From networks models to geometry: a new view on Hamiltonian systems

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1. Review on classical Hamiltonian systems
2. Power-conserving interconnections and Dirac structures
3. Port-Hamiltonian systems
4. Composition of Dirac structures
5. Control in the port-Hamiltonian framework
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7. Conclusions and outlook

Joint work with B. Maschke, R. Ortega, G. Golo, ...

Common view on Hamiltonian systems

Classical Hamiltonian equations of motion

$$\begin{aligned} \dot{q}_i &= \quad \frac{\partial H}{\partial p_i}(q,p) \\ & \qquad\qquad\qquad i = 1, \cdots, n \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}(q,p) \end{aligned}$$

where

- $q = (q_1, \cdots, q_n)$ are the configuration coordinates,
- $p = (p_1, \cdots, p_n)$ are the generalized momenta,
- $H(q,p)$ is the total energy of the system.

Geometrically (coordinate-free) this is usually described by the triple

$$(T^*Q, \omega, H)$$

where

- Q is the configuration manifold
  (with local coordinates $q = (q_1, \cdots, q_n)$)
- ω is canonical symplectic form on the cotangent bundle T^*Q
  (in local coordinates given by $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$)
- $H : T^*Q \to \mathbb{R}$.

The Hamiltonian dynamics is defined by the vector field X_H satisfying

$$\omega(X_H, -) = -dH$$

Further generalizations: 1) Replace T^*Q by a general *symplectic manifold* (M, ω), 2) *Infinite-dimensional* case.

Equivalently, let $\{,\}$ denote the canonical Poisson bracket on T^*Q, in canonical coordinates for T^*Q given by

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right),$$

then X_H is determined by the requirement

$$X_H(F) = \{F, H\}$$

for all $F : T^*Q \to \mathbb{R}$.

In an arbitrary set of local coordinates x the dynamics takes the form

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$$

where $J = -J^T$ is the structure matrix of the Poisson bracket with elements

$$J_{ij} = \{x_i, x_j\}, \quad i, j = 1, \cdots, n$$

Note that in the finite-dimensional case J has **full rank**.

On the other hand, it is well-known that many dynamical equations of physical interest are **not** precisely of this form, although they should be regarded as **Hamiltonian in a generalized sense**.

Typical example are the *Euler equations for the rigid body*

$$\begin{pmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{pmatrix} = \begin{pmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial p_x} \\ \frac{\partial H}{\partial p_y} \\ \frac{\partial H}{\partial p_z} \end{pmatrix}$$

with $p = (p_x, p_y, p_z)$ the body angular momentum vector along the three principal axes, and $H(p) = \frac{1}{2}\left(\frac{p_x^2}{I_x} + \frac{p_y^2}{I_y} + \frac{p_z^2}{I_z}\right)$ the kinetic energy (I_x, I_y, I_z principal moments of inertia.)

In general, many systems are of the Hamiltonian form

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$$

with $J = -J^T$, but **not** of full rank.

Hamiltonian systems obtained by symmetry reduction

The Euler equations can be regarded as the **reduction** of classical Hamiltonian equations on a cotangent bundle.
Reduced space is the orbit space of the action of a Lie group that leaves the Hamiltonian invariant.

In fact, $Q = SO(3)$ and the cotangent bundle $T^*SO(3)$ can be reduced by the action of $SO(3)$ on $T^*SO(3)$ into $so(3)^*$,
while the Hamiltonian is *invariant* under this action.

This holds in many situations, both in the finite- and infinite-dimensional case
("Marsden-Weinstein *reduction by symmetry* program").

A different point of view:

Network modeling of physical systems

Prevailing trend in *modeling and simulation* of lumped-parameter systems (multi-body systems, electrical circuits, electro-mechanical systems, robotic systems, cell-biological systems, etc.).

Advantages of network modeling:

- Systematic modeling procedure, which offers structural insight.
- Flexibility. Re-usability of components. Suited to design/control.
- Multi-physics approach.
- "Modularity can beat complexity."

Originates from engineering, and calls for **mathematical theory of networks and systems**.

Possible *disadvantage* of network modeling: it generally leads to a large set of differential and *algebraic* equations (DAEs), **seemingly without any structure**.
This is a serious obstacle for analysis and control; especially for nonlinear models.

Aim: to identify the underlying Hamiltonian structure of network models of physical systems, and to use it for analysis, simulation and control.

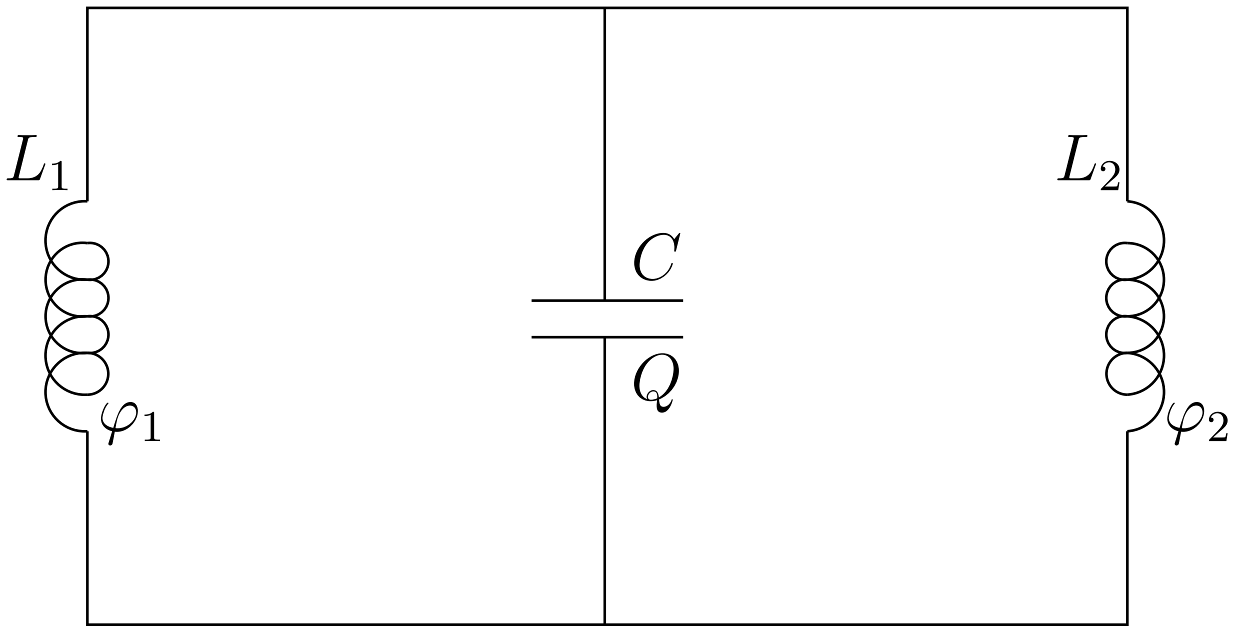
Port-based network modeling

Interaction between ideal system components is modeled by **power-ports** modeling the *energy exchange* between the components.

Associated to every power-port there are *conjugate pairs* of variables (called **flows** f and **efforts** e), *whose product* $e^T f$ *equals power*.
E.g., voltages and currents, generalized forces and velocities, pressure and volume change, etc.

This leads to a (generalized) Hamiltonian description of multi-physics systems.

Example Two inductors with magnetic energies $H_1(\varphi_1), H_2(\varphi_2)$ (φ_1 and φ_2 magnetic flux linkages), and capacitor with electric energy $H_3(Q)$ (Q charge).



Question: *How to write this LC-circuit as a Hamiltonian system in a modular way?*

Storage equations for the components of the LC-circuit:

Inductor 1 $\quad \dot{\varphi}_1 = f_1$ (voltage)

(current) $e_1 = \frac{\partial H_1}{\partial \varphi_1}$

Inductor 2 $\quad \dot{\varphi}_2 = f_2$ (voltage)

(current) $e_2 = \frac{\partial H_2}{\partial \varphi_2}$

Capacitor $\quad \dot{Q} = f_3$ (current)

(voltage) $e_3 = \frac{\partial H_3}{\partial Q}$

If the energy functions H_i are *quadratic*, e.g., $H_3(Q) = \frac{1}{2C}Q^2$, then the elements are *linear*, e.g., voltage over capacitor $= \frac{\partial H_3}{\partial Q} = \frac{Q}{C}$, and similarly for the inductors.

Kirchhoff's voltage and current laws are

$$\begin{pmatrix} -f_1 \\ -f_2 \\ -f_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

Substitution of eqns. of components yields Hamiltonian system

$$\begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{Q} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial \varphi_1} \\ \frac{\partial H}{\partial \varphi_2} \\ \frac{\partial H}{\partial Q} \end{pmatrix}$$

with $H(\varphi_1, \varphi_2, Q) := H_1(\varphi_1) + H_2(\varphi_2) + H_3(Q)$ total energy.

Preliminary conclusions

- The structure matrix J is completely determined by the **interconnection structure** of the system (in this case, Kirchhoff's current and voltage laws).
- Skew-symmetry of J corresponds to the interconnection being **power-conserving**. (Tellegen's theorem for Kirchhoff's laws.)
- There is **no** clear underlying co-tangent bundle or symplectic manifold !
- Building blocks of our theory should be **open** dynamical systems, instead of **closed** dynamical systems (the *systems point of view*).
- Complex Hamiltonian systems are obtained by *interconnecting* open Hamiltonian systems.

The Hamiltonian equations for a **closed** dynamical system

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x), \quad J(x) = -J^T(x), \quad x \in \mathcal{X}$$

is extended to **open** dynamical systems:

$$\begin{array}{rcll} \dot{x} & = & J(x)\frac{\partial H}{\partial x}(x) + g(x)f, & f \in \mathbb{R}^m \\ & & & x \in \mathcal{X} \text{ state space} \\ e & = & g^T(x)\frac{\partial H}{\partial x}(x), & e \in \mathbb{R}^m \end{array}$$

where the external ports defined by the matrix $g(x)$, and $f \in \mathbb{R}^m, e \in \mathbb{R}^m$ are the power-variables at the external ports (open to interconnection to other systems).
By skew-symmetry of J we obtain for any g the *energy-balance*

$$\frac{dH}{dt}(x(t)) = e^T(t)f(t) = \text{ power supplied to the system}$$

Interconnection of two open Hamiltonian systems

$$\dot{x}_i = J_i(x_i)\frac{\partial H_i}{\partial x_i}(x_i) + g_i(x_i)f_i$$

$$x_i \in \mathcal{X}_i, \quad i = 1, 2$$

$$e_i = g_i^T(x)\frac{\partial H_i}{\partial x_i}(x_i)$$

via the *feedback interconnection*
(power-conserving since $f_1 e_1 + f_2 e_2 = 0$!)

$$f_1 = -e_2, \quad f_2 = e_1$$

yields the Hamiltonian system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} J_1(x_1) & -g_1(x_1)g_2^T(x_2) \\ g_2(x_2)g_1^T(x_1) & J_2(x_2) \end{pmatrix}}_{J_{int}(x_1,x_2)} \begin{pmatrix} \frac{\partial H_1}{\partial x_1}(x_1) \\ \frac{\partial H_2}{\partial x_2}(x_2) \end{pmatrix}$$

with state space $\mathcal{X}_1 \times \mathcal{X}_2$, and total Hamiltonian $H_1(x_1) + H_2(x_2)$.

However, this class of Hamiltonian open systems is **not** closed under arbitrary interconnection:

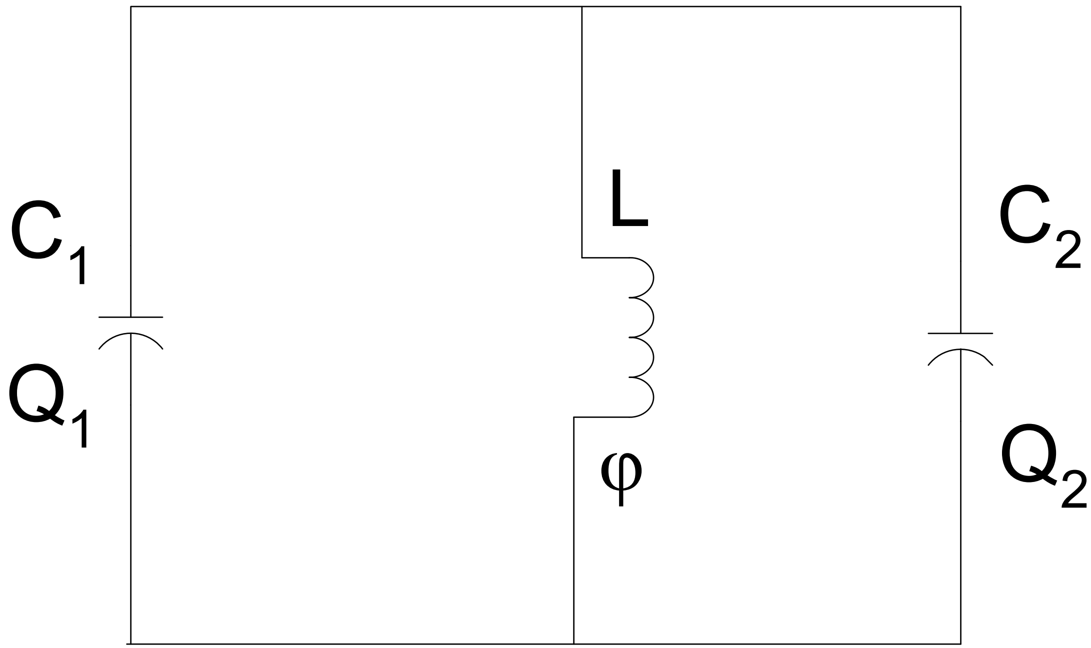


Figure 1: Capacitors and inductors swapped.

Composition leads to algebraic constraints between the state variables; in this case Q_1 and Q_2.

What is the appropriate generalization of the Poisson structure J ?

Answer: **Dirac structures**
('From skew-symmetric *mappings* to skew-symmetric *relations*')

Power is defined by

$$P = e(f) =: < e \mid f >, \quad (f, e) \in \mathcal{V} \times \mathcal{V}^*.$$

where the linear space $\mathcal{V}$ is called the space of *flows* f (e.g. currents), and $\mathcal{V}^*$ the space of *efforts* e (e.g. voltages).

Symmetrized form of power is the indefinite *bilinear form* $\ll, \gg$ on $\mathcal{V} \times \mathcal{V}^*$:

$$\ll (f^a, e^a), (f^b, e^b) \gg \quad := \quad < e^a \mid f^b > + < e^b \mid f^a >,$$

$$(f^a, e^a), (f^b, e^b) \in \mathcal{V} \times \mathcal{V}^*.$$

Definition 1 (Weinstein, Courant, Dorfman) *A (constant) Dirac structure is a subspace*

$$\mathcal{D} \subset \mathcal{V} \times \mathcal{V}^*$$

such that

$$\mathcal{D} = \mathcal{D}^{\perp},$$

where $\perp$ *denotes orthogonal complement with respect to the bilinear form* $\ll, \gg$.

For a *finite-dimensional* linear space $\mathcal{V}$ this is equivalent to

(i) $< e \mid f >= 0$ for all $(f, e) \in \mathcal{D}$,

(ii) $\dim \mathcal{D} = \dim \mathcal{V}$.

Examples

Mathematical

(a) Let $J : \mathcal{V}^* \to \mathcal{V}$ be a skew-symmetric mapping. Then its graph $\{(f, e) \in \mathcal{V} \times \mathcal{V}^* \mid f = Je\}$ is a Dirac structure.

(b) Let $\omega : \mathcal{V} \to \mathcal{V}^*$ be a skew-symmetric mapping. Then graph $\omega \subset \mathcal{V} \times \mathcal{V}^*$ is a Dirac structure.

(c) Let $\mathcal{W} \subset \mathcal{V}$ be a subspace, and let $ann\,\mathcal{W}$ be its annihilating subspace of $\mathcal{V}^*$. Then $\mathcal{W} \times ann\,\mathcal{W} \subset \mathcal{V} \times \mathcal{V}^*$ is a Dirac structure.

Physical

(a) Kirchhoff's laws

(b) Transformers and gyrators

(c) Kinematic pairs

(d) Ideal (workless) constraints

For many systems, especially those with 3-D mechanical components, the interconnection structure will be *modulated* by the energy or geometric variables.

This leads to the notion of *non-constant* Dirac structures on *manifolds*.

Definition 2 *Consider a smooth manifold* M*. A Dirac structure on* M *is a vector subbundle* $\mathcal{D} \subset TM \oplus T^*M$ *such that for every* $x \in M$ *the vector space*

$$\mathcal{D}(x) \subset T_xM \times T_x^*M$$

is a Dirac structure as before.

Geometric definition of a port-Hamiltonian system

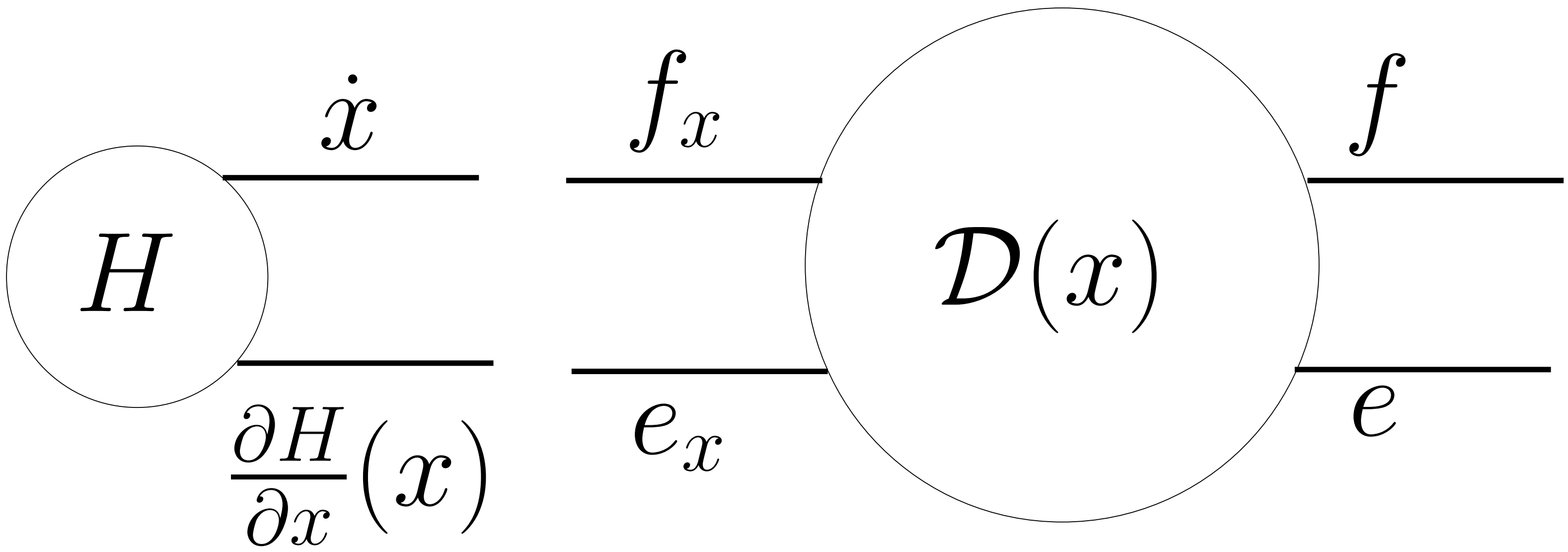


Figure 2: Port-Hamiltonian system

The dynamical system defined by the relations

$$(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), f(t), e(t)) \in \mathcal{D}(x(t)), \quad t \in \mathbb{R}$$

is called a **port-Hamiltonian system**.

So we have generalized from (M, ω, H) to $(\mathcal{X}, \mathcal{D}, \mathcal{F}, \mathcal{H})$.

Particular case is a Dirac structure $\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F} \times \mathcal{F}^*$ given as the graph of the skew-symmetric map

$$\begin{bmatrix} f_x \\ e \end{bmatrix} = \begin{bmatrix} -J(x) & -g(x) \\ g^T(x) & 0 \end{bmatrix} \begin{bmatrix} e_x \\ f \end{bmatrix},$$

leading ($f_x = -\dot{x}, e_x = \frac{\partial H}{\partial x}(x)$) to a Hamiltonian open system as before

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)f, \quad x \in \mathcal{X}, f \in \mathbb{R}^m \\ e &= g^T(x)\frac{\partial H}{\partial x}(x), \qquad\qquad e \in \mathbb{R}^m \end{aligned}$$

In general, the equations of a port-Hamiltonian system are **DAEs**.

Energy-dissipation is included by *terminating* some of the ports by resistive elements

$$f_R = -F(e_R),$$

where the mapping F is such that

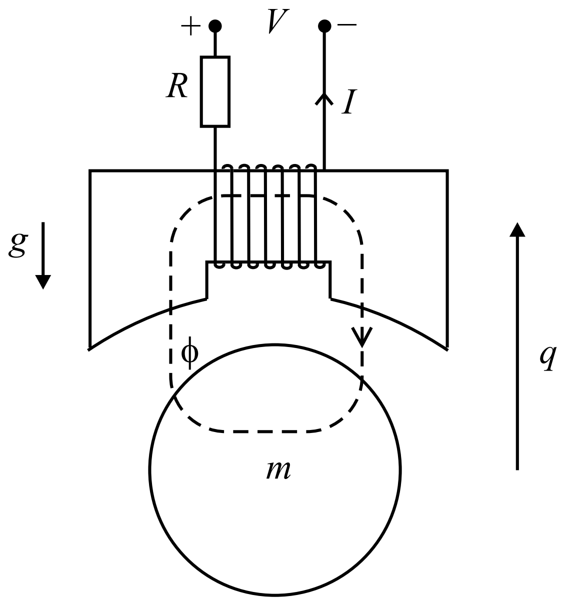
$$e_R^T F(e_R) \geq 0, \quad \text{for all} \quad e_R$$

Then the energy balance $\frac{d}{dt}H = e^T f$ is replaced by

$$\frac{d}{dt}H \leq e^T f$$

Hence, the system is **passive** if $H \geq 0$.
Theory of port-Hamiltonian systems extends theory of passive systems in control theory.

Electro-mechanical system (magnetically levitated ball)



$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \varphi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} V, \quad I = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \varphi} \end{bmatrix}$$

Coupling of electrical and mechanical domain via the Hamiltonian.

Example: Mechanical systems with kinematic constraints

Constraints on the generalized velocities $\dot{q}$:

$$A^T(q)\dot{q} = 0.$$

This leads to *constrained* Hamiltonian equations

$$\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p}(q,p) \\
\dot{p} &= -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda + B(q)f \\
0 &= A^T(q)\frac{\partial H}{\partial p}(q,p) \\
e &= B^T(q)\frac{\partial H}{\partial p}(q,p)
\end{aligned}$$

with $H(q,p)$ total energy, and λ the constraint forces.
Dirac structure is defined by the Poisson structure on T^*Q together with constraints $A^T(q)\dot{q} = 0$ and external force matrix $B(q)$.

Can be extended to general *multi-body systems*.

Jacobi identity and holonomic constraints

There is an important notion of *integrability* of a Dirac structure on a manifold.

Definition 3 (Dorfman, Courant) *A Dirac structure $\mathcal{D}$ on a manifold M is called integrable if*

$$< L_{X_1}\alpha_2 \mid X_3 > + < L_{X_2}\alpha_3 \mid X_1 > + < L_{X_3}\alpha_1 \mid X_2 >= 0$$

for all $(X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3) \in \mathcal{D}$.

For *constant* Dirac structures the integrability condition is automatically satisfied.

The Dirac structure $\mathcal{D}$ defined by the canonical symplectic structure and kinematic constraints $A^T(q)\dot{q} = 0$ satisfies the *integrability condition* if and only if the constraints are *holonomic*; that is, can be integrated to geometric constraints $\phi(q) = 0$.

Special cases; see Dalsmo & vdS for more info.

(a) Let J be a (pseudo-)Poisson structure on M, defining a skew-symmetric mapping $J : T^*M \to TM$. Then graph $J \subset T^*M \oplus TM$ is a Dirac structure.
Integrability is equivalent to the *Jacobi-identity* for the Poisson structure.

(b) Let ω be a (pre-)symplectic structure on M, defining a skew-symmetric mapping $\omega : TM \to T^*M$. Then graph $\omega \subset TM \oplus T^*M$ is a Dirac structure.
Integrability is equivalent to the *closedness* of the symplectic structure.

(c) Let K be a constant-dimensional distribution on M, and let $ann\, K$ be its annihilating co-distribution. Then $K \times ann\, K \subset TM \oplus T^*M$ is a Dirac structure.
Integrability is equivalent to the *involutivity* of distribution K.

Integrability of the Dirac structure is equivalent to the existence of **canonical coordinates**:

If the Dirac structure $\mathcal{D}$ on $\mathcal{X}$ is *integrable* then there exist coordinates (q,p,r,s) for $\mathcal{X}$ such that

$$\mathcal{D}(x) = \{(f_q, f_p, f_r, f_s, e_q, e_p, e_r, e_s) \in T_x\mathcal{X} \times T_x^*\mathcal{X}\}$$

$$\begin{cases} f_q = -e_p, & f_p = e_q \\ \\ f_r = 0, & 0 = e_s \end{cases}$$

Hence the Hamiltonian system corresponding to $\mathcal{D}$ and $H : \mathcal{X} \to \mathbb{R}$ is

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p,r,s) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p,r,s) \\ \dot{r} &= 0 \\ 0 &= \frac{\partial H}{\partial s}(q,p,r,s) \end{aligned}$$

Port-Hamiltonian systems are more than energy-conserving or passive.

For any Dirac structure $\mathcal{D}$ define

$$G_1 \quad := \quad \{f_x \mid \exists e_x, f, e \text{ s.t. } (f_x, e_x, f, e) \in \mathcal{D}\} \quad \subset T_x\mathcal{X}$$

$$P_1 \quad := \quad \{e_x \mid \exists f_x, f, e \text{ s.t. } (f_x, e_x f, e) \in \mathcal{D}\} \quad \subset T_x^*\mathcal{X}$$

The space G_1 expresses the set of admissible flows, and therefore the **Casimir functions**:

• C is a Casimir function iff $\frac{\partial C}{\partial x}(f_x) = 0$ for all $f_x \in G_1$. Indeed, then $\frac{dC}{dt} = \frac{\partial C}{\partial x}(x(t))\dot{x}(t) = 0$ for all solutions.

P_1 determines the set of admissible efforts, and therefore the **algebraic constraints**:

• x should satisfy the equations $dH(x) \in P_1(x)$.

Composition of Dirac structures The *composition* of two finite-dimensional Dirac structures with *partially shared* variables is *again* a Dirac structure:

$$\mathcal{D}_A \subset \mathcal{V}_1 \times \mathcal{V}_1^* \times \mathcal{V}_2 \times \mathcal{V}_2^*$$

$$\mathcal{D}_B \subset \mathcal{V}_2 \times \mathcal{V}_2^* \times \mathcal{V}_3 \times \mathcal{V}_3^*$$

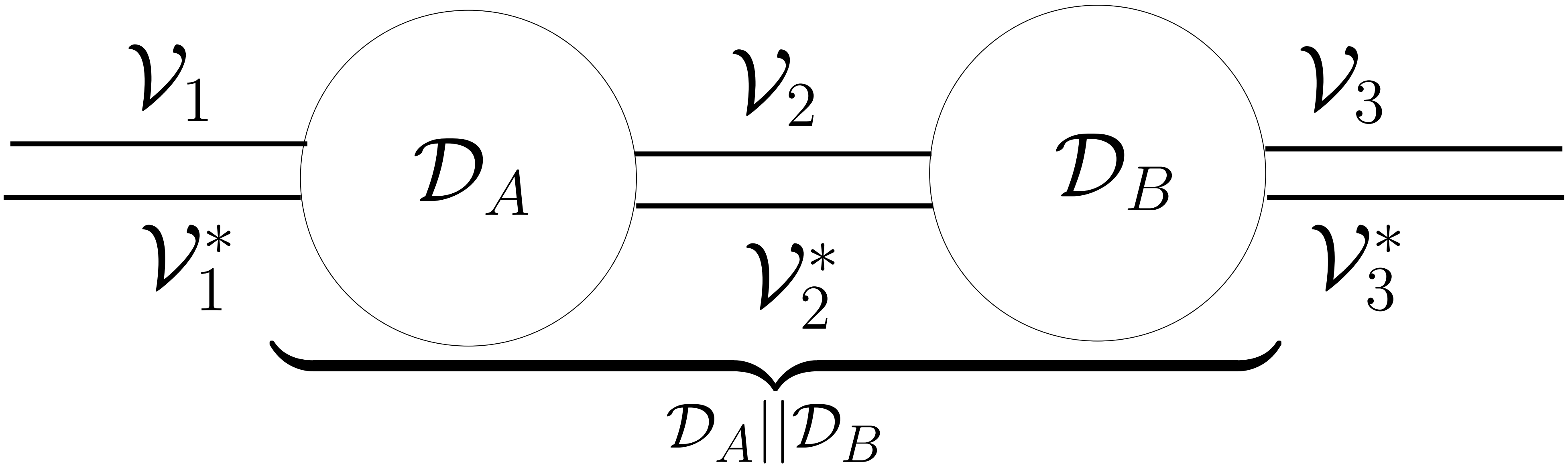


Figure 3: Composed Dirac structure

This implies that the interconnection of port-Hamiltonian systems is *again a port-Hamiltonian system*.

Starting point for control:

Connect the given *plant* port-Hamiltonian system to a to-be-designed *controller* port-Hamiltonian system

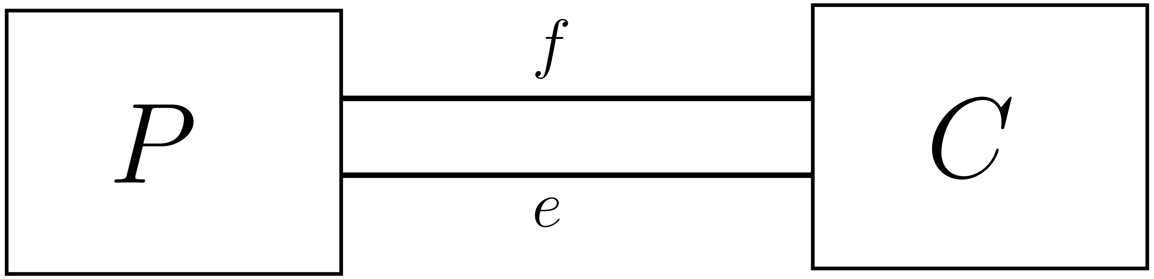


Figure 4: Control by Interconnection

Interconnected system is again a port-Hamiltonian system with total energy $H_{tot} = H_P + H_C$, and composed Dirac structure D_{comp} derived from D_P and D_C.

Control problem 1: Stabilization

By deliberate choice of D_C we may generate new conserved quantities K *Casimir functions* for the interconnected system, and use the candidate Lyapunov function (even for unstable plant systems!)

$$V := H_P + H_C + K$$

Addition of *energy-dissipating elements* may result in asymptotic stabilization.
By additional feedback loops we may introduce *virtual subsystems*: "IDA-PBC control theory" or theory of 'Controlled Lagrangians".

Control problem 2: Energy transfer control

Consider two port-Hamiltonian systems Σ_i in input-state-output form

$$\begin{aligned} \dot{x}_i &= J_i(x_i)\frac{\partial H_i}{\partial x_i} + g_i(x_i)u_i \\ y_i &= g_i^T(x_i)\frac{\partial H_i}{\partial x_i}, \qquad i = 1, 2 \end{aligned}$$

Suppose we want to **transfer** the energy from the port-Hamiltonian system Σ_1 to the port-Hamiltonian system Σ_2, while keeping the total energy $H_1 + H_2$ constant.

This can be done e.g. by the feedback

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & -y_1 y_2^T \\ y_2 y_1^T & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

By skew-symmetry it follows the interconnected system is Hamiltonian, that is $\frac{d}{dt}(H_1 + H_2) = 0$.

However, for the individual energies

$$\frac{d}{dt} H_1 = -y_1^T y_1 y_2^T y_2 = -||y_1||^2 ||y_2||^2 \leq 0$$

implying that H_1 is decreasing. On the other hand,

$$\frac{d}{dt} H_2 = y_2^T y_2 y_1^T y_1 = ||y_2||^2 ||y_1||^2 \geq 0$$

implying that H_2 is increasing at the same rate.

Control problem 3: Impedance control

Given the plant port-Hamiltonian system, **design** a controller port-Hamiltonian system such that the *behavior at the interaction port* of the plant port-Hamiltonian system is a desired one.

Applications e.g. in robotics ('interaction with the environment').

This problem raises the fundamental question:

Given the plant port-Hamiltonian system, and the controller port-Hamiltonian system to be **arbitrarily designed**, what are the **achievable** behaviors of the interconnected plant-controller system at an interaction port of the plant?

Sub-question: What are the achievable Dirac structures ?

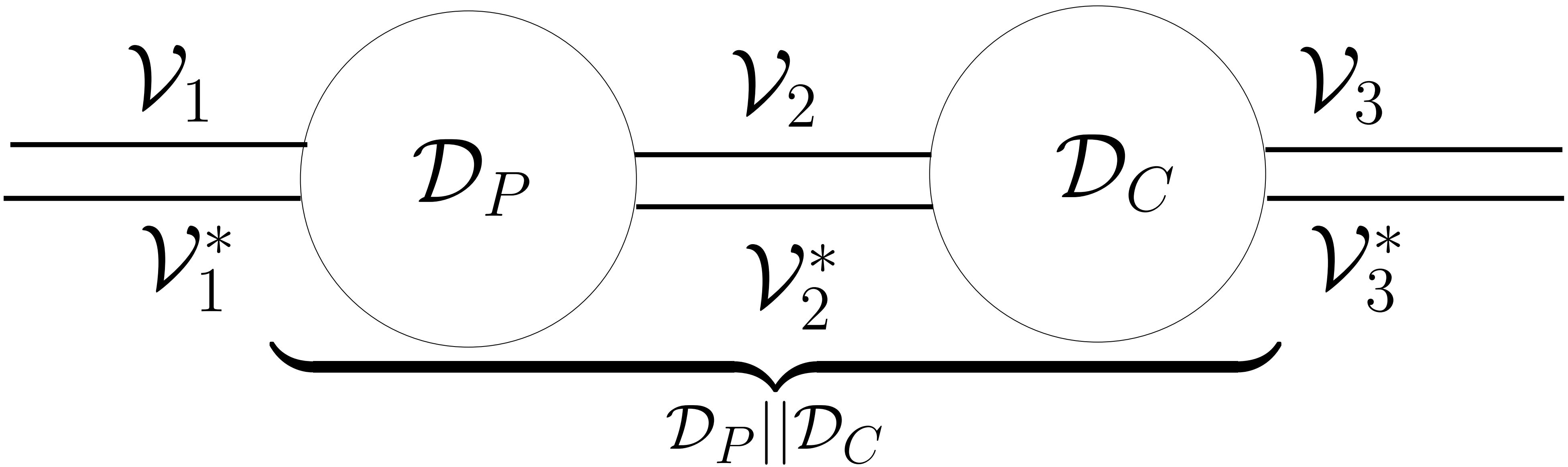


Figure 5: Achievable Dirac structures

$$
\begin{aligned}
\mathcal{D}_P^0 &:= \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathcal{D}_P\} \\
\mathcal{D}_P^\pi &:= \{(f_1, e_1) \mid \exists (f_2, e_2) : (f_1, e_1, f_2, e_2) \in \mathcal{D}_P\} \\
\mathcal{D}^0 &:= \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathcal{D}\} \\
\mathcal{D}^\pi &:= \{(f_1, e_1) \mid \exists (f_3, e_3) : (f_1, e_1, f_3, e_3) \in \mathcal{D}\}
\end{aligned}
$$

Theorem 4 *Given a plant Dirac structure $\mathcal{D}_P$, and desired Dirac structure $\mathcal{D}$. Then there exists a controller Dirac structure $\mathcal{D}_C$ such that $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_C$ if and only if one of the following equivalent conditions is satisfied*

$$\mathcal{D}_P^0 \quad \subset \quad D^0$$

$$D^\pi \quad \subset \quad \mathcal{D}_P^\pi$$

Sufficiency is shown using the controller Dirac structure

$$\mathcal{D}_C := \mathcal{D}_P^* \parallel \mathcal{D}$$

resulting in closed-loop Dirac structure $\mathcal{D}_P \parallel \mathcal{D}_P^* \parallel \mathcal{D} = \mathcal{D}$.
Network/systems interpretation ??

How to incorporate distributed-parameter components ?

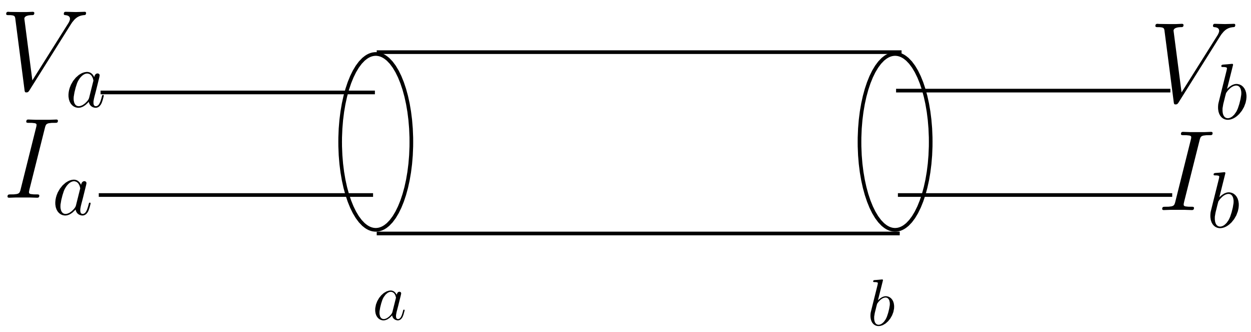


Figure 6: **Transmission line**

Telegrapher's equations define the boundary control system

$$
\begin{aligned}
\frac{\partial Q}{\partial t}(z,t) &= -\frac{\partial}{\partial z} I(z,t) &= -\frac{\partial}{\partial z}\frac{\phi(z,t)}{L(z)} \\
\frac{\partial \phi}{\partial t}(z,t) &= -\frac{\partial}{\partial z} V(z,t) &= -\frac{\partial}{\partial z}\frac{Q(z,t)}{C(z)} \\
V_a(t) &= V(a,t), \quad I_a(t) &= I(a,t) \\
V_b(t) &= V(b,t), \quad I_b(t) &= I(b,t)
\end{aligned}
$$

Defines an infinite-dimensional port-Hamiltonian system

Define *internal* flows (f_E, f_M) and efforts (e_E, e_M):

electric flow	$f_E : [a,b] \to \mathbb{R}$
magnetic flow	$f_M : [a,b] \to \mathbb{R}$
electric effort	$e_E : [a,b] \to \mathbb{R}$
magnetic effort	$e_M : [a,b] \to \mathbb{R}$

together with *external* boundary flows $f = (f_a, f_b)$ and boundary efforts $e = (e_a, e_b)$. Define the *infinite-dimensional Dirac structure*

$$\begin{bmatrix} f_E \\ f_M \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} e_E \\ e_M \end{bmatrix}$$

$$\begin{bmatrix} f_{a,b} \\ e_{a,b} \end{bmatrix} = \begin{bmatrix} e_{E|a,b} \\ e_{M|a,b} \end{bmatrix}$$

This defines an infinite-dimensional Dirac structure on the space of *internal* flows and efforts and *boundary* flows and efforts (use *integration by parts* !).

Substituting (as in the lumped-parameter case)

$$\left.\begin{array}{rcl} f_E & = & -\frac{\partial Q}{\partial t} \\ f_M & = & -\frac{\partial \varphi}{\partial t} \end{array}\right\} f_x = -\dot{x}$$

$$\left.\begin{array}{rcl} e_E & = & \frac{Q}{C} = \frac{\partial \mathcal{H}}{\partial Q} \\ e_M & = & \frac{\varphi}{L} = \frac{\partial \mathcal{H}}{\partial \varphi} \end{array}\right\} e_x = \frac{\partial H}{\partial x}$$

with quadratic energy density

$$\mathcal{H}(Q,\varphi) = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}\frac{\varphi^2}{L}$$

we recover the telegrapher's equations.

This extends e.g. to Maxwell's equations on a domain with boundary,
to flexible beam models,
and to boundary-controlled fluid dynamics.

In all these cases the infinite-dimensional Dirac structure is determined by a set of **conservation laws**.
(Like Kirchhoff's laws in the finite-dimensional circuit case !)

Typical case

$$\frac{\partial \alpha_1}{\partial t} + \frac{\partial \beta_2}{\partial z} = 0$$

$$\frac{\partial \alpha_2}{\partial t} + \frac{\partial \beta_1}{\partial z} = 0$$

with $\beta_i = \frac{\partial \mathcal{H}}{\partial \alpha_i}, \, i = 1, 2$. This defines a port-Hamiltonian system with energy density $\mathcal{H}$ and power-variables at the boundary determined by β_i.

Interconnection of finite- and infinite-dimensional port-Hamiltonian system again defines a port-Hamiltonian system.

- Opportunities for analysis, simulation, and control of mixed ODEs and PDEs.
- Spatial discretization of infinite-dimensional components to finite-dimensional port-Hamiltonian systems using mixed finite-element methods.
- What can we say about well-posedness ? Relations with passivity theory (Staffans et al.). Shock waves in nonlinear case.
- Composition of Dirac structures on Hilbert spaces, and relations with scattering representations (Kurula et al.).

Conclusions

- Port-Hamiltonian systems provide a unified framework for *modeling, analysis, and simulation* of complex multi-physics systems.
- Inclusion of distributed-parameter components.
- Starting point for control: 'control by interconnection', IDA-PBC control. Suggests new control paradigms.
- Port-Hamiltonian systems with variable network topology (power systems, robotic systems, 'embedded systems').

- Extensions to thermodynamic systems and chemical reaction networks.
- Model reduction of port-Hamiltonian systems.
- Merging network (graph) information with dynamics. Dirac structure as a mixture of *algebraic-graph* and *geometric* object.
- Reconciliation with co-tangent bundle point of view and variational calculus.

See proceedings for further info and references.
Also see website www.geoplex.cc of European IST project Geoplex for applications in various domains.