

Conservation Laws and Lumped System Dynamics

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1 Introduction

Physical systems modeling, aimed at network modeling of complex multi-physics systems, has especially flourished in the fifties and sixties of the 20-th century, see e.g. [11, 4] and references provided therein. With the reinforcement of the 'systems' legacy in Systems & Control, the growing recognition that 'control' is not confined to developing algorithms for processing the measurements of the system into control signals (but instead is concerned with the design of the total controlled system), and facing the complexity of modern technological and natural systems, systematic methods for physical systems modeling of large-scale lumped- and distributed-parameter systems capturing their basic physical characteristics are needed more than ever.

In this paper we are concerned with the development of a systematic framework for modeling multi-physics systems which is directly based on *conservation laws*. Modeling based on conservation laws is prevalent in a distributed-parameter context in areas such as fluid dynamics and hydraulic systems, chemical and thermodynamical systems [2], as well as electromagnetism, but is also underlying the basic structure of lumped-parameter systems such as electrical circuits. While the natural framework for formulating Kirchhoff's laws for electrical circuits is the circuit *graph* we will show in this paper how distributed-parameter conservation laws can be discretized by using the proper generalization of the notion of graph to 'higher-dimensional networks', called *k-complexes* in algebraic topology. Furthermore, we show how these discretized conservation laws define a power-conserving intercon-

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nection structure, called a Dirac structure, which, when combined with the (discretized) constitutive relations, defines a finite-dimensional *port-Hamiltonian system* [14, 13, 5].

In previous work [15] we have laid down a framework for formulating conservation laws described by partial differential equations as infinite-dimensional port-Hamiltonian systems. Furthermore, in [8] we have shown how such infinite-dimensional port-Hamiltonian systems can be spatially discretized to finite-dimensional port-Hamiltonian systems by making use of mixed finite-element methods. In this paper we show how alternatively we can directly spatially 'lump' the dynamics described by conservation laws in a structure-preserving manner, again obtaining a finite-dimensional port-Hamiltonian system description. This approach also elucidates the concept of the spatial system boundary, and leads to the notion of *distributed terminals*.

This paper is a follow-up of our previous paper [16]. Older references in this spirit include [10, 12].

2 Kirchhoff's laws on graphs and circuit dynamics

In this section we recall the abstract formulation of Kirchhoff's laws on graphs, dating back to the historical work of Kirchhoff [9], as can be found e.g. in [1, 3]. In order to deal with *open* electrical circuits we define *open graphs*, and we show how Kirchhoff's laws on open graphs define a power-conserving interconnection structure, called a Dirac structure, between the currents through and the voltages over the edges of the graph, and the boundary currents and potentials. This enables us to describe the circuit dynamics as a port-Hamiltonian system.

2.1 Graphs

An *oriented graph*¹ \mathcal{G} , see e.g. [3], consists of a finite set \mathcal{V} of *vertices* and a finite set \mathcal{E} of directed *edges*, together with a mapping from \mathcal{E} to the set of ordered pairs of \mathcal{V} . Thus to any branch $e \in \mathcal{E}$ there corresponds an ordered pair $(v, w) \in \mathcal{V}^2$ representing the initial vertex v and the final vertex w of this edge. An oriented graph is completely specified by its *incidence matrix* B , which is an $\bar{v} \times \bar{e}$ matrix, \bar{v} being the number of vertices and \bar{e} being the number of edges, with (i, j) -th element b_{ij} equal to 1 if the j -th edge is an edge towards vertex i , equal to -1 if the j -th edge is an edge originating from vertex i , and 0 otherwise.

Given an oriented graph we define its *vertex space* Λ_0 as the real vector space of all functions from \mathcal{V} to \mathbb{R} . Clearly Λ_0 can be identified with $\mathbb{R}^{\bar{v}}$. Furthermore, we

¹ In fact, we will be considering *multi-graphs* since we allow for the existence of multiple branches between the same pair of vertices.

define its *edge space* Λ_1 as the vector space of all functions from \mathcal{E} to \mathbb{R} . Again, Λ_1 can be identified with $\mathbb{R}^{\mathcal{E}}$.

In the context of an electrical circuit Λ_1 will be the vector space of currents *through* the edges in the circuit. The dual space of Λ_1 will be denoted by Λ^1 , and defines the vector space of voltages *across* the edges. (We have highlighted the words 'through' and 'across' to refer to the classical use of 'through' and 'across' variables, see e.g. [11].) Furthermore, the duality product $\langle V|I \rangle = V^T I$ of a vector of currents $I \in \Lambda_1$ with a vector of voltages $V \in \Lambda^1$ is the total power over the circuit. Similarly, the dual space of Λ_0 is denoted by Λ^0 and defines the vector space of potentials at the vertices.

Remark 1. Since Λ_0 and Λ_1 have a canonical basis corresponding to the individual vertices, respectively edges, there is a standard Euclidean inner product on both spaces, and thus both Λ^0 and Λ^1 can be identified with Λ_0 , respectively Λ_1 , such that the duality product becomes this standard inner product. In situations to be treated later on this will not necessarily be the case.

The incidence matrix B can be also regarded as the matrix representation of a linear map (denoted by the same symbol)

$$B : \Lambda_1 \rightarrow \Lambda_0$$

called the *incidence operator* or (*boundary operator*). Its adjoint map is denoted in matrix representation as

$$B^T : \Lambda^0 \rightarrow \Lambda^1,$$

and is called the *co-incidence* (or *co-boundary*) operator.

2.2 Kirchhoff's laws for graphs

Consider an oriented graph \mathcal{G} specified by its incidence operator B . Kirchhoff's laws associated with the graph are expressed as follows. Kirchhoff's current laws (KCL) are given as

$$I \in \ker B, \tag{1}$$

while Kirchhoff's voltage laws (KVL) take the form

$$V \in \text{im } B^T. \tag{2}$$

A graph theoretic interpretation of Kirchhoff's current and voltage laws can be given as follows [3]. The kernel of the incidence operator B is the *cycle space* $Z \subset \Lambda_1$ of the graph, while the image $U \subset \Lambda^1$ of the co-incidence operator B^T is its *cut space* (or, co-cycle space). Since $\ker B = (\text{im } B^T)^\perp$ (with $^\perp$ denoting the orthogonal complement with respect to the duality product between the dual spaces Λ_1 and Λ^1) the cycle space is the orthogonal complement of the cut space.

This leads to the equivalent way of formulating Kirchhoff's current laws as the fact that the total current I along any cut is equal to zero, since $I \in \ker B$ is equivalent to I being orthogonal to the cut space U . The simplest elements of the cut space U (which in fact are spanning the linear space U) are the cuts given by all edges starting from or terminating on a single vertex v . Kirchhoff's current laws for these cut sets mean nothing else than the expression that the currents entering or leaving any vertex v sum up to zero. Indeed, if v is numbered as the i -th vertex then the i -th equation in the linear set of equations $BI = 0$ is precisely this.

On the other hand, since $V \in \text{im} B^T$ is equivalent to V being orthogonal to the cycle space Z , Kirchhoff's voltage laws can be equivalently described as the fact that the total voltage over every cycle is zero.

The difference between Kirchhoff's current and voltage laws is also reflected by writing Kirchhoff's voltage laws as

$$V = B^T \psi \quad (3)$$

for some vector $\psi \in \Lambda^0$, which has the physical interpretation of being the vector of *potentials* at every vertex. Hence Kirchhoff's voltage laws express that the voltage distribution V over the edges of the graph corresponds to a potential distribution over the vertices.

Of course, Tellegen's theorem automatically follows from Kirchhoff's laws. Indeed, take any current distribution I satisfying Kirchhoff's current laws $BI = 0$, and any voltage distribution V satisfying Kirchhoff's voltage laws $V = B^T \psi$. Then,

$$V^T I = \psi^T BI = 0 \quad (4)$$

In particular, Tellegen's theorem implies that for any *actual* current and voltage distribution over the circuit the total power $V^T I$ is equal to zero.

We summarize the Kirchhoff behavior $\mathcal{B}_K(\mathcal{G})$ of a graph \mathcal{G} with incidence matrix B as

$$\mathcal{B}_K(\mathcal{G}) := \{(I, V) \in \Lambda_1 \times \Lambda^1 \mid I \in \ker B, V \in \text{im} B^T\} \quad (5)$$

It immediately follows that the Kirchhoff behavior defines a *Dirac structure*. Recall [6, 14, 13] that a subspace $D \subset V \times V^*$ for some vector space V defines a Dirac structure if $D = D^{\text{orth}}$ where orth denotes the orthogonal complement with respect to the indefinite inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on $V \times V^*$ defined as

$$\langle\langle (v_1, v_1^*), (v_2, v_2^*) \rangle\rangle := \langle v_1^* | v_2 \rangle + \langle v_2^* | v_1 \rangle,$$

with $v_1, v_2 \in V, v_1^*, v_2^* \in V^*$, where $\langle | \rangle$ denotes the duality product between V and V^* .

2.3 Kirchhoff's laws for open graphs

Although in Kirchhoff's original treatment of circuits and graphs external currents entering the vertices of the graph were an indispensable notion, this has not been articulated very well in the subsequent formalization of circuits and graphs². Hence a reinforcement of this *systems point of view* is definitely in order.

We will do so by extending the notion of graph to *open graph*. An open graph \mathcal{G} is obtained from an ordinary graph with set of vertices \mathcal{V} by identifying a subset $\mathcal{V}_b \subset \mathcal{V}$ of *boundary vertices*. The interpretation of \mathcal{V}_b is that these are the vertices that are open to interconnection (e.g., with other graphs). The remaining subset $\mathcal{V}_i := \mathcal{V} - \mathcal{V}_b$ are the *internal vertices* of the open graph.

Remark 2. Another way of defining open graphs is by identifying some of the *edges* to be the *boundary edges* (open to interconnection). Such a definition is straightforward, and we will not elaborate on this. The distinction between the definitions of an open graph using boundary vertices or boundary edges is analogous to the difference between *boundary control* of distributed-parameter systems and *distributed control*; see also Section 3.

Kirchhoff's current laws apply to an open graph \mathcal{G} in a different manner than to an ordinary graph, since the ordinary Kirchhoff's current laws would imply that the sum of the currents over all edges incident on a boundary vertex is *zero*, which is *not* what we want for interconnection. Furthermore, by Tellegen's theorem, the ordinary KCL would imply that the total power in the circuit is equal to zero, thus implying that there cannot be any ingoing or outgoing power flow. Hence we have to modify Kirchhoff's current laws by requiring that the incidence operator B maps the vector of currents I to a vector that has zero components corresponding to the internal vertices, while for the boundary vertices the image is equal to (minus) the *boundary current* I_b . Decomposing the incidence operator B as $\begin{bmatrix} B_i \\ B_b \end{bmatrix}$ with B_i the part of the incidence operator corresponding to the internal vertices, and B_b the part corresponding to the boundary vertices, we thus arrive at

$$B_i I = 0, \quad B_b I = -I_b, \quad \text{KCL} \quad (6)$$

Here the vector I_b is belonging to the vector space Λ_b of functions from the boundary vertices \mathcal{V}_b to \mathbb{R} (which is identified with $\mathbb{R}^{\bar{v}_b}$, with \bar{v}_b the number of boundary vertices)³.

Kirchhoff's voltage laws (KVL) remain unchanged, and will be written as

² Unfortunately, this holds for many formalizations of physical theories over the last century. A proper theory of *mechanics* should include external forces from the very start, instead of restricting itself to closed mechanical systems. *Thermodynamics* cannot be properly formalized without taking interaction with other systems into account.

³ Alternatively, open graphs can be defined by attaching 'one-sided open edges' (properly called *leaves*) to every boundary vertex in \mathcal{V}_b . Then the elements of the vector I_b are the currents through these leaves, see also [17].

$$V = B^T \psi = B_i^T \psi_i + B_b^T \psi_b, \quad \text{KVL} \quad (7)$$

where ψ_i denotes the vector of the potentials at the internal vertices and ψ_b the vector of potentials at the boundary vertices. Note that $\psi_b \in \Lambda^b$ (where we define Λ^b to be the dual of the space of boundary currents Λ_b). This results in the following Kirchhoff behavior for an open graph \mathcal{G} :

$$\begin{aligned} B_K(\mathcal{G}) := \{ & (I, V, I_b, \psi_b) \in \Lambda_1 \times \Lambda^1 \times \Lambda_b \times \Lambda^b \mid \\ & B_i I = 0, B_b I = -I_b, \exists \psi_i \text{ s.t. } V = B_i^T \psi_i + B_b^T \psi_b \} \end{aligned} \quad (8)$$

By computing as before, cf. (4), the total power over the graph we now obtain

$$V^T I = \psi_i^T B_i I + \psi_b^T B_b I = -\psi_b^T I_b \quad (9)$$

Thus, for open graphs the total power $V^T I$ is equal to the outgoing power $-\psi_b^T I_b$. This will lead to the following characterization of the Kirchhoff behavior of open graphs as Dirac structures.

Proposition 1. *Let \mathcal{G} be an open graph with incidence matrix $B = \begin{bmatrix} B_i \\ B_b \end{bmatrix}$. Then its Kirchhoff behavior $\mathcal{B}_K(\mathcal{G})$ is a Dirac structure.*

Proof. As shown in (9), $V^T I + \psi_b^T I_b = 0$. By considering $I = I_1 + I_2, V = V_1 + V_2, I_b = I_{b1} + I_{b2}, \psi = \psi_1 + \psi_2$, with $(I_j, V_j, I_{bj}, \psi_j) \in \mathcal{B}_K(\mathcal{G}), j = 1, 2$, it follows that

$$\ll (I_1, V_1, I_{b1}, \psi_1), (I_2, V_2, I_{b2}, \psi_2) \gg = 0,$$

which implies that $\mathcal{B}_K(\mathcal{G}) \subset (\mathcal{B}_K(\mathcal{G}))^{\text{orth}}$.

For showing the reverse inclusion, consider a quadruple $(I, V, I_b, \psi_b) \in (\mathcal{B}_K(\mathcal{G}))^{\text{orth}}$, that is,

$$V^T \bar{I} + \bar{V}^T I + \psi_i^T \bar{I}_b + \bar{\psi}_b^T I_b = 0 \quad (10)$$

for all $(\bar{I}, \bar{V}, \bar{I}_b, \bar{\psi}_b)$ satisfying

$$B_i \bar{I} = 0, B_b \bar{I} = -\bar{I}_b, \bar{V} = B_i^T \bar{\psi}_i + B_b^T \bar{\psi}_b, \quad \text{for some } \bar{\psi}_i.$$

Writing out (10) we obtain

$$\begin{aligned} 0 &= V^T \bar{I} + \bar{\psi}_i^T B_i I + \bar{\psi}_b^T B_b I - \bar{\psi}_b^T B_i I + \bar{\psi}_b^T I_b \\ &= (V - B_b^T \psi_b)^T \bar{I} + \bar{\psi}_i^T B_i I + \bar{\psi}_b^T (B_b I + I_b) \end{aligned}$$

for all $\bar{\psi}_i, \bar{\psi}_b$ and all \bar{I} satisfying $B_i \bar{I} = 0$. It follows that $B_i I = 0, B_b I + I_b = 0$, and $V - B_b^T \psi_b \in \text{im } B_i^T$, or equivalently $V - B_b^T \psi_b = B_i^T \psi_i$ for some ψ_i . \square

2.4 Constraints on boundary currents and invariance of boundary potentials

It is a well-known property [3] of any incidence matrix B that

$$\mathbb{1}^T B = 0 \quad (11)$$

where $\mathbb{1}$ denotes the vector with all components equal to 1. From this property it follows that the rank of the incidence matrix B is at most $\bar{v} - 1$. In fact, the rank is given as [3] $\text{rank} B = \bar{v} - k_{\mathcal{G}}$, where $k_{\mathcal{G}}$ is the number of components⁴ of the graph \mathcal{G} . (Thus $\text{rank} B = \bar{v} - 1$ for a connected graph.) By (6) it follows that

$$0 = \mathbb{1}^T B I = \mathbb{1}_b^T B_b I = -\mathbb{1}_b^T I_b = -\sum_{v_b} I_{v_b} \quad (12)$$

with $\mathbb{1}_b$ denoting the vector with all ones of dimension equal to the number of boundary vertices, and where the summation is over all boundary vertices $v_b \in \mathcal{V}_b$. Hence the boundary part of the Kirchhoff behavior of an open graph is constrained by the obvious fact that all boundary currents sum up to zero. Dually, we may always add to the vector of potentials ψ the vector $\mathbb{1}$ leaving invariant the vector of voltages $V = B^T \psi$. Hence, to the vector of boundary potentials ψ_b we may always add the vector $\mathbb{1}_b$. Summarizing we arrive at a similar statement as in [17]):

Proposition 2. *Consider an open graph \mathcal{G} with Kirchhoff behavior $\mathcal{B}_K(\mathcal{G})$. Then for each $(I, V, I_b, \psi_b) \in \mathcal{B}_K(\mathcal{G})$ it holds that*

$$\mathbb{1}_b^T I_b = 0$$

while for any constant $c \in \mathbb{R}$

$$(I, V, I_b, \psi_b + c \mathbb{1}_b) \in \mathcal{B}_K(\mathcal{G})$$

This implies that we may restrict the dimension of the space of external variables $\Lambda_b \times \Lambda^b$ by two. Indeed, we may define

$$\Lambda_{b\text{red}} := \{I_b \in \Lambda_b \mid I_b \in \ker \mathbb{1}_b^T\}$$

and its dual space

$$\Lambda_{\text{red}}^b := (\Lambda_{b\text{red}})^* = \Lambda^b / \text{im } \mathbb{1}_b$$

It is rather straightforward to show that the Kirchhoff behavior $\mathcal{B}_K(\mathcal{G})$ reduces to a linear subspace of the reduced space $\Lambda_1 \times \Lambda^1 \times \Lambda_{b\text{red}} \times \Lambda_{\text{red}}^b$, which is also a Dirac structure. A circuit interpretation of this reduction is that we may consider one of the boundary vertices, say the first one, to be the reference ground vertex, and that we may reduce the vector of boundary potentials $\psi_b = (\psi_{b1}, \dots, \psi_{b\bar{v}_b})$ to a vector

⁴ A component is a maximal subgraph which is connected, that is, every two vertices are linked by a path of, -non-oriented-, edges.

of voltages $(\psi_{b_2} - \psi_{b_1}, \dots, \psi_{b_{\bar{v}_b}} - \psi_{b_1})$. A graph-theoretical interpretation is that instead of the incidence operator B we consider the *restricted* incidence operator [1].

For a graph \mathcal{G} with more than one connected component the above holds for each connected component. It follows that there are as many independent constraints on the boundary currents I_b as the number of the connected components of the open graph \mathcal{G} . Dually, the space of allowed boundary potentials ψ_b is invariant under translation by as many independent vectors $\mathbb{1}_b$ as the number of connected components.

A complementary view on the fact that the sum of the boundary currents is equal to zero and the boundary potentials are invariant under translation along $\mathbb{1}_b$ is the fact that we may *close* an open graph \mathcal{G} to an ordinary graph $\tilde{\mathcal{G}}$. Consider first the case that \mathcal{G} is connected. Then we may add one virtual ('ground') vertex v_0 , and virtual edges from this virtual vertex to every boundary vertex $v_b \in \mathcal{V}_e$, in such a manner that the Kirchhoff behavior of this graph $\tilde{\mathcal{G}}$ *extends* the Kirchhoff behavior of the open graph \mathcal{G} . In fact, to the virtual vertex v_0 we may associate an arbitrary potential ψ_{v_0} (a ground-potential), and we may rewrite the righthand-side of (9) as (since $\sum_{v_b} I_{v_b} = 0$)

$$-\sum_{v_b} (\psi_{v_b} - \psi_{v_0}) I_{v_b} = -\sum_{v_b} V_{v_b} I_{v_b} \quad (13)$$

where $V_{v_b} := \psi_{v_b} - \psi_{v_0}$ and I_{v_b} denotes the voltage across and the current through the virtual edge towards the boundary vertex v_b .

If the open graph \mathcal{G} consists of more than one component, then one extends the graph by adding a virtual vertex to *every* component containing boundary vertices.

2.5 Interconnection of open graphs

Consider two open graphs \mathcal{G}^j with boundary operators $B^j = \begin{bmatrix} B_i^j \\ B_b^j \end{bmatrix}$, $j = 1, 2$. *Interconnection* is done by identifying some of their boundary vertices, and equating (up to a minus sign) the boundary potentials and currents corresponding to these boundary vertices.

For simplicity we consider the case that we can equate *all* their boundary vertices with each other, resulting in an ordinary (closed) graph with set of vertices $\mathcal{V}_i^1 \cup \mathcal{V}_i^2 \cup \mathcal{V}$, where $\mathcal{V}_i := \mathcal{V}_b^1 = \mathcal{V}_b^2$ denotes the set of shared boundary vertices. The incidence operator B of this interconnected graph is given as

$$B = \begin{bmatrix} B_i^1 & 0 \\ 0 & B_i^2 \\ B_b^1 & B_b^2 \end{bmatrix}, \quad (14)$$

corresponding to the following interconnection constraints on the boundary potentials and currents

$$\psi_b^1 = \psi_b^2, \quad I_b^1 + I_b^2 = 0 \quad (15)$$

Of course, several extensions are possible. For example, one may still retain the shared vertices $\mathcal{V}_b := \mathcal{V}_b^1 = \mathcal{V}_b^2$ as being boundary vertices (instead of internal vertices as above) by extending (15) to

$$\psi_b^1 = \psi_b^2 = \psi_b, \quad I_b^1 + I_b^2 + I_b = 0 \quad (16)$$

with I_b, ψ_b the boundary currents and potentials of the interconnected graph.

2.6 Constitutive relations and port-Hamiltonian circuit dynamics

The dynamics of an RLC-circuit is defined, on top of Kirchhoff's laws for its circuit graph, by the constitutive relations of its elements (in this case, capacitors, inductors and resistors). They specify for each edge e a relation between the current I_e through and the voltage V_e across the edge. The simplest case is a *resistive* relation between I_e and V_e such that $V_e I_e \leq 0$. In particular, a linear resistor at edge e is specified by a relation $V_e = -R_e I_e$ with $R_e \geq 0$. In the case of a *capacitive* relation one defines an additional energy variable Q_e (denoting the charge) together with a real function $H_{C_e}(Q_e)$ denoting the electric energy stored in the capacitor. The constitutive relations for a capacitor at edge e are given by

$$\dot{Q}_e = -I_e, \quad V_e = \frac{dH_{C_e}}{dQ_e}(Q_e) \quad (17)$$

Alternatively, in the case of an inductor one specifies the magnetic energy $H_{L_e}(\Phi_e)$, where Φ_e denotes the magnetic flux linkage, together with the dynamic relations

$$\dot{\Phi}_e = -V_e, \quad I_e = \frac{dH_{L_e}}{d\Phi_e}(\Phi_e) \quad (18)$$

Substituting these constitutive relations into the Kirchhoff behavior $\mathcal{B}_K(\mathcal{G})$ (which is a Dirac structure) results in a port-Hamiltonian⁵ system, see e.g. [14, 13], given by

$$((I_C, I_L, I_R), (V_C, V_L, V_R), I_b, \psi_b) \in \mathcal{B}_K(\mathcal{G})$$

where the vectors $I_C, I_L, I_R, V_C, V_L, V_R$ denote the currents, respectively voltages, corresponding to the capacitors, inductors, and resistors, related as

⁵ Strictly speaking, the terminology 'port-Hamiltonian' is not completely appropriate since this assumes that the system boundary consists of *ports*, that is pairs of vertices with boundary variables being the currents through and the voltages across the edge corresponding to each port. Nevertheless, the mathematical structure and system description remains the same. Furthermore, by reducing the Kirchhoff behavior as above, or alternatively by extending it through the addition of a ground vertex, the boundary variables become true port variables.

$$I_C = -\dot{Q}, V_C = \frac{\partial H_C}{\partial Q}(Q), \quad V_L = -\dot{\Phi}, I_L = \frac{\partial H_L}{\partial \Phi}(\Phi),$$

where Q denotes the vector of charges at the capacitors, Φ denotes the vector of fluxes at the inductors, H_C and H_L denote the total electric and magnetic energies, and where moreover the vectors I_R, V_R satisfy a resistive relation.

Example 1. Let us consider an LC-circuit (for simplicity without boundary vertices). We will start by decomposing the circuit graph \mathcal{G} as the interconnection of a graph corresponding to the capacitors and a graph corresponding to the inductors. Define \mathcal{V} as the set of all vertices that are adjacent to at least one capacitor *as well as* to at least one inductor. Then split the circuit graph into an open circuit graph \mathcal{G}^C corresponding to the capacitors and an open circuit graph \mathcal{G}^L corresponding to the inductors, both with set of boundary vertices \mathcal{V} . Denote the incidence matrices of these two circuit graphs by

$$B^C := \begin{bmatrix} B_i^C \\ B_b^C \end{bmatrix}, B^L := \begin{bmatrix} B_i^L \\ B_b^L \end{bmatrix}$$

Assuming for simplicity that all capacitors and inductors are linear we arrive at the following equations for the C -circuit

$$\begin{aligned} B_b^C \dot{Q} &= I_b^C, B_i^C \dot{Q} = 0 \\ B_b^{CT} \psi_b^C &= C^{-1}Q - B_i^{CT} \psi_i^C \end{aligned}$$

with C the diagonal matrix with diagonal elements corresponding to the capacitances of the capacitors, and for the L -circuit

$$\begin{aligned} \dot{\Phi} &= B_b^{LT} \psi_b^L + B_i^{LT} \psi_i^L \\ 0 &= B_i^L L^{-1} \Phi \\ I_b^L &= -B_i^L \Phi \end{aligned}$$

with L the diagonal matrix of inductances.

The equations of the LC -circuit are obtained by imposing the interconnection constraints $\psi_b^C = \psi_b^L =: \psi_i$ and $I_b^C + I_b^L = 0$. By eliminating the boundary currents I_b^C, I_b^L one arrives at the equations

$$\begin{aligned} \begin{bmatrix} B_i^C & 0 \\ 0 & B_i^L \\ B_b^C & B_b^L \end{bmatrix} \begin{bmatrix} -\dot{Q} \\ L^{-1}\Phi \end{bmatrix} &= 0 \\ \begin{bmatrix} C^{-1}Q \\ -\dot{\Phi} \end{bmatrix} &= \begin{bmatrix} B_i^{CT} & 0 & B_b^{CT} \\ 0 & B_i^{LT} & B_b^{LT} \end{bmatrix} \begin{bmatrix} \psi_i^C \\ \psi_i^L \\ \psi_i \end{bmatrix} \end{aligned}$$

3 Conservation laws on higher-dimensional complexes

In this section we will extend the formalization of conservation laws on graphs as expressed by Kirchhoff's laws to higher-dimensional networks. In particular, this will allow us to systematically spatially *discretize* distributed-parameter physical systems to finite-dimensional lumped-parameter systems, represented in port-Hamiltonian form.

3.1 Kirchhoff behavior on k -complexes

An oriented graph with incidence matrix B is a typical example of what is called in algebraic topology a 1-*complex*. Indeed, the sequence

$$\Lambda_1 \xrightarrow{B} \Lambda_0 \xrightarrow{\mathbb{1}} \mathbb{R}$$

satisfies the property $\mathbb{1} \circ B = 0$. In general, a k -complex Λ is specified by a sequence of real linear spaces⁶ $\Lambda_0, \Lambda_1, \dots, \Lambda_k$, together with a sequence of incidence operators

$$\Lambda_k \xrightarrow{\partial_k} \Lambda_{k-1} \xrightarrow{\partial_{k-1}} \dots \Lambda_1 \xrightarrow{\partial_1} \Lambda_0$$

with the property that $\partial_{j-1} \circ \partial_j = 0$, $j = 2, \dots, k$. The vector spaces Λ_j , $j = 0, 1, \dots, k$, are called the spaces of j -chains. Each Λ_j is generated by a finite set of j -cells (like edges and vertices for graphs) in the sense that Λ_j is the set of functions from the j -cells to \mathbb{R} . A typical example of a k -complex is the triangularization of a k -dimensional manifold, with the j -cells, $j = 0, 1, \dots, k$, being the sets of vertices, edges, faces, etc..

Example 2. Consider the triangularization of a 2-dimensional sphere by a tetrahedron with 4 faces, 6 edges, and 4 vertices. The matrix representation of the incidence operator ∂_2 (from the faces of the tetrahedron to its edges) is

$$\begin{array}{ccccc} & \langle v_1 v_2 v_3 \rangle & \langle v_1 v_3 v_4 \rangle & \langle v_1 v_4 v_2 \rangle & \langle v_2 v_4 v_3 \rangle \\ \langle v_1 v_2 \rangle & 1 & 0 & -1 & 0 \\ \langle v_1 v_3 \rangle & -1 & 1 & 0 & 0 \\ \langle v_1 v_4 \rangle & 0 & -1 & 1 & 0 \\ \langle v_2 v_3 \rangle & 1 & 0 & 0 & -1 \\ \langle v_2 v_4 \rangle & 0 & 0 & -1 & 1 \\ \langle v_3 v_4 \rangle & 0 & 1 & 0 & -1 \end{array}$$

where the expressions $\langle v_1 v_2 v_3 \rangle, \dots$ denote the faces (with corresponding orientation), and $\langle v_1 v_2 \rangle, \dots$ are the edges.

⁶ In algebraic topology [7] one usually starts with *abelian groups* Λ_j .

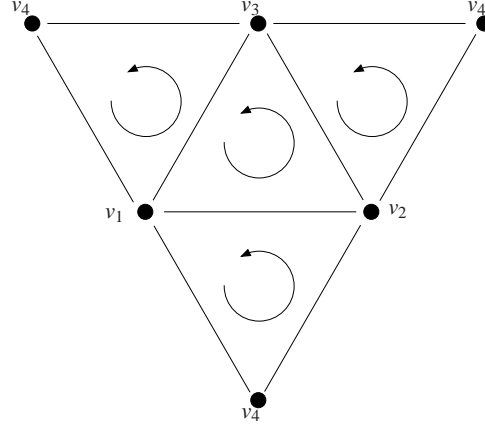


Fig. 1 Tetrahedron triangularizing a sphere

The matrix representation of the incidence operator ∂_1 (from edges to vertices) is given as

$$\begin{array}{rcccccc}
 & \langle v_1 v_2 \rangle & \langle v_1 v_3 \rangle & \langle v_1 v_4 \rangle & \langle v_2 v_3 \rangle & \langle v_2 v_4 \rangle & \langle v_3 v_4 \rangle \\
 \langle v_1 \rangle & -1 & -1 & -1 & 0 & 0 & 0 \\
 \langle v_2 \rangle & 1 & 0 & 0 & -1 & -1 & 0 \\
 \langle v_3 \rangle & 0 & 1 & 0 & 1 & 0 & -1 \\
 \langle v_4 \rangle & 0 & 0 & 1 & 0 & 1 & 1
 \end{array}$$

It can be verified that $\partial_1 \circ \partial_2 = 0$.

Denoting the dual linear spaces by Λ^j , $j = 0, 1, \dots, k$, we obtain the following dual sequence

$$\Lambda^0 \xrightarrow{d_1} \Lambda^1 \xrightarrow{d_2} \Lambda^2 \dots \Lambda^{k-1} \xrightarrow{d_k} \Lambda^k$$

where the adjoint maps d_j , $j = 0, 1, \dots, k$, satisfy the analogous property $d_j \circ d_{j-1} = 0$, $j = 2, \dots, k$. The elements of Λ^j are called j -cochains.

Consider any k -complex Λ , with k -chains $\alpha \in \Lambda_k$ and k -cochains $\beta \in \Lambda^k$. We define, similarly as in the case of a graph (1-complex) its Kirchhoff behavior as

$$\begin{aligned}
 B_K(\Lambda) &:= \{(\alpha, \beta) \in \Lambda_k \times \Lambda^k \mid \\
 &\quad \partial_k \alpha = 0, \exists \phi \in \Lambda^{k-1} \text{ s.t. } \beta = d_k \phi\}
 \end{aligned} \tag{19}$$

We will still refer to $\partial_k \alpha = 0$ as Kirchhoff's current laws (KCL), and to $\beta = d_k \psi$ as Kirchhoff's voltage laws (KVL). As before, it is immediately seen that $B_K(\Lambda) \subset \Lambda_k \times \Lambda^k$ is a *Dirac structure*. In particular, it follows that $\langle \beta \mid \alpha \rangle_k = 0$ for every $(\alpha, \beta) \in B_K(\Lambda)$, where $\langle \cdot \mid \cdot \rangle_k$ denotes the duality product between the dual linear spaces Λ_k and Λ^k .

3.2 Open k -complexes

Next we consider an *open k -complex*, by identifying a subset $\mathcal{V}_{(k-1)}^b$ of the set of all $(k-1)$ -cells, called the *boundary $(k-1)$ -cells*⁷, while the remaining $(k-1)$ -cells are denoted as the *internal $(k-1)$ -cells*. Define the linear space of functions from this subset of $(k-1)$ -cells to \mathbb{R} as $\Lambda_b \subset \Lambda_{k-1}$ with dual space denoted as Λ^b . Decompose correspondingly $\partial_k : \Lambda_k \rightarrow \Lambda_{k-1}$ as $\partial_k = (\partial_k^i, \partial_k^b)$, with adjoint mapping $d_k = (d_k^i, d_k^b)$. As before, Kirchhoff's voltage laws remain unchanged

$$\beta = d_k \psi = d_k^i \psi_i + d_k^b \psi_b, \quad (20)$$

where ψ_b is the vector of potentials at the boundary $(k-1)$ -cells and ψ_i is the vector of potentials at the internal $(k-1)$ -cells. On the other hand, Kirchhoff's current laws are modified into

$$\partial_k^i \alpha = 0, \quad \partial_k^b \alpha = -\alpha_b \quad (21)$$

where α_b denotes the vector of external 'currents' entering the boundary $(k-1)$ -cells. By computing as before the total power we obtain for any α and β satisfying (20, 21)

$$\begin{aligned} \langle \beta | \alpha \rangle_k &= \langle d_k \psi | \alpha \rangle_k = \langle d_k^i \psi_i + d_k^b \psi_b | \alpha \rangle_k = \\ &= \langle \psi_i | \partial_k^i \alpha \rangle_k + \langle \psi_b | \partial_k^b \alpha \rangle_k = - \langle \psi_b | \alpha_b \rangle_{k-1} \end{aligned} \quad (22)$$

The space of boundary variables $(\alpha_b, \psi_b) \in \Lambda_b \times \Lambda^b$ describes the *distributed terminals* of the open k -complex.

Similar to Proposition 1 it is shown that the Kirchhoff behavior of an open k -complex Λ defined as

$$\begin{aligned} \mathcal{B}_K(\Lambda) &:= \{(\alpha, \beta, \alpha_b, \psi_b) \in \Lambda_k \times \Lambda^k \times \Lambda_b \times \Lambda^b | \\ &\partial_k^i \alpha = 0, \partial_k^b \alpha = -\alpha_b, \exists \psi_i \text{ s.t. } \beta = d_k^i \psi_i + d_k^b \psi_b\} \end{aligned} \quad (23)$$

is a Dirac structure.

Analogously to graphs, Kirchhoff current laws for open k -complexes imply certain constraints on the incoming 'currents' α_b . Indeed, by the fact that $\partial_{k-1} \circ \partial_k = 0$ it follows that $\partial_{(k-1)b} \alpha_b = 0$, where $\partial_{(k-1)b}$ denotes the $(k-1)$ -th incidence operator restricted to $\Lambda_b \subset \Lambda_{k-1}$. (Note that in the case of a graph the role of $\partial_{(k-1)b}$ is played by the linear map $\mathbb{1}_b^T$.) As in the case of graphs, this allows us to *reduce* the Kirchhoff behavior to a space that is still a Dirac structure, or, alternatively, to *close* the open k -complex. This is done by completing the open k -complex Λ with space of boundary currents Λ_b by an additional set of $(k-1)$ -cells and k -cells.

Also the *interconnection* of open k -complexes is defined similar to the case of open graphs.

⁷ One could also consider as boundary cells subsets of the j -th cells for $j \neq k-1$. In particular, choosing $j = k$ would correspond to 'distributed interaction'. The choice $j = k-1$ corresponds to the important case of 'boundary' interaction.

4 Port-Hamiltonian dynamics on k -complexes

Consider an open k -complex Λ , together with its Kirchoff behavior $\mathcal{B}_K(\Lambda)$. Dynamics on the k -complex can be defined in various ways. Similar to the case of electrical circuits we could define constitutive relations for every k -cell, by specifying a relation between every component of Λ_k and Λ^k . As in the case of an electrical circuit this can be a relation of static resistive type, or a dynamic relation (of capacitive or inductive nature).

In this section we will define dynamics in a different way by specifying one type of dynamical relations between Λ_k and Λ^k , together with resistive relations between Λ_{k-1} and Λ^{k-1} . This will define a port-Hamiltonian dynamics, which is of *relaxation* type since there is only one type of physical energy (and thus no oscillations between different types of physical energy occur).

On the k -complex Λ , with $\partial_k : \Lambda_k \rightarrow \Lambda_{k-1}$ and $d_k : \Lambda^{k-1} \rightarrow \Lambda^k$, we define the following relations

$$\begin{aligned} f_x &= -d_k e, & f_x \in \Lambda^k, e \in \Lambda^{k-1} \\ f &= \partial_k e_x, & e_x \in \Lambda_k, f \in \Lambda_{k-1} \end{aligned} \quad (24)$$

It is checked [16] that this defines a Dirac structure $\mathcal{D} \subset \Lambda^k \times \Lambda_k \times \Lambda^{k-1} \times \Lambda_{k-1}$. This allows us to define a port-Hamiltonian dynamics by imposing the following constitutive relations. First we associate to every k -cell an energy storage, leading to

$$\dot{x} = -f_x, \quad e_x = \frac{\partial H}{\partial x}(x), \quad x \in \Lambda^k \quad (25)$$

with $H(x)$ the total stored energy, and $x \in \Lambda^k$ the total vector of energy variables. Furthermore, we associate to every $(k-1)$ -cell a (linear) resistive relation, leading to

$$e = -Rf, \quad R = R^T \geq 0 \quad (26)$$

Substituted in (24) this yields the relaxation dynamics

$$\dot{x} = d_k e = -d_k R f = -d_k R \partial_k \frac{\partial H}{\partial x}(x), \quad x \in \Lambda^k \quad (27)$$

with the property that

$$\frac{dH}{dt} = -(\partial_k \frac{\partial H}{\partial x}(x))^T R \partial_k \frac{\partial H}{\partial x}(x) = -f^T R f \leq 0 \quad (28)$$

For an *open* complex with boundary $(k-1)$ -cells the definition is modified as follows. Instead of (24) we consider

$$\begin{aligned}
f_x &= -d_k \begin{bmatrix} e \\ e_b \end{bmatrix}, & f_x \in \Lambda^k, \begin{bmatrix} e \\ e_b \end{bmatrix} \in \Lambda^{k-1}, e_b \in \Lambda^b \\
\begin{bmatrix} f_i \\ f_b \end{bmatrix} &= \partial_k e_x, & e_x \in \Lambda_k, \begin{bmatrix} f \\ f_b \end{bmatrix} \in \Lambda_{k-1}, f_b \in \Lambda_b
\end{aligned} \tag{29}$$

with f_b, e_b corresponding to the *boundary* $(k-1)$ -cells, and f, e corresponding to the *internal* cells. Imposing the same storage relations (25) and resistive relations (26) we arrive at

$$\begin{aligned}
\dot{x} &= -d_k^r R \partial_k^r \frac{\partial H}{\partial x}(x) + d_k^b e_b \\
f_b &= \partial_k^b \frac{\partial H}{\partial x}(x)
\end{aligned} \tag{30}$$

where we have split d_k as $d_k = [d_k^r \ d_k^b]$ and $\partial_k = \begin{bmatrix} \partial_k^r \\ \partial_k^b \end{bmatrix}$ (according to the division of the $(k-1)$ -cells into internal cells corresponding to resistive behavior and boundary cells). This defines a port-Hamiltonian system with inputs e_b and outputs f_b .

4.1 Example: Heat transfer on a 2-complex

The above formulation of systems of conservation laws and port-Hamiltonian systems on k -complexes will be illustrated with the model of heat transfer in a 2-dimensional medium (for instance a plate). Instead of first considering the pde-model and then discretizing, we will directly consider the dynamics on a 2-complex as arising from a triangulation of the 2-dimensional spatial domain. We assume the medium to be undeformable (hence mechanical work is neglected) and that there is no mass transfer.

We will write the heat transfer in terms of the conservation of internal energy. First we identify the physical variables as chains and cochains of the given 2-complex. The components of the internal energy vector $u \in \Lambda^2$ denote the energy of each face. The heat conduction is given by the *heat flux* $f \in \Lambda^1$ whose components equal the heat flux through every edge. Hence the basic conservation law (conservation of energy) is given as

$$\frac{du}{dt} = d_2 f$$

The thermodynamic properties are defined by Gibbs' relation, and generated by the *entropy function* $s = s(u) \in C^\infty(\Lambda^2)$ as thermodynamic potential. Since we consider transformations which are isochore and without mass transfer, Gibbs' relation reduces to the definition of the vector of intensive variables $e_u \in \Lambda_2$ which is (entropy-)conjugated to the vector of extensive variables u :

$$e_u = \frac{\partial s}{\partial u}(u)$$

The components e_u are equal to the reciprocal of the temperature at each 2-face.

Since the temperature is varying over the faces, there is a *thermodynamic driving force* vector $e \in \Lambda_1$ given as the vector of differences

$$e = \partial_2 e_u$$

By Fourier's law the heat flux is determined by the thermodynamic driving force vector as

$$f = R(e_u) e, \quad (31)$$

with $R(e_u) = R^T(e_u) \geq 0$ depending on the heat conduction coefficients. (Note the sign-difference with (26).) The resulting system is a port-Hamiltonian system (of relaxation type), with vector of state variables x given by the internal energy vector u , and 'Hamiltonian' $s(u)$. By (28) the entropy $s(u)$ satisfies

$$\frac{ds}{dt} = (\partial_2 \frac{\partial s}{\partial u}(u))^T R(e_u) \partial_2 \frac{\partial s}{\partial u}(u) = f^T R(e_u) f \geq 0$$

expressing the fact that the entropy is monotonously *increasing*. (Note again the sign-difference with the treatment above, where the Hamiltonian H was decreasing.)

The exchange of heat through the boundary of the system is incorporated as above, cf. (29, 30), by splitting the edges (1-cells) into internal edges with the resistive relation (31) and boundary edges. This leads to

$$\frac{ds}{dt} = (\partial_2 \frac{\partial s}{\partial u}(u))^T R(e_u) \partial_2 \frac{\partial s}{\partial u}(u) + e_b f_b$$

with f_b, e_b denoting the heat flux, respectively, thermodynamical driving force, through the boundary edges.

5 Conclusions

A framework has been laid down for the formulation of open physical systems on k -complexes, generalizing the graph-theoretic formulation of electrical circuit dynamics with terminals. It has been shown that Kirchhoff's laws can be generalized to open k -complexes, defining a Dirac structure involving boundary currents and potentials, thus generalizing the concept of 'terminal' to the distributed case. This has been illustrated on the example of heat transfer on a 2-complex. This simple example already shows how one can directly define a finite-dimensional port-Hamiltonian dynamics, capturing the physical meaning of the involved variables and retaining the conservation laws, without the need to formulate the dynamics as a set of pde's (and possibly to discretize the pde's later on).

In future work we will apply and extend the framework to different classes of port-Hamiltonian systems on k -complexes (corresponding to different physical settings), and employ these models for boundary control.

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