Non-Gibbsian states

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1 Gibbs states

When we consider statistical mechanical models on a finite subset V of an infinite lattice L, e.g. $\mathbb{Z}^d$, Gibbs states for (not-too-long-range) interactions Φ are given by probability distributions or density operators (density matrices) of the canonical Gibbsian form $\frac{exp(-\beta H_V^\Phi)}{Z_V}$, where $\beta H_V^\Phi = \sum_{X \subset V} \Phi_X$.

The equivalent objects for infinite systems are also known as Gibbs states. These are linear functionals on a set of observables which can be either random variables (for classical systems) or operators (for quantum-mechanical systems). These Gibbs states describe the system in equilibrium, at inverse temperature β. For a given interaction there may exist more than one Gibbs state; if this is the case these different Gibbs states then describe different possible equilibrium phases of the system.

For infinite classical systems, according to the Dobrushin-Lanford-Ruelle theory, the appropriate Gibbs states are probability measures which are defined by the property that their conditional probabilities are of the above-mentioned Gibbsian form.

More precisely, if for a classical lattice system the single-spin space Ω_0 is compact (and in particular we will concentrate below on the even simpler situation when it is finite), the configuration space for the infinite lattice is $\Omega = {\Omega_0}^L$. In this situation the theory has been developed in the most complete form, and the distinction between Gibbs measures and non-Gibbsian measures is unambiguous.

An interaction Φ then is a collection of bounded functions Φ_X on Ω, such that Φ_X only depends on the coordinates in X (it is measurable with respect to the associated σ-algebra of events in $\Omega_X = {\Omega_0}^X$, thus it can be viewed also as a function on Ω_X), labeled by the finite subsets X of L, such that $sup_i \sum_{i \in X} ||\Phi_X||_\infty$ is finite (the interaction is then absolutely summable). The total energy of the system (the infinite-volume Hamiltonian H^Φ) intuitively is given by

$$\beta H^\Phi = \sum_{X \subset L} \Phi_X \tag{1}$$

However, this infinite sum tends to be ill-defined, as it takes the values plus or minus infinity for most configurations. (An infinite system at positive temperature typically contains an infinite amount of energy).

The correct version of this intuition is as follows:
A Gibbs state or Gibbs measure μ for interaction Φ is a probability measure

on Ω, such that

$$\frac{\mu(\omega_Y^1|\omega_{Y^c})}{\mu(\omega_Y^2|\omega_{Y^c})} = exp - [\sum_{X\subset L} \Phi_X(\omega_Y^1\omega_{Y^c}) - \Phi_X(\omega_Y^2\omega_{Y^c})] \tag{2}$$

holds, for any choice of $Y \subset L$, configurations $\omega_Y^{1,2}$ in Y, and μ-almost all external configurations ω_{Y^c} in the complement of Y. The infinite-volume configurations $\omega_Y^i\omega_{Y^c}$ (for $i = 1, 2$) are equal to ω_Y^i inside Y and equal to ω_{Y^c} outside Y.

In the case where Ω_0 is not finite, but still compact, one has a similar identity for conditional probability densities instead of conditional probabilities.

Due to the summability the of the interaction, the infinite sum of differences (which describes the difference between two "infinite energies") in the exponent now is well-defined. We note that as a function of ω_{Y^c}, the above expression is "quasilocal" (sometimes called "almost Markovian"), that is the dependence of coordinates far away from Y becomes weak; in the case where Ω_0 is finite, this property of quasilocality coincides with continuity in the product topology. Moreover, these measures satisfy a finite-energy condition; this means that any configuration ω_Y in Y occurs with a positive probability and this probability has a lower bound which is uniform in the external configuration ω_{Y^c}. For an extensive treatment of the theory of Gibbs measures, see [3]. See also [5], for a treatment which includes quantum models.

2 Non-Gibbsian States

As Gibbs measures have many useful properties (e.g. they satisfy a variational principle, and they possess nice large-deviation properties), it is often attempted in physics to write a measure which one comes across as a Gibbs measure for some a priori unknown effective interaction. For finite systems one can almost always simply do this, by taking the logarithm of the Gibbs state. The logarithms of the probabilities then give the energies, with the temperature incorporated. (Similarly in quantum mechanics the logarithm of a density operator provides an energy operator.) However, the problem becomes less trivial for infinite systems.

One of the main classes of examples from equilibrium statistical mechanics where one tries to find such an interaction are the measures obtained by a renormalization-group (coarse-graining) transformation acting on some given Gibbs measure (which in probabilistic language is just a marginal measure), so as to define a renormalization-group map from Hamiltonians to Hamiltonians [4].

Other examples where one tries this occur in non-equlibrium statistical mechanics, whether they are obtained by a stochastic evolution acting on an initial Gibbs measure (to describe the system in the transient regime), or for an invariant measure – non-equilibrium steady state – for such a stochastic evolution (to obtain a MacLennan-Zubarev description). If the space-time description is such that it can be written in a Gibbsian form, as a measure on histories -say for discrete time-, again the question is one of considering a marginal measure of a Gibbs measure, but now the marginal means restricting to the random variables in a lower-dimensional system.

Yet other examples are Fortuin-Kasteleyn random-cluster measures, joint measures of quenched disordered systems for which one aims to find a Morita "grand potential", and Hidden Markov Fields.

It has been found, however, that in many of such situations these measures lack the quasilocality property, and thus are "non-Gibbsian". This finding originally came as something of a surprise. See [2, 7] for a more extensive discussion.

In fact, it should not have been that surprising; in the case of translation invariant (translation-ergodic) measures, it has been proven by Israel [7] that most of these measures (that is to say, the set of all such measures is a dense G_δ) are non-Gibbsian in this sense.

This non-Gibbsianness in principle may invalidate many perturbative or heuristic calculations in the physics literature in which these effective interactions were computed.

In particular, non-Gibbsianness of a measure often means that some spin configurations exist, and sometimes can be identified, which act as (nonremovable) points of discontinuity -in the product topology- for a particular spin observable expectation, conditioned on those configurations.

In the proofs of the non-Gibbsian property, often such a point of discontinuity acts as a conditioning for, or provides a term in a Hamiltonian of, an auxiliary model, and the existence of long-range order (associated to the existence of more than one Gibbs measure which defines a kind of first-order phase transition) for this auxiliary model can be translated into the violation of the quasilocality property, that is the non-Gibbsianness, of the original model.

Once one knows that a measure is non-Gibbsian, one can try to make finer distinctions, by considering how large the set of these discontinuity points is. Such considerations have led to the notions of Almost Gibbsian and Weakly Gibbsian measures. Almost Gibbsian measures, in which the set of discontinuity points has measure zero, and which also which satisfy the finite-energy condition share a number of "nice" properties with Gibbs measures. Weakly Gibbsian measures in which the set of points where the formal interaction diverges has measure zero, form a wider class; in general they can be more pathological.

Some further issues related to non-Gibbsianness and possible extensions of this notion which have been considered in the literature are mentioned below:

1) For mean-field models a notion of non-Gibbsianness can be developed in which continuity in the external configuration with respect to the product topology is replaced by continuity as a function of empirical averages (of spins, e.g.), see for example the paper of Häggström and Külske in [7].

2) For non-compact-spin models, the definition of Gibbsianness is somewhat more arbitrary than for compact spins, as quasilocality does not hold for Gibbs measures for a natural class of infinite-range interactions. (For a proposal and some discussion on this point see for example the paper of Dereudre and Roelly in [7].

3) For quantum models a Gibbs state is identified with a KMS state on some C^*-algebra of spin operators, for some C^*-dynamics. Classical non-Gibbsian measures for fininte-spin systems can be extended to quantum states which are non-Gibbsian, and there exist a few nonclassical examples of non-Gibbsian quantum states (see e.g. [6]), but not much is known in general.

4) In the thermodynamic formalism of dynamical systems, SRB (Sinai-Ruelle-Bowen) measures tend to be Gibbs measures on some restricted configuation space, for one-dimensional systems with a rather short-range potential, in which the interaction energy between two infinite half-lines is finite. Such measures carry the name "(Bowen-)Gibbs measures". In this dynamical systems literature the term "non-Gibbsianness" is used when this short-range property is violated [1, 8]. This notion of non-Gibbsianness is not equivalent to the one mentioned above.

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