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Asymptotic stabilization via control by interconnection of port-Hamiltonian systems[☆]

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ABSTRACT

We study the asymptotic properties of *control by interconnection*, a passivity-based controller design methodology for stabilization of port-Hamiltonian systems. It is well-known that the method, in its basic form, imposes some unnatural controller initialization to yield asymptotic stability of the desired equilibrium. We propose two different ways to overcome this restriction, one based on adaptation ideas, and the other one adding an extra damping injection to the controller. The analysis and design principles are illustrated through an academic example.

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1. Introduction

Recently, port-Hamiltonian (PH) models (van der Schaft, 2000) have been a focus of attention in the control community (e.g. Cheng, Astolfi, and Ortega (2005), Fujimoto, Sukurama, and Sugie (2003), Ortega, van der Schaft, Maschke, and Escobar (2002) and Wang, Feng, and Cheng (2007)). There are, at least, two reasons for their appeal: first, that they describe a wide class of physical systems, including (but not limited to), systems described by Euler–Lagrange equations. Second, that PH models directly reveal the fundamental role of the physical concepts of energy, dissipation and interconnection—making passivity-based control (PBC) (Ortega & Spong, 1989; van der Schaft, 2000) a suitable candidate to regulate the behavior of PH systems.

In this paper, we are interested in the stabilization of PH systems using control by interconnection (Cbi) (Ortega, van der Schaft, Mareels, & Maschke, 2001; Ortega et al., 2002). Similarly to other PBC techniques, the objective in Cbi is to render the closed-loop passive with respect to a desired energy (storage) function. This is accomplished in Cbi selecting the controller to also be a PH system which, connected to the plant through a power-preserving interconnection, results in a closed-loop that is again PH with energy function equal to the sum of the plant's and the controller's energies.

In its original formulation, applicability of Cbi is stymied by the so-called dissipation obstacle (Ortega et al., 2001), a problem that appears when the dissipation of the open-loop is different from zero at the desired equilibrium. In Ortega, van der Schaft, Castaños, and Astolfi (2008), this problem has been solved, generating different passive outputs giving rise to the so-called power shaping Cbi. Both methods, standard and power shaping Cbi, rely on the creation of functions, called Casimirs, which are independent of the energy function. The existence of these invariants presents an obstruction to the *asymptotic* stabilization of the desired equilibrium. The main contribution of this paper is to propose two modifications to the existing Cbi to overcome this problem. The first modification is motivated by adaptation principles, while the second one is based on the addition of an extra damping injection to the controller. As an additional by-product of the analysis performed, the two versions of Cbi are unified.

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To make the paper self-contained, we begin the following section with a brief description of Cbl and refer the reader to Ortega et al. (2008) for more details. Section 3 contains specific guidelines to apply Cbl for equilibrium stabilization. The modifications to achieve asymptotic stability are then presented in Section 4. Finally, we state some concluding remarks in Section 5.

Notation. The arguments of the functions are omitted once they are defined and there is no possibility of confusion. All vectors defined in the paper are column vectors, even the gradient of a scalar function, denoted with the operator $\nabla \triangleq \frac{\partial}{\partial x}$. We also define $\nabla^2 \triangleq \frac{\partial^2}{\partial x^2}$. Given a vector x and a matrix $K = K^T > 0$, $\|x\|$ denotes the Euclidean norm and $\|x\|_K$ the norm $x^T K x$.

2. Preliminaries

Although this note deals with PH systems (van der Schaft, 2000) only, it will be useful to consider first a general nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^m$ is the output, with $m \leq n$. The functions f, g and h are smooth and of appropriate dimensions and the matrix g is full rank, uniformly in x .

2.1. Cyclo-passivity

Definition 1. System (1) is said to be *cyclo-passive* if there exists a differentiable function $H : \mathbb{R}^n \rightarrow \mathbb{R}$ (called the storage function) that satisfies the power balance inequality

$$\dot{H} \leq y^T u \tag{2}$$

when evaluated along the trajectories of (1).

Recall that a system is passive if (2) holds and H is bounded from below. Because of this additional restriction, every passive system is cyclo-passive but the converse is not true. In terms of energy exchange, cyclo-passive systems exhibit a net absorption of energy along closed trajectories (Hill & Moylan, 1980), while passive systems absorb energy along any trajectory that starts from a state of minimal energy $x(0) = \arg \min H(x)$.

According to Hill–Moylan’s Theorem (Hill & Moylan, 1980), system (1) is cyclo-passive (with storage function $H(x)$) if and only if, for some $q \in \mathbb{N}$, there exists a function $l : \mathbb{R}^n \rightarrow \mathbb{R}^q$ such that

$$\nabla H^T f = -\|l\|^2 \tag{3a}$$

$$h = g^T \nabla H. \tag{3b}$$

Setting the dissipation $d \triangleq \|l\|^2$ and differentiating H leads to the power balance

$$\dot{H} = y^T u - d. \tag{4}$$

We now focus on PH systems

$$\Sigma : \begin{cases} \dot{x} = F \nabla H + g u \\ y = g^T \nabla H \end{cases} \tag{5}$$

where $F : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, with $F + F^T \leq 0$. It can be easily verified that (5) is cyclo-passive with storage function H and dissipation $d \triangleq -\nabla H^T F \nabla H$.

For future reference let us compute the assignable equilibria of (5) as the elements of the set

$$\mathcal{E}_x \triangleq \{x \mid g^T F \nabla H = 0\}, \tag{6}$$

with $g^\perp : \mathbb{R}^n \rightarrow \mathbb{R}^{(n-m) \times n}$ a full rank left-annihilator of g , that is, $g^\perp g = 0$ and $\text{rank } g = n - m$. Associated to each $x_* \in \mathcal{E}_x$ there is a uniquely defined constant control given by

$$u_* \triangleq -g^+(x_*) F(x_*) \nabla H(x_*), \tag{7}$$

where g^+ is the Moore-Penrose pseudo-inverse of g , that is, $g^+ \triangleq [g^T g]^{-1} g^T$. Note that g^+ is well-defined since g is assumed full rank, implying that the inverse of $g^T g$ always exists.

2.2. Example

The system described by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x_1 + x_2 \\ -x_2^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} - x_2^2 \\ x_2^3 \end{pmatrix} u \tag{8}$$

can be written in the PH form (5) with

$$F = \begin{pmatrix} -\frac{1}{2} & x_2 \\ 0 & -x_2^2 \end{pmatrix}, \quad H = \frac{1}{2}x_1^2 + x_2, \quad g = \begin{pmatrix} \frac{1}{2} - x_2^2 \\ x_2^3 \end{pmatrix} \tag{9}$$

and output

$$y = g^T \nabla H = x_1 \left(\frac{1}{2} - x_2^2 \right) + x_2^3.$$

Notice that Eq. (4) does not yield any information about the stability of the open-loop equilibrium $(0, 0)$, since H is not bounded from below. Actually, it can be readily seen that with $u = 0$ the equilibrium is unstable and that the trajectories of the open-loop system exhibit finite escape time. Moreover, the origin cannot be stabilized by any continuous feedback.

The set of assignable equilibria for this system is

$$\mathcal{E}_x = \{(x_1, x_2) \mid x_2^2(1 - x_1 x_2) = 0\}. \tag{10}$$

2.3. Control by interconnection

In Cbl a PH controller of the form

$$\Sigma_c : \begin{cases} \dot{\xi} = u_c \\ y_c = \nabla H_c(\xi) \end{cases} \tag{11}$$

is proposed. $\xi \in \mathbb{R}^m$ is the state of the controller, u_c, y_c are the input and the output of the controller, respectively, and $H_c : \mathbb{R}^m \rightarrow \mathbb{R}$ is a to-be-designed controller storage function. See Ortega et al. (2008) and van der Schaft (2000) for a justification of this choice of controller structure.

Control by interconnection comes in two basic variants. In the standard version, Σ and Σ_c are coupled using the classical unitary feedback power-preserving interconnection

$$\Sigma_I : \begin{cases} \begin{pmatrix} u \\ u_c \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ y_c \end{pmatrix} + \begin{pmatrix} v \\ 0 \end{pmatrix}, \end{cases} \tag{12}$$

where v is a new virtual input.¹ It is well-known (van der Schaft, 2000) that the PH structure is invariant under power-preserving interconnection; this pattern leading to the interconnected PH system

$$\Sigma_{Ts} : \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} F & -g \\ g^T & 0 \end{pmatrix} \nabla H_T + \begin{pmatrix} g \\ 0 \end{pmatrix} v \\ y_{Ts} = \begin{pmatrix} g^T & 0 \end{pmatrix} \nabla H_T \end{cases} \tag{13}$$

¹ We recall that an interconnection of PH systems is power preserving if it satisfies $y^T u + y_c^T u_c = y^T v$.

with

$$H_T(x, \xi) \triangleq H(x) + H_c(\xi) \quad (14)$$

the new total energy.

A new version of Cbl has been recently introduced in Ortega et al. (2008) that, being related to the power shaping procedure of Ortega, Jeltsema, and Scherpen (2003), is called power shaping Cbl. In this case, F is assumed to be non-singular and a modified PH system with a new passive output is generated as

$$\Sigma_{ps} : \begin{cases} \dot{x} = F\nabla H + gu \\ y_{ps} = -g^T F^{-T} (F\nabla H + gu). \end{cases} \quad (15)$$

Noticing that $y_{ps} = -g^T F^{-T} \dot{x}$ it is easy to show (Ortega et al., 2003) that (15) satisfies $\dot{H} \leq u^T y_{ps}$. The interconnection is then given by

$$\Sigma_{lps} : \begin{cases} \begin{pmatrix} u \\ u_c \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{ps} \\ y_c \end{pmatrix} + \begin{pmatrix} v \\ 0 \end{pmatrix}, \end{cases} \quad (16)$$

that yields the PH closed-loop system²

$$\Sigma_{Tps} : \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} F & -g \\ -g^T F^{-T} F & g^T F^{-T} g \end{pmatrix} \nabla H_T \\ + \begin{pmatrix} g \\ -g^T F^{-T} g \end{pmatrix} v \\ y_{Tps} = \begin{pmatrix} g^T & -g^T F^{-1} g \end{pmatrix} \nabla H_T. \end{cases} \quad (17)$$

So far, we have constructed interconnected systems which are cyclo-passive with storage function H_T . Since H_c can be modified at will, it seems reasonable to use it to “shape” the total storage function. We are interested in shaping H_T along the x coordinates, but unfortunately, H_c is a function of ξ , so this idea cannot be applied directly. One way to get around this, is to relate x and ξ in the following way.

Assumption 2. There exist a differentiable mapping $C : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the Jacobian of which has rank m and at least one of the following conditions is satisfied.

(1) (Standard Cbl)

$$\begin{pmatrix} g^T \\ F \end{pmatrix} \nabla C = - \begin{pmatrix} 0 \\ g \end{pmatrix}. \quad (18)$$

(2) (Power shaping Cbl) $\det F(x) \neq 0$ and

$$F\nabla C = -g. \quad (19)$$

Assumption 2 is made throughout the paper. That is, it is assumed that, for the given F and g , a solution of the partial differential equations (18) or (19) is known. Also, to simplify the presentation, it is assumed that F is full rank. The power shaping Cbl presented above is called “Basic Cbl-PS” in Ortega et al. (2008). In that paper we present another version of Cbl that generates a new, full-rank matrix to replace F .

In Ortega et al. (2008) it is shown that condition 1 (resp., 2) of Assumption 2 ensures that, for any $\kappa \in \mathbb{R}^m$, the manifolds $\mathcal{M}_\kappa = \{(x, \xi) \mid C(x) - \xi = \kappa\}$ are invariant³ under the flow of the system (13) (resp., (17)). As discussed in Ortega et al. (2008, 2001) and van der Schaft (2000), and also shown below, the construction

² To verify that it is indeed PH, notice that for any x and ξ ,

$$\begin{aligned} & \begin{pmatrix} x^T & \xi^T \end{pmatrix} \begin{pmatrix} F & -g \\ -g^T F^{-T} F & g^T F^{-T} g \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} \\ & = (x - F^{-1}g\xi)^T F(x - F^{-1}g\xi) \leq 0. \end{aligned}$$

³ That is, $C(x(t)) - \xi(t) = C(x_0) - \xi_0 \forall t$, where $(x_0, \xi_0) \triangleq (x(0), \xi(0))$.

of this, so-called, *Casimir function* $C(x) - \xi$ is the key step of Cbl that allows to shape the storage function in the state coordinates x . In order to reveal this property and, at the same time, provide a unified framework to study both versions of Cbl, we find it convenient to define the PH system

$$\Sigma_T : \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = F_T \nabla H_T + g_T v \\ y_T = g_T^T \nabla H_T \end{cases} \quad (20)$$

where

$$F_T \triangleq \begin{pmatrix} I \\ \nabla C^T \end{pmatrix} (F \quad -g), \quad g_T \triangleq \begin{pmatrix} I \\ \nabla C^T \end{pmatrix} g. \quad (21)$$

Notice that (20) describes the behavior of both closed-loop systems, (13) and (18), or (17) and (19). In the sequel we deal only with (20) on the understanding that, depending on which condition of Assumption 2 is satisfied, we are referring to either one of the Cbl controllers.

The proposition below opens the possibility of creating appropriate storage functions that can be shaped along x .

Proposition 3. The PH system (20) is cyclo-passive with storage function

$$W(x, \xi) \triangleq H_T(x, \xi) + \Phi(C(x) - \xi), \quad (22)$$

for any differentiable $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}$.

Proof. Compute $\dot{W} = \dot{H}_T + \dot{\Phi}$. Since Σ_T is cyclo-passive with storage function H_T and dissipation $d_T \triangleq -\nabla H_T^T F_T \nabla H_T$, we have

$$\begin{aligned} \dot{W} & = v^T y_T - d_T + \nabla^T \Phi \begin{pmatrix} \nabla C^T & -I \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} \\ & = v^T y_T - d_T \end{aligned}$$

where the last equality follows from (20), (21) and

$$\begin{pmatrix} \nabla C^T & -I \end{pmatrix} \begin{pmatrix} I \\ \nabla C^T \end{pmatrix} = 0. \quad \square$$

3. Stabilization

In this section we show how Proposition 3 can be used for stabilization of an arbitrary element of the assignable equilibrium set \mathcal{E}_x , defined in (6). We propose functions H_c and Φ and give conditions on C that ensure the stabilization requirement.

As a first step, define the set of equilibria \mathcal{E} for the system (20) in open-loop (i.e., with $v = 0$). According to (20) and (21)

$$\mathcal{E} = \{(x, \xi) \mid F\nabla H - g\nabla H_c = 0\}. \quad (23)$$

In the previous section it has been shown that W satisfies

$$\dot{W} = y_T^T v - d_T. \quad (24)$$

with $d_T \geq 0$. It follows from standard Lyapunov theory that if W has a strict minimum at a point $(x_*, \xi_*) \in \mathcal{E}$ and we set $v = 0$, then (x_*, ξ_*) is stable. Our goal is thus, to find appropriate Φ and H_c , and impose conditions on C , such that

$$(x_*, \xi_*) = \arg \min W(x, \xi). \quad (25)$$

Clearly, negativity of \dot{W} can be reinforced by setting

$$v = -K_v y_T, \quad K_v = K_v^T > 0. \quad (26)$$

This damping injection (also called $L_g V$) approach is usually adopted in PBC to try to make the equilibrium *asymptotically* stable, which is the case if y_T is a detectable output (van der Schaft, 2000). Unfortunately, we will show below that the latter condition is not satisfied for Cbl and we must adopt another strategy, which will be presented in Section 4. But first we propose a solution to the problem of stabilization of an arbitrary element of \mathcal{E}_x .

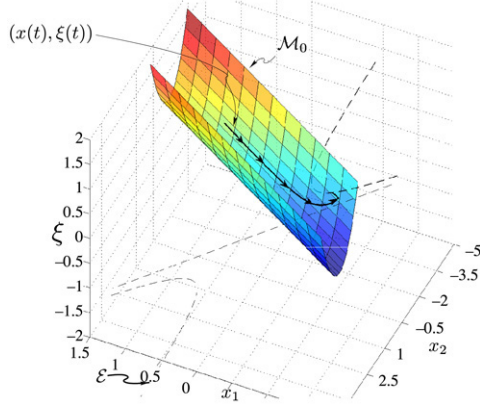


Fig. 1. The invariant manifold \mathcal{M}_0 , the equilibria locus \mathcal{E} and the simulated response.

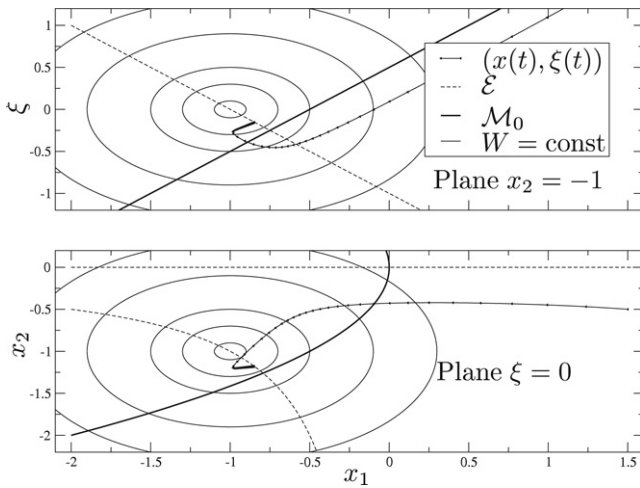


Fig. 2. Level sets of W and invariant manifold \mathcal{M}_0 , equilibria locus \mathcal{E} and simulated response, projected into the planes $x_2 = -1$ (above) and $\xi = 0$ (below).

Note that $\mathcal{E} \cap \mathcal{M}_0 = \{(-0.85, -1.18, -0.15), (-0.5, 0, -0.5)\}$. Fig. 1 shows \mathcal{M}_0 , \mathcal{E} and the trajectory starting at $(x_0, \xi_0) = (\frac{3}{2}, -\frac{1}{2}, \frac{13}{8})$ and converging to $(-0.85, -1.18, -0.15)$. Fig. 2 shows the intersection of \mathcal{M}_0 and the level sets of W with the planes $x_2 = x_{2*} = -1$ and $\xi = \xi_* = 0$. The projections of \mathcal{E} and the trajectory are also shown.

4.2. Adaptive Cbl

It is clear that another way to satisfy the constraint (32) is by shifting away from zero the desired value of ξ to the new value

$$\xi_* = C(x_*) - C(x_0) + \xi_0. \quad (33)$$

This amounts to changing H_c to

$$H_c(\xi) = \frac{1}{2} \|\xi - \xi_* - K_c^{-1}u_*\|_{K_c}^2, \quad (34)$$

so that $\nabla H_c(\xi_*) = -u_*$. Geometrically, we are shifting the equilibrium locus \mathcal{E} along ξ , so that it intersects the manifold where the trajectory starts, that is, \mathcal{M}_{κ_0} , with

$$\kappa_0 \triangleq C(x_0) - \xi_0, \quad (35)$$

at the desired x_* .

In principle, this scheme still hinges on knowledge of the initial condition, but this issue can be removed by reformulating it as a parameter estimation problem. We try first a classical certainty-equivalence adaptive control approach viewing ξ_* as the unknown

parameter. This is indeed possible because the plant is linear in u and, for quadratic H_c , ξ_* enters also linearly in u . Define a new storage function for the controller (11) as

$$\tilde{H}_c(\xi, \hat{\xi}_*) \triangleq \frac{1}{2} \|\xi - \hat{\xi}_* - K_c^{-1}u_*\|_{K_c}^2,$$

where $\hat{\xi}_*$ denotes the estimate of ξ_* . Let us compute

$$\begin{aligned} \nabla_{\xi} \tilde{H}_c &= K_c(\xi - \hat{\xi}_*) - u_* = K_c(\xi - \xi_*) - u_* - K_c \tilde{\xi}_* \\ &= \nabla H_c - K_c \tilde{\xi}_*, \end{aligned}$$

where we have defined the parameter error $\tilde{\xi}_* \triangleq \hat{\xi}_* - \xi_*$. The control signal then becomes $u = -\nabla_{\xi} \tilde{H}_c = -\nabla H_c + K_c \tilde{\xi}_*$. The closed-loop system is still of the form (20) with v replaced by $v + K_c \tilde{\xi}_*$. Since the invariance of the manifolds \mathcal{M}_{κ} is preserved, the power balance equation (24) is still satisfied with the “new v ”. Proceeding with the classical adaptive control design we would propose a candidate Lyapunov function $V(x, \xi, \hat{\xi}_*) = W(x, \xi) + \frac{1}{2} \|\tilde{\xi}_*\|_{\Gamma}^2$, $\Gamma = \Gamma^T > 0$, and an estimation law of the form $\dot{\hat{\xi}}_* = -\Gamma K_c y_T$, which would make $\dot{V} = \dot{W} \leq 0$. Unfortunately, this simple scheme will not solve our problem. Indeed, since the derivative of the new Lyapunov function has not changed, the lack of detectability problem is still present. The only way to achieve the desired objective is to ensure parameter convergence, that is, $\lim_{t \rightarrow \infty} \tilde{\xi}_*(t) = 0$, which is not satisfied due to existence of a manifold of equilibria.

It turns out that if we estimate the parameter κ_0 (instead of ξ_*) and use the invariance of the manifold \mathcal{M}_{κ_0} we can design a scheme that ensures parameter convergence. The result is summarized in the proposition below, which is the adaptive version of Proposition 4.

Proposition 5. Consider the PH system Σ (resp., Σ_{ps}) given in (5) (resp., (15)) interconnected through Σ_I (12) (resp., Σ_{Ips} (16)) with the adaptive controller

$$\hat{\Sigma}_c : \begin{cases} \dot{\hat{\xi}} = u_c \\ \dot{\hat{\kappa}}_0 = -\Gamma (\hat{\kappa}_0 - C(x) + \xi) \\ y_c = \nabla_{\xi} \hat{H}_c(\xi, \hat{\kappa}_0) \end{cases}$$

where $\hat{H}_c(\xi, \hat{\kappa}_0) \triangleq \frac{1}{2} \|\xi - C(x_*) + \hat{\kappa}_0 - K_c^{-1}u_*\|_{K_c}^2$, u_* is defined in (7) and $v = -K_v y_T$.

- (i) Exponential parameter convergence is ensured, more precisely $\|\hat{\kappa}_0(t) - \kappa_0\| \leq e^{-\lambda_{\min}(\Gamma)t} \|\hat{\kappa}_0(0) - \kappa_0\|$ for all $t \geq 0$.
- (ii) For any $x_* \in \mathcal{E}_x$ the point $(x_*, \xi_*, 0)$, where ξ_* is given in (33), is a stable equilibrium if (28) holds.
- (iii) The orbits of the residual dynamics are confined to the set $\mathcal{Z} \times \{\xi = \tilde{\xi}\}$, where $\tilde{\xi}$ is a constant and

$$\mathcal{Z} \triangleq \left\{ x \mid \begin{pmatrix} \nabla H \\ \nabla C^T \end{pmatrix} \cdot [F \nabla H - g(K_c(C(x) - C(x_*)) - u_*)] = 0 \right\}.$$

- (iv) Suppose no trajectory $x(t)$ can stay identically in \mathcal{Z} , other than isolated points. Then, $(x_*, \xi_*, 0)$ is an asymptotically stable equilibrium. It will be globally asymptotically stable if it is the only point in \mathcal{Z} and if W is radially unbounded.

Proof. Define $\tilde{\kappa}_0 \triangleq \hat{\kappa}_0 - \kappa_0$. From invariance of the manifold \mathcal{M}_{κ_0} we have that $\kappa_0 = C(x_0) - \xi_0 = C(x(t)) - \xi(t)$. Consequently, $\dot{\tilde{\kappa}}_0 = -\Gamma \tilde{\kappa}_0$, from which claim (i) follows immediately.

Proceeding as done for the standard adaptive controller above, one has that $\nabla_{\xi} \hat{H}_c = \nabla H_c - K_c \tilde{\kappa}_0$, $u = -\nabla H_c + K_c \tilde{\kappa}_0$, and the power balance equation becomes

$$\dot{W} = y_T^T (v - K_c \tilde{\kappa}_0) - d_T. \quad (36)$$

Consider the Lyapunov function candidate $V(x, \xi, \tilde{\kappa}_0) = W + \frac{1}{2} \|\tilde{\kappa}_0\|_{\mu, \Gamma}^2$, with $\mu > 0$. Differentiation with respect to time and some standard bounding shows that, for all K_v, K_c, Γ , there exists μ such that

$$\dot{V} \leq -d_T - \epsilon(\|y_T\|^2 + \|\tilde{\kappa}_0\|^2) \quad (37)$$

holds for some $\epsilon > 0$, which shows that V is a Lyapunov function, so the equilibrium is stable establishing (ii).

Now, we apply LaSalle's Theorem (La Salle & Lefschetz, 1961) and conclude from (37) that d_T and y_T tend to zero as $t \rightarrow \infty$. The residual dynamics are obtained imposing to the system the restrictions $d_T = 0, y_T = 0$ and $\tilde{\kappa}_0 = 0$. First, note that with $\tilde{\kappa}_0 = 0$ the dynamics reduce to Σ_T . Second, $y_T = 0$ implies $v = 0$ and $\dot{\xi} = 0$, consequently $\xi = \bar{\xi}$. Furthermore, from the equation of $\dot{\xi}$, we have

$$0 = \dot{\xi} = \nabla C^\top [F \nabla H - g \nabla H_c(\bar{\xi})]. \quad (38)$$

Now, recall that the dissipation is

$$\begin{aligned} 0 = d_T &= -\nabla H_T^\top F_T \nabla H_T \\ &= -(\nabla H^\top \quad \nabla H_c^\top) \begin{pmatrix} I \\ \nabla C^\top \end{pmatrix} (F \quad -g) \begin{pmatrix} \nabla H \\ \nabla H_c \end{pmatrix} \\ &= (\nabla H^\top + \nabla H_c^\top \nabla C^\top) (F \nabla H - g \nabla H_c), \end{aligned} \quad (39)$$

which combined with (38) yields,

$$\nabla H^\top [F(x) \nabla H(x) - g(x) \nabla H_c(\bar{\xi})] = 0. \quad (40)$$

The proof of (iii) is completed noting that $C(x) - \bar{\xi} = \kappa_0$ and evaluating ∇H_c at $\bar{\xi}$.

The proof of (iv) is a direct consequence of the celebrated theorem by Barbashin and Krasovskii (1952). \square

4.3. Example (continued)

We now apply adaptive Cbl to the example. Except for points on the hyperbola $x_1 x_2 = 1$, the matrix

$$\begin{pmatrix} \nabla H^\top \\ \nabla C^\top \end{pmatrix} = \begin{pmatrix} x_1 & 1 \\ 1 & x_2 \end{pmatrix} \quad (41)$$

is non-singular, so the orbits of the residual dynamics are confined to equilibrium points $\bar{x} \in \mathcal{E}$ satisfying

$$F(\bar{x}) \nabla H(\bar{x}) - g(\bar{x}) (C(\bar{x}) - C(x_*) + u_*) = 0.$$

For all $x_1^* < -\frac{1}{2}$ the only solutions of the above equation are⁶ $\bar{x}^l = \text{col}(x_{1*}, x_{2*})$ and $\bar{x}^h = \text{col}(x_{1*} + \frac{1}{4} x_{2*}^2, 0)$. When $x_1 x_2 = 1$, the vector $\text{col}(x_2, -1)$ is an eigenvector associated to the zero eigenvalue of the matrix (41), so points \bar{x} satisfying

$$F(\bar{x}) \nabla H(\bar{x}) - g(\bar{x}) (C(\bar{x}) - C(x_*) + u_*) = \begin{pmatrix} \bar{x}_2 \\ -1 \end{pmatrix} \psi(\bar{x})$$

for some function $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ can also contain the orbits of the residual dynamics. Since $\bar{x}_1 \bar{x}_2 = 1$ implies $g^\perp(\bar{x}) F(\bar{x}) \nabla H(\bar{x}) = 0$ (see Appendix for details), then one obtains $g^\perp(\bar{x}) \text{col}(\bar{x}_2, -1) = 0$. The solution set of the previous equation is empty, which implies that

$$\mathcal{Z} = \left\{ \text{col}(x_{1*}, x_{2*}), \text{col} \left(x_{1*} + \frac{1}{4} x_{2*}^2, 0 \right) \right\}.$$

Fig. 3 shows that now \mathcal{M}_0 and \mathcal{E} intersect at the desired x_* . Convergence towards the desired value is achieved with the adaptive scheme.

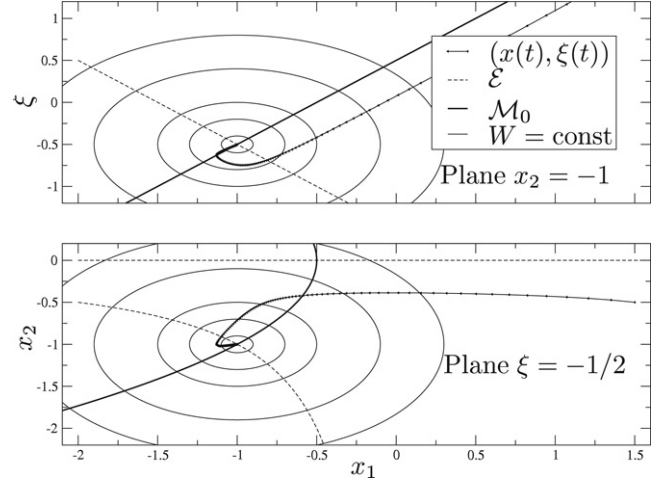


Fig. 3. Level sets of W and invariant manifold \mathcal{M}_0 , all intersected with the planes $x_2 = -1$ (above) and $\xi = -\frac{1}{2}$ (below). Equilibrium set \mathcal{E} and simulated response, both projected into the planes $x_2 = -1$ and $\xi = -\frac{1}{2}$.

4.4. Controller damping injection Cbl

Another possible way to achieve convergence is to destroy the invariance of the Casimirs, adding a damping injection to the controller. The idea is to go back to the previous controller storage function (27), that we repeat here for ease of reference

$$H_c(\xi) = \frac{1}{2} \|\xi - K_c^{-1} u_*\|_{K_c}^2, \quad (42)$$

but add an extra virtual input $w \in \mathbb{R}^m$ to the controller through the interconnection, that is,

$$\Sigma_{1w} : \begin{cases} \begin{pmatrix} u \\ u_c \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_T \\ y_c \end{pmatrix} + \begin{pmatrix} v \\ w \end{pmatrix}. \end{cases} \quad (43)$$

The interconnected system takes the form

$$\Sigma_{T w} : \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = F_T \nabla H_T + g_T v + \begin{pmatrix} 0 \\ I \end{pmatrix} w \\ y_T = g_T^\top \nabla H_T \\ z = \begin{pmatrix} 0 & I \end{pmatrix} \nabla W \end{cases} \quad (44)$$

where we have defined the corresponding conjugate output z . Notice that, for all $w \neq 0$, the invariance of the manifolds \mathcal{M}_k has been destroyed because $\dot{C} - \dot{\xi} = -w$. However, the time derivative of W is

$$\dot{W} = -d_T + y_T^\top v + z^\top w, \quad (45)$$

so the new system is also cyclo-passive with the same storage function W and port variables $((y_T, z), (v, w))$.

Proposition 6. Consider $\Sigma_{T w}$ with H_c given by (42), with u_* defined in (7), v by (26) and

$$w = -K_w z, \quad K_w = K_w^\top > 0. \quad (46)$$

- (i) For any $x_* \in \mathcal{E}_x$ the point $(x_*, 0)$ is a stable equilibrium if (28) holds.
- (ii) The orbits of the residual dynamics are confined to the set $\mathcal{Z}_w \times \{\xi = 0\}$, where

$$\mathcal{Z}_w = \left\{ x \mid \begin{pmatrix} \nabla H^\top \\ \nabla C^\top \end{pmatrix} [F \nabla H - g u_*] = 0 \right\}.$$

⁶ The details are not shown, but this fact can be verified by looking at the discriminant of the resulting cubic polynomial.

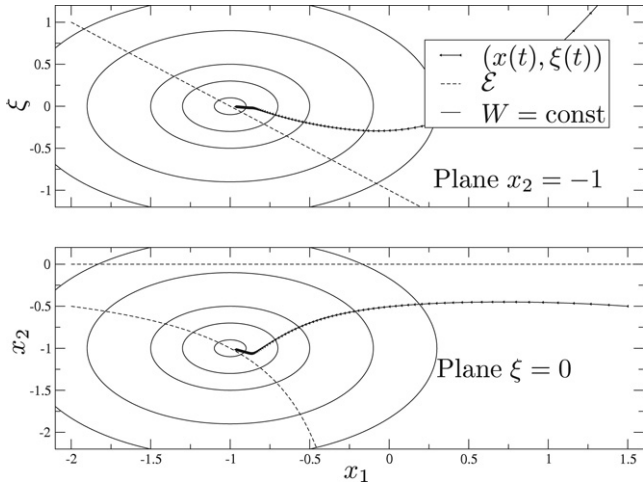


Fig. 4. Level sets of W intersected with the planes $x_2 = -1$ (above) and $\xi = 0$ (below). Equilibria locus \mathcal{E} and simulated response, both projected into the planes $x_2 = -1$ and $\xi = 0$.

(iii) If no trajectory $x(t)$ can stay identically in \mathcal{Z}_w , other than isolated points, $(x_*, 0)$ is an asymptotically stable equilibrium. It will be globally asymptotically stable if it is the only point in \mathcal{Z}_w and if W is radially unbounded.

Proof. Take W as a candidate Lyapunov function. Eq. (45), (26) and (46) imply that it is a Lyapunov function and (i) follows. Applying LaSalle's Theorem gives that d_T , y_T and z tend to zero as $t \rightarrow \infty$. The residual dynamics are those of Σ_{T_w} with the restrictions $d_T = 0$, $y_T = 0$ and $z = 0$. From the latter it follows that $\nabla_{\xi} W = 0$, which implies $\nabla H_c = \nabla \Phi(C(x) - \xi) = u_*$, which in turn implies $\xi = 0$. From the equation of $\dot{\xi}$, with $\xi = v = w = 0$, we get $0 = \dot{\xi} = \nabla C^T(x) [F(x)\nabla H(x) - g(x)u_*] = 0$, which is the second row in \mathcal{Z}_w . From this equation and (39) one is lead to conclude that $\nabla H^T(x) [F(x)\nabla H(x) - g(x)u_*] = 0$, that gives the first row, and completes the proof of (ii).

Point (iii) follows from Barbashin–Krasovskii's Theorem. \square

4.5. Example (continued)

We now apply controller damping injection Cbl to the system of the example. The analysis follows along the same lines as in the adaptive Cbl scenario. In this case $\mathcal{Z}_w = \{\text{col}(x_{1*}, x_{2*}), \text{col}(x_{1*}, 0)\}$. Fig. 4 shows the trajectories of the system for $K_w = 2$. These are no longer restricted to \mathcal{M}_0 . Again, convergence to x_* is achieved.

Simulations show that for the initial condition $(x_0, \xi_0) = (-1/2, 1/2, 0)$, convergence of the state of Σ_T is towards $(x_{1*} + x_{2*}^2/4, 0) = (-3/4, 0)$ for the adaptive Cbl and towards $(x_{1*}, 0) = (-1, 0)$ for the controller damping injection Cbl. Indeed, since \mathcal{Z} and \mathcal{Z}_w contain more than one point, stability is global but convergence is not. Notice, however, that in the controller damping injection scenario, the exact value of the unwanted equilibrium is known. This, together with the fact that the Lyapunov function W is non-decreasing over time, allows to obtain an estimate of the region of attraction: the open ball centered at $\text{col}(x_{1*}, x_{2*}, 0)$ and of radius $\|\text{col}(x_{1*}, x_{2*}, 0) - \text{col}(x_{1*}, 0, 0)\| = |x_{2*}|$.

5. Conclusions

We have shown that the existence of the Casimir functions, inherent in the Cbl design methodology, present an obstacle for asymptotic convergence of the state towards a desired equilibrium. In order to surmount this obstacle, two variations of the method have been developed. Paradoxically, once the modified versions

are used, the same Casimir functions narrow the possible limit sets, thus contributing to the desired asymptotic behavior. The Casimir functions also simplify the analysis of such limit sets, as they provide m algebraic constraints that, as shown in the example, can sometimes obviate the need to differentiate the output to obtain the residual dynamics. Interestingly, each method generates a different limit set.

It is clear that the selection of a quadratic function for H_c renders the controller linear, more precisely, a linear PI (for a suitably defined plant output). The results in the paper may be then interpreted as identification of a class of nonlinear PH systems that are asymptotically stabilizable via linear PI. Although the choice of a linear PI may be restrictive for some academic examples, it is certainly a family of controllers of practical interest. It should be, furthermore, pointed out that the general framework of Cbl does not impose this restriction on H_c , and it is made here to obtain easily interpretable general results. We are currently exploring other controller structures for which similar results can be established.

Appendix. The set \mathcal{E}

Consider an arbitrary point $(\bar{x}, \bar{\xi}) \in \mathcal{E}$. From Ortega et al. (2008, Lemma 2), we know that the conditions that define the set (23) are equivalent to

$$g^\perp(\bar{x})F(\bar{x})\nabla H(\bar{x}) = 0 \quad (\text{A.1a})$$

$$\nabla H_c(\bar{\xi}) = g^+(\bar{x})F(\bar{x})\nabla H(\bar{x}). \quad (\text{A.1b})$$

From (A.1a) we get that $\frac{\bar{x}_2^2}{2}(1 - \bar{x}_1\bar{x}_2) = 0$. In other words, if a given \bar{x} is in \mathcal{E} , then it must satisfy

$$\bar{x} \in \{(\bar{x}_1, 1/\bar{x}_1) \mid \bar{x}_1 \neq 0\} \cup \{(\bar{x}_1, 0) \mid \bar{x}_1 \in \mathbb{R}\}. \quad (\text{A.2})$$

Note that

$$g^+(\bar{x})F(\bar{x})\nabla H(\bar{x}) = \frac{(\frac{1}{2}\bar{x}_1 - \bar{x}_2)(\bar{x}_2^2 - \frac{1}{2}) + \bar{x}_2^5}{(\bar{x}_2^2 - \frac{1}{2})^2 + \bar{x}_2^6},$$

hence, because of (A.2),

$$g^+(\bar{x})F(\bar{x})\nabla H(\bar{x}) = \begin{cases} -\frac{1}{\bar{x}_2}(\bar{x}_2^2 - \frac{1}{2})^2 + \bar{x}_2^6 & = -\frac{1}{\bar{x}_2} \quad \bar{x}_2 \neq 0 \\ -\bar{x}_1 & \bar{x}_2 = 0. \end{cases}$$

In any case, $g^+(\bar{x})F(\bar{x})\nabla H(\bar{x}) = -\bar{x}_1$. Finally, from (A.1b) and the fact that $u_* = -1$ we get that $\nabla H_c(\bar{\xi}) = \bar{\xi} + 1 = -\bar{x}_1$ or, equivalently, $\bar{\xi} = -\bar{x}_1 - 1$, so

$$\mathcal{E} = \left\{ \text{col} \left(\bar{x}_1, \frac{1}{\bar{x}_1}, -\bar{x}_1 - 1 \right) \mid \bar{x}_1 \neq 0 \right\} \cup \{ \text{col}(\bar{x}_1, 0, -\bar{x}_1 - 1) \mid \bar{x}_1 \in \mathbb{R} \}. \quad (\text{A.3})$$

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