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1 Griffiths singularities

Griffiths singularities (sometimes called Griffiths-McCoy singularities [5, 8]) occur in the statistical mechanics of disordered systems. Consider for example a lattice spin model in which the interaction Hamiltonian is itself a random object, and consider the behaviour of the free energy density for a typical realization of this Hamiltonian. In situations where this free energy and similar thermodynamic functions of these quenched disordered systems are non-analytic, but typically C^∞ (infinitely often differentiable) in the thermodynamic (infinite-volume) limit, one speaks of a Griffiths singularity. In other words, in this *Griffiths phase*, an infinite-order phase transition is present in which there occur *essential* singularities. This singular behaviour is due to the occurrence with low but finite density in the infinite volume of arbitrarily large - "rare" - regions where all interaction terms are strong and cooperative enough that one is almost at a phase transition.

Usually these thermodynamic functions are self-averaging, so one can either consider them for a typical realization, or one can study the disorder-averaged free energy density.

The original work of Griffiths applied to low-temperature quenched dilute Ising models, when the occupation probability lies beneath the percolation threshold. Then the system breaks apart into Ising models on finite clusters. Griffiths, by analyzing the Yang-Lee zeros of the partition functions which come arbitrarily close to the real axis for larger and larger volumes, proved that, although due to the almost-sure finiteness of these clusters no first-order transition with ferromagnetic long-range order can occur, in the paramagnetic phase below the critical temperature of the non-dilute Ising model the free energy and magnetisation are non-analytic as a function of the magnetic field h at the value $h = 0$.

Later, Sütö [9] proved the infinite differentiability as a function of h. Since then, for many classical and quantum disordered lattice models with independent disorder it has been shown that an expected singularity of the free energy density as a function of various parameters in the Hamiltonian cannot be worse than C^∞.

Under a stochastic dynamics of Glauber (spin-flip) type, in the regime of

Griffiths singularities the time autocorrelation functions are expected to have slow (non-exponential) decay. Some bounds on the possible decay rate have been derived, and in the few situations where Griffiths singularities have been proved to exist, these bounds have been shown to be satisfied. These slow dynamics are in fact the most promising for observing in either experiments or computer simulations the presence of a Griffiths phase.

A selection of mathematically rigorous versions of these results, mostly proven by multiscale methods, can be found in [3, 4, 1, 2].

Moreover, there is an extensive nonrigorous physics literature discussing the Griffiths phase, see e.g. [10].

Although it is expected that Griffiths singularities occur in great generality, the proof of their existence in a wider class of quenched random models which are not dilute, but are of more general random-bond or random-field type, has proved to be elusive so far.

A similar rare-region effect occurs in the theory of disordered electron systems, where it is responsible for the appearance of Lifshitz band tails (Lifshitz singularities) [6, 7].

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