Conservation Laws and Interconnection of Systems

at the occasion of Jan Willems' 70th birthday

Arjan van der Schaft, University of Groningen, the Netherlands

in collaboration with Bernhard Maschke, Lyon, France

Very much inspired by ongoing discussions with Jan: the famous **terminal** versus **port** debate.
Preliminary 'conclusion': interconnection based on **conservation laws** (as in electrical circuits) is captured by terminals. However, 'ports' can be derived from 'terminals', and **are** very useful for multi-physics systems.

Oriented graphs and Kirchhoff's laws[a]

Figure 1: Kirchhoff

An **oriented graph** $\mathcal{G}$ consists of a finite set $\mathcal{V}$ of **vertices** and a finite set $\mathcal{E}$ of directed **edges**, together with a mapping from $\mathcal{E}$ to the set of ordered pairs of $\mathcal{V}$.
Thus to any edge $e \in \mathcal{E}$ there corresponds an ordered pair $(v, w) \in \mathcal{V}^2$ representing the initial vertex v and the final vertex w of this edge.

[a] G. Kirchhoff, *Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der Linearen Verteilung galvanische Ströme geführt wird*, Ann. Phys. Chem. 72, pp. 497–508, 1847.

An oriented graph is specified by its **incidence matrix** B, which is an $\bar{v} \times \bar{e}$ matrix, $\bar{v}$ being the number of vertices and $\bar{e}$ being the number of edges.
The (i,j)-th element b_{ij} equal to 1 if the j-th edge is an edge towards vertex i, equal to -1 if the j-th edge is an edge originating from vertex i, and 0 otherwise.

Given an oriented graph we define its **vertex space** Λ_0 as the real vector space of all functions from $\mathcal{V}$ to $\mathbb{R}$. Clearly Λ_0 can be identified with $\mathbb{R}^{\bar{v}}$.

Furthermore, we define its **edge space** Λ_1 as the vector space of all functions from $\mathcal{E}$ to $\mathbb{R}$. Again, Λ_1 can be identified with $\mathbb{R}^{\bar{e}}$.

In the context of an electrical circuit Λ_1 will be the vector space of **currents** *through* the edges in the circuit. The dual space of Λ_1 will be denoted by Λ^1, and defines the vector space of **voltages** *across* the edges.

The **duality** product $< V|I >= V^T I$ of the vector of currents $I \in \Lambda_1$ with a vector of voltages $V \in \Lambda^1$ is the total **power** over the circuit.

Similarly, the dual space of Λ_0 is denoted by Λ^0 and defines the vector space of **potentials** at the vertices.

The incidence matrix B defines linear map

$$B : \Lambda_1 \to \Lambda_0$$

with adjoint map

$$B^T : \Lambda^0 \to \Lambda^1,$$

called the **co-incidence** operator.

Kirchhoff's **current laws** are given as

$$I \in \ker B,$$

while Kirchhoff's **voltage laws** take the form

$$V \in \operatorname{im} B^T,$$

or equivalently

$$V = B^T \psi$$

for some vector $\psi \in \Lambda^0$ (**potentials**). Hence Kirchhoff's voltage laws express that the voltage distribution V over the edges of the graph corresponds to a potential distribution over the vertices.

Tellegen's theorem immediately follows from Kirchhoff's laws.

Indeed, take any current distribution I satisfying Kirchhoff's current laws $BI = 0$, and any voltage distribution V satisfying Kirchhoff's voltage laws $V = B^T\psi$. Then,

$$V^T I = \psi^T B I = 0$$

Something stronger holds: define on the space of voltages and currents the indefinite inner product

$$<< (V_1, I_1), (V_2, I_2) >>= V_1^T I_2 + V_2^T I_1$$

Then

$$D := \{(I, V) \mid BI = 0, V \in \operatorname{im} B^T\}$$

satisfies $D = D^\perp$, thus the subspace of allowed currents and voltages is a **Dirac structure**.

Kirchhoff's laws for open graphs

An **open graph** $\mathcal{G}$ is obtained from an ordinary graph by selecting a subset $\mathcal{V}_b \subset \mathcal{V}$ of **boundary vertices**. The remaining subset consists of the **internal vertices** of the open graph.

Decomposing the incidence matrix B as $\begin{bmatrix} B_i \\ B_b \end{bmatrix}$ we arrive at

$$B_i I = 0, \quad B_b I = -I_b, \quad \text{KCL}$$

with I_b belonging to the vector space Λ_b of functions from the boundary vertices $\mathcal{V}_b$ to $\mathbb{R}$ (which is identified with $\mathbb{R}^{\bar{v}_b}$, with $\bar{v}_b$ the number of boundary vertices).

Alternatively, open graphs can be defined by attaching 'one-sided open edges' (properly called *leaves*) to every boundary vertex in $\mathcal{V}_b$. Then I_b are the currents through these leaves, cf. Jan's CSM paper.

Kirchhoff's voltage laws (KVL) become

$$V = B^T \psi = B_i^T \psi_i + B_b^T \psi_b, \quad \text{KVL}$$

where ψ_i denotes the potentials at the internal and ψ_b the potentials at the boundary vertices. Note that $\psi_b \in \Lambda^b$ (dual of the space of boundary currents Λ_b).

This results in the **Kirchhoff behavior** for an open graph $\mathcal{G}$:

$$\begin{aligned} B_K(\mathcal{G}) \quad := \quad & \{(I, V, I_b, \psi_b) \in \Lambda_1 \times \Lambda^1 \times \Lambda_b \times \Lambda^b \mid \\ & B_i I = 0, B_b I = -I_b, \, \exists \psi_i \text{ s.t. } V = B_i^T \psi_i + B_b^T \psi_b\} \end{aligned} \tag{1}$$

$B_K(\mathcal{G})$ is a **Dirac structure**. In particular,

$$V^T I = \psi_i^T B_i I + \psi_b^T B_b I = -\psi_b^T I_b \tag{2}$$

and thus the total power is equal to zero.

It is a well-known property of any incidence matrix B that

$$\mathbb{1}^T B = 0$$

where $\mathbb{1}$ denotes the vector with all components equal to 1. It follows that

$$0 = \mathbb{1}^T B I = \mathbb{1}^T B_b I = -\mathbb{1}_b^T I_b = -\sum_{v_b} I_{v_b}$$

Hence for each $(I, V, I_b, \psi_b) \in \mathcal{B}_K(\mathcal{G})$ it holds that

$$\mathbb{1}_b^T I_b = 0$$

while for any constant $c \in \mathbb{R}$

$$(I, V, I_b, \psi_b + c\mathbb{1}_b) \in \mathcal{B}_K(\mathcal{G})$$

See again Jan's Control Systems Magazine paper.

This implies that we may restrict the dimension of the space of external variables $\Lambda_b \times \Lambda^b$ by two. Indeed, we may define

$$\Lambda_{b\mathrm{red}} := \{I_b \in \Lambda_b \mid I_b \in \ker \mathbb{1}_b^T\}$$

and its dual space

$$\Lambda^b_{\mathrm{red}} := (\Lambda_{b\mathrm{red}})^* = \Lambda^b / \operatorname{im} \mathbb{1}_b$$

It is straightforward to show that the Kirchhoff behavior $\mathcal{B}_K(\mathcal{G})$ reduces to a linear subspace of the reduced space $\Lambda_1 \times \Lambda^1 \times \Lambda_{b\mathrm{red}} \times \Lambda^b_{\mathrm{red}}$, which is also a Dirac structure. A circuit interpretation of this reduction is that we may consider one of the boundary vertices, say the first one, to be the reference ground vertex, and that we may reduce the vector of boundary potentials $\psi_b = (\psi_{b1}, \cdots, \psi_{b\bar{v}_b})$ to a vector of voltages $(\psi_{b2} - \psi_{b1}, \cdots, \psi_{b\bar{v}_b} - \psi_{b1})$. A graph-theoretical interpretation is that instead of the incidence matrix B we consider the *restricted* incidence matrix.

For a graph $\mathcal{G}$ with more than one connected component the above holds for each connected component.

A complementary view is that we may **close** an open graph $\mathcal{G}$ to an ordinary graph $\bar{\mathcal{G}}$.

If $\mathcal{G}$ is connected then this is done by adding one virtual ('ground') vertex v_0, and virtual edges from this virtual vertex to every boundary vertex $v_b \in \mathcal{V}_b$. To the virtual vertex v_0 we may associate an arbitrary potential ψ_{v_0} (a ground-potential), and we may rewrite the power balance as

$$-\sum_{v_b}(\psi_{v_b} - \psi_{v_0})I_{v_b} = -\sum_{v_b} V_{v_b} I_{v_b}$$

where $V_{v_b} := \psi_{v_b} - \psi_{v_0}$ and I_{v_b} denotes the voltage across and the current through the virtual edge towards the boundary vertex v_b.

If the open graph $\mathcal{G}$ consists of more than one component, then add a virtual vertex to component containing boundary vertices.

Interconnection of open graphs

Consider two open graphs $\mathcal{G}^j$ with incidence matrices

$$B^j = \begin{bmatrix} B_i^j \\ B_b^j \end{bmatrix}, j = 1, 2$$

Interconnection is done by identifying some of their boundary vertices, and equating (up to a minus sign) the corresponding boundary potentials and currents.

If *all* boundary vertices are identified, a closed graph results with vertices $\mathcal{V}_i^1 \cup \mathcal{V}_i^2 \cup \mathcal{V}$, where $\mathcal{V}_i := \mathcal{V}_b^1 = \mathcal{V}_b^2$ denotes the set of shared boundary vertices. The incidence matrix B of this interconnected graph is given as

$$B = \begin{bmatrix} B_i^1 & 0 \\ 0 & B_i^2 \\ B_b^1 & B_b^2 \end{bmatrix},$$

Port-Hamiltonian dynamics on open graphs

RLC circuits: specify for all edges constitutive relations from

- *Resistor*: Relation between I_e and V_e such that $V_e I_e \leq 0$. A linear resistor is specified by $V_e = -R_e I_e$ with $R_e \geq 0$.
- *Capacitor*: Define an energy variable Q_e together with a function $H_{Ce}(Q_e)$ denoting the electric energy:

$$\dot{Q}_e = -I_e, \quad V_e = \frac{dH_{Ce}}{dQ_e}(Q_e)$$

- *Inductor*: Specify the magnetic energy $H_{Le}(\Phi_e)$, where Φ_e denotes the magnetic flux linkage:

$$\dot{\Phi}_e = -V_e, \quad I_e = \frac{dH_{Le}}{d\Phi_e}(\Phi_e)$$

Substituting these constitutive relations into $\mathcal{B}_K(\mathcal{G})$ one obtains a **port-Hamiltonian system**.

There are other possibilities to define a port-Hamiltonian system on open graphs.

For example, consider the relations

$$V = -B^T \begin{bmatrix} \psi_i \\ \psi_b \end{bmatrix}, \quad \begin{bmatrix} \xi_i \\ \xi_b \end{bmatrix} = BI, \quad \begin{bmatrix} \xi_i \\ \xi_b \end{bmatrix} \in \Lambda_0,$$

together with the usual inductor relations for each edge e:

$$\dot{\Phi}_e = -V_e, \quad I_e = \frac{dH_{Le}}{d\Phi_e}(\Phi_e)$$

and 'resistive' relations for all the vertices:

$$\psi_i = -R\xi_i, \quad R = R^T \geq 0$$

See later on !

This leads to a different type of port-Hamiltonian dynamics:

$$
\begin{aligned}
\dot{\Phi} &= -V = B^T \begin{bmatrix} \psi_i \\ \psi_b \end{bmatrix} = -B_i^T R B_i I + B_b^T \psi_b \\
&= -B_i^T R B_i \tfrac{dH_L}{d\Phi}(\Phi) + B_b^T \psi_b \\
\xi_b &= B_b \tfrac{dH_L}{d\Phi}(\Phi)
\end{aligned}
$$

which can be regarded as 'some kind of RL circuit',
with external (input and output) variables ψ_b, ξ_b.

Extension to k-complexes

An oriented graph with incidence matrix B is a typical example of what is called in algebraic topology a 1-**complex**. Indeed, the sequence

$$\Lambda_1 \xrightarrow{B} \Lambda_0 \xrightarrow{\mathbb{1}} \mathbb{R}$$

satisfies the property $\mathbb{1} \circ B = 0$.

In general, a k-complex Λ is specified by a **sequence** of real linear spaces $\Lambda_0, \Lambda_1, \cdots, \Lambda_k$, together with a sequence of incidence operators

$$\Lambda_k \xrightarrow{\partial_k} \Lambda_{k-1} \xrightarrow{\partial_{k-1}} \cdots \Lambda_1 \xrightarrow{\partial_1} \Lambda_0$$

with the property that

$$\partial_{j-1} \circ \partial_j = 0,\ j = 2, \cdots, k.$$

The vector spaces Λ_j, $j = 0, 1 \cdots, k$, are called the spaces of j-**chains**.

Each Λ_j is generated by a finite set of j-cells (like edges and vertices for graphs) in the sense that Λ_j is the set of functions from the j-cells to $\mathbb{R}$.

A typical example of a k-complex is the triangularization of a k-dimensional manifold, with the j-cells, $j = 0, 1, \cdots, k$, being the sets of vertices, edges, faces, etc..

Consider the triangularization of a 2-dimensional sphere by a tetrahedron with 4 faces, 6 edges, and 4 vertices.

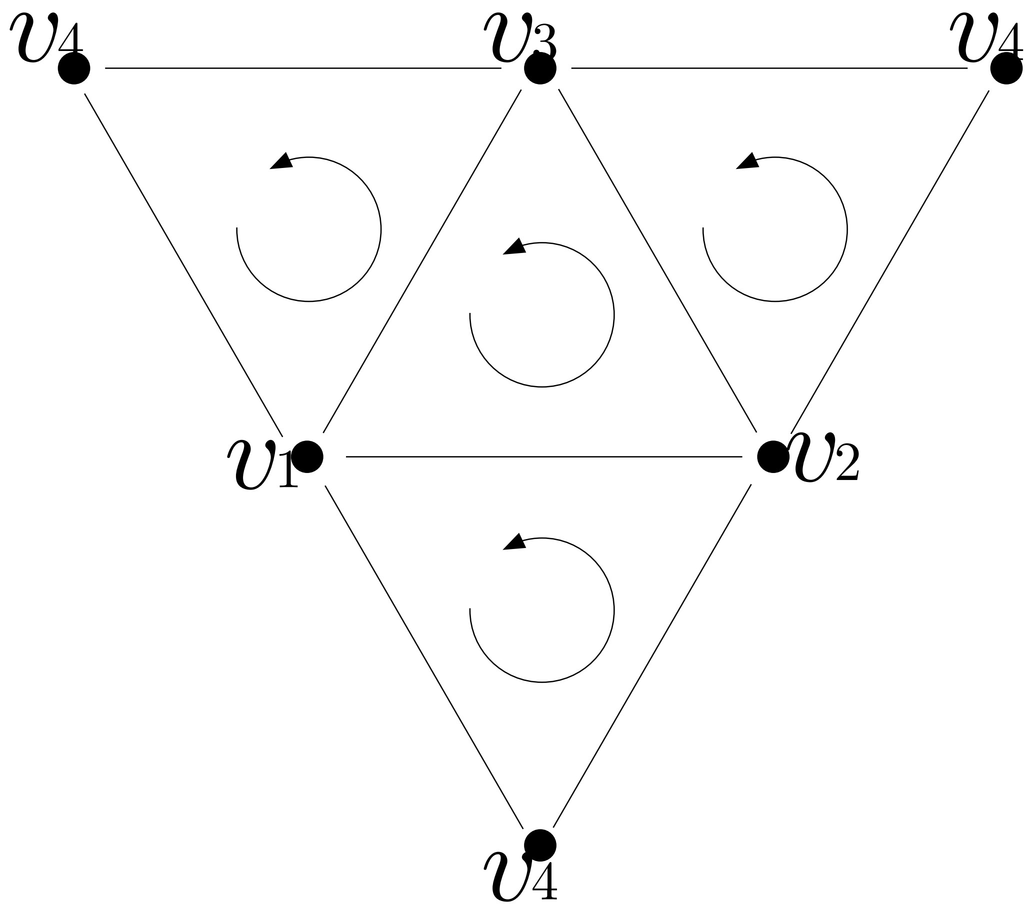

Figure 2: Tetrahedron triangularizing a sphere

The matrix representation of the incidence operator ∂_2 (from the **faces** of the tetrahedron to its **edges**) is

	$< v_1v_2v_3 >$	$< v_1v_3v_4 >$	$< v_1v_4v_2 >$	$< v_2v_4v_3 >$
$< v_1v_2 >$	1	0	-1	0
$< v_1v_3 >$	-1	1	0	0
$< v_1v_4 >$	0	-1	1	0
$< v_2v_3 >$	1	0	0	-1
$< v_2v_4 >$	0	0	-1	1
$< v_3v_4 >$	0	1	0	-1

where the expressions $< v_1v_2v_3 >, \ldots$ denote the faces (with corresponding orientation), and $< v_1v_2 >, \ldots$ are the edges.

The matrix representation of the incidence operator ∂_1 (from **edges** to **vertices**) is given as

	$< v_1 v_2 >$	$< v_1 v_3 >$	$< v_1 v_4 >$	$< v_2 v_3 >$	$< v_2 v_4 >$	$< v_3 v_4 >$
$< v_1 >$	-1	-1	-1	0	0	0
$< v_2 >$	1	0	0	-1	-1	0
$< v_3 >$	0	1	0	1	0	-1
$< v_4 >$	0	0	1	0	1	1

It can be verified that $\partial_1 \circ \partial_2 = 0$.

Denoting the dual linear spaces by Λ^j, $j = 0, 1 \cdots, k$, we obtain the dual sequence

$$\Lambda^0 \stackrel{d_1}{\longrightarrow} \Lambda^1 \stackrel{d_2}{\longrightarrow} \Lambda^2 \cdots \Lambda^{k-1} \stackrel{d_k}{\longrightarrow} \Lambda^k$$

where the adjoint maps d_j, $j = 0, 1 \cdots, k$, satisfy the analogous property

$$d_j \circ d_{j-1} = 0, \; j = 2, \cdots, k.$$

The elements of Λ^j are called j-**cochains**.

Open k-complexes

Split the $(k-1)$-cells into **boundary** and **internal** cells. Denote the linear space of functions from the boundary $(k-1)$-cells to $\mathbb{R}$ by $\Lambda_b \subset \Lambda_{k-1}$, with dual space denoted as Λ^b. Decompose correspondingly $\partial_k : \Lambda_k \to \Lambda_{k-1}$ as

$$\partial_k = \begin{bmatrix} \partial_k^i \\ \partial_k^b \end{bmatrix},$$

with adjoint mapping $d_k = d_k^i + d_k^b$. Kirchhoff's **voltage laws** are

$$\beta = d_k \psi = d_k^i \psi_i + d_k^b \psi_b,$$

where ψ_b is the vector of 'potentials' at the boundary $(k-1)$-cells. Kirchhoff's **current laws** become

$$\partial_k^i \alpha = 0, \quad \partial_k^b \alpha = -\alpha_b$$

where α_b denotes the vector of boundary 'currents'.

By computing the total power we obtain

$$< \beta \mid \alpha >_k = < d_k \psi \mid \alpha >_k = < d_k^i \psi_i + d_k^b \psi_b \mid \alpha >_k =$$

$$< \psi_i \mid \partial_k^i \alpha >_k + < \psi_b \mid \partial_k^b \alpha >_k = - < \psi_b \mid \alpha_b >_{k-1}$$

The space of boundary variables $(\alpha_b, \psi_b) \in \Lambda_b \times \Lambda^b$ describes the **distributed terminals** of the open k-complex.

It is shown that the Kirchhoff behavior of an open k-complex Λ defined as

$$\begin{aligned} \mathcal{B}_K(\Lambda) \quad := \quad & \{(\alpha, \beta, \alpha_b, \psi_b) \in \Lambda_k \times \Lambda^k \times \Lambda_b \times \Lambda^b \mid \\ & \partial_k^i \alpha = 0,\ \partial_k^b = -\alpha_b,\ \exists \psi_i \text{ s.t. } \beta = d_k^i \psi_i + d_k^b \psi_b\} \end{aligned}$$

is again a **Dirac structure**.

Analogously to graphs, Kirchhoff current laws for open k-complexes imply certain constraints on the boundary 'currents' α_b. Indeed, by the fact that

$$\partial_{k-1} \circ \partial_k = 0$$

it follows that

$$\partial_{(k-1)b}\alpha_b = 0,$$

where $\partial_{(k-1)b}$ denotes the $(k-1$-th incidence operator restricted to $\Lambda_b \subset \Lambda_{k-1}$.

Dually, we may add to any ψ_b an arbitrary element in

$$\operatorname{im} d_{(k-1)b}$$

As in the case of graphs, this allows us to *reduce* the Kirchhoff behavior, or, alternatively, to *close* the open k-complex by completing the open k-complex Λ by an additional set of $(k-1)$-cells and k-cells.

Port-Hamiltonian systems on open k-complexes

One possibility:
On the k-complex Λ, with $\partial_k : \Lambda_k \to \Lambda_{k-1}$ and $d_k : \Lambda^{k-1} \to \Lambda^k$, define the following relations

$$f_x = -d_k e, \qquad f_x \in \Lambda^k,\, e \in \Lambda^{k-1}$$

$$f = \partial_k e_x, \qquad e_x \in \Lambda_k,\, f \in \Lambda_{k-1}$$

It is checked that this defines a Dirac structure

$$\mathcal{D} \subset \Lambda^k \times \Lambda_k \times \Lambda^{k-1} \times \Lambda_{k-1}$$

In particular

$$< f_x \mid e_x > + < e \mid f >= 0$$

Associate now to every k-cell an **energy storage**, leading to

$$\dot{x} = -f_x, \quad e_x = \frac{\partial H}{\partial x}(x), \quad x \in \Lambda^k$$

with $H(x)$ the total stored energy, and $x \in \Lambda^k$ the vector of energy variables.

Furthermore, associate to every $(k-1)$-cell a (linear) **resistive** relation, leading to

$$e = -Rf, \quad R = R^T \geq 0$$

This yields the port-Hamiltonian dynamics

$$\dot{x} = d_k e = -d_k\, R\, f = -d_k\, R\, \partial_k \frac{\partial H}{\partial x}(x), \quad x \in \Lambda^k \tag{3}$$

For an *open* complex with boundary $(k-1)$-cells the definition is modified as follows. Consider

$$
f_x = -d_k \begin{bmatrix} e \\ e_b \end{bmatrix}, \qquad f_x \in \Lambda^k, \begin{bmatrix} e \\ e_b \end{bmatrix} \in \Lambda^{k-1}, \, e_b \in \Lambda^b
$$

$$
\begin{bmatrix} f \\ f_b \end{bmatrix} = \partial_k e_x, \qquad e_x \in \Lambda_k, \begin{bmatrix} f \\ f_b \end{bmatrix} \in \Lambda_{k-1}, \, f_b \in \Lambda_b
$$

with f_b, e_b corresponding to the *boundary* $(k-1)$-cells. Imposing the same storage and resistive relations we arrive at

$$
\dot{x} = -d_k^r \, R \, \partial_k^r \frac{\partial H}{\partial x}(x) + d_k^b e_b
$$

$$
f_b = \partial_k^b \frac{\partial H}{\partial x}(x)
$$

This is a port-Hamiltonian system with inputs e_b and outputs f_b.

Typical example: **Diffusion systems**

Consider a diffusion system on a 3-dimensional spatial domain $Z \subset \mathbb{R}^3$ with smooth boundary ∂Z. The state is described by a smooth map $x : Z \to \mathbb{R}$.

$$
\begin{aligned}
f_x &= \operatorname{div} e \\
f &= \operatorname{grad} e_x \\
f_b &= -e \cdot n \quad \text{on } \partial Z \\
e_b &= e_x \quad \text{on } \partial Z
\end{aligned}
$$

where n denotes the normal vector to ∂Z. This defines a Dirac structure on the space of variables

$$(f_x, e_x, f, e, f_b, e_b)$$

where f_x, e_x, are mappings from Z to $\mathbb{R}$, f, e are mappings from Z to $\mathbb{R}^3$, and f_b and e_b are functions from ∂Z to $\mathbb{R}$.

Next, consider the constitutive relations for the energy storage

$$\frac{\partial}{\partial t} x(z,t) = -f_x(z), \quad e_x(z) = \frac{\partial H}{\partial x}(x(z))$$

for some energy density $H(x(z))$ (and Hamiltonian $\mathcal{H}$ being the integral of H over Z.). Furthermore, power dissipation is

$$e(z) = -R(z) f(z), \quad R(z) = R^T(z) \geq 0$$

This yields the **infinite-dimensional** port-Hamiltonian system

$$\tfrac{\partial}{\partial t} x(z,t) \quad = \quad \mathrm{div}[R(z) \mathrm{grad} \tfrac{\partial H}{\partial x}(x(z,t))]$$

plus the boundary control conditions on ∂Z:

$$f_b(z,t) = -R(z) \mathrm{grad} \frac{\partial H}{\partial x}(x(z,t)) \cdot n, \quad e_b(z,t) = \frac{\partial H}{\partial x}(x(z,t))$$

These equations imply the energy balance

$$\frac{d\mathcal{H}}{dt} = -\int_Z e_x f_x = -\int_Z f^T R(z) f dz + \int_{\partial Z} e_b f_b$$

Discretized diffusion equations

Consider a triangularization of the spatial domain Z:

$$\underset{\textbf{tetrahedra}}{\Lambda_3} \overset{\partial_3}{\longrightarrow} \underset{\textbf{faces}}{\Lambda_2} \overset{\partial_2}{\longrightarrow} \underset{\textbf{edges}}{\Lambda_1} \overset{\partial_1}{\longrightarrow} \underset{\textbf{vertices}}{\Lambda_0} \overset{\partial_0}{\longrightarrow} \mathbb{R}$$

Define the Dirac structure, and constitutive relations

$$\begin{aligned} f_x &= d_3 e + d_{3b} e_b, \quad f_x \in \Lambda^3, e \in \Lambda^2 \\ f &= -\partial_3 e_x, \quad e_x \in \Lambda_3, f \in \Lambda_2 \\ f_b &= \partial_{3b} e_x \end{aligned}$$

$$\begin{aligned} \dot{x} &= -f_x, \quad e_x = \frac{\partial H}{\partial x}(x), \quad x \in \Lambda^3 \\ e &= -Rf, \quad R = R^T \geq 0 \end{aligned}$$

This leads to the finite-dimensional port-Hamiltonian system

$$\begin{aligned} \dot{x} &= -d_3 R \partial_3 \frac{\partial H}{\partial x}(x) + d_{3b} e_b \\ e_b &= \partial_{3b} \frac{\partial H}{\partial x}(x) \end{aligned}$$

Conclusions

- Going back to Kirchhoff: open graphs. Typical example of: Get the physics **and** the mathematics right before doing analysis and computation.
- Port-Hamiltonian systems on graphs.
- From graphs to k-complexes. Discretizing spatial domains by k-complexes.
- Extending Kirchhoff's laws to open k-complexes. Definitions of port-Hamiltonian dynamics on open k-complexes.
- Compare with structure preserving discretization of port-Hamiltonian PDE systems (by mixed finite element methods). Example: diffusion systems.
- Outlook: Generalization of network dynamics to dynamics on k-complexes.