Interconnection of port-Hamiltonian systems and composition of Dirac structures

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Abstract

Port-based network modeling of physical systems leads to a model class of nonlinear systems known as port-Hamiltonian systems. Port-Hamiltonian systems are defined with respect to a geometric structure on the state space, called a Dirac structure. Interconnection of port-Hamiltonian systems results in another port-Hamiltonian system with Dirac structure defined by the composition of the Dirac structures of the subsystems. In this paper the composition of Dirac structures is being studied, both in power variables and in wave variables (scattering) representation. This latter case is shown to correspond to the Redheffer star product of unitary mappings. An equational representation of the composed Dirac structure is derived. Furthermore, the regularity of the composition is being studied. Necessary and sufficient conditions are given for the achievability of a Dirac structure arising from the standard feedback interconnection of a plant port-Hamiltonian system and a controller port-Hamiltonian system, and an explicit description of the class of achievable Casimir functions is derived.

Key words: Network modeling, composition, scattering, star product, Casimirs

1 Introduction

Port-based network modeling of complex physical systems (with components stemming from different physical domains) leads to a class of nonlinear systems, called *port-Hamiltonian systems*, see e.g. [11,24,12,13,3,20,25]. Port-Hamiltonian systems are defined by a Dirac structure (formalizing the power-conserving interconnection structure of the system), an energy function (the Hamiltonian), and a resistive relation. A key property of Dirac structures is that the power-conserving interconnection of Dirac structures again defines a Dirac structure, see [13,21]. This implies that any power-conserving interconnection of port-Hamiltonian systems is again a port-Hamiltonian system, with the Dirac structure being the composition of the Dirac structures of its constituent parts, Hamiltonian the sum of the Hamiltonians, and resistive relations determined by the individual resistive relations. As a result power-conserving interconnections of port-Hamiltonian systems can be studied to a considerable extent in terms of the composition of their Dirac structures. In particular, the feedback interconnection of a plant port-Hamiltonian system with a controller port-Hamiltonian system can be studied from the vantage-ground of the composition of a plant Dirac structure with a controller Dirac structure.

In this work we present some fundamental results about the composition of Dirac structures. First, we derive expressions for the composition of Dirac structures, and we study its regularity properties. Secondly, we describe the composition in wave variables (scattering representation). We show how this leads to the Redheffer star product of unitary operators. Thirdly, we extend the re-

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sults concerning the achievable 'closed-loop' Dirac structures obtained in [21,13], and we derive an explicit characterization of the obtainable Casimir functions of the closed-loop system. In previous publications, see e.g. [26,3,20,15], these closed-loop Casimirs have shown to be instrumental in problems of stabilization of port-Hamiltonian systems. Partial and preliminary versions of the material covered in this paper have been presented in [1,23].

2 Dirac structures and port-Hamiltonian systems

2.1 Dirac structures

Let us briefly recall the definition of a Dirac structure. We start with a space of *power variables* $\mathcal{F} \times \mathcal{F}^*$, for some linear space $\mathcal{F}$, with *power* defined by

$$P = < f^* \mid f >, \quad (f, f^*) \in \mathcal{F} \times \mathcal{F}^*, \tag{1}$$

where $< f^* \mid f >$ denotes the duality product, that is, the linear functional $f^* \in \mathcal{F}^*$ acting on $f \in \mathcal{F}$. We call $\mathcal{F}$ the space of *flows* f, and $\mathcal{F}^*$ the space of *efforts* $e = f^*$, with $< e \mid f >$ the power of the pair $(f, e) \in \mathcal{F} \times \mathcal{F}^*$. Typical examples of power variables are pairs of voltages and currents (with, say, the vector of currents being the flow vector, and the vector of voltages being the effort vector), or conjugated pairs of generalized velocities and forces in the mechanical domain. By symmetrizing the definition of power we define a *bilinear form* $\ll, \gg$ on the space of power variables $\mathcal{F} \times \mathcal{F}^*$, given as

$$\begin{aligned} \ll (f^a, e^a), (f^b, e^b) \gg := < e^a \mid f^b > + < e^b \mid f^a >, \\ (f^a, e^a), (f^b, e^b) \in \mathcal{F} \times \mathcal{F}^*. \end{aligned} \tag{2}$$

Definition 1 *[2,4] A (constant) Dirac structure on $\mathcal{F} \times \mathcal{F}^*$ is a subspace*

$$\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$$

such that $\mathcal{D} = \mathcal{D}^\perp$, where $\perp$ denotes orthogonal complement with respect to the indefinite bilinear form $\ll, \gg$.

It follows that $< e \mid f >= 0$ for all $(f, e) \in \mathcal{D}$, and hence any Dirac structure is *power-conserving*. Furthermore, if $\mathcal{F}$ is finite-dimensional, then any Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ satisfies $\dim \mathcal{D} = \dim \mathcal{F}$.

Remark 2 *For many systems, especially those with $3-D$ mechanical components, the interconnection structure is actually modulated by the energy or geometric variables. This leads to the notion of non-constant Dirac structures on manifolds, see e.g. [2,4,3,20,19]. For simplicity of exposition we focus in the current paper on the constant case, although everything can be extended to the case of Dirac structures on manifolds.*

Dirac structures on finite-dimensional linear spaces admit different *representations*. Here we just mention one type that will be used in the sequel. Every Dirac structure $\mathcal{D}$ can be represented in *kernel representation* as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid Ff + Ee = 0\} \tag{3}$$

for linear maps $F : \mathcal{F} \to \mathcal{V}$ and $E : \mathcal{F}^* \to \mathcal{V}$ satisfying

$$\begin{aligned} (i) \quad & EF^* + FE^* = 0, \\ (ii) \quad & \operatorname{rank} F + E = \dim \mathcal{F}, \end{aligned} \tag{4}$$

where $\mathcal{V}$ is a linear space with the same dimension as $\mathcal{F}$, and where $F^* : \mathcal{V}^* \to \mathcal{F}^*$ and $E^* : \mathcal{V}^* \to \mathcal{F}^{**} = \mathcal{F}$ are the adjoint maps of F and E, respectively. It follows that $\mathcal{D}$ can be also written in *image representation* as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid f = E^*\lambda, e = F^*\lambda, \lambda \in \mathcal{V}^*\} \tag{5}$$

Sometimes it will be useful to relax the choice of the linear mappings F and E by allowing $\mathcal{V}$ to be a linear space of dimension greater than the dimension of $\mathcal{F}$. In this case we shall speak of *relaxed* kernel and image representations.
Matrix kernel and image representations are obtained by choosing linear coordinates for $\mathcal{F}$, $\mathcal{F}^*$ and $\mathcal{V}$. Indeed, take any basis $f_1, \cdots, f_n$ for $\mathcal{F}$ and the *dual basis* $e_1 = f_1^*, \cdots, e_n = f_n^*$ for $\mathcal{F}^*$, where $\dim \mathcal{F} = n$. Furthermore, take any set of linear coordinates for $\mathcal{V}$. Then the linear maps F and E are represented by $n \times n$ matrices F and E satisfying $EF^T + FE^T = 0$ and $\operatorname{rank}[F|E] = \dim \mathcal{F}$. In the case of a relaxed kernel and image representation F and E will be $n' \times n$ matrices with $n' > n$.

2.2 Port-Hamiltonian systems

Consider a lumped-parameter physical system given by a power-conserving interconnection defined by a constant Dirac structure $\mathcal{D}$, and a number of energy-storing elements with total vector of energy-variables x. For simplicity we assume that the energy-variables are living in a *linear* space $\mathcal{X}$, although everything can be generalized to the case of manifolds (see Remark 2). The constitutive relations of the energy-storing elements are specified by their individual stored energies, leading to a *total energy* (or Hamiltonian) $H(x)$.

The space of flow variables for the Dirac structure $\mathcal{D}$ is split into $\mathcal{X} \times \mathcal{F}$ with $f_x \in \mathcal{X}$ the flows corresponding to the energy-storing elements, and $f \in \mathcal{F}$ denoting the remaining flows (corresponding to dissipative elements and external ports). Correspondingly, the space of effort variables is split as $\mathcal{X}^* \times \mathcal{F}^*$, with $e_x \in \mathcal{X}^*$ the efforts corresponding to the energy-storing elements and $e \in \mathcal{F}^*$ the remaining efforts. Thus $\mathcal{D} \subset \mathcal{X} \times \mathcal{X}^* \times \mathcal{F} \times \mathcal{F}^*$.
On the other hand, the vector of *flows* of the energy-storing elements is given by $\dot{x}$, and the vector of *efforts*

is given by $\frac{\partial H}{\partial x}(x)$. (We will write both vectors throughout as *column* vectors; in particular, $\frac{\partial H}{\partial x}(x)$ is the column vector with $i-$th component given by $\frac{\partial H}{\partial x_i}(x)$.) Indeed, the energy storing elements satisfy the total energy balance $\frac{dH}{dt}(x(t)) = \frac{\partial^T H}{\partial x}(x(t))\dot{x}(t)$. The flows and efforts of the energy-storing elements are interconnected by setting $f_x = -\dot{x}$ (the minus sign is included to have a consistent power flow direction; see the discussion in the next section) and $e_x = \frac{\partial H}{\partial x}(x)$. By substitution of the interconnection constraints into the specification of the Dirac structure $\mathcal{D}$, that is, $(f_x, e_x, f, e) \in \mathcal{D}$, this leads to the dynamical system

$$(-\dot{x}(t), \frac{\partial H}{\partial x}(t), f(t), e(t)) \in \mathcal{D}, \tag{6}$$

called a *port-Hamiltonian system*. Because of the power-conserving property of Dirac structures we immediately obtain the *power balance*

$$\begin{aligned}\frac{dH(x(t))}{dt} = \frac{\partial^T H}{\partial x}(x(t))\dot{x}(t) = \\ - < e_x(t) \mid f_x(t) >=< e(t) \mid f(t) >, \end{aligned}\tag{7}$$

expressing that the increase of of internal energy of the port-Hamiltonian system is equal to the externally supplied power.

Equational representations of the port-Hamiltonian system (6) are obtained by choosing a specific representation of the Dirac structure $\mathcal{D}$. For example, if $\mathcal{D}$ is given in matrix kernel representation

$$\begin{aligned}\mathcal{D} = \{(f_x, e_x, f, e) \in \mathcal{X} \times \mathcal{X}^* \times \mathcal{F} \times \mathcal{F}^* \\ \mid F_x f_x + E_x e_x + Ff + Ee = 0\},\end{aligned}\tag{8}$$

with $E_x F_x^T + F_x E_x^T + EF^T + FE^T = 0$ and rank $[F_x \vdots E_x \vdots F \vdots E] = \dim(\mathcal{X} \times \mathcal{F})$, then the port-Hamiltonian system is given by the equations

$$F_x \dot{x}(t) = E_x \frac{\partial H}{\partial x}(x(t)) + Ff(t) + Ee(t) \tag{9}$$

consisting in general of differential equations *and* algebraic equations in the state variables x (DAEs), together with equations relating the state variables to the external port variables f, e.

An important special case of port-Hamiltonian systems is the class of *input-state-output port-Hamiltonian systems*, where there are no algebraic constraints on the state variables, and the flow and effort variables f and e are split into power-conjugate input-output pairs (u, y):

$$\begin{aligned}\dot{x} &= J(x)\tfrac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\tfrac{\partial H}{\partial x}(x)\end{aligned} \quad x \in \mathcal{X} \tag{10}$$

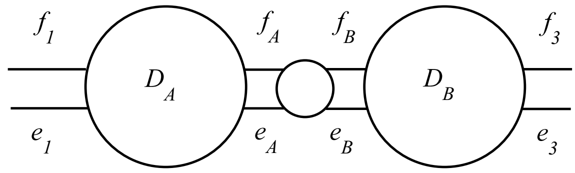

Fig. 1. The composition of $\mathcal{D}_A$ and $\mathcal{D}_B$.

where the matrix $J(x)$ is skew-symmetric, that is $J(x) = -J^T(x)$. The Dirac structure of the system is given by the graph of the skew-symmetric map

$$\begin{bmatrix} -J(x) & -g(x) \\ g^T(x) & 0 \end{bmatrix} \tag{11}$$

3 Composition of Dirac structures

First we study the composition of *two* Dirac structures with partially shared variables. Consider a Dirac structure $\mathcal{D}_A$ on a product space $\mathcal{F}_1 \times \mathcal{F}_2$ of two linear spaces $\mathcal{F}_1$ and $\mathcal{F}_2$, and another Dirac structure $\mathcal{D}_B$ on a product space $\mathcal{F}_2 \times \mathcal{F}_3$, with $\mathcal{F}_3$ being an additional linear space. The space $\mathcal{F}_2$ is the space of shared flow variables, and $\mathcal{F}_2^*$ the space of shared effort variables; see Figure 1. Since the *incoming* power in $\mathcal{D}_A$ due to the power variables in $\mathcal{F}_2 \times \mathcal{F}_2^*$ should equal the *outgoing* power from $\mathcal{D}_B$ we cannot simply equate the flows f_A and f_B and the efforts e_A and e_B, but instead we define the interconnection constraints as

$$\begin{aligned} f_A &= -f_B \in \mathcal{F}_2 \\ e_A &= \ \ e_B \ \ \in \mathcal{F}_2^* \end{aligned} \tag{12}$$

Thus the *composition* of the Dirac structures $\mathcal{D}_A$ and $\mathcal{D}_B$, denoted by $\mathcal{D}_A \parallel \mathcal{D}_B$, is defined as

$$\begin{aligned}\mathcal{D}_A \parallel \mathcal{D}_B := \{(f_1, e_1, f_3, e_3) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^* \mid \\ \exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{F}_2^* \text{ s.t. } (f_1, e_1, f_2, e_2) \in \mathcal{D}_A \\ \text{and } \ (-f_2, e_2, f_3, e_3) \in \mathcal{D}_B\}\end{aligned} \tag{13}$$

The fact that the composition of two Dirac structures is again a Dirac structure has been proved in [21,3]. Here we provide a simpler alternative proof (inspired by a result in [14]), which provides a constructive way to derive the equations of the composed Dirac structure from the equations of the individual Dirac structures. Furthermore, this proof will also allow us to study the *regularity* of the composition in the next subsection.

Theorem 3 *Let $\mathcal{D}_A$, $\mathcal{D}_B$ be Dirac structures as in Definition 1 (defined with respect to $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$, respectively $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$, and their bilinear forms). Then $\mathcal{D}_A \parallel \mathcal{D}_B$ is a Dirac structure with respect to the bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$.*

Proof We make use of the following basic fact from linear algebra:

$$[(\exists \lambda \text{ s.t. } A\lambda = b)] \Leftrightarrow [\forall \alpha \text{ s.t. } \alpha^T A = 0 \Rightarrow \alpha^T b = 0]$$

Consider $\mathcal{D}_A$, $\mathcal{D}_B$ given in matrix image representation as

$$\begin{aligned} \mathcal{D}_A &= \operatorname{im} \begin{bmatrix} E_1 & F_1 & E_{2A} & F_{2A} & 0 & 0 \end{bmatrix}^T \\ \mathcal{D}_B &= \operatorname{im} \begin{bmatrix} 0 & 0 & E_{2B} & F_{2B} & E_3 & F_3 \end{bmatrix}^T \end{aligned} \tag{14}$$

Then,

$$\begin{gathered} (f_1, e_1, f_3, e_3) \in \mathcal{D}_A \parallel \mathcal{D}_B \Leftrightarrow \\ \Leftrightarrow \exists \lambda_A, \lambda_B \text{ s.t. } \begin{bmatrix} f_1 & e_1 & 0 & 0 & f_3 & e_3 \end{bmatrix}^T = \\ = \begin{bmatrix} E_1 & F_1 & E_{2A} & F_{2A} & 0 & 0 \\ 0 & 0 & E_{2B} & -F_{2B} & E_3 & F_3 \end{bmatrix}^T \begin{bmatrix} \lambda_A \\ \lambda_B \end{bmatrix} \Leftrightarrow \\ \Leftrightarrow \forall (\beta_1, \alpha_1, \beta_2, \alpha_2, \beta_3, \alpha_3) \text{ s.t.} \\ (\beta_1^T \alpha_1^T \beta_2^T \alpha_2^T \beta_3^T \alpha_3^T) \begin{bmatrix} E_1 & F_1 & E_{2A} & F_{2A} & 0 & 0 \\ 0 & 0 & E_{2B} & -F_{2B} & E_3 & F_3 \end{bmatrix}^T = 0, \\ \beta_1^T f_1 + \alpha_1^T e_1 + \beta_3^T f_3 + \alpha_3^T e_3 = 0 \Leftrightarrow \\ \Leftrightarrow \forall (\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3) \text{s.t.} \\ \begin{bmatrix} F_1 & E_1 & F_{2A} & E_{2A} & 0 & 0 \\ 0 & 0 & -F_{2B} & E_{2B} & F_3 & E_3 \end{bmatrix} \begin{bmatrix} \alpha_1^T & \beta_1^T & \alpha_2^T & \beta_2^T & \alpha_3^T & \beta_3^T \end{bmatrix} = 0, \\ \beta_1^T f_1 + \alpha_1^T e_1 + \beta_3^T f_3 + \alpha_3^T e_3 = 0 \Leftrightarrow \\ \Leftrightarrow \forall (\alpha_1, \beta_1, \alpha_3, \beta_3) \in \mathcal{D}_A \parallel \mathcal{D}_B, \\ \beta_1^T f_1 + \alpha_1^T e_1 + \beta_3^T f_3 + \alpha_3^T e_3 = 0 \Leftrightarrow \\ \Leftrightarrow (f_1, e_1, f_3, e_3) \in (\mathcal{D}_A \parallel \mathcal{D}_B)^\perp \end{gathered}$$

Thus $\mathcal{D}_A \parallel \mathcal{D}_B = (\mathcal{D}_A \parallel \mathcal{D}_B)^\perp$ and is a Dirac structure. □

In the following theorem an explicit expression for the composition of two Dirac structures in terms of matrix kernel/image representations is given.

Theorem 4 *Let $\mathcal{F}_i, i = 1, 2, 3,$ be finite-dimensional linear spaces with $\dim \mathcal{F}_i = n_i$. Consider Dirac structures $\mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$, $n_A = \dim \mathcal{F}_1 \times \mathcal{F}_2 = n_1 + n_2$, $\mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$, $n_B = \dim \mathcal{F}_2 \times \mathcal{F}_3 = n_2 + n_3$, given by relaxed matrix kernel/image representations $(F_A, E_A) = ([F_1|F_{2A}], [E_1|E_{2A}])$, (F_A, E_A) $n'_A \times n_A$ matrices, $n'_A \geq n_A$, respectively $(F_B, E_B) = ([F_{2B}|F_3], [E_{2B}|E_3])$, (F_B, E_B) $n'_B \times n_B$ matrices, $n'_B \geq n_B$. Define the $(n'_A + n'_B) \times 2n_2$ matrix*

$$M = \begin{bmatrix} F_{2A} & E_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix} \tag{15}$$

and let L_A, L_B be $m \times n'_A$, respectively $m \times n'_B$, matrices with

$$L = [L_A | L_B], \qquad \ker L = \operatorname{im} M \tag{16}$$

Then

$$\begin{aligned} F &= [L_A F_1 | L_B F_3] \\ E &= [L_A E_1 | L_B E_3] \end{aligned} \tag{17}$$

is a relaxed matrix kernel representation of $\mathcal{D}_A \parallel \mathcal{D}_B$.

Proof. Consider the notation corresponding to Figure 1 where for any $\lambda_A \in \mathbb{R}^{n'_A}$, $\lambda_B \in \mathbb{R}^{n'_B}$ their associated elements in $\mathcal{D}_A$, respectively $\mathcal{D}_B$, are given by

$$\begin{bmatrix} f_1 \\ e_1 \\ f_A \\ e_A \end{bmatrix} = \begin{bmatrix} E_1^T \\ F_1^T \\ E_{2A}^T \\ F_{2A}^T \end{bmatrix} \lambda_A; \qquad \begin{bmatrix} f_3 \\ e_3 \\ f_B \\ e_B \end{bmatrix} = \begin{bmatrix} E_3^T \\ F_3^T \\ E_{2B}^T \\ F_{2B}^T \end{bmatrix} \lambda_B \tag{18}$$

Since

$$\begin{gathered} \begin{bmatrix} E_{2A}^T \\ F_{2A}^T \end{bmatrix} \lambda_A = \begin{bmatrix} f_A \\ e_A \end{bmatrix} = \begin{bmatrix} -f_B \\ e_B \end{bmatrix} = \begin{bmatrix} -E_{2B}^T \\ F_{2B}^T \end{bmatrix} \lambda_B \\ \Leftrightarrow \begin{bmatrix} \lambda_A \\ \lambda_B \end{bmatrix} \in \ker M^T \end{gathered} \tag{19}$$

it follows that $(f_1, f_3, e_1, e_3) \in \mathcal{D}_A \parallel \mathcal{D}_B$ if and only if $\exists \begin{bmatrix} \lambda_A^T \lambda_B^T \end{bmatrix}^T \in \ker M^T$ such that (18) holds. By (16) we can write $\begin{bmatrix} \lambda_A^T \lambda_B^T \end{bmatrix}^T \in \ker M^T$ as

$$\begin{bmatrix} \lambda_A \\ \lambda_B \end{bmatrix} = \begin{bmatrix} L_A^T \\ L_B^T \end{bmatrix} \lambda, \ \lambda \in \mathbb{R}^m \tag{20}$$

Substituting (20) in (18) we obtain

$$\begin{aligned} \mathcal{D}_A \parallel \mathcal{D}_B = \{ & (f_1, e_1, f_3, e_3) \mid \\ & \begin{bmatrix} f_1 \\ e_1 \\ f_3 \\ e_3 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} E_1^T \\ F_1^T \end{bmatrix} L_A^T \\ \begin{bmatrix} E_3^T \\ F_3^T \end{bmatrix} L_B^T \end{bmatrix} \lambda, \ \lambda \in \mathbb{R}^m \} \end{aligned} \tag{21}$$

which corresponds to the representation (17). □

Remark 5 *The minimal number of rows m in the definition of the matrix L in (16) is given as $m = \dim \ker M^T$ (since $\ker L = \operatorname{im} M$ is equivalent to $\operatorname{im} L^T = \ker M^T$).*

Remark 6 *The relaxed kernel/image representation (17) can be readily understood by premultiplying the* *equations characterizing the composition of $\mathcal{D}_A$ with $\mathcal{D}_B$*

$$\begin{bmatrix} F_1 & E_1 & F_{2A} & E_{2A} & 0 & 0 \\ 0 & 0 & -F_{2B} & E_{2B} & F_3 & E_3 \end{bmatrix} \begin{bmatrix} f_1 \\ e_1 \\ f_2 \\ e_2 \\ f_3 \\ e_3 \end{bmatrix} = 0, \qquad (22)$$

by the matrix $L = [L_A | L_B]$. Since $LM = 0$ this results as in (17) in the relaxed kernel representation

$$L_A F_1 f_1 + L_A E_1 e_1 + L_B F_3 f_3 + L_B E_3 e_3 = 0 \qquad (23)$$

It readily follows that the power-conserving interconnection of *any* number of Dirac structures is again a Dirac structure; see also [13,21]). Indeed, consider ℓ Dirac structures $\mathcal{D}_k \subset \mathcal{F}_k \times \mathcal{F}_k^* \times \mathcal{F}_{Ik} \times \mathcal{F}_{Ik}^*$, $k = 1, \cdots, \ell$, interconnected to each other via a Dirac structure $\mathcal{D}_I \subset \mathcal{F}_{I1} \times \mathcal{F}_{I1}^* \times \cdots \times \mathcal{F}_{I\ell} \times \mathcal{F}_{I\ell}^*$. This can be regarded as the composition of the *product* Dirac structure $\mathcal{D}_1 \times \cdots \times \mathcal{D}_\ell$ with the interconnection Dirac structure $\mathcal{D}_I$. Hence by the above theorem the result is again a Dirac structure.

Furthermore, it is immediate that the composition of Dirac structures is *associative* in the following sense. Given two Dirac structures $\mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ and $\mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$, and their composition $\mathcal{D}_A \parallel \mathcal{D}_B$. Now compose the composed Dirac structure $\mathcal{D}_A \parallel \mathcal{D}_B$ with a *third* Dirac structure $\mathcal{D}_C \subset \mathcal{F}_3 \times \mathcal{F}_3^* \times \mathcal{F}_4 \times \mathcal{F}_4^*$, resulting in the composition $(\mathcal{D}_A \parallel \mathcal{D}_B) \parallel \mathcal{D}_C$. It is immediately checked that the same composed Dirac structure results from first composing $\mathcal{D}_B$ with $\mathcal{D}_C$, and then composing the outcome with $\mathcal{D}_A$, that is

$$(\mathcal{D}_A \parallel \mathcal{D}_B) \parallel \mathcal{D}_C = \mathcal{D}_A \parallel (\mathcal{D}_B \parallel \mathcal{D}_C)$$

Hence we may as well omit the brackets, and simply write $\mathcal{D}_A \parallel \mathcal{D}_B \parallel \mathcal{D}_C$.

Remark 7 *Instead of the canonical interconnection $f_A = -f_B$, $e_A = e_B$ another standard power-conserving interconnection is the 'gyrative' interconnection*

$$f_A = e_B, \quad f_B = -e_A \qquad (24)$$

(The standard feedback interconnection, regarding f_A, f_B as inputs, and e_A, e_B as outputs, is of this type.) Composition of two Dirac structures $\mathcal{D}_A, \mathcal{D}_B$ by this gyrative interconnection also results in a Dirac structure, since it equals the interconnection $\mathcal{D}_A \parallel \mathcal{D}_I \parallel \mathcal{D}_B$, where $\mathcal{D}_I$ is the 'symplectic' Dirac structure given by

$$f_{IA} = -e_{IB}, \quad f_{IB} = e_{IA} \qquad (25)$$

interconnected to $\mathcal{D}_A$ and $\mathcal{D}_B$ via the canonical interconnections $f_{IA} = -f_A, e_{IA} = e_A$, $f_{IB} = -f_B, e_{IB} = e_B$.

3.1 Regularity of compositions

In this subsection we study a particular property in the composition of Dirac structures, namely the property that the variables corresponding to the ports through which the connection takes place (the *internal* power variables) are uniquely determined by the values of the *external* power variables.

Definition 8 *Given two Dirac structures $\mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ and $\mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$. Their composition is said to be regular if the values of the power variables in $\mathcal{F}_2 \times \mathcal{F}_2^*$ are uniquely determined by the values of the power variables in $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$; that is, the following implication holds*

$$\begin{aligned} &(f_1, e_1, f_2, e_2) \in \mathcal{D}_A,\ (-f_2, e_2, f_3, e_3) \in \mathcal{D}_B \\ &(f_1, e_1, \tilde{f}_2, \tilde{e}_2) \in \mathcal{D}_A,\ (-\tilde{f}_2, \tilde{e}_2, f_3, e_3) \in \mathcal{D}_B \end{aligned} \Longrightarrow$$
$$\Longrightarrow f_2 = \tilde{f}_2, e_2 = \tilde{e}_2 \qquad (26)$$

Proposition 9 *The composition of two Dirac structures $\mathcal{D}_A$ and D_B given in matrix kernel representation by $([F_1|F_{2A}], [E_1|E_{2A}])$ and $([F_3|F_{2B}], [E_3|E_{2B}])$, respectively, is a regular composition if and only if the $(n_1 + 2n_2 + n_3) \times 2n_2$ matrix M defined in (15) is of full rank ($= 2n_2$).*

Proof Let $(f_1, e_1, f_3, e_3) \in \mathcal{D}_A \parallel \mathcal{D}_B$, and let (f_2, e_2) be such that $(f_1, e_1, f_2, e_2) \in \mathcal{D}_A$, $(f_3, e_3, -f_2, e_2) \in \mathcal{D}_B$. Then (f_2', e_2') satisfies

$$(f_1, e_1, f_2', e_2') \in \mathcal{D}_A, (f_3, e_3, -f_2', e_2') \in \mathcal{D}_B \Leftrightarrow$$
$$(\tilde{f}_2, \tilde{e}_2) := (f_2 - f_2', e_2 - e_2') \ \text{ satisfies } \begin{cases} (0, 0, \tilde{f}_2, \tilde{e}_2) \in \mathcal{D}_A \\ (0, 0, -\tilde{f}_2, \tilde{e}_2) \in \mathcal{D}_B \end{cases}$$
$$\Leftrightarrow \begin{cases} [F_1|E_1|F_{2A}|E_{2A}] \begin{bmatrix} 0 & 0 & \tilde{f}_2^T & \tilde{e}_2^T \end{bmatrix}^T = 0 \\ [F_3|E_3| - F_{2B}|E_{2B}] \begin{bmatrix} 0 & 0 & \tilde{f}_2^T & \tilde{e}_2^T \end{bmatrix}^T = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} [F_{2A}|E_{2A}][\tilde{f}_2^T \tilde{e}_2^T]^T = 0 \\ [F_{2B}| - E_{2B}][\tilde{f}_2^T \tilde{e}_2^T]^T = 0 \end{cases} \Leftrightarrow [\tilde{f}_2^T \tilde{e}_2^T]^T \in \ker M$$

Hence $\tilde{f}_2 = 0, \tilde{e}_2 = 0$ if and only if $\ker M = 0$. □

Other ways to interpret regularity immediately follow. In view of the image representations of the Dirac structures $\mathcal{D}_A$ and $\mathcal{D}_B$ the matrix M has full rank if and only if

$$\mathcal{D}_A^\pi + \mathcal{D}_B^\pi = \mathcal{F}_2 \times \mathcal{F}_2^* \qquad (27)$$

where the projections $\mathcal{D}_A^\pi, \mathcal{D}_B^\pi$ are defined as $\mathcal{D}_A^\pi = \{(f_2, e_2) \in \mathcal{F}_2 \times \mathcal{F}_2^* \mid \exists f_1, e_1 \text{ s.t. } (f_1, e_1, f_2, e_2) \in \mathcal{D}_A\}$ and similarly for $\mathcal{D}_B^\pi$. Hence the composition $\mathcal{D}_A \parallel \mathcal{D}_B$

is regular if and only if (27) holds, which means that every value $(f_2, e_2) \in \mathcal{F}_2 \times \mathcal{F}_2^*$ can be achieved as a linear combination $(f_2', e_2') + (f_2'', e_2'')$ by properly selecting (f_1, e_1) and (f_3, e_3) satisfying $(f_1, e_1, f_2', e_2') \in \mathcal{D}_A$ and $(f_2'', e_2'', f_3, e_3) \in \mathcal{D}_B$.

Furthermore, we note that if the matrix M has full rank $(= 2n_2)$ then $\dim \ker M = (n_1 + 2n_2 + n_3) - 2n_2 = n_1 + n_3$, and hence the matrix L as defined in Theorem 4 has $n_1 + n_3$ rows. Thus if we start in Theorem 4 from ordinary (that is, non-relaxed) matrix kernel representations of $\mathcal{D}_A$ and $\mathcal{D}_B$ then the matrix kernel/image representation F, E of the composition $\mathcal{D}_A \parallel \mathcal{D}_B$ defined in (17) is again an ordinary kernel/image representation. In fact, it follows that the matrix kernel/image representation defined in (17) is ordinary *if and only if* the composition $\mathcal{D}_A \parallel \mathcal{D}_B$ is regular. Summarizing

Proposition 10 *The composition $\mathcal{D}_A \parallel \mathcal{D}_B$ is regular if and only if (27) holds, if and only if the matrix kernel/image representation defined in (17) (starting from ordinary kernel/image representations for $\mathcal{D}_A$ and $\mathcal{D}_B$) is ordinary.*

Finally, still another way to characterize regularity is to consider the *independency* of the equations describing $\mathcal{D}_A$ and $\mathcal{D}_B$. (A similar notion of regularity of interconnection is employed in the behavioral theory of interconnection of dynamical systems, c.f. [28].)

Proposition 11 *The composition of two Dirac structures $\mathcal{D}_A$ and $\mathcal{D}_B$, whose individual matrix kernel representations define a set of $n_1 + n_2$, respectively $n_2 + n_3$, independent equations, is regular if and only if the resulting $n_1 + 2n_2 + n_3$ equations obtained by taking together the equations of $\mathcal{D}_A$ and $\mathcal{D}_B$ are independent.*

Proof The $n_1 + 2n_2 + n_3$ equations (22) are independent if and only if the dimension of the kernel of the matrix in (22) is equal to $2n_1 + 2n_2 + 2n_3 - (n_1 + 2n_2 + n_3) = n_1 + n_3$. Because the dimension of $\mathcal{D}_A \parallel \mathcal{D}_B$ is equal to $n_1 + n_3$ (since $\mathcal{D}_A \parallel \mathcal{D}_B$ is a Dirac structure) it follows that the equations (22) are independent if and only if (f_2, e_2) in (22) is determined by (f_1, e_1) and (f_3, e_3). □

Example 12 *A simple example of a non-regular composition is a port-Hamiltonian system with dependent output constraints. Indeed, consider an input-state-output port-Hamiltonian system (10). In kernel representation its Dirac structure is given as*

$$\mathcal{D}_A = \{(f_x, e_x, u, y) \mid \begin{bmatrix} I \\ 0 \end{bmatrix} f_x + \begin{bmatrix} J \\ g^T \end{bmatrix} e_x + \begin{bmatrix} g \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -I \end{bmatrix} y = 0\} \quad (28)$$

Consider the composition with the Dirac structure $\mathcal{D}_B$ corresponding to the zero-output constraint $y = g^T \frac{\partial H}{\partial x} = 0$, i.e., $\mathcal{D}_B = \{(u, y) \mid y = 0\}$. The matrix M in this case is given by $M = \begin{bmatrix} g & 0 \\ 0 & -I \\ 0 & I \end{bmatrix}$, which has full rank if and only if $\operatorname{rank} g = \dim y$. Hence if $\operatorname{rank} g < \dim y$, the composition is not regular, and the input variable u is not uniquely determined. This irregularity is common in mechanical systems where dependent kinematic constraints lead to non-uniqueness of the constraint forces. (A typical example is a table with four legs standing on the ground.)

4 Scattering representation

In this section we show how by using in the total space of power variables $\mathcal{F} \times \mathcal{F}^*$ a *different splitting* than the 'canonical' duality splitting (in flows $f \in \mathcal{F}$ and efforts $e \in \mathcal{F}^*$), we may obtain other useful representations of Dirac structures (and port-Hamiltonian systems).

Consider the space of power variables given in general form as $\mathcal{F} \times \mathcal{F}^*$, for some finite-dimensional linear space $\mathcal{F}$. The duality product $< e \mid f >$ defines the instantaneous *power* of the signal $(f, e) \in \mathcal{F} \times \mathcal{F}^*$. The basic idea of a *scattering representation* is to rewrite the power as the *difference* between two non-negative terms, that is, the difference between an *incoming* power and an *outgoing* power. This is accomplished by the introduction of new coordinates for the total space $\mathcal{F} \times \mathcal{F}^*$, based on the canonical bilinear form (2). From a matrix representation of $\ll, \gg$ it immediately follows that $\ll, \gg$ is an *indefinite* bilinear form, which has n singular values $+1$ and n singular values -1 ($n = \dim \mathcal{F}$).

A pair of subspaces $\Sigma^+, \Sigma^- \subset \mathcal{F} \times \mathcal{F}^*$ is called a pair of *scattering subspaces* if

(i) $\Sigma^+ \oplus \Sigma^- = \mathcal{F} \times \mathcal{F}^*$
(ii) $\ll \sigma_1^+, \sigma_2^+ \gg > 0$ for all $\sigma_1^+, \sigma_2^+ \in \Sigma^+$ unequal to 0.
$\ll \sigma_1^-, \sigma_2^- \gg < 0$ for all $\sigma_1^-, \sigma_2^- \in \Sigma^-$ unequal to 0.
(iii) $\ll \sigma^+, \sigma^- \gg = 0$ for all $\sigma^+ \in \Sigma^+, \sigma^- \in \Sigma^-$.

It is readily seen that any pair of scattering subspaces (Σ^+, Σ^-) satisfies $\dim \Sigma^+ = \dim \Sigma^- = \dim \mathcal{F}$. The collection of pairs of scattering subspaces can be characterized as follows.

Lemma 13 *Let (Σ^+, Σ^-) be a pair of scattering subspaces. Then there exists an invertible linear map $R : \mathcal{F} \to \mathcal{F}^*$, with*

$$< (R + R^*) f \mid f >> 0, \qquad \textit{for all } 0 \neq f \in \mathcal{F}, \quad (29)$$

such that

$$\begin{aligned} \Sigma^+ &:= \{(R^{-1} e, e) \in \mathcal{F} \times \mathcal{F}^* \mid e \in \mathcal{F}^*\} \\ \Sigma^- &:= \{(f, -R^* f) \in \mathcal{F} \times \mathcal{F}^* \mid f \in \mathcal{F}\} \end{aligned} \quad (30)$$

Conversely, for any invertible linear map $R : \mathcal{F} \to \mathcal{F}^*$ satisfying (29) the pair (Σ^+, Σ^-) defined in (30) is a pair of scattering subspaces.

Proof. Let (Σ^+, Σ^-) be a pair of scattering subspaces. Since $\ll, \gg$ is positive definite on $\Sigma^+, \Sigma^+ \cap (\mathcal{F} \times 0) = 0$ and $\Sigma^+ \cap (0 \times \mathcal{F}^*) = 0$. Hence we can write Σ^+ as in (30) for some invertible linear map R. Checking positive-definiteness of $\ll, \gg$ on Σ^+ then yields (29). Similarly, $\Sigma^- \cap (\mathcal{F} \times 0) = 0, \Sigma^- \cap (0 \times \mathcal{F}^*) = 0$. Orthogonality of Σ^- with respect to Σ^+ (condition (iii)) implies that Σ^- is given as in (30). Conversely, a direct computation shows that (Σ^+, Σ^-) defined in (30) for R satisfying (29) defines a pair of scattering subspaces. □

The fundamental relation between the representation in terms of power vectors $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ and the scattering representation is given by the following. Let (Σ^+, Σ^-) be a pair of scattering subspaces. Then every pair of power vectors $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ can be also represented as

$$(f, e) = \sigma^+ + \sigma^- \tag{31}$$

for uniquely defined $\sigma^+ \in \Sigma^+, \sigma^- \in \Sigma^-$, called the *wave vectors*. Using orthogonality of Σ^+ w.r.t. Σ^- it immediately follows that for all $(f_i, e_i) = \sigma_i^+ + \sigma_i^-, i = 1, 2$

$$\ll (f_1, e_1), (f_2, e_2) \gg = < \sigma_1^+, \sigma_2^+ >_{\Sigma^+} - < \sigma_1^-, \sigma_2^- >_{\Sigma^-} \tag{32}$$

where $<,>_{\Sigma^+}$ denotes the inner product on Σ^+ defined as the restriction of $\ll, \gg$ to Σ^+, and $<,>_{\Sigma^-}$ denotes the inner product on Σ^- defined as *minus* the restriction of $\ll, \gg$ to Σ^-. Taking $f_1 = f_2 = f, e_1 = e_2 = e$ and thus $\sigma_1^+ = \sigma_2^+ = \sigma^+, \sigma_1^- = \sigma_2^- = \sigma^-$, leads to

$$\begin{aligned} &< e \mid f > = \tfrac{1}{2} \ll (f, e), (f, e) \gg = \\ &\tfrac{1}{2} < \sigma^+, \sigma^+ >_{\Sigma^+} - \tfrac{1}{2} < \sigma^-, \sigma^- >_{\Sigma^-} \end{aligned} \tag{33}$$

Equation (33) yields the following interpretation of the wave vectors. The vector σ^+ can be regarded as the *incoming wave* vector, with half times its squared norm being the *incoming power*, and the vector σ^- is the *outgoing wave* vector, with half times its squared norm being the *outgoing power*.

Remark 14 *Note that the incoming wave vector σ^+ corresponding to (f, e) is zero if and only if $e = -R^* f$. The physical interpretation of this condition is that the incoming wave vector is zero if the port is terminated on the 'matching' resistive relation $e_R = R^* f_R$ (with the standard interconnection $e_R = e, f_R = -f$).*

Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ be a Dirac structure, that is, $\mathcal{D} = \mathcal{D}^\perp$ with respect to $\ll, \gg$. What is its representation in wave vectors? Since $\ll, \gg$ is zero restricted to $\mathcal{D}$ it follows that for every pair of scattering subspaces (Σ^+, Σ^-)

$$\mathcal{D} \cap \Sigma^+ = 0, \quad \mathcal{D} \cap \Sigma^- = 0, \tag{34}$$

and hence (see also [20]) $\mathcal{D}$ can be represented as the graph of an invertible linear map $\mathcal{O} : \Sigma^+ \to \Sigma^-$

$$\mathcal{D} = \{\sigma^+ + \sigma^- \mid \sigma^- = \mathcal{O}\sigma^+, \ \ \sigma^+ \in \Sigma^+\} \tag{35}$$

Furthermore, by (32) $< \sigma_1^+, \sigma_2^+ >_{\Sigma^+} = < \mathcal{O}\sigma_1^+, \mathcal{O}\sigma_2^+ >_{\Sigma^-}$ for every $\sigma_1^+, \sigma_2^+ \in \Sigma^+$, and thus

$$\mathcal{O} : (\Sigma^+, <,>_{\Sigma^+}) \to (\Sigma^-, <,>_{\Sigma^-}) \tag{36}$$

is a unitary map (isometry). Conversely, every unitary map $\mathcal{O}$ as in (36) defines a Dirac structure by (35). Thus for every pair of scattering subspaces (Σ^+, Σ^-) we have a one-to-one correspondence between unitary maps (36) and Dirac structures $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$.

4.1 Inner product scattering representations

A particular useful class of scattering subspaces (Σ^+, Σ^-) are those defined by an invertible map $R : \mathcal{F} \to \mathcal{F}^*$ satisfying (29) such that $R = R^*$. In this case R is determined by the *inner product* on $\mathcal{F}$ defined as

$$< f_1, f_2 >_R := < R f_1 \mid f_2 > = < R f_2 \mid f_1 > \tag{37}$$

or equivalently by the inner product on $\mathcal{F}^*$ defined as

$$< e_1, e_2 >_{R^{-1}} := < e_2 \mid R^{-1} e_1 > = < e_1 \mid R^{-1} e_2 > \tag{38}$$

In this case we may define an explicit representation of the pair of scattering subspaces (Σ^+, Σ^-) as follows. Define for every $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ the pair s^+, s^- by

$$\begin{aligned} s^+ &:= \tfrac{1}{\sqrt{2}}(e + Rf) \in \mathcal{F}^* \\ s^- &:= \tfrac{1}{\sqrt{2}}(e - Rf) \in \mathcal{F}^* \end{aligned} \tag{39}$$

Let s_i^+, s_i^- correspond to $(f_i, e_i), i = 1, 2$. Then by direct computation

$$\begin{aligned} 2 < s_1^+, s_2^+ >_{R^{-1}} &= < e_1, e_2 >_{R^{-1}} + < f_1, f_2 >_R \\ &\quad + \ll (f_1, e_1), (f_2, e_2) \gg \\ 2 < s_1^-, s_2^- >_{R^{-1}} &= < e_1, e_2 >_{R^{-1}} + < f_1, f_2 >_R \\ &\quad - \ll (f_1, e_1), (f_2, e_2) \gg \end{aligned} \tag{40}$$

Hence, if $(f_i, e_i) \in \Sigma^+$, or equivalently $s_i^- = e_i - R f_i = 0$, then $2 < s_1^+, s_2^+ >_{R^{-1}} = 2 \ll (f_1, e_1), (f_2, e_2) \gg$, while if $(f_i, e_i) \in \Sigma^-$, or equivalently $s_i^+ = e_i + R f_i = 0$, then

$2 < s_1^-, s_2^- >_{R^{-1}} = -2 \ll (f_1, e_1), (f_2, e_2) \gg$. Thus the mappings

$$\begin{array}{ll} \sigma^+ = (f,e) \in \Sigma^+ \longmapsto s^+ = \frac{1}{\sqrt{2}}(e + Rf) \in \mathcal{F}^* \\ \sigma^- = (f,e) \in \Sigma^- \longmapsto s^- = \frac{1}{\sqrt{2}}(e - Rf) \in \mathcal{F}^* \end{array} \quad (41)$$

are isometries (with respect to the inner products on Σ^+ and Σ^-, and the inner product on $\mathcal{F}^*$ defined by (38)). Hence we may identify the wave vectors σ^+, σ^- with s^+, s^-.

Let us now consider the representation of a Dirac structure $\mathcal{D}$ in terms of the wave vectors (s^+, s^-) (see also the treatment in [20], § 4.3.3). For every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ there exist linear mappings $F : \mathcal{F} \to \mathcal{V}$ and $E : \mathcal{F}^* \to \mathcal{V}$ satisfying (4). Thus for any $(f, e) \in \mathcal{D}$ the wave vectors (s^+, s^-) defined by (41) are given as

$$\begin{array}{lll} s^+ &=& \frac{1}{\sqrt{2}}(F^*\lambda + RE^*\lambda) = \frac{1}{\sqrt{2}}(F^* + RE^*)\lambda \\ s^- &=& \frac{1}{\sqrt{2}}(F^*\lambda - RE^*\lambda) = \frac{1}{\sqrt{2}}(F^* - RE^*)\lambda, \quad \lambda \in \mathcal{V}^* \end{array}$$

The mapping $F^* + RE^*$ is invertible. Indeed, suppose that $(F^* + RE^*)(\lambda) = 0$. By (4(i)) also $EF^*\lambda + FE^*\lambda = 0$. It follows that $ERE^*\lambda + FR^{-1}F^*\lambda = 0$, and hence by positive-definiteness of R and (4(ii)) $\lambda = 0$. Therefore

$$s^- = (F^* - RE^*)(F^* + RE^*)^{-1}s^+ \quad (42)$$

Hence the unitary map $\mathcal{O} : \mathcal{F}^* \to \mathcal{F}^*$ associated with the Dirac structure (recall that we identify Σ^+ and Σ^- with $\mathcal{F}^*$ by (41)) is given as

$$\mathcal{O} = (F^* - RE^*)(F^* + RE^*)^{-1} \quad (43)$$

By adding $EF^* + FE^* = 0$ it follows that

$$(FR^{-1} + E)(F^* + RE^*) = (FR^{-1} - E)(F^* - RE^*)$$

and hence also (since similarly as above it can be shown that $FR^{-1} - E$ is invertible)

$$\mathcal{O} = (FR^{-1} - E)^{-1}(FR^{-1} + E) \quad (44)$$

From here it can be verified that $\mathcal{O}^* R^{-1} \mathcal{O} = R^{-1}$, showing that indeed (as proved before by general considerations) $\mathcal{O} : \mathcal{F}^* \to \mathcal{F}^*$ is a unitary mapping.

Given a kernel/image representation (F, E) for a Dirac structure $\mathcal{D}$, it is obvious that for any invertible map $C : \mathcal{V} \to \mathcal{V}'$ also $\mathcal{D} = \ker C(F + E) = \ker(CF + CE)$. Hence there are infinitely many (F, E) pairs representing $\mathcal{D}$ in kernel/image representation, corresponding to only one $\mathcal{O}$ map in the chosen scattering representation.

Theorem 15 *Consider any inner product R on $\mathcal{F}$ and the resulting scattering representation. The set of (F, E) pairs representing a given Dirac structure $\mathcal{D}$ on $\mathcal{F} \times \mathcal{F}^*$ with scattering representation $\mathcal{O}$ is given as*

$$\{(F, E) \mid F = X(\mathcal{O} + I)R,\ E = X(\mathcal{O} - I), X : \mathcal{F}^* \to \mathcal{V} \textit{ invertible} \} \quad (45)$$

Proof Obviously, any (F, E) pair corresponding to $\mathcal{D}$ can be expressed as $F = (A + B)R, E = A - B$, where $A = \frac{1}{2}(FR^{-1} + E), B = \frac{1}{2}(FR^{-1} - E)$. By (44) the mappings A and B are invertible, while $\mathcal{O} = B^{-1}A$. Hence substituting $A = B\mathcal{O}$ F and E can be expressed as $F = B(\mathcal{O} + I)R, E = B(\mathcal{O} - I)$, and taking $C = B^{-1}$ the following 'canonical' kernel representation for $\mathcal{D}$ is found

$$\begin{cases} F' = (\mathcal{O} + I)R \\ E' = \mathcal{O} - I \end{cases} \quad (46)$$

yielding the parametrization of $\mathcal{D}$ given in (45). □

4.2 Composition in scattering representation

Recall that composition in power vector representation is simply given by the interconnection constraints

$$f_A = -f_B \in \mathcal{F}, e_A = e_B \in \mathcal{F}^* \quad (47)$$

Now consider the scattering representation of the power vectors (f_A, e_A) with respect to an inner product R_A as given by the wave vectors

$$\begin{array}{lll} s_A^+ &:=& \frac{1}{\sqrt{2}}(e_A + R_A f_A) \in \mathcal{F}^* \\ s_A^- &:=& \frac{1}{\sqrt{2}}(e_A - R_A f_A) \in \mathcal{F}^* \end{array} \quad (48)$$

and analogously the scattering representation of the power vectors (f_B, e_B) with respect to another inner product R_B, given by

$$\begin{array}{lll} s_B^+ &:=& \frac{1}{\sqrt{2}}(e_B + R_B f_B) \in \mathcal{F}^* \\ s_B^- &:=& \frac{1}{\sqrt{2}}(e_B - R_B f_B) \in \mathcal{F}^* \end{array} \quad (49)$$

Then the interconnection constraints (47) on the power vectors yield the following interconnection constraints on the wave vectors

$$\begin{array}{lll} s_A^+ - s_B^- &:=& \frac{1}{\sqrt{2}}(R_A - R_B)f_A \\ s_B^+ - s_A^- &:=& \frac{1}{\sqrt{2}}(R_A - R_B)f_A \end{array} \quad (50)$$

together with

$$\begin{array}{lll} s_A^+ - s_A^- &:=& \sqrt{2}R_A f_A \\ s_B^- - s_B^+ &:=& \sqrt{2}R_B f_A \end{array} \quad (51)$$

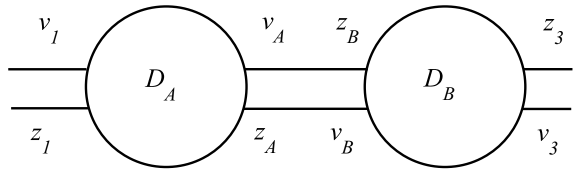

Fig. 2. Composition of $\mathcal{D}_A$ and $\mathcal{D}_B$ using wave vectors.

leading to

$$\begin{aligned} s_A^+ - s_B^- &= s_B^+ - s_A^- \\ R_A^{-1}(s_A^+ - s_A^-) + R_B^{-1}(s_B^+ - s_B^-) &= 0 \end{aligned} \tag{52}$$

The first equation of (52) can be interpreted as a *power balance* of the wave vectors. Indeed, in our convention for power flow s^+ are *incoming* wave vectors for the system and thus *outgoing* wave vectors for the point of interconnection, while s^- are *outgoing* wave vectors for the system and thus *incoming* wave vectors for the point of interconnection. Hence the first equation of (52) states that the loss (= difference) between the outgoing wave vector s_A^+ and the incoming wave vector s_B^- is equal to the loss between the outgoing wave vector s_B^+ and the incoming wave vector s_A^-. The second equation expresses a balance between the loss as seen from A and the loss as seen from B.

The scattering at A is said to be *matching* with the scattering at B if $R_A = R_B$. In this case (47) is equivalent to the following interconnection constraints between the wave vectors:

$$\begin{aligned} s_A^+ &= s_B^- \\ s_B^+ &= s_A^-, \end{aligned} \tag{53}$$

simply expressing that the outgoing wave vector for A equals the incoming wave vector for B, and conversely. In the rest of this section we restrict ourselves to the matching case $R_A = R_B = R$. Also, in order to simplify computations, we consider a coordinate representation such that R is given by the identity matrix (= Euclidean inner product). Furthermore, for ease of notation we denote s_A^+, s_B^+ by v_A, v_B and s_A^-, s_B^- by z_A, z_B. Thus we consider the composition as in Figure 2 of two Dirac structures $\mathcal{D}_A, \mathcal{D}_B$ by the interconnection equations (in scattering representation) $v_A = z_B, z_A = v_B$. By redrawing Figure 2 in standard *feedback interconnection* form as in Figure 3 it is readily seen that this corresponds to the well-known *Redheffer star product* (see e.g. [18]) of $\mathcal{O}_\mathcal{A}$ and $\mathcal{O}_\mathcal{B}$.

Proposition 16 *The scattering representation of $\mathcal{D}_A \parallel \mathcal{D}_B$ is given by $\mathcal{O}_A \star \mathcal{O}_B$, with the unitary mappings $\mathcal{O}_A$ and $\mathcal{O}_B$ being the scattering representation of $\mathcal{D}_A$ and $\mathcal{D}_B$ respectively, and $\star$ denoting the Redheffer star product.*

Note that this immediately yields that the Redheffer star product of two unitary mappings is again a unitary mapping (since $\mathcal{D}_A \parallel \mathcal{D}_B$ is again a Dirac structure.) Explicit formulas for $\mathcal{O}_A \star \mathcal{O}_B$ have been recently obtained in [9] (see also [1]).

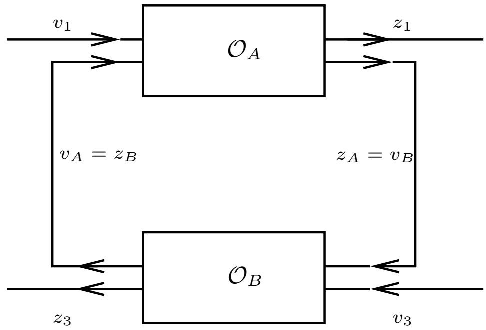

Fig. 3. Figure 2 redrawn as the Redheffer star product of $\mathcal{O}_\mathcal{A}$ and $\mathcal{O}_\mathcal{B}$.

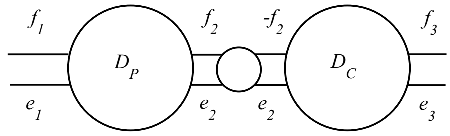

Fig. 4. $\mathcal{D}_P \parallel \mathcal{D}_C$

5 Achievable Dirac structures

A main question in the theory of 'control by interconnection' of port-Hamiltonian systems is to investigate which closed-loop port-Hamiltonian systems can be achieved by interconnecting a *given* plant port-Hamiltonian system P with a *to-be-designed* controller port-Hamiltonian system C. Desired properties of the closed-loop system may e.g. include the internal stability of the closed-loop system and its behavior at an interaction port. The *Impedance Control* problem as formulated in e.g. [8] as the problem of designing the controller system in such a way that the closed-loop system has a desired 'impedance' at the interaction port may be approached from this point of view.

Within the framework of the current paper the control by interconnection problem of port-Hamiltonian systems is restricted to the investigation of the *achievable Dirac structures* of the closed-loop system. That is, given the Dirac structure $\mathcal{D}_P$ of the plant system P and the to-be-designed Dirac structure $\mathcal{D}_C$ of the controller system C, what are the achievable Dirac structures $\mathcal{D}_P \parallel \mathcal{D}_C$ (see Figure 4.

Theorem 17 *Given a plant Dirac structure $\mathcal{D}_P$ with port variables f_1, e_1, f_2, e_2, and a desired Dirac structure $\mathcal{D}$ with port variables f_1, e_1, f_3, e_3. Then there exists a controller Dirac structure $\mathcal{D}_C$ such that $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_C$ if and only if one of the following two equivalent conditions*

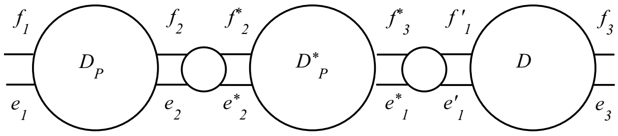

Fig. 5. $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_P^* \parallel \mathcal{D}$.

is satisfied

$$\mathcal{D}_P^0 \subset \mathcal{D}^0 \tag{54}$$
$$\mathcal{D}^\pi \subset \mathcal{D}_P^\pi \tag{55}$$

where

$$\begin{cases} \mathcal{D}_P^0 := \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathcal{D}_P\} \\ \mathcal{D}_P^\pi := \{(f_1, e_1) \mid \exists (f_2, e_2) \ s.t. \ (f_1, e_1, f_2, e_2) \in \mathcal{D}_P\} \\ \mathcal{D}^0 := \{(f_1, e_1) \mid (f_1, e_1, 0, 0) \in \mathcal{D}\} \\ \mathcal{D}^\pi := \{(f_1, e_1) \mid \exists (f_3, e_3) \ s.t. \ (f_1, e_1, f_3, e_3) \in \mathcal{D}\} \end{cases} \tag{56}$$

Remark 18 *A partial version of this theorem was given in [21].*

The following simple proof of Theorem 17 (using an idea from [14]; compare with the proof given in [21]) is based on the following, partially sign-reversed, copy (or 'internal model') $\mathcal{D}_P^*$ of the plant Dirac structure $\mathcal{D}_P$

$$\mathcal{D}_P^* := \{(f_1, e_1, f_2, e_2) \mid (-f_1, e_1, -f_2, e_2) \in \mathcal{D}_P\}, \tag{57}$$

which is easily seen to be a Dirac structure if and only if $\mathcal{D}_P$ is a Dirac structure.

Proof of Theorem 17. First we will show that there exists a controller Dirac structure $\mathcal{D}_C$ such that $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_C$ if and only if the two conditions (54) and (55) are satisfied. At the end we will prove that conditions (54) and (55) are actually equivalent.

Necessity of (54) and (55) is obvious. *Sufficiency* is shown using the controller Dirac structure

$$\mathcal{D}_C := \mathcal{D}_P^* \parallel \mathcal{D}$$

(see Figure 5). To check that $\mathcal{D} \subset \mathcal{D}_P \parallel \mathcal{D}_C$, consider $(f_1, e_1, f_3, e_3) \in \mathcal{D}$. Because $(f_1, e_1) \in \mathcal{D}^\pi$, applying (55) yields that $\exists (f_2, e_2)$ such that $(f_1, e_1, f_2, e_2) \in \mathcal{D}_P$. It follows that $(-f_1, e_1, -f_2, e_2) \in \mathcal{D}_P^*$. Recall the following interconnection constraints in Figure 5:

$$f_2 = -f_2^*, e_2 = e_2^*, f_1^* = -f_1', e_1^* = e_1'$$

By taking $(f_1', e_1') = (f_1, e_1)$ in Figure 5 it follows that $(f_1, e_1, f_3, e_3) \in \mathcal{D}_P \parallel \mathcal{D}_C$. Therefore, $\mathcal{D} \subset \mathcal{D}_P \parallel \mathcal{D}_C$.

To check that $\mathcal{D}_P \parallel \mathcal{D}_C \subset \mathcal{D}$, consider $(f_1, e_1, f_3, e_3) \in \mathcal{D}_P \parallel \mathcal{D}_C$. Then there exist $f_2 = -f_2^*, e_2 = e_2^*$, $f_1^* = -f_1', e_1^* = e_1'$ such that

$$(f_1, e_1, f_2, e_2) \in \mathcal{D}_P \tag{58}$$
$$(f_1^*, e_1^*, f_2^*, e_2^*) \in \mathcal{D}_P^* \Longleftrightarrow (f_1', e_1', f_2, e_2) \in \mathcal{D}_P \tag{59}$$
$$(f_1', e_1', f_3, e_3) \in \mathcal{D} \tag{60}$$

Subtracting (59) from (58), making use of the the linearity of $\mathcal{D}_P$, we get

$$(f_1 - f_1', e_1 - e_1', 0, 0) \in \mathcal{D}_P \Longleftrightarrow (f_1 - f_1', e_1 - e_1') \in \mathcal{D}_P^0 \tag{61}$$

Using (61) and (54) we get

$$(f_1 - f_1', e_1 - e_1', 0, 0) \in \mathcal{D} \tag{62}$$

Finally, adding (60) and (62), we obtain $(f_1, e_1, f_3, e_3) \in \mathcal{D}$, and so $\mathcal{D}_P \parallel \mathcal{D}_C \subset \mathcal{D}$.

Finally we show that conditions (54) and (55) are equivalent. In fact, we prove that $(\mathcal{D}^0)^\perp = \mathcal{D}^\pi$ and the same for $\mathcal{D}_P$. Here, $^\perp$ denotes the orthogonal complement with respect to the canonical bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^*$ defined as

$$\ll (f_1^a, e_1^a), (f_1^b, e_1^b) \gg := < e^a \mid f^b > + < e^b \mid f^a >,$$

for $(f_1^a, e_1^a), (f_1^b, e_1^b) \in \mathcal{F}_1 \times \mathcal{F}_1^*$. Then since $\mathcal{D}_P^0 \subset \mathcal{D}^0$ implies $(\mathcal{D}^0)^\perp \subset (\mathcal{D}_P^0)^\perp$ the equivalence between (54) and (55) is immediate.

In order to show $(\mathcal{D}^0)^\perp = \mathcal{D}^\pi$ first take $(f_1, e_1) \in (\mathcal{D}^\pi)^\perp$, implying that

$$\ll (f_1, e_1), (\tilde{f}_1, \tilde{e}_1) \gg = < e_1 | \tilde{f}_1 > + < \tilde{e}_1 | f_1 > = 0$$

for all $(\tilde{f}_1, \tilde{e}_1)$ for which there exists $\tilde{f}_3, \tilde{e}_3$ such that $(\tilde{f}_1, \tilde{e}_1, \tilde{f}_3, \tilde{e}_3) \in \mathcal{D}$. This immediately implies that $(f_1, e_1, 0, 0) \in \mathcal{D}^\perp = \mathcal{D}$, and thus that $(f_1, e_1) \in \mathcal{D}^0$. Hence, $(\mathcal{D}^\pi)^\perp \subset \mathcal{D}^0$ and thus $(\mathcal{D}^0)^\perp \subset \mathcal{D}^\pi$. To prove the converse inclusion, take $(f_1, e_1) \in \mathcal{D}^\pi$, implying that there exists (f_3, e_3) such that $(f_1, e_1, f_3, e_3) \in \mathcal{D} = \mathcal{D}^\perp$. Hence,

$$< e_1 | \tilde{f}_1 > + < \tilde{e}_1 | f_1 > + < e_3 | \tilde{f}_3 > + < \tilde{e}_3 | f_3 > = 0$$

for all $(\tilde{f}_1, \tilde{e}_1, \tilde{f}_3, \tilde{e}_3) \in \mathcal{D}$ implying $< e_1 | \tilde{f}_1 > + < \tilde{e}_1 | f_1 > = 0$ for all $(\tilde{f}_1, \tilde{e}_1, 0, 0) \in \mathcal{D}$, and thus $(f_1, e_1) \in (\mathcal{D}^0)^\perp$. □

Remark 19 *By allowing $\mathcal{D}_P$ to be interconnected to an interconnection structure $\mathcal{K}_C$ that is not necessarily a Dirac structure we do not gain anything for the set of achievable Dirac structures. Indeed, let $\mathcal{K}_C$ be any subspace (not necessarily a Dirac structure) of the space of variables f_2, e_2, f_3, e_3 and suppose that $\mathcal{D}_P \parallel \mathcal{K}_C = \mathcal{D}$ (where the composition $\mathcal{D}_P \parallel \mathcal{K}_C$ is defined in the same*

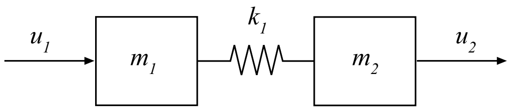

Fig. 6. Port-Hamiltonian plant system P.

way as for Dirac structures). Then, as in the necessity part of the proof of Theorem 17, this implies that (54, 55) are satisfied, and thus, by the sufficiency part of the proof, there also exists a Dirac structure $\mathcal{D}_C$ such that $\mathcal{D}_P \parallel \mathcal{D}_C = \mathcal{D}$. This means that if we want to realize a power-conserving interconnection structure there is no loss of generality in restricting to 'controller' interconnection structures that are power-conserving.

The proof of Theorem 17 immediately provides us with a closed expression for a 'canonical' controller Dirac structure $\mathcal{D}_C$ such that $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_C$:

Proposition 20 *Given a plant Dirac structure $\mathcal{D}_P$, and $\mathcal{D}$ satisfying the conditions of Theorem 17. Then $\mathcal{D}_C := \mathcal{D}_P^* \parallel \mathcal{D}$, with $\mathcal{D}_P^*$ defined as in (57), achieves $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_C$.*

Example 21 *Consider the plant system P*

$$\begin{bmatrix} \dot{q}_1 \\ \dot{p}_1 \\ \dot{q}_2 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_P}{\partial q_1} \\ \frac{\partial H_P}{\partial p_1} \\ \frac{\partial H_P}{\partial q_2} \\ \frac{\partial H_P}{\partial p_2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (63)$$

(see Figure 6), composed by two masses m_1 and m_2, linked by a spring k_1, subject to external forces u_1 and u_2. The state of the plant system is $x_P = (q_1, p_1, q_2, p_2)$, with q_i denoting the positions of both masses and p_i the corresponding momenta, $i = 1, 2$. The Hamiltonian of the plant system P is $H_P(x_P) = \frac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + k_1(q_2 - q_1)^2\right)$ and the Dirac structure $\mathcal{D}_P \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ of P is given in kernel/image representation by (see [20] for an explicit computation)

$$F_P = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&1&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}, \; E_P = \begin{bmatrix} 0&1&0&0&0&0 \\ -1&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&-1&0&0&0 \\ 0&-1&0&0&1&0 \\ 0&0&0&-1&0&1 \end{bmatrix}$$

The desired port-Hamiltonian system Q (Figure 7) is the same as P with the second mass m_2 connected to an extra

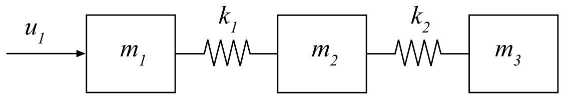

Fig. 7. Port-Hamiltonian desired system Q.

mass m_3 by a spring k_2. The equations of Q are given as

$$\begin{bmatrix} \dot{q}_1 \\ \dot{p}_1 \\ \dot{q}_2 \\ \dot{p}_2 \\ \dot{\Delta q_3} \\ \dot{p}_3 \end{bmatrix} = \begin{bmatrix} 0&1&0&0&0&0 \\ -1&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&-1&0&1&0 \\ 0&0&0&-1&0&1 \\ 0&0&0&0&-1&0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_Q}{\partial q_1} \\ \frac{\partial H_Q}{\partial p_1} \\ \frac{\partial H_Q}{\partial q_2} \\ \frac{\partial H_Q}{\partial p_2} \\ \frac{\partial H_Q}{\partial \Delta q_3} \\ \frac{\partial H_Q}{\partial p_3} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_1$$

with u_1 the external force. The state of Q is $x_Q = (q_1, p_1, q_2, p_2, \Delta q_3, p_3)$, with $q_i, i = 1, 2$, denoting as before the position of masses m_1, m_2 and Δq_3 the elongation of spring k_2. Furthermore, $p_i, i = 1, 2, 3$, denote the momenta of the three masses. The Hamiltonian of Q is $H_Q(x_Q) = \frac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \frac{p_3^2}{m_3} + k_1(q_2 - q_1)^2 + k_2(\Delta q_3)^2\right)$ while the Dirac structure $\mathcal{D} \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$ of Q is given in kernel/image representation as

$$F = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&1 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0 \end{bmatrix}, E = \begin{bmatrix} 0&1&0&0&0&0&0 \\ -1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&-1&0&1&0&0 \\ 0&0&0&-1&0&1&0 \\ 0&0&0&0&-1&0&0 \\ 0&-1&0&0&0&0&1 \end{bmatrix}$$

By construction, $\mathcal{D}$ is trivially achievable from $\mathcal{D}_P$ by interconnection. In the following, this will be formally checked as an illustration of Theorem 17. Furthermore, we will explicitly compute the controller Dirac structure $\mathcal{D}_C$ as defined in Proposition 20 and show how this corresponds to the Dirac structure of the extra mass-spring system, and that the desired Dirac structure $\mathcal{D}$ is indeed obtained by composition of $\mathcal{D}_P$ with $\mathcal{D}_C$. According to Theorem 17, conditions (54) or (55) should be satisfied. This can be most easily checked as follows. Since $\mathcal{D}_P$ is given in kernel representation as $\ker[F_P|E_P]$ it follows that $\mathcal{D}_P^0 = \ker[F_P^0|E_P^0]$, where F_P^0 and E_P^0 are obtained from F_P, respectively E_P, by deleting the columns corresponding to $\mathcal{F}_2$, respectively $\mathcal{F}_2^*$. Similarly, $\mathcal{D}^0$ is obtained from $\mathcal{D} = \ker[F|E]$ as $\mathcal{D}^0 = \ker[F^0|E^0]$, where F^0 and E^0 are obtained from F, respectively E, by deleting the columns corresponding to $\mathcal{F}_3$, respectively $\mathcal{F}_3^*$. Checking the condition (54) $\mathcal{D}_P^0 \subset \mathcal{D}^0$ now amounts to checking that the rows of $[F^0|E^0]$ are linear combinations of the rows of $[F_P^0|E_P^0]$, which is easily seen to be the case *for the Dirac structures $\mathcal{D}_P$ and $\mathcal{D}$ at hand. Proposition 20 defines the controller Dirac structure $\mathcal{D}_C$ as $\mathcal{D}_P^* \parallel \mathcal{D}$ (whose composition with $\mathcal{D}_P$ should be equal to $\mathcal{D}$). Note that $\mathcal{D}_P^*$ is simply given by $F_P^* = -F_P$ and $E_P^* = E_P$. Application of Theorem 4 yields after some calculations that $\mathcal{D}_C$ is given in kernel/image representation as*

$$F_C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{64}$$

This is directly seen to be the Dirac structure of the controller system

$$\begin{aligned} \dot{\Delta q_3} &= \frac{p_3}{m_3} + v \\ \dot{p_3} &= -k_2 \Delta q_3 \\ F &= k_2 \Delta q_3 \end{aligned} \tag{65}$$

which can be identified with a mass-spring system with mass m_3 and spring k_2, with v denoting the velocity of the left-end of the spring k_2 and F the spring force at this point. It directly follows that $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_C$.

In *scattering representation* Proposition 20 takes the following form. First note that if $\mathcal{O}_P$ is the scattering representation of $\mathcal{D}_P$, then the scattering representation of $\mathcal{D}_P^*$ is given by $\mathcal{O}_P^{-1}$. Indeed, if we substitute in (39) $-f$ for f, then s^+ becomes s^- and conversely. Thus the unitary map corresponding to $\mathcal{D}_P^*$ is the *inverse* of the map $\mathcal{O}_P$ corresponding to $\mathcal{D}_P$.

Corollary 22 *Given a plant Dirac structure $\mathcal{D}_P$, and $\mathcal{D}$ satisfying the conditions of Theorem 17, in scattering representation given by $\mathcal{O}_P$, respectively $\mathcal{O}$. Then $\mathcal{D}_C$ with scattering representation $\mathcal{O}_C$ defined by $\mathcal{O}_C := \mathcal{O}_P^{-1} \star \mathcal{O}$ achieves $\mathcal{O} = \mathcal{O}_P \star \mathcal{O}_C$. Hence, under the conditions of Theorem 17, $\mathcal{O} = \mathcal{O}_P \star \mathcal{O}_P^{-1} \star \mathcal{O}$.*

5.1 Achievable Casimirs and constraints

An important application of Theorem 17 concerns the characterization of the *Casimir functions* which can be achieved for the closed-loop system by interconnecting a given plant port-Hamiltonian system with associated Dirac structure $\mathcal{D}_P$ with a controller port-Hamiltonian system with associated Dirac structure $\mathcal{D}_C$. This constitutes a cornerstone for passivity-based control of port-Hamiltonian systems as developed e.g. in [15,16]. Dually, we characterize the achievable *algebraic constraints* for the closed-loop system. In order to explain these notions consider first a port-Hamiltonian system without external (controller or interaction) ports. Also assume for simplicity that there is no resistive port. Thus we consider a state space $\mathcal{X}$ with Dirac structure $\mathcal{D} \subset \mathcal{X} \times \mathcal{X}^*$. Then the following subspaces of $\mathcal{X}$, respectively $\mathcal{X}^*$, are of importance

$$\begin{aligned} G_1 &:= \{f_x \in \mathcal{X} \mid \exists e_x \in \mathcal{X}^* \text{ such that } (f_x, e_x) \in \mathcal{D}\} \\ P_1 &:= \{e_x \in \mathcal{X}^* \mid \exists f_x \in \mathcal{X} \text{ such that } (f_x, e_x) \in \mathcal{D}\} \end{aligned} \tag{66}$$

The subspace G_1 expresses the set of *admissible flows*, and P_1 the set of *admissible efforts.*

A *Casimir function* $K : \mathcal{X} \to \mathbb{R}$ of the port-Hamiltonian system is defined to be a function which is constant along all trajectories of the port-Hamiltonian system, irrespectively of the Hamiltonian H. Since $f_x = -\dot{x}(t) \in G_1$, it follows that $K : \mathcal{X} \to \mathbb{R}$ is a Casimir function if $\frac{dK}{dt}(x(t)) = \frac{\partial^T K}{\partial x}(x(t))\dot{x}(t) = 0$ for all $\dot{x}(t) \in G_1$. Equivalently, this can be formulated by defining the following subspace of the dual space of efforts

$$P_0 = \{e_x \in \mathcal{X}^* \mid (0, e_x) \in \mathcal{D}\} \tag{67}$$

It can be readily seen that $G_1 = P_0^{\perp}$ where $\perp$ denotes orthogonal complement with respect to the duality product $< | >$. Hence K is a Casimir function iff $\frac{\partial K}{\partial x}(x) \in P_0$.

Dually, the algebraic constraints for the port-Hamiltonian system are determined by the space P_1, since necessarily $\frac{\partial^T H}{\partial x}(x) \in P_1$, which will induce constraints on the state variables x. Similar to the above it can be seen that $P_1 = G_0^{\perp}$ where the subspace of flows G_0 is given as

$$G_0 = \{f_x \in \mathcal{X} \mid (f_x, 0) \in \mathcal{D}\} \tag{68}$$

Let us now consider the question of characterizing the set of achievable Casimirs for the closed-loop system $\mathcal{D}_P \parallel \mathcal{D}_C$, where $\mathcal{D}_P$ is the given Dirac structure of the plant port-Hamiltonian system with Hamiltonian H_P, and $\mathcal{D}_C$ is the (to-be-designed) controller Dirac structure. In this case, the Casimirs will depend on the plant state x as well as on the controller state ξ. Since the controller Hamiltonian $H_C(\xi)$ is at our own disposal we will be primarily interested in the dependency of the Casimirs on the plant state x. (Since we want to use the Casimirs for shaping the total Hamiltonian $H + H_C$ to a Lyapunov function, cf. [15,16].)

Consider the notation given in Figure 4, and assume that the ports (f_1, e_1) are connected to the (given) energy storing elements of the plant port-Hamiltonian system (that is, $f_1 = -\dot{x}, e_1 = \frac{\partial H_P}{\partial x}$), while (f_3, e_3) are connected to the (to-be-designed) energy storing elements of a controller port-Hamiltonian system (that is, $f_3 = -\dot{\xi}, e_3 = \frac{\partial H_C}{\partial \xi}$). Note that the number of ports (f_3, e_3) can be freely chosen. The achievable Casimir functions are characterized as follows. $K(x, \xi)$ is an achievable Casimir function if there exists a controller Dirac structure $\mathcal{D}_C$ such that

$$(0, \frac{\partial K}{\partial x}(x,\xi), 0, \frac{\partial K}{\partial \xi}(x,\xi)) \in \mathcal{D}_P \parallel \mathcal{D}_C \qquad (69)$$

Hence for every achievable Casimir function $K(x,\xi)$ the partial gradient $\frac{\partial K}{\partial x}(x,\xi)$ belongs to the space

$$P_{Cas} = \{e_1 \mid \exists \mathcal{D}_C \text{ s.t. } \exists e_3 : (0, e_1, 0, e_3) \in \mathcal{D}_P \parallel \mathcal{D}_C\} \qquad (70)$$

and, conversely (under integrability conditions) for any $e_1 \in P_{Cas}$ there will exist an achievable Casimir function $K(x,\xi)$ such that $\frac{\partial K}{\partial x}(x,\xi) = e_1$. Thus the question of characterizing the achievable Casimirs of the closed-loop system, with respect to their dependence on the plant state x, is translated to finding a characterization of the space P_{Cas}. This is answered by the following theorem.

Theorem 23 *The space P_{Cas} defined in (70) is equal to*

$$\tilde{P} := \{e_1 \mid \exists (f_2, e_2) \text{ such that } (0, e_1, f_2, e_2) \in \mathcal{D}_P\}$$

Proof $P_{Cas} \subset \tilde{P}$ trivially. By using the controller Dirac structure $\mathcal{D}_C = \mathcal{D}_P^*$, we immediately obtain $\tilde{P} \subset P_{Cas}$. □

Dually, the *achievable constraints* of the the interconnection of the plant system with Dirac structure $\mathcal{D}_P$ and Hamiltonian $H_P(x)$ with a controller system with Dirac structure $\mathcal{D}_C$ and Hamiltonian $H_C(\xi)$ are given as

$$(\frac{\partial H_P}{\partial x}(x), \frac{\partial H_C}{\partial \xi}(\xi)) \in P_1,$$

where P_1 is the subspace of efforts as described above with respect to the Dirac structure $\mathcal{D}_P \parallel \mathcal{D}_C$. It follows that the plant state x satisfies the constraints $\frac{\partial^T H_P}{\partial x}(x) f_1 = -\frac{\partial^T H_C}{\partial \xi}(\xi) f_3$ for all f_1, f_3 such that $(f_1, 0, f_3, 0) \in \mathcal{D}_P \parallel \mathcal{D}_C$. The possible flow vectors f_1 in this expression are given by the space

$$G_{Alg} = \{f_1 \mid \exists \mathcal{D}_C \text{ s.t. } \exists f_3 \text{ for which } (f_1, 0, f_3, 0) \in \mathcal{D}_P \parallel \mathcal{D}_C\} \qquad (71)$$

Theorem 24 *The space G_{Alg} defined in (71) is equal to*

$$\tilde{G} := \{f_1 \mid \exists (f_2, e_2) \text{ such that } (f_1, 0, f_2, e_2) \in \mathcal{D}_P\}$$

Example 25 *Consider the input-state-output port-Hamiltonian plant system with inputs f_2 and outputs e_2*

$$\begin{aligned} \dot{x} &= J(x)\tfrac{\partial H_P}{\partial x}(x) + g(x) f_2, \; x \in \mathcal{X}, f_2 \in \mathbb{R}^m \\ e_2 &= g^T(x)\tfrac{\partial H_P}{\partial x}(x), \qquad e_2 \in \mathbb{R}^m \end{aligned} \qquad (72)$$

It is easily seen that

$$P_{Cas} = \tilde{P} = \{e_1 \mid \exists f_2 \text{ such that } 0 = J(x)e_1 + g(x)f_2\},$$

implying that the achievable Casimirs $K(x,\xi)$ are such that $e_1 = \frac{\partial K}{\partial x}(x,\xi)$ satisfies $J(x)\frac{\partial K}{\partial x}(x,\xi) \in \operatorname{im} g(x)$ for all ξ, that is, K as a function of x (for any fixed ξ) is a Hamiltonian function corresponding to a Hamiltonian vector field contained in the distribution spanned by the input vector fields given by the columns of $g(x)$. Similarly

$$G_{Alg} = \tilde{G} = \{f_1 \mid \exists f_2 \text{ s.t. } f_1 = -g(x)f_2\} = \operatorname{im} g(x),$$

which implies that the achievable algebraic constraints are of the form $\frac{\partial^T H_P}{\partial x}(x) g(x) = \frac{\partial^T H_C}{\partial \xi}(\xi) f_3$. This means that the outputs $e_2 = g^T(x)\frac{\partial H_P}{\partial x}(x)$ can be constrained in any way by interconnecting the system with a suitable controller port-Hamiltonian system.

6 Conclusions

The results obtained in this paper raise a number of questions. Port-based network modeling of multi-body systems (see e.g. [13]) lead to a (large number of) implicit equations describing the dynamics and the interconnection constraints. It is of interest to work out the equational representation as obtained in Section 3 in this case, and to give effective algorithms to reduce the obtained (relaxed) kernel/image representation to a maximally explicit form, making use of the available additional structure. Also in other modeling contexts it is profitable to have an explicit algorithm for the minimal representation of the complex composed Dirac structure arising from a network interconnection of Dirac structures at hand (combining graph-theoretical tools with the geometric theory of Dirac structures).

Another venue for research concerns the extension of the results obtained in this paper to *infinite-dimensional* Dirac structures. Some results concerning the composition of finite-dimensional Dirac structures with infinite-dimensional Dirac structures of a special type, namely the Stokes-Dirac structures as defined in [25], have been obtained in [17]. For general Dirac structures on Hilbert spaces in [6] a counterexample has been provided showing that the composition of infinite-dimensional Dirac structures may not always result in another Dirac structure. Recently in [9], making use of scattering representations, necessary and sufficient general conditions have been derived for the composition of infinite-dimensional Dirac structures to define again a Dirac structure.

The interpretation of the canonical controller Dirac structure as obtained in Section 5 deserves further study. In fact, the definition of the canonical controller Dirac structure achieving a certain desired closed-loop

Dirac structure suggests an 'internal model' interpretation, with ensuing robustness properties. (Note that $\mathcal{D}_C$ as constructed in Proposition 5.4 contains a copy of the plant Dirac structure. Its construction can thus be seen as a, static, network analogue of the usual, dynamic, internal model principle, where (a part of) the plant dynamics (or of the extended plant, that is, plant system together with exosystem) is copied in the controller system.) Finally, the characterization of the achievable Dirac structures is only a first step towards characterizing the achievable port-Hamiltonian closed-loop behaviors.

References

[1] J. Cervera, A.J. van der Schaft, A. Banos, On composition of Dirac structures and its implications for control by interconnection, pp. 55–64 in *Nonlinear and Adaptive Control*, Eds. A. Zinober, D. Owens, Springer Lect. Notes in Control and Information Sciences 281, Springer, London, 2002.

[2] T.J. Courant. Dirac manifolds. *Trans. American Math. Soc.*, 319:631–661, 1990.

[3] M. Dalsmo and A.J. van der Schaft. On representations and integrability of mathematical structures in energy-conserving physical systems. *SIAM J. Control and Optimization*, 37:54–91, 1999.

[4] I. Dorfman. *Dirac Structures and Integrability of Nonlinear Evolution Equations.* John Wiley, Chichester, 1993.

[5] G. Escobar, A.J. van der Schaft, R. Ortega, A Hamiltonian viewpoint in the modelling of switching power converters. *Automatica, Special Issue on Hybrid Systems, vol.35*, pp.445–452, 1999.

[6] G. Golo, *Interconnection Structures in Port-Based Modelling: Tools for Analysis and Simulation* Ph.D. Thesis, University of Twente, The Netherlands, 2002.

[7] G. Golo, A. van der Schaft, P.C. Breedveld, B.M. Maschke, Hamiltonian formulation of bond graphs. *Nonlinear and Hybrid Systems in Automotive Control* Eds. R. Johansson, A. Rantzer, pp. 351–372, Springer London, 2003.

[8] N. Hogan, Impedance Control: An approach to manipulation. *Journal of Dynamical Systems, Measurements and Control* 107(1), pp.1–24, 1985.

[9] M. Kurula, A.J. van der Schaft, H. Zwart. Composition of infinite-dimensional linear Dirac-type structures. accepted for publication at MTNS 2006, Kyoto, Japan.

[10] A. Macchelli, S. Stramigioli, A.J. van der Schaft, C. Melchiorri. Scattering for infinite-dimensional port Hamiltonian systems. *Proc. 41st IEEE Conf. on Decision and Control*, Las Vegas, Nevada, December 2002.

[11] B.M. Maschke, A.J. van der Schaft, and P.C. Breedveld. An intrinsic Hamiltonian formulation of network dynamics: Non-standard Poisson structures and gyrators. *J. Franklin Inst.*, 329:923–966, 1992.

[12] B.M. Maschke, A.J. van der Schaft. Interconnection of systems: the network paradigm. *Proceedings of the 38th IEEE Conference on Decision and Control* 1, pp.207–212, New York, NY, USA, 1997.

[13] B.M. Maschke and A.J. van der Schaft. Interconnected mechanical systems, Part I and II. In A. Astolfi, D.J.N Limebeer, C. Melchiorri, A. Tornambè, and R.B. Vinter, editors, *Modelling and Control of Mechanical Systems*, pages 1–30. Imperial College Press, 1997.

[14] H. Narayanan. Some applications of an implicit duality theorem to connections of structures of special types including Dirac and reciprocal structures. *Systems & Control Letters*, 45: 87–96, 2002.

[15] R. Ortega, A.J. van der Schaft, I. Mareels, and B.M. Maschke. Putting energy back in control. *Control Systems Magazine*, 21:18–33, 2001.

[16] R. Ortega, A.J. van der Schaft, B.M. Maschke, and G. Escobar. Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems. *Automatica*, 38:585–596, 2002.

[17] R. Pasumarthy A.J. van der Schaft. On interconnections of infinite dimensional port-Hamiltonian systems. *Proceedings of the Sixteenth International Symposium on Mathematical Theory of Networks and Systems* (MTNS2004), Leuven, Belgium, July 5–9, 2004.

[18] R.M. Redheffer, On a certain linear fractional transformation. *J. Math. and Physics*, 39: 269–286, 1960.

[19] A.J. van der Schaft. Implicit Hamiltonian systems with symmetry. *Rep.Math.Phys.*, 41:203–221, 1998.

[20] A.J. van der Schaft. *L_2-Gain and Passivity Techniques in Nonlinear Control.* Springer-Verlag, 2000.

[21] A.J. van der Schaft. Interconnection and Geometry. in *The Mathematics of Systems and Control: from Intelligent Control to Behavioral Systems*, Editors J.W. Polderman, H.L. Trentelman, Groningen, pp. 203-218, 1999.

[22] A.J. van der Schaft. Port-Hamiltonian systems: network modeling and control of nonlinear physical systems. pp. 127–168 in *Advanced Dynamics and Control of Structures and Machines* CISM Courses and lectures No. 444, CISM International Centre for Mechanical Sciences, Udine, Italy, April 15–19, 2002 (Eds. H. Irshik, K. Schlacher), Springer, Wien, New York, 2004.

[23] A.J. van der Schaft, J. Cervera. Composition of Dirac structures and control of port-Hamiltonian systems. *Proceedings 15th International Symposium on Mathematical Theory of Networks and Systems (MTNS2002)*, Eds. D.S. Gilliam, J. Rosenthal, South Bend, August 12-16, 2002.

[24] A.J. van der Schaft and B.M. Maschke. The Hamiltonian formulation of energy conserving physical systems with external ports. *Archiv fur Elektronik und Übertragungstechnik*, 49:362–371, 1995.

[25] A.J. van der Schaft and B.M. Maschke. Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *Journal of Geometry and Physics, vol.42*, pp.166-194, 2002.

[26] S. Stramigioli, B.M. Maschke, and A.J. van der Schaft. Passive output feedback and port interconnection. In H.J.C. Huijberts, H. Nijmeijer, A.J. van der Schaft, and J.M.A. Scherpen, editors, *Proceedings 4th IFAC NOLCOS*, pages 613–618. Elsevier, Amsterdam, 1998.

[27] S. Stramigioli, A.J. van der Schaft, B.M. Maschke, C. Melchiorri. Geometric scattering in robotic telemanipulation", *IEEE Transactions on Robotics and Automation*, 18:588–596, 2002.

[28] J.C. Willems. On interconnections, control, and feedback. *IEEE Transactions on Automatic Control*, 42:326–339, 1997.