = 1, 2, \cdots, N. \) Then

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (c_{ij} - \bar{c})(e_i - e_j)^T H(e_i - e_j) \\
= 2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (c_{ij} - \bar{c}) e_i^T H(e_i - e_j).
\]

Thus

\[
\dot{V}_2(e, c) \\
= 2 \sum_{i=1}^{N} e_i^T Q^{-1} A e_i \\
- 2 \sum_{i=1}^{N} e_i^T Q^{-1} B B^T Q^{-1} \sum_{j=1}^{N} a_{ij} c_{ij}(e_i - e_j) \\
- \frac{2}{N} \sum_{i=1}^{N} e_i^T Q^{-1} E \sum_{j=1}^{N} (\phi(Cx_i, t) - \phi(Cx_j, t)) \\
+ 2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (c_{ij} - \bar{c}) e_i^T Q^{-1} B B^T Q^{-1}(e_i - e_j) \\
= 2 \sum_{i=1}^{N} e_i^T Q^{-1} A e_i \\
- 2 \sum_{i=1}^{N} e_i^T Q^{-1} \sum_{j=1}^{N} (\phi(Cx_i, t) - \phi(Cx_j, t)) \\
- 2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e_i^T Q^{-1} B B^T Q^{-1}(e_i - e_j) \\
= 2 \sum_{i=1}^{N} e_i^T Q^{-1} A e_i \\
- \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{i=1}^{N} (c_{ij} - \bar{c})^2 \\
- 2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e_i^T Q^{-1} B B^T Q^{-1}(e_i - e_j) \\
= 2 \sum_{i=1}^{N} e_i^T Q^{-1} A e_i \\
- \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} (c_{ij} - \bar{c})^2 (\phi(Cx_i, t) - \phi(Cx_j, t)) \\
- 2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e_i^T Q^{-1} B B^T Q^{-1}(e_i - e_j) \\
= 2 \sum_{i=1}^{N} e_i^T Q^{-1} A e_i \\
- \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} (c_{ij} - \bar{c})^2 (\phi(Cx_i, t) - \phi(Cx_j, t)) \\
- 2 \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e_i^T Q^{-1} B B^T Q^{-1}(e_i - e_j)
\]

Let \( \mathcal{U} \) be an orthogonal matrix such that \( \mathcal{U}^T \mathcal{U} = \Lambda \) as defined in Section 2. Let \( \tilde{e} = \{ \mathcal{U}^T \otimes I_\lambda \} c \) and \( \tilde{c} = \{ \mathcal{U}^T \otimes I_\lambda \} c \), where \( \tilde{e} = [\tilde{e}_1^T, \cdots, \tilde{e}_N^T]^T \) and \( \tilde{c} = [\tilde{c}_1^T, \cdots, \tilde{c}_N^T]^T \). It is obvious that \( \tilde{c}_i = \frac{1}{\sqrt{\lambda}} \tilde{e}_i \). Then for all trajectories of \( e \) and \( c \) we get

\[
\dot{V}_2(e, c) \\
\leq 2 \tilde{c}^T \left( I_{N-1} \otimes Q^{-1} A - \tilde{c} \Lambda \otimes Q^{-1} B B^T Q^{-1} \right) \tilde{e} \\
= \sum_{i=2}^{N} \tilde{c}_i^T \left( A^T Q^{-1} + Q^{-1} A - 2 \tilde{c}_i \Lambda Q^{-1} B B^T Q^{-1} \right) \tilde{e}_i.
\]

By Finsler’s lemma, (10) implies that there exists \( \dot{\overline{c}} > 0 \) such that \( AQ + QA^T - 2\lambda \tilde{c} \Lambda B B^T < 0 \). Obviously,

\[
AQ + QA^T - 2\lambda \tilde{c} \Lambda B B^T \leq AQ + QA^T - 2\lambda \tilde{c} \Lambda B B^T < 0
\]

for all \( i = 2, \cdots, N \). Thus \( \dot{V}_2(e, c) \leq 0 \). Hence \( \dot{V}_2(e, c) \) is bounded and so is each \( c_{ij} \). Note that \( H \geq 0 \). This implies that \( c_{ij} \) is nondecreasing. It follows that \( c_{ij} \) converges to certain finite values. Let \( S := \{(e(t), c(t)) \mid V_2(e, c) \equiv 0\} \). Note that \( \dot{V}_2(e, c) \equiv 0 \) implies that \( \bar{c} \equiv 0 \). We also know that \( \dot{c}_i \equiv 0 \). Therefore, by LaSalle’s invariance principle, \( \dot{c}(t) \rightarrow 0 \) and \( c(t) \rightarrow 0 \) as \( t \rightarrow \infty \). This completes the proof. \( \square \)

4.2 Incrementally Sector Bounded Nonlinearities

In this subsection we consider the incrementally sector bounded nonlinearity case.

**Theorem 4** Assume that the uncertain functions \( \phi(\cdot, t) \) satisfy (13) for all \( y_1, y_2 \in \mathbb{R}^s \) and \( t \in \mathbb{R}^1 \). If there exist a matrix \( Q > 0 \) and a positive real number \( \rho \) such that (15) holds, then the network of agents (1) with the adaptive protocol (18), where \( F = -B^T Q^{-1} \) and \( H = Q^{-1} B B^T Q^{-1} \), is robustly synchronized, i.e. the Lur’e network (19) is synchronized for all incrementally sector bounded \( \phi(\cdot, t) \).

**Proof.** As in the proof of Theorem 3, let \( e_i = x_i - \frac{1}{N} \sum_{j=1}^{N} x_j \) and \( e = [e_1^T, e_2^T, \cdots, e_N^T]^T \). Then again we
obtain (20). Choose the same Lyapunov function candidate $V_2(e, c)$. The time derivative of $V_2(e, c)$ along the trajectories of (20) is given by

$$
\dot{V}_2(e, c) = 2 \sum_{i=1}^{N} e_i Q^{-1} A e_i - 2 \sum_{i=1}^{N} e_i^T Q^{-1} B B^T Q^{-1} \sum_{j=1}^{N} a_{ij} e_i \cdot e_j - 2 \sum_{i=1}^{N} e_i^T Q^{-1} E \phi(Cx_i, t) + \frac{2}{N} \left( \sum_{i=1}^{N} e_i^T \right) Q^{-1} \sum_{j=1}^{N} E \phi(Cx_i, t) + \frac{2}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} (e_i - \bar{e}) e_i^T Q^{-1} B B^T Q^{-1} (e_i - e_j).
$$

Since

$$
\sum_{i=1}^{N} e_i = \sum_{i=1}^{N} \left( x_i - \frac{1}{N} \sum_{j=1}^{N} x_j \right) = 0,
$$
we obtain

$$
\dot{V}_2(e, c) = 2 \sum_{i=1}^{N} e_i Q^{-1} A e_i - 2 \sum_{i=1}^{N} e_i^T Q^{-1} E \phi(Cx_i, t) - 2e \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e_i^T Q^{-1} B B^T Q^{-1} (e_i - e_j) = e^T \left[ I_N \otimes (Q^{-1} A + A^T Q^{-1}) \right] e - 2e^T \left( I_N \otimes Q^{-1} E \right) \Phi(y, t) - 2e^T \left( I_N \otimes Q^{-1} B B^T Q^{-1} \right) e
$$

$$
= \begin{bmatrix}
    e \\
    \Phi(y, t)
\end{bmatrix}^T
\begin{bmatrix}
    I_N \otimes (Q^{-1} A + A^T Q^{-1}) & -I_N \otimes Q^{-1} E \\
    -2e \otimes Q^{-1} B B^T Q^{-1} & -I_N \otimes E^T Q^{-1}
\end{bmatrix}
\begin{bmatrix}
    e \\
    \Phi(y, t)
\end{bmatrix}.
$$

Let \( \bar{e} = (\mathcal{U}^T \otimes I_e) e \) and \( \bar{e} = (\mathcal{U}_{\bar{e}}^T \otimes I_e) e \), where \( \bar{e} = [\bar{e}_1^T, \bar{e}_2^T, \ldots, \bar{e}_N^T]^T \) and \( \bar{e} = [\bar{e}_2^T, \ldots, \bar{e}_N^T]^T \). Then we get

$$
\dot{V}_2(e, c) = \begin{bmatrix}
    \bar{e} \\
    \Phi(y, t)
\end{bmatrix}^T
\begin{bmatrix}
    I_{N-1} \otimes (Q^{-1} A + A^T Q^{-1}) & -I_{N-1} \otimes Q^{-1} E \\
    -2e \lambda_2 Q^{-1} B B^T Q^{-1} & -I_{N-1} \otimes E^T Q^{-1}
\end{bmatrix}
\begin{bmatrix}
    \bar{e} \\
    \Phi(y, t)
\end{bmatrix}.
$$

It is easily seen that

$$
\bar{e} = \begin{bmatrix}
    I_N - \frac{1}{N} 1_N 1_N^T
\end{bmatrix} \otimes I_n x = (\mathcal{U} \mathcal{U}_e^T \otimes I_n) x.
$$

As \( \bar{e} = (\mathcal{U}^T \otimes I_e) e \), we have \( \bar{e} = (\mathcal{U}_{\bar{e}}^T \otimes I_e) x = \bar{x} \). Thus, from the proof of Lemma 4, we have the following

$$
\begin{bmatrix}
    \bar{e} \\
    \Phi(y, t)
\end{bmatrix}^T
\begin{bmatrix}
    \bar{e} \\
    \Phi(y, t)
\end{bmatrix} \leq 0.
$$

By taking the Schur complement and applying Finsler’s lemma, (15) implies that there exists \( \bar{e} > 0 \) such that

$$
\begin{bmatrix}
    Q^{-1} A + A^T Q^{-1} & -Q^{-1} E \\
    -2e \lambda_2 Q^{-1} B B^T Q^{-1} & +\tau C^T (S_1 + S_2)
\end{bmatrix} < 0,
$$

where \( \tau = \frac{1}{\rho} \). Therefore

$$
\begin{bmatrix}
    Q^{-1} A + A^T Q^{-1} & -Q^{-1} E \\
    -2e \lambda_2 Q^{-1} B B^T Q^{-1} & +\tau C^T (S_1 + S_2)
\end{bmatrix} < 0
$$

for all \( i = 2, \ldots, N \). It follows that \( \dot{V}_2(e, c) \leq 0 \) for all incrementally sector bounded \( \Phi(\cdot, t) \). Following a simi-
lar analysis as in the proof of Theorem 3, the proof is completed. □

5 Examples and simulations

In this section, we present some numerical simulations to illustrate the theoretical results obtained in this paper. We consider Chua’s circuit, which is described by the following system of nonlinear ordinary differential equations [16]:

\[
\begin{align*}
\dot{x}_1 &= 10.0(-x_1 + x_2 - f(x_1)) \\
\dot{x}_2 &= x_1 - x_2 + x_3 \\
\dot{x}_3 &= -14.87x_2
\end{align*}
\]

where \(x_1(t), x_2(t), x_3(t) \in \mathbb{R}\). \(f(x_1)\) is a piecewise linear function that represents the change in resistance versus current across Chua’s diode, which is given by \(f(x_1) = -0.68x_1 - 0.295(|x_1 + 1| - |x_1 - 1|)\). It is possible to rewrite (21) in the form of a Lur’e system with control input \(u \in \mathbb{R}\):

\[
\begin{align*}
\dot{z} &= Ax + Bu + Ez \\
y &= Cx \\
z &= -\phi_1(y)
\end{align*}
\]

where \(x = [x_1, x_2, x_3]^T, A = [-3.2, 10.0; 1, -1.0, -14.87, 0], B = [1; 1; 0], C = [1, 0, 0], E = [-2.95, 0; 0, 0]\) and \(\phi_1(x_1) = |x_1 + 1| - |x_1 - 1|\). \(B^+\) exists, and the pair \((A, B)\) is controllable. It is easily seen that \(\phi_1(\cdot)\) is incrementally passive, but also incrementally sector bounded with \(S_1 = 0\) and \(S_2 = 2\) (see Fig. 2). Taking (22) as the individual agent dynamics, a network of 9 such agents is shown in Fig. 3, in which the interconnection topology is undirected and connected. Let the graph be unweighted, i.e. \(a_{ij} = 1\) when the edge \((i, j)\) (or \((j, i)\)) exists. We compute the second smallest Laplacian eigenvalue of the graph to be \(\lambda_2 = 0.4822\).

Example 1. Since \(\phi_1(\cdot)\) is incrementally passive, we first attempt to apply Theorem 1 to find a static distributed protocol that robustly synchronizes the network. Unfortunately, the conditions in Theorem 1 are not satisfied since there does not exist a positive definite matrix \(Q\) such that \(CQ = E^T\). We have the dynamical system \(\dot{x}_1 = -x_1^2 + x_2 + u, \dot{x}_2 = -x_2 + u\) to be an available example for Theorem 1. A suitable positive definite matrix \(Q\) could be \(Q = [1, 0; 0, q]\), where \(q\) is any positive real number. Due to space limitations, the simulation plots for this example are omitted.

Example 2. As we have noted before, \(\phi_1(\cdot)\) is incrementally sector bounded with \(S_1 = 0\) and \(S_2 = 2\). Therefore we proceed by trying to find an available synchronization protocol by means of Theorem 2. Using the LMI Control Toolbox in Matlab, we find that the positive definite matrix

\[
Q = \begin{bmatrix}
540.7 & 248.5 & 56.1 \\
248.5 & 413.3 & 23.0 \\
56.1 & 23.0 & 2457.7
\end{bmatrix},
\]

and the positive real number \(\rho = 689.9865\) satisfy condition (15) in Theorem 2, and \(k = 41893\) satisfies condition (16). The corresponding feedback gain matrix is computed as

\[
F = -kB^TQ^{-1} = \begin{bmatrix}
-42.8694 & -75.6846 & 1.6868
\end{bmatrix}.
\]

For \(i = 1, 2, \cdots, 9\), let \(x_i = [x_{i1}, x_{i2}, x_{i3}]^T\) be the state of agent \(i\). Denote \(X_j := [x_{1j}, x_{2j}, \cdots, x_{9j}]^T\) for \(j = 1, 2, 3\). Choose the initial states as \(x_i(0) = 0.1|\cdot; i|\), \(i = 1, 2, \cdots, 9\). The first component of the trajectories of the network (3), i.e. \(X_1\), is plotted in Fig. 4. Clearly, the network reaches synchronization. In order to illustrate the robustness of the synchronization, we have also considered the nonlinearity \(\phi_2(x_1) = \arctan(x_1)\).
tal passivity and incremental sector boundedness con-
in this paper we have discussed the roles of incremen-
tal weights.
Example 2, we have tested the protocol for the nonlin-
erity \( \phi_2 \). In this example we apply the results of Sec-
tion 4 to Chua’s circuit as we did in Example 2. We pro-
vide simulation results for the incremental sector bound-
edness case. As the positive definite matrix \( Q \) in Theo-
rem 4 is equal to \( Q \) in Theorem 2, we obtain
\[
F = -B^TQ^{-1} = \begin{bmatrix} -0.0010 & -0.0018 & 0.0000 \end{bmatrix},
\]
\[
H = Q^{-1}BB^TQ^{-1} = \begin{bmatrix} 10470 & 18490 & -410 \\ 18490 & 32640 & -730 \\ -410 & -730 & 20 \end{bmatrix}.
\]
We choose the same initial states as in Example 2. Similarly, the first component of the trajectories of the network \( \Sigma \), i.e. \( X_1 \), is plotted in Fig. 6. The network indeed reaches synchronization. The time-varying cou-
ping weights \( c_{ij} \) are shown in Fig. 7. These converge to finite steady-state values. Similar to the results of Example 2, we have tested the protocol for the nonlinearity \( \phi_2 = \arctan(\cdot) \), and have found that the network is synchronized as well. Due to space limitations the simulation results are omitted.

6 Conclusions

In this paper we have discussed the roles of incremen-
tal passivity and incremental sector boundedness condi-
tions in robust synchronization of homogeneous Lur’e
networks. Sufficient conditions for the existence of static distributed protocols to robustly synchronize Lur’e networks have been given. The protocols can be imple-
mented by each agent in a fully distributed fashion. The required feedback gain matrices are computed by solv-
ing LMI’s, which can be easily done using the LMI Control Toolbox in Matlab. As a possible topic for future research we intend to investigate robust synchronization problems for networks of Lur’e systems against hetero-
geneous uncertain nonlinearities within identical sector bounds.

Acknowledgements

The first author would like to thank the China Scholar-
ship Council (CSC) for the financial assistance to perform his research at the University of Groningen. The authors want to thank the anonymous reviewers for their constructive suggestions that helped to improve the quality of this paper.

References


