On Balancing of Passive and port-Hamiltonian Systems

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Main question;

Network modeling of complex physical systems, or spatial discretization of distributed-parameter physical systems, usually leads to high-dimensional models. For analysis and control we would like to approximate these models by low-dimensional models, **retaining** the main physical characteristics, such as energy conservation and other conservation laws (mass conservation, charge conservation, etc.).

Even more, we would like to replace a high-dimensional component in a complex system by a lower-dimensional approximation, while retaining the same physical structure (e.g. by replacing a high-dimensional mass-spring system by a low-dimensional one.)

We start with the question of retaining passivity (energy conservation), and then continue with other conserved quantities as captured by the port-Hamiltonian formulation.

Contents

Consider a nonlinear system

$$\Sigma : \begin{array}{rcl} \dot{x} & = & a(x) + b(x)u \\ \\ y & = & c(x) + d(x)u \end{array}$$

where $u, y \in \mathbb{R}^m$, and $x \in \mathbb{R}^n$ are local coordinates for an n-dimensional state space manifold $\mathcal{X}$. Assume the existence of an *equilibrium* x_0, that is, $a(x_0) = 0, c(x_0) = 0$.
Σ is called *lossless* if there exists a *storage function* $H : \mathcal{X} \to \mathbb{R}$ with $H(x_0) = 0$ and $H(x) \geq 0$ for every $x \neq x_0$, such that (Willems, 1972)

$$H(x(t_2)) - H(x(t_1)) = \int_{t_1}^{t_2} u^T(t)y(t)dt$$

for all solutions $(u(\cdot), x(\cdot), y(\cdot))$ and all time instants $t_1 \leq t_2$.
The system is *passive* if $=$ is replaced by $\leq$.
'Lossless' is usually an idealization of 'passive'.

First concentrate on lossless systems.

If H is *differentiable* then the property of being lossless is equivalent to:

$$\begin{aligned} \frac{\partial^T H}{\partial x}(x)a(x) &= 0 \\ c(x) &= b^T(x)\frac{\partial H}{\partial x}(x) \\ d(x) &= -d^T(x) \end{aligned}$$

If additionally H is *positive definite*, that is, $H(x) > 0$ for every $x \neq x_0$, then it immediately follows that x_0 is *stable*, but *not* asymptotically stable.

Remark 1 *The linear system*

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

with equilibrium $x_0 = 0$ *is lossless if there exists a quadratic storage function* $H(x) = \frac{1}{2}x^T Q x$ *with* $Q = Q^T \geq 0$ *satisfying*

$$A^T Q + QA = 0, \quad C = B^T Q, \quad D = -D^T$$

Standard **open-loop balancing** for a nonlinear systems is based on computing the observability function

$$O(x) := \int_0^\infty \frac{1}{2} \parallel y(t) \parallel^2 dt$$

where $u = 0$ and $x(0) = x$. This presumes *asymptotic stability*. Next compute the *controllability function*

$$C(x) := \inf_u \int_{-\infty}^0 \frac{1}{2} \parallel u(t) \parallel^2 dt$$

for final condition $x(0) = x$.

In the **linear** case

$$\begin{array}{rcll} O(x) & = & \frac{1}{2}x^T M x, & \qquad A^T M + MA = -C^T C \\ C(x) & = & \frac{1}{2}x^T W^{-1} x, & \qquad AW + WA^T = -BB^T \end{array}$$

M is called the **observability Gramian** and W is called the **controllability Gramian**.

Balancing in the linear case consists of finding linear state coordinates $x = (x_1, x_2, \cdots, x_n)$ such that

$$M = W = \text{diag } (\sigma_1, \sigma_2, \cdots, \sigma_n)$$

with

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$$

This means that the **last** state components, when compared to the **first** ones, are **BOTH** more difficult to observe and to reach. The numbers $\sigma_1, \sigma_2, \cdots, \sigma_n$ are equal to the singular values of the Hankel map defined by the linear system.

Remark 2 *Note that in balanced coordinates the observability and controllability functions are*

$$\begin{aligned} O(x) &= \tfrac{1}{2}(\sigma_1 x_1^2 + \sigma_2 x_2^2 + \cdots + \sigma_n x_n^2) \\ C(x) &= \tfrac{1}{2}(\tfrac{x_1^2}{\sigma_1} + \tfrac{x_2^2}{\sigma_2} + \cdots + \tfrac{x_n^2}{\sigma_n}) \end{aligned}$$

Since a lossless system is never asymptotically stable, standard open-loop balancing can**not** be applied.

However, we may switch to the so-called *scattering representation* Σ_s of Σ, which is obtained by the following transformation of the external variables (inputs and outputs) u and y:

$$\begin{array}{rcl} v & = & \frac{1}{\sqrt{2}}(u+y) \\ z & = & \frac{1}{\sqrt{2}}(-u+y) \end{array}$$

with inverse

$$\begin{array}{rcl} u & = & \frac{1}{\sqrt{2}}(v-z) \\ y & = & \frac{1}{\sqrt{2}}(v+z) \end{array}$$

Substitution of these expressions into Σ (assume for simplicity $d(x) = 0$) yields the scattering representation Σ_s

$$\Sigma_s : \begin{array}{rcl} \dot{x} & = & a(x) - b(x)c(x) + \sqrt{2}b(x)v \\ \\ z & = & \sqrt{2}c(x) - v \end{array}$$

which can be regarded as an input-state-output system with input v (the 'incoming wave') and output z (the 'outgoing wave').

The following equalities relate the *power variables* u, y to the *wave variables* v, z:

$$\begin{array}{rcl} \frac{1}{2} \parallel v \parallel^2 - \frac{1}{2} \parallel z \parallel^2 & = & u^T y \\ \frac{1}{2} \parallel u \parallel^2 + \frac{1}{2} \parallel y \parallel^2 & = & \parallel z \parallel^2 + u^T y = \parallel v \parallel^2 - u^T y \\ \parallel v \parallel^2 + \parallel z \parallel^2 & = & \parallel u \parallel^2 + \parallel y \parallel^2 \ \ (\textit{parallelogram identity}) \end{array}$$

The first equality implies the following characterization of losslessness in terms of the scattering representation Σ_s:

$$H(x(t_2)) - H(x(t_1)) = \int_{t_1}^{t_2} \frac{1}{2} \parallel v(t) \parallel^2 - \frac{1}{2} \parallel z(t) \parallel^2 dt$$

for all solution trajectories $(v(\cdot), x(\cdot), z(\cdot))$.
We may regard the term $\frac{1}{2} \parallel v(t) \parallel^2$ as the incoming power associated to the incoming wave v, and $\frac{1}{2} \parallel z(t) \parallel^2$ as the outgoing power corresponding to the outgoing wave z.

Generally, x_0 becomes an *asymptotically stable* equilibrium for Σ_s with $v = 0$. Indeed, if H is positive definite then for $v = 0$

$$\frac{d}{dt} H(x(t)) = -\frac{1}{2} \parallel z(t) \parallel^2 = - \parallel y(t) \parallel^2$$

ensuring asymptotic stability if the system Σ_s for $v = 0$ is *zero-state detectable* (with x_0 representing the zero-state). Similarly, the *time-reversed* system Σ_s for $z = 0$ satisfies

$$\frac{d}{d(-t)} H(x(t)) = -\frac{1}{2} \parallel v(t) \parallel^2 = - \parallel y(t) \parallel^2$$

Assumption 3 *The equilibrium* x_0 *is globally asymptotically stable for* Σ_s *with* $v = 0$*, and globally asymptotically stable for the time-reversed system* Σ_s *with* $z = 0$*.*

Remark 4 *For a linear system* Σ *with* $D = 0$ *the scattering representation reduces to*

$$\begin{aligned} \dot{x} &= (A - BC)x + \sqrt{2}Bv \\ z &= \sqrt{2}Cx - v \end{aligned}$$

and the standing assumption is satisfied if and only if the pair (C, A) *is detectable.*

Open-loop balancing of the scattering representation Σ_s

This involves the computation of the *observability function*

$$O_s(x) := \int_0^\infty \frac{1}{2} \parallel z(t) \parallel^2 dt$$

where $v = 0$ and the integral is taken with initial condition $x(0) = x$. Because $\frac{1}{2} \parallel z(t) \parallel^2$ is the outgoing power, the observability function $O_s(x)$ equals the *outgoing physical energy*.

For a lossless system Σ_s we obtain

$$H(x(T)) - H(x(0)) = \int_0^T \frac{1}{2} \parallel v(t) \parallel^2 - \frac{1}{2} \parallel z(t) \parallel^2 dt$$

and since $H(x_0) = 0$ it follows that $O_s(x) = H(x)$.

Secondly, open-loop balancing involves the computation of the *controllability function*

$$C_s(x) := \inf_v \int_{-\infty}^{0} \frac{1}{2} \parallel v(t) \parallel^2 dt$$

where the infimum is taken over all input functions $v : (-\infty, 0) \to \mathbb{R}^m$ taking the state from x_0 at $t = -\infty$ to x at $t = 0$. Thus $C_s(x)$ is the minimal physical energy that is needed to transfer the state from x_0 to x. We see that

$$\begin{array}{rcl} C_s(x) & = & \inf_v [\int_{-\infty}^{0} \frac{1}{2} \parallel z(t) \parallel^2 dt + \int_{-\infty}^{0} u^T(t) y(t) dt] \\ & = & \inf_v [\int_{-\infty}^{0} \frac{1}{2} \parallel z(t) \parallel^2 dt + H(x)], \end{array}$$

leading to the optimal input v being such that $z = 0$, while $C_s(x) = H(x)$.

We conclude that for a lossless system

$$O_s = H = C_s$$

and balancing does not yield information about the relative importance of state components.

For a linear lossless system in scattering representation Σ_s the observability Gramian M_s is the unique solution to

$$(A - BC)^T M_s + M_s(A - BC) = -2C^T C,$$

and the controllability Gramian W_s is the unique solution to

$$(A - BC)W_s + W_s(A - BC)^T = -2BB^T,$$

It follows that $M_s = W_s^{-1} = Q$. Hence $M_s W_s$ equals the identity matrix, and the Hankel singular values of a linear lossless system in scattering representation are all equal to one.

Secondly, we consider **LQG balancing**, or **closed-loop balancing** and its extension to the nonlinear case.

Define the *future energy function* E_f as

$$E_f(x) := \inf_u \int_0^\infty \frac{1}{2} \parallel u(t) \parallel^2 + \frac{1}{2} \parallel y(t) \parallel^2 dt$$

where the infimum is taken over all input functions $u : (0, \infty) \to \mathbb{R}^m$ taking the system from state x at $t = 0$ to x_0 at time $t = \infty$. It follows that

$$\begin{aligned} E_f(x) &= \inf_u [\textstyle\int_0^\infty (\parallel v(t) \parallel^2 - u^T(t) y(t)) dt] \\ &= \inf_u [\textstyle\int_0^\infty \parallel v(t) \parallel^2 dt] + H(x) - H(x(\infty)) \\ &= \inf_u [\textstyle\int_0^\infty \parallel v(t) \parallel^2 dt] + H(x) \end{aligned}$$

since $x(\infty) = x_0$ and $H(x_0) = 0$.

This minimization has the obvious solution u being such that $v = 0$, leading to the equality $E_f(x) = H(x)$.

Secondly we define the *past energy function* E_p as

$$E_p(x) := \inf_u \int_{-\infty}^{0} \frac{1}{2} \parallel u(t) \parallel^2 + \frac{1}{2} \parallel y(t) \parallel^2 dt$$

where the infimum is taken over all input functions $u : (-\infty, 0) \to \mathbb{R}^m$ taking the system from state x_0 at $t = -\infty$ to x at time $t = 0$.

It follows that

$$\begin{aligned} E_p(x) &= \inf_u [\textstyle\int_{-\infty}^{0} (\parallel z(t) \parallel^2 + u^T(t) y(t)) dt] \\ &= \inf_u [\textstyle\int_{-\infty}^{0} \parallel z(t) \parallel^2 dt + \int_{-\infty}^{0} u^T(t) y(t) dt] \\ &= \inf_u [\textstyle\int_{-\infty}^{0} \parallel z(t) \parallel^2 dt] + H(x) \end{aligned}$$

since $x(-\infty) = x_0$ and $H(x_0) = 0$. This last minimization has the obvious solution u being such that $z = 0$, leading to $E_p(x) = H(x)$.

In conclusion, for a lossless system

$$E_f = H = E_p$$

Therefore, again no information is obtained about the relative importance of the state components.
For a linear lossless system $E_f(x) = \frac{1}{2}x^T P x$, with P the stabilizing solution to the Control Algebraic Riccati Equation (CARE)

$$A^T P + PA + C^T C - PBB^T P = 0$$

and $E_p(x) = \frac{1}{2}x^T S^{-1} x$, with S the stabilizing solution to the Filter Algebraic Riccati Equation (FARE)

$$AS + SA^T + BB^T - SC^T CS = 0$$

It follows that $P = S^{-1} = Q$, and thus $PS = I$ and the LQG similarity invariants are all equal to 1.

Remark 5 *We could also apply (nonlinear) closed-loop balancing to the scattering representation* Σ_s*. However, due to the the parallelogram identity*

$$\| v \|^2 + \| z \|^2 = \| u \|^2 + \| y \|^2$$

the future and past energy functions E_f *and* E_p *for the scattering representation are equal to the future and past energy functions for the power variable representation.*

Finally, the same equality result follows if we apply *positive real balancing* to Σ. The available storage $S_a(x)$ at x is given as

$$S_a(x) = \sup_{u,T\geq 0} - \int_0^T u^T(t)y(t)dt$$

while the required supply $S_r(x)$ to reach x at $t=0$ starting from x_0 equals

$$S_r(x) = \inf_{u,T\geq 0, x(-T)=x_0} \int_{-T}^0 u^T(t)y(t)dt$$

It follows that $S_r(x_0)=0$, while also $S_a(x_0)=0$. Furthermore (Willems, 1972)

$$S_a(x) \leq H(x) \leq S_r(x)$$

for all x. In fact, it follows that $S_a \leq S \leq S_r$ for *all* storage functions S, and S_a is the *minimal* and S_r the *maximal* storage function. However, for a lossless system S_a and S_r are generally equal:

$$S_a = H = S_r$$

What is happening for passive systems that are *not* lossless ?

A system is *passive* with differentiable storage function H if

$$\begin{bmatrix} 2\frac{\partial^T H}{\partial x}(x)a(x) & \frac{\partial^T H}{\partial x}(x) - c^T(x) \\ b^T(x)\frac{\partial H}{\partial x}(x) - c(x) & -d(x) + d^T(x) \end{bmatrix} \leq 0$$

For $d(x) = 0$ this reduces to

$$\begin{array}{rcl} \frac{\partial^T H}{\partial x}(x)a(x) & \leq & 0 \\ c(x) & = & b^T(x)\frac{\partial H}{\partial x}(x) \end{array}$$

In general there is now a *gap* between the available storage S_a and the required supply S_r:

$$S_a \leq H \leq S_r$$

For a linear passive system Σ with $D = 0$ the available storage S_a is given as $\frac{1}{2} x^T Q_a x$ where Q_a is the *minimal* solution to the Linear Matrix Inequality (LMI)

$$A^T Q + QA \leq 0, \quad B^T Q = C$$

while the required supply is $\frac{1}{2} x^T Q_r x$ where Q_r is the *maximal* solution to this same LMI. The 'singular values' are the eigenvalues of $Q_a Q_r^{-1}$ and balancing amounts to a coordinate transformation such that $Q_a = Q_r^{-1} = D$.

For a passive system that is not lossless we do *not* always get strict inequalities !

Consider a mass-spring damper system

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{m} \\ -k & -c \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad u = \text{ force} \\ y &= \begin{bmatrix} 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \text{ velocity} \end{aligned}$$

with physical energy $H(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$.

The LMI takes the form

$$
\begin{bmatrix} 0 & -k \\ \frac{1}{m} & -c \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} + \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{m} \\ -k & -c \end{bmatrix} \leq 0
$$

$$
\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \end{bmatrix}
$$

The last equation yields $q_{12} = 0$ as well as $q_{22} = \frac{1}{m}$. Substituted in the inequality this yields as unique solution $q_{11} = k$, corresponding to a unique storage function, being the physical energy $H(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$.

Main results:

In the **lossless** case we found

$$E_f = O_s = S_a = H = S_r = C_s = E_p$$

implying that the 'singular values' corresponding to every 'balancing pair' $(S_a, S_r), (O_s, C_s), (E_f, E_p)$ are all 1.
In the general **passive** case we can show

$$E_f \leq O_s \leq S_a \leq H \leq S_r \leq C_s \leq E_p$$

Hence the 'singular values' corresponding to every balancing pair (S_a, S_r), (O_s, C_s), (E_f, E_p) are now all ≤ 1, and balancing *does* give information.

On the other hand, the gaps in the inequality are critically depending on the amount of internal energy dissipation, and may not be very robust. Balancing seems to compare mainly the **energy dissipation** of the state components.

Structure-preserving balancing and model reduction ?

Known fact: Positive real-balancing and model reduction by truncation leads to a lower-dimensional system that is again passive.

How to proceed ?

Almost every linear passive system can be rewritten as a *port-Hamiltonian system*

$$\begin{aligned} \dot{x} &= (J-R)Qx + (G-P)u \\ y &= (G+P)^T Qx + (M+S)u, \end{aligned}$$

with Hamiltonian $H(x) = \frac{1}{2}x^T Q x$, where $Q = Q^T$. J is a skew-symmetric $n \times n$ matrix, M is a skew-symmetric $m \times m$ matrix and G is an $n \times m$ matrix, specifying together the *interconnection structure*.

The matrices R, S, P, with R a symmetric $n \times n$ matrix, S a symmetric $m \times m$ matrix and P an $n \times m$ matrix, specify the resistive relation

$$\begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \geq 0$$

In the 'normal' case $P = 0$ this reduces to $R \geq 0$, $S \geq 0$.

Theorem 6 (Passive linear systems are port-Hamiltonian)
(1). If the linear system is passive with quadratic storage function $\frac{1}{2}x^TQx$ *satisfying* $Q \geq 0$*, and* $\ker Q \subset \ker A$*, then it is port-Hamiltonian.*
(2). Any port-Hamiltonian linear system with $Q \geq 0$ *is passive.*

Note that the condition $\ker Q \subset \ker A$ is automatically satisfied if $Q > 0$.

Structure-preserving model reduction of port-Hamiltonian systems ?

Model reduction by truncation based on balancing can be regarded as an approximate version of Kalman decomposition.

Thus first step in structure-preserving model reduction for port-Hamiltonian systems is to study the Kalman-decomposition of port-Hamiltonian systems.

Let us concentrate on the **linear** case.

Consider a port-Hamiltonian system

$$\begin{aligned} \dot{x} &= FQx + Bu, \quad J = -J^T,\ R = R^T \geq 0 \\ y &= B^T Q x, \qquad\quad Q = Q^T \geq 0 \end{aligned} \tag{1}$$

where $F := J - R$, satisfying $F + F^T \leq 0$. Suppose the system is **not** controllable. Take linear coordinates $x = (x_1, x_2)$ such that the upper part of

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} B_1^T & B_2^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is the reachability subspace R.

By invariance of R this implies

$$\begin{aligned} F_{21}Q_{11} + F_{22}Q_{21} &= 0 \\ B_2 &= 0 \end{aligned}$$

If follows that the dynamics restricted to R is given as

$$\begin{aligned} \dot{x}_1 &= (F_{11}Q_{11} + F_{12}Q_{21})x_1 + B_1 u \\ y &= B_1^T Q_{11} x_1 \end{aligned}$$

First assume that F_{22} in invertible. Then we may solve for Q_{21} as $Q_{21} = -F_{22}^{-1}F_{21}Q_{11}$. Substitution yields

$$\begin{aligned} \dot{x}_1 &= (F_{11} - F_{12}F_{22}^{-1}F_{21})Q_{11}x_1 + B_1 u \\ y &= B_1^T Q_{11} x_1 \end{aligned}$$

which is again a port-Hamiltonian system. Indeed, $F + F^T \leq 0$ implies that the Schur complement $\bar{F} = F_{11} - F_{12}F_{22}^{-1}F_{21}$ satisfies $\bar{F} + \bar{F}^T \leq 0$.

Suppose the port-Hamiltonian system is **not observable**. Then there exist coordinates $x = (x_1, x_2)$ such that the *lower* part is the unobservability subspace N. By invariance of N it follows that

$$\begin{aligned} F_{11}Q_{12} + F_{12}Q_{22} &= 0 \\ B_1^T Q_{12} + B_2^T Q_{22} &= 0 \end{aligned}$$

Then the dynamics on the quotient space $\mathcal{X}/N$ is

$$\begin{aligned} \dot{x}_1 &= (F_{11}Q_{11} + F_{12}Q_{21})x_1 + B_1 u \\ y &= B_1^T Q_{11} x_1 + B_2^T Q_{21} x_1 \end{aligned}$$

Assuming invertibility of Q_{22} it follow that $F_{12} = -F_{11}Q_{12}Q_{22}^{-1}$ and $B_2^T = -B_1^T Q_{12}Q_{22}^{-1}$. Substitution yields

$$\begin{aligned} \dot{x}_1 &= F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + B_1 u \\ y &= B_1^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 \end{aligned}$$

which is again a port-Hamiltonian system with Hamiltonian $\bar{H} = \frac{1}{2}x_1^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1$.

Remark Note that $(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}) \geq 0$ if $Q \geq 0$.

Both decompositions can be combined into a port-Hamiltonian Kalman-decomposition.

This suggests two approaches for structure-preserving model-reduction for port-Hamiltonian systems. Suppose by some sort of balancing we end up with the decomposition

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\
y &= \begin{bmatrix} B_1^T & B_2^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}$$

where the lower part of the system corresponds to a dynamics that can be “neglected”.

Then there are two canonical ways for structure-preserving model reduction:

- Set $x_2 = \dot{x}_2 = 0$. This yields the port-Hamiltonian approximation

$$\begin{aligned} \dot{x}_1 &= (F_{11} - F_{12}F_{22}^{-1}F_{21})Q_{11}x_1 + B_1 u \\ y &= B_1^T Q_{11} x_1 \end{aligned}$$

- Set $e_2 := Q_{21}x_1 + Q_{22}x_2 = 0$. This leads to

$$\begin{aligned} \dot{x}_1 &= F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + B_1 u \\ y &= B_1^T (Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 \end{aligned}$$

Conclusions

1. For lossless systems all balancing approaches lead to pairs of functions that are equal to the physical energy.
2. For passive systems the singular values corresponding to the three balancing pairs are ≤ 1, but crucially depend on the damping.
3. Kalman-decomposition of port-Hamiltonian systems suggests two canonical ways of structure-preserving model reduction.

See www.math.rug.nl/~arjan for further information.