

Modeling for control of an inflatable space reflector, the linear 1-D case^{*}

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Abstract: In this paper we develop a mathematical model of the dynamics for an inflatable space reflector, which can be used to design a controller for the shape of the inflatable structure. Inflatable structures have very nice properties, suitable for aerospace applications. We can construct e.g. a huge light weight reflector for a satellite which consumes very little space in the rocket because it can be inflated when the satellite is in the orbit. So with this technology we can build inflatable reflectors which are about 100 times bigger than solid ones. But to be useful for telescopes we have to actively control the surface of the inflatable to achieve the desired surface accuracy. The starting point of the control design is modeling for control, in our case port-Hamiltonian (pH) modeling. We will show how to derive an infinite and also finite dimensional port-Hamiltonian model of a 1-D Euler-Bernoulli beam with piezo actuation. In the future we will also focus on 2-D models.

Keywords: flexible structure, port-Hamiltonian, distributed, discretization

1 Introduction

Inflatable structures are a very promising technology for space applications [4]. With this emerging technology we are able to build up to 100 times bigger space crafts, which are up to 10 times cheaper in terms of costs but still use the same space in the orbiting device. But inflatable structures have one big disadvantage

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and that is the lack of stiffness and weight of the material.

Due to the fact that any inflatable structure is build of a polymer casing, it is clear that an inflatable structure is not able to have the same surface accuracy as a rigid body. This disadvantage makes it at the moment hard to use inflatable structures in high accuracy situations.

The solution for this problem is to use smart materials which have the possibility to change their properties on demand, e.g. piezoelectric polymers [9]. Because these materials are made of polymers it is possible to build extremely thin actuators which then can be bonded to the casing of our inflatable structure.

In this paper we show how to develop a model for a 1-D flexible structure with a piezoelectric element as actuator in the port-Hamiltonian (pH) modeling framework [1]. The here proposed approach is somewhat different to [5, 10], because we aim at different configurations of the piezoelectric composite, we also propose how to derive a lumped model for a small piece of the beam (local model) and show how to interconnect these local models to derive a pH-model of the complete beam. We approach the problem with to purpose to extend it to the 2-D and 3-D cases in the future.

In Section 2 we introduce the basic physical relations which we use to formulate our model. After this is accomplished we define in Section 3 a distributed pH model for a piezoelectric beam which is based on the ideas of [5], but we focus on a specific beam model (Euler-Bernoulli beam). Additionally we also show how to discretize the distributed model in a finite differences approach. The result of the discretization is a lumped model which describes the dynamics of a small part of the beam. To achieve a lumped model which represents the full dynamics of the beam we have to interconnect the local models, see Section 4. Finally in Section 5 we show how the proposed model can be used to define a model for a piezoelectric composite, which will be a possible actuator for the shape control of an inflatable structure.

The proposed model can also be used for modeling other structure, namely any flexible structure with a piezo actuation e.g. for vibration control in civil engineering.

2 Background on Continuum dynamics and the piezoelectric effect

In this section we briefly introduce the physics we use in the following sections. In this paper we focus only on linear materials and small/linear strains [2, 7].

We first take a look at the constitutive equations of our model. If we consider a beam without a piezo actuation we know from Hooke's Law that the stress-strain relations can be described as

$$\sigma = C^E \varepsilon,$$

where we used the common matrix notation instead of the tensor notation. Here σ is the stress, ε the related strain and C^E a matrix which relates the stress and the strain. In general σ and ε are second order tensors of dimension 3, e.g. σ_{ij} describes the stress in the i,j direction ($i, j \in \{1, 2, 3\}$), and C^E is a fourth order tensor. The subscripts corresponds to directions in our coordinate system, 1 corresponds to x ,

2 to y and 3 to the z direction.

For piezoelectric material we additionally have that the piezo effect induces an additional strain in the material which is caused by an electrical field (actuation property). Similarly the deformation of the piezoelectric element also changes the electrical field in our element (sensing property). So the coupled constitutive relations for piezo electric material [8] can be described as

$$\begin{bmatrix} \sigma \\ D \end{bmatrix} = \begin{bmatrix} C^E & -e^T \\ e & \epsilon^e \end{bmatrix} \begin{bmatrix} \varepsilon \\ E \end{bmatrix}. \quad (1)$$

Here D is the electrical displacement and E is the electrical field in the piezo element, ϵ^e is the electrical permittivity and e is the piezoelectric constant of the material.

The strain ε in our beam is related to the deformation \mathbf{u} of the beam. The electrical field and the electrical displacement can be described by Maxwell's equations. So we can state the compatibility equations as

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i} \right), \quad E_i = -\frac{\partial \varphi}{\partial x_i}, \quad \rho_e = \sum_i \frac{\partial D_i}{\partial x_i}, \quad (2)$$

here φ is the electrical potential and ρ_e is the electrical charge density. Note: $x_1 = x$, $x_2 = y$ and $x_3 = z$.

The dynamical equilibrium of piezoelectric material can be described by Newton's laws (balance of mechanical forces). We can state it as

$$\rho \ddot{\mathbf{u}}_i = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \quad (3)$$

where ρ is the density of the material and f_i is a component of the body force acting on the material.

3 Port-Hamiltonian modeling of an piezoelectric Euler-Bernoulli beam

In this section we want to introduce a port-Hamiltonian (pH) model, see [1, 6, 5], for a flexible piezoelectric beam, described in the Euler-Bernoulli framework. We assume that a body force is acting on the beam (f_i). We will first derive an infinite dimensional pH model and then spatially discretize this model to get a discretized pH model. The discretization we use here is based on finite differences, but there are also other discretization methods for distributed pH systems, see [3].

3.1 Infinite dimensional model

To derive the distributed pH model for our beam, we first have to define the displacements which take place in our beam. For an Euler-Bernoulli beam it is in general assumed that the displacement takes place in the x and z direction only. So we have pure bending. The displacement vector can be described as

$$\mathbf{u}(x, z) = [u_0(x) - z\phi(x), 0, w(x)]^T$$

where $u_0(x)$ is the displacement of a material point at the neutral line of the beam and $w(x)$ describes the deflection of the beam from the undeformed configuration, $\phi(x)$ is defined to be the slope of the beam so $\phi(x) = \frac{\partial w}{\partial x}$, see Figure 1.

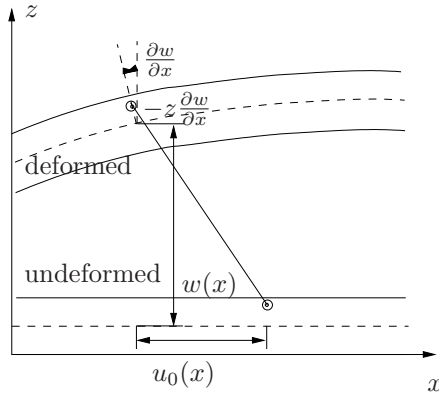


Figure 1. Deformation of a beam under external influences

A non-uniform width of the beam is needed to be able to tune the induced strain depending on the position along the x -axis. So we see that the cross sectional area (in the yz -plane) of the beam is depending on the position along the beam. In the sequel, we denote $A(x)$ as A , but we have to keep in mind the dependency on x , see Figure 2.

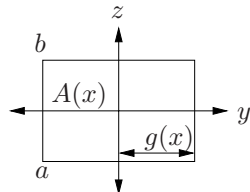


Figure 2. Cross sectional area of the beam

Now we also state some assumptions for the electrical field E of the piezoelectric beam. To be able to connect the beam to an electrical power source the upper and on the lower side of the beam are covered with an electrode. Due to the applied potential an electrical field will be created. We assume that the electrical field has only a z -component and varies linearly over the thickness of the piezo ($E_z \neq 0, E_x = E_y = 0$), see [7]. Therefore, we can define E_z to be

$$E_z = E_2 z + E_1.$$

Because of Maxwell's equations ($E_z = -\frac{\partial}{\partial z}\varphi_z$) it follows that the electrical potential between the two electrodes is quadratic in z direction

$$\varphi_z = \frac{1}{2}\varphi_2 z^2 + \varphi_1 z + \varphi_0.$$

Hence we have the following relation between the applied potential and the induced electrical field,

$$E_1 = -\varphi_1, E_2 = -\varphi_2.$$

Now we formulate equations for the energy stored in the beam. First we take a look at the kinetic energy of our beam. If we define $\mathbf{p} = \rho \dot{\mathbf{u}}$ to be the moment at a specific point in the beam we can express the kinetic energy in the beam as

$$K = \int_0^L \int_A \frac{\|\mathbf{p}\|^2}{\rho} dA dx.$$

It is easy to see that the kinetic energy defined as a volume integral, but \mathbf{p} depends only on x . In consequence, we can express the kinetic energy as a line integral if we integrate first over the cross sectional area of the beam,

$$\int_A \frac{\|\mathbf{p}\|^2}{\rho} dA = \int_A (\dot{u}_0 - z\dot{\phi})^2 + \dot{w}^2 dA.$$

Then we can rewrite the kinetic energy as

$$K = \frac{1}{2} \int_0^L \rho \left(A\dot{u}_0^2 - 2I_0\dot{u}_0\dot{\phi} + I\dot{\phi}^2 + A\dot{w}^2 \right) dx = \frac{1}{2} \int_0^L \tilde{\mathbf{p}}^T \mathbf{M}^{-1} \tilde{\mathbf{p}} dx,$$

where

$$\tilde{\mathbf{p}} = \begin{bmatrix} \rho A & -\rho I_0 & 0 \\ -\rho I_0 & \rho I & 0 \\ 0 & 0 & \rho A \end{bmatrix} \begin{bmatrix} \dot{u}_0 \\ \dot{\phi} \\ \dot{w} \end{bmatrix} = \mathbf{M} \frac{\partial}{\partial t} \tilde{\mathbf{u}},$$

where $\tilde{\mathbf{p}} = [p_1 \ p_2 \ p_3]^T$, $\tilde{\mathbf{u}} = [u_0 \ \phi \ w]^T$.

Note that $I = \int_A z^2 dA \neq 0$ and $I_0 = \int_A z dA = 0 \ \forall x \in [0, L]$, if we choose centroidal coordinates ($a = -b$).

The potential energy stored in our beam consist of mechanical and electrical energy. The potential energy can be described as

$$P = \frac{1}{2} \int_0^L \int_A \begin{bmatrix} \varepsilon \\ E \end{bmatrix}^T \begin{bmatrix} C^E & e \\ -e & \epsilon^e \end{bmatrix}^T \begin{bmatrix} \varepsilon \\ E \end{bmatrix} dA dx$$

As we did for the kinetic energy we now calculate the integral over the cross sectional area of the potential energy to define the potential energy as a line integral.

We first put our attention to the mechanical potential energy,

$$\begin{aligned} \int_A \sigma \varepsilon dA &= \int_A \sigma dA \varepsilon_0 + \int_A -z \sigma dA \kappa = \kappa (C^E A \varepsilon_0 - C^E I_0 \kappa - e A E_1 - e I_0 E_2) \varepsilon_0 \\ &\quad + (-C^E I_0 \varepsilon_0 + C^E I \kappa + e I_0 E_1 + e I E_2) \kappa. \end{aligned}$$

The same we do for the electrical potential energy,

$$\begin{aligned} \int_A D E dA &= \int_A D z dA E_2 + \int_A D dA E_1 = (e A e_0 - e I_0 \kappa + \varepsilon^e A E_1 + \varepsilon^e I_0 E_2) E_1 \\ &\quad + (e I_0 \varepsilon_0 - e I \kappa + \varepsilon^e I_0 E_1 + \varepsilon^e I E_2) E_2. \end{aligned}$$

So we can rewrite our constitutive equations (1) for the 1D case as

$$\begin{bmatrix} \tilde{\sigma} \\ \tilde{D} \end{bmatrix} = \begin{bmatrix} C^E \mathbf{N}_1 & -e \mathbf{N}_2^T \\ e \mathbf{N}_2 & \epsilon^e \mathbf{N}_3 \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon} \\ \tilde{E} \end{bmatrix},$$

where

$$\tilde{\sigma} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}, \tilde{D} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \tilde{\varepsilon} = \begin{bmatrix} \varepsilon_0 \\ \kappa \end{bmatrix}, \mathbf{N}_1 = \begin{bmatrix} A & -I_0 \\ -I_0 & I \end{bmatrix},$$

$$\mathbf{N}_2 = \begin{bmatrix} A & -I_0 \\ I_0 & -I \end{bmatrix}, \mathbf{N}_3 = |\mathbf{N}_1|.$$

With this definition we are able to rewrite our energy function in the following way

$$P = \frac{1}{2} \int_0^L \begin{bmatrix} \tilde{\varepsilon} \\ \tilde{E} \end{bmatrix}^T \begin{bmatrix} C^E \mathbf{N}_1 & -e \mathbf{N}_2^T \\ e \mathbf{N}_2 & \epsilon^e \mathbf{N}_3 \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon} \\ \tilde{E} \end{bmatrix} dx.$$

So the Hamiltonian of a 1D piezoelectric beam is given as

$$H(\tilde{\mathbf{p}}, \tilde{\varepsilon}, \tilde{E}) = K(\tilde{\mathbf{p}}) + P(\tilde{\varepsilon}, \tilde{E}).$$

Together with the equations of motion (3) we can define the distributed port-Hamiltonian model as

$$\begin{bmatrix} \dot{\tilde{\varepsilon}} \\ \dot{\tilde{\mathbf{p}}} \\ \dot{\tilde{E}} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \mathbf{I}_2 & 0 \\ \frac{\partial}{\partial x} \mathbf{I}_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \nabla_{\tilde{\varepsilon}} H \\ \nabla_{\tilde{\mathbf{p}}} H \\ \nabla_{\tilde{E}} H \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbf{I}_3 & 0 \\ 0 & -\mathbf{I}_2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 0 & 0 \\ \mathbf{I}_3 & 0 \\ 0 & -\mathbf{I}_2 \end{bmatrix}^T \begin{bmatrix} \nabla_{\tilde{\varepsilon}} H \\ \nabla_{\tilde{\mathbf{p}}} H \\ \nabla_{\tilde{E}} H \end{bmatrix},$$

with \mathbf{I}_n is the unit matrix of size n .

For this equation we used the fact that $\dot{z} = 0$ because the cross sectional area of our beam does not vary in time so

$$\dot{E} = \dot{E}_1 + z \dot{E}_2 = -\frac{\partial \dot{\varphi}}{\partial z} = -\dot{\varphi}_1 - z \dot{\varphi}_2.$$

3.2 Spatial discretization

Next we want to derive a finite dimensional pH model which describes the dynamics for an element at spatial position x_L of length Δx based on the forces acting on the boundary's of the element. To do this we first define the following flows and effort at the boundary's. We define the right boundary to $x_R = x_L + \Delta x$. Here the subscripts L and R are used to identify a state at the left and right boundary, respectively. The discretization in this chapter will be done via finite differences $\frac{\partial}{\partial x} f(x) \approx \frac{1}{\Delta x} (f(x + \Delta x) - f(x))$.

flows		efforts	
$\dot{u}_0(x_L) = f_d^L$	$\dot{u}_0(x_R) = f_d^R$	$\sigma_1^a(x_L) + \Delta x f_1 = e_d^L + e_{f_1}$	$\sigma_1^a(x_R) = e_d^R$
$\dot{\phi}(x_L) = f_r^L$	$\dot{\phi}(x_R) = f_r^R$	$\sigma_2^a(x_L) + \Delta x f_2 = e_r^L + e_{f_2}$	$\sigma_2^a(x_R) = e_r^R$
$\dot{w} = f_p$	$\rho_1^e = f_1^e$	$f_3 \Delta x = e_p$	$\phi_2^e = e_2^e$
$\rho_2^e = f_2^e$		$\phi_1^e = e_1^e$	

We know that the strains ε_0 and κ are defined as spatial derivatives of u_0 and ϕ , respectively. So we define the following discretized equations of motion

$$\dot{\kappa}^a \dot{\varepsilon}_0^a = \dot{u}_0^R - \dot{u}_0^L = f_d^R - f_d^L, \dot{\kappa}^a = \dot{\phi}^R - \dot{\phi}^L = f_r^R - f_r^L.$$

With this formulation of strain we have to rewrite the constitutive equations in the following way,

$$\begin{bmatrix} \tilde{\sigma}^a \\ \tilde{D}^a \end{bmatrix}^R = \begin{bmatrix} \frac{1}{\Delta x} C^E \mathbf{N}_1 & -e \mathbf{N}_2 \\ \frac{1}{\Delta x} e \mathbf{N}_2 & \epsilon^e \mathbf{N}_1 \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon}^a \\ \tilde{E}^a \end{bmatrix}.$$

If we define the discretized moments \mathbf{p}^a to be

$$\tilde{\mathbf{p}}^a = \Delta x \mathbf{M} \mathbf{u},$$

we can define the equations of moments for them to be

$$\dot{\tilde{\mathbf{p}}}^a = \begin{bmatrix} \tilde{\sigma}^a \\ 0 \end{bmatrix}^R + \begin{bmatrix} \tilde{\sigma}^a \\ 0 \end{bmatrix}^L + \Delta x \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} e_d^R \\ e_r^R \\ 0 \end{bmatrix} - \begin{bmatrix} e_d^L \\ e_r^L \\ 0 \end{bmatrix} + \begin{bmatrix} e_{f1} \\ e_{f2} \\ e_{f3} \end{bmatrix}.$$

Then we have everything we need to define the port-Hamiltonian model, which is given as

$$\begin{bmatrix} \dot{\tilde{\varepsilon}}^a \\ \dot{\tilde{\mathbf{p}}}^a \\ \dot{\tilde{E}}^a \end{bmatrix} = \begin{bmatrix} 0 & -I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \nabla_{\mathbf{x}} H + \begin{bmatrix} I_2 & 0 & 0 & 0 & 0 \\ 0 & -I_2 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -I_2 \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} I_2 & 0 & 0 & 0 & 0 \\ 0 & -I_2 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -I_2 \end{bmatrix}^T \nabla_{\mathbf{x}} H,$$

where

$$\nabla_{\mathbf{x}} H = [e_d^R, e_r^R, f_d^L, f_r^L, f_p, f_1^e, f_2^e]^T, \mathbf{u} = [f_d^R, f_r^R, e_d^L, e_r^L, e_{f1}, e_{f2}, e_{f3}, e_1^e, e_2^e]^T$$

According to the discretization, the Hamiltonian is given as

$$H = \frac{1}{2\Delta x} \tilde{\mathbf{p}}^a M^{-1} \tilde{\mathbf{p}}^a + \frac{1}{2} \begin{bmatrix} \tilde{\varepsilon}^a \\ \tilde{E}^a \end{bmatrix}^T \begin{bmatrix} \frac{1}{\Delta x} C^E \mathbf{N}_1 & -e \mathbf{N}_2 \\ \frac{1}{\Delta x} e \mathbf{N}_2 & \epsilon^e \mathbf{N}_1 \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon}^a \\ \tilde{E}^a \end{bmatrix}$$

4 Interconnection of the lumped port-Hamiltonian model

Next we construct a lumped port-Hamiltonian model for the full piezoelectric beam. The model derived in Section 3 represents the dynamics of a small element at a specific point x . To derive a model which represents the full beam we first divide

our beam in n sub parts with length $\Delta x = \frac{L}{n}$. Then we define n local pH models which we have to interconnect such that we have the global model of the beam.

To be able to interconnect the model we identify the inputs and outputs of our local system to be able to connect them to the left and right local model. The inputs and outputs of the local piezoelectric beam model are given as

Inputs		Outputs	
$f_d^R = \dot{u}_0(x_R)$	$e_{f_2} = \Delta x f_2(x_L)$	$e_d^R = \sigma_1^a(x_R)$	$f_r^L = \dot{\phi}(x_L)$
$f_r^R = \dot{\phi}(x_R)$	$e_p = f_3 \Delta x$	$e_r^R = \sigma_2^a(x_R)$	$f_p = \dot{w}$
$e_d^L = \sigma_1^a(x_L)$	$\phi_1^e = e_1^e$	$f_d^L = \dot{u}_0(x_L)$	$\rho_1^e = f_1^e$
$e_r^L = \sigma_2^a(x_L)$	$\phi_2^e = e_2^e$	$f_r^L = \dot{\phi}(x_L)$	$\rho_2^e = f_2^e$
$e_{f_1} = \Delta x f_1(x_L)$		$f_d^L = \dot{u}_0(x_L)$	

First we identify the ports which are due to external influences or due to internal influences (induced by port-Hamiltonian model at x_{i-1} or x_{i+1}). It is obvious that the external power ports, induced by external mechanical and electrical sources, are

External Inputs	External Outputs
$e_{f_1} = \Delta x f_1(x)$	$f_d^L = \dot{u}_0(x)$
$e_{f_2} = \Delta x f_2(x)$	$f_r^L = \dot{\phi}(x)$
$e_p = f_3 \Delta x$	$f_p = \dot{w}$
$\phi_1^e = e_1^e$	$\rho_1^e = f_1^e$
$\phi_2^e = e_2^e$	$\rho_2^e = f_2^e$

Through these ports we can exchange energy with the outside world.

The internal ports of the system at x_i are used to exchange energy with the systems at x_{i-1} and x_{i+1} . It is easy to see that the output of the system at x_i is the input for the system at x_{i+1} and vice versa. Similarly the output of the system x_{i-1} is the input of x_i . This gives us the following ports to the system at x_{i+1} and x_{i-1} .

Inputs from		Outputs for	
x_{i+1}	x_{i-1}	x_{i+1}	x_{i-1}
$f_d^R = \dot{u}_0(x_R)$	$e_d^L = N(x_L)$	$e_d^L = N(x_L)$	$f_d^L = \dot{u}_0(x_L)$
$f_r^R = \dot{\phi}(x_R)$	$e_r^L = M(x_L)$	$e_r^L = M(x_L)$	$f_r^L = \dot{\phi}(x_L)$

The schematics of the described power exchange are shown in Figure 3.

5 Modeling of a symmetric piezoelectric composite

In this section we want to define a system that describes the dynamics of a piezoelectric composite. The composite consists of a base layer to which on both sides a piezoelectric layer is bonded. The model of a single layer was already given in Section 3. The reason why we chose a piezoelectric composite which has a piezoelectric patches at both sides is that with this we can induce twice the actuation force, if we let deform both patches in the same way. It is also possible to use one piezoelectric

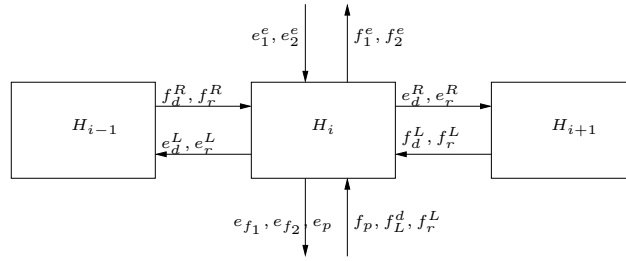


Figure 3. Interconnection of discretized pH-models

patch as a sensor and the other as an actuator. This are big advantages compared to a composite where we have only one piezoelectric patch attached to the base layer. Since the material of the two piezoelectric layers is the same, the constitutive equations for them are the same, see (1).

Therefore, we first define the connection between the layers. Because in our final system the piezo-electric layers are bonded to the base layer, the strains in all 3 layers are the same. These constraints assure the perfect bonding so

$$\varepsilon_b = \varepsilon_{p_1} = \varepsilon_{p_2}.$$

In the sequel we will use the subscript b to identify the base layer, the subscript p_1 for the upper piezo-electric layer, and the subscript p_2 for the lower piezo-electric layer. From the continuity of strain it also automatically follows that the $\mathbf{u}_b = \mathbf{u}_{p_1} = \mathbf{u}_{p_2} \Rightarrow \dot{\mathbf{u}}_b = \dot{\mathbf{u}}_{p_1} = \dot{\mathbf{u}}_{p_2}$.

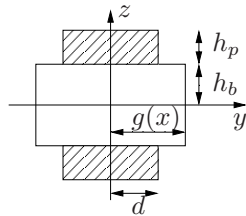


Figure 4. Cross sectional area of the composite

Before we try to express the total stored energy as a line integral we have to define the geometry of our system, see Figure 4. We assume that the base layer has a constant thickness ($2d$) and a constant height $2h_b$ while the length is L . We also define that the origin of the yz -Plane is in the center of mass of the base layer. So the cross sectional area of the base layer A_b is $[-d, d] \times [-h_b, h_b]$. With this it follows that $I_{b,0} = \int_{A_b} z dA_b = 0$. On top of the base layer the piezo-electric layer is bonded. The height of the layer is h_p and the width is depending on x ($2g(x)$). We also assume that the width is symmetric with the x -axis. Then we define the cross sectional area of the piezo-electric layer $A_{p_1}(x)$ as $[-g(x), g(x)] \times [h_b, h_b + h_p]$. Under the base layer we also have attached a piezo electric patch which has the same geometry so that the cross sectional area A_{p_2} is $[-g(x), g(x)] \times [-h_b, -h_b - h_p]$. With this given geometry we can formulate the total stored energy as a line integral.

Remark 1: If we take a look at the constants I , I_0 and A for every layer we find the following relations, $I_{b,0} = 0$, $A_{p_1} = A_{p_2}$, $I_{p_1,0} = -I_{p_2,0}$, $I_{p_1} = I_{p_2}$.

The energy stored in the composite will be the sum of the energy's stored in the three layers,

$$H_{tot} = H_b + H_{p_1} + H_{p_2}.$$

In Section 3 we already defined the model for a piezo-electric beam as a line integral. The model for the base layer is the same except that all electrical terms are zero. And thus we can now combine these models to derive a model which describes the dynamics of the piezoelectric composite. First we find a global expression for the total kinetic energy as a line integral. The total kinetic energy is given as

$$K_{tot} = \frac{1}{2} \int_0^L \tilde{\mathbf{p}}_b^T \mathbf{M}_b^{-1} \tilde{\mathbf{p}}_b + \tilde{\mathbf{p}}_{p_1}^T \mathbf{M}_{p_1}^{-1} \tilde{\mathbf{p}}_{p_1} + \tilde{\mathbf{p}}_{p_2}^T \mathbf{M}_{p_2}^{-1} \tilde{\mathbf{p}}_{p_2} dx,$$

where

$$\tilde{\mathbf{p}}_b = \mathbf{M}_b \frac{\partial}{\partial t} \tilde{\mathbf{u}}, \tilde{\mathbf{p}}_{p_i} = \mathbf{M}_{p_i} \frac{\partial}{\partial t} \tilde{\mathbf{u}}, \mathbf{M}_b = \begin{bmatrix} \rho_b A_b & 0 & 0 \\ 0 & \rho_b I_b & 0 \\ 0 & 0 & \rho_b A_b \end{bmatrix}$$

$$\mathbf{M}_{p_i} = \begin{bmatrix} \rho_p A_{p_i} & -\rho_p I_{p_i,0} & 0 \\ -\rho_p I_{p_i,0} & \rho_p I_{p_i} & 0 \\ 0 & 0 & \rho_p A_{p_i} \end{bmatrix}.$$

Now we can combine the kinetic energy in the following way

$$K_{tot} = \frac{1}{2} \int_0^L \tilde{\mathbf{p}}_{tot}^T \mathbf{M}_{tot}^{-1} \tilde{\mathbf{p}}_{tot} dx,$$

with

$$\tilde{\mathbf{p}}_{tot} = \mathbf{M}_{tot} \frac{\partial}{\partial t} \tilde{\mathbf{u}}, \mathbf{M}_{tot} = \mathbf{M}_b + \mathbf{M}_{p_1} + \mathbf{M}_{p_2}.$$

Next we do the same for the mechanical potential energy. It is the sum of the mechanical potential energy's of the three layers, so

$$P_{tot} = P_b + P_{p_1} + P_{p_2}.$$

From Section 3 we already have an expression as a line integral for each potential energy. So we have to combine these expression to get the total potential energy. So

$$P_{tot} = \frac{1}{2} \int_0^L \tilde{\boldsymbol{\varepsilon}}^T C_b^E \mathbf{N}_{b,1} \tilde{\boldsymbol{\varepsilon}} + \sum_{i=1}^2 \begin{bmatrix} \tilde{\boldsymbol{\varepsilon}} \\ \tilde{E}_{p_i} \end{bmatrix}^T \begin{bmatrix} C_p^E \mathbf{N}_{p_i,1}^T & e \mathbf{N}_{p_i,2} \\ -e \mathbf{N}_{p_i,2}^T & \epsilon^e \mathbf{N}_{p_i,3} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\varepsilon}} \\ \tilde{E}_{p_i} \end{bmatrix},$$

where

$$\mathbf{N}_{b,1} = \begin{bmatrix} I_b & 0 \\ 0 & A_b \end{bmatrix}, \mathbf{N}_{p_i,1} = \begin{bmatrix} I_{p_i} & -I_{p_i,0} \\ -I_{p_i,0} & A_{p_i} \end{bmatrix}, \mathbf{N}_{p_{1/2},2} = \begin{bmatrix} A_{p_i} & -I_{p_i,0} \\ I_{p_i,0} & -I_{p_i} \end{bmatrix}$$

$$\mathbf{N}_{p_i,3} = |\mathbf{N}_{p_i,2}|, \tilde{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_0 \\ \kappa \end{bmatrix}, \tilde{E}_{p_1} = \begin{bmatrix} E_{p_1,1} \\ E_{p_1,2} \end{bmatrix}, \tilde{E}_{p_2} = \begin{bmatrix} E_{p_2,1} \\ E_{p_2,2} \end{bmatrix}.$$

We are able to rewrite this potential energy if we use the following constitutive equations

$$\begin{bmatrix} \tilde{\boldsymbol{\sigma}} \\ \tilde{D}_{p_1} \\ \tilde{D}_{p_2} \end{bmatrix} = \begin{bmatrix} C_{11} & -C_{21}^T & -C_{31}^T \\ C_{21} & C_{22} & 0 \\ C_{31} & 0 & C_{33} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\varepsilon}} \\ \tilde{E}_{p_1} \\ \tilde{E}_{p_2} \end{bmatrix},$$

where

$$C_{11} = C_b^E \mathbf{N}_{b,1} + 2C_p^E (\mathbf{N}_{p1,1} + \mathbf{N}_{p2,1}), C_{21} = e \mathbf{N}_{p1,1},$$

$$C_{22} = \epsilon^e \mathbf{N}_{p1,3}, C_{31} = e \mathbf{N}_{p2,1}, C_{33} = \epsilon^e \mathbf{N}_{p2,3}.$$

With this definition we are able to rewrite our energy function as

$$H = \frac{1}{2} \int_0^L \tilde{\mathbf{p}}_{tot}^T \mathbf{M} \tilde{\mathbf{p}}_{tot} + \begin{bmatrix} \tilde{\epsilon} \\ \tilde{E}_{p1} \\ \tilde{E}_{p2} \end{bmatrix}^T \begin{bmatrix} C_{11}^T & C_{21}^T & C_{31}^T \\ -C_{21}^T & C_{22}^T & 0 \\ -C_{31}^T & 0 & C_{33}^T \end{bmatrix} \begin{bmatrix} \tilde{\epsilon} \\ \tilde{E}_{p1} \\ \tilde{E}_{p2} \end{bmatrix} dx$$

The equations of motion for our system are defined in the same way as in Section 3 except that we now have two electrical fields. With this in mind we can formulate the following port-Hamiltonian model

$$\begin{bmatrix} \dot{\tilde{\epsilon}} \\ \dot{\tilde{\mathbf{p}}}_{tot} \\ \dot{\tilde{E}}_{p1} \\ \dot{\tilde{E}}_{p2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \mathbf{I}_2 & 0 \\ \frac{\partial}{\partial x} \mathbf{I}_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \nabla_{\tilde{\epsilon}} H \\ \nabla_{\tilde{\mathbf{p}}_{tot}} H \\ \nabla_{\tilde{E}_{p1}} H \\ \nabla_{\tilde{E}_{p2}} H \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mathbf{I}_3 & 0 \\ 0 & -\mathbf{I}_4 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \dot{\varphi}_{p1,1} \\ \dot{\varphi}_{p1,2} \\ \dot{\varphi}_{p2,1} \\ \dot{\varphi}_{p2,2} \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} 0 & 0 \\ \mathbf{I}_3 & 0 \\ 0 & -\mathbf{I}_4 \end{bmatrix}^T \begin{bmatrix} \nabla_{\tilde{\epsilon}} H \\ \nabla_{\tilde{\mathbf{p}}_{tot}} H \\ \nabla_{\tilde{E}_{p1}} H \\ \nabla_{\tilde{E}_{p2}} H \end{bmatrix}.$$

Remark 2: The discretization can be done in the same way as it is done in Section 3.2, therefore we will not state it here.

6 Concluding remarks

In this paper we have determined a model for an inflatable structure in an pH framework. The modeling was done in a pH formulation in such a way that it can be used for an energy based control methods. The achieved model is a linear pH model which can easily be used to represent the dynamics of a piezo electric composite beam. Also the fact that the system can be expressed as an interconnection of subsystems simplifies the way to express a more complex system.

For the future we aim at including large/nonlinear deformations in the pH model. Additionally we want to derive a 2D-model.

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