

# Interpolation-based $\mathcal{H}_2$ Model Reduction for port-Hamiltonian Systems

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**Abstract**—Port network modeling of physical systems leads directly to an important class of passive state space systems: port-Hamiltonian systems. We consider here methods for model reduction of large scale port-Hamiltonian systems that preserve port-Hamiltonian structure and are capable of yielding reduced order models that satisfy first-order optimality conditions with respect to an  $\mathcal{H}_2$  system error metric. The methods we consider are closely related to rational Krylov methods and variants are described using both energy and co-energy system coordinates. The resulting reduced models have port-Hamiltonian structure and therefore are guaranteed passive, while still retaining the flexibility to interpolate the true system transfer function at any (complex) frequency points that are desired.

## I. INTRODUCTION

Port network modeling of physical systems leads directly to their representation as *port-Hamiltonian systems* which are, if the *Hamiltonian* is non-negative, an important class of *passive state space systems*. At the same time modeling of physical systems often leads to high-dimensional dynamical models. State space dimensions tend to become large as well if distributed-parameter (PDE) models are spatially discretized, and so *model reduction* for such high-dimensional systems becomes an important concern. The goal of this work is to demonstrate that specific model reduction techniques for linear states-space systems can be also applied to port-Hamiltonian systems in such a way as to preserve the *port-Hamiltonian structure* in the reduced order models, which preserves, as a consequence, *passivity* as well.

There are a variety of methods for model reduction of large-scale linear dynamical systems. *Moment matching* methods are an important class and are based on matching a specific number of the coefficients of the Laurent series expansion of the transfer function of the full order model with that of the reduced order model at certain points in the complex plane, i.e. the reduced order transfer function interpolates the original and possibly its derivatives, at selected interpolation points (“shifts”). Since the moments are numerically ill-conditioned, the goal is to match moments without explicitly computing them. This is achieved in practice by utilizing numerically efficient rational Krylov methods; model reduction by moment matching is also called model reduction by rational Krylov methods. In the last decade, rational Krylov model reduction has become the

method of choice in large-scale settings where the number of state-variables extend to the hundreds of thousands.

The main drawback of moment matching for model reduction has been, until recently, the largely *ad hoc* choice of interpolation points. This problem has been recently resolved by Gugercin *et al.* [?], [?], [7] who introduced a shift selection strategy leading to  $\mathcal{H}_2$ -optimal reduced order models. See also [?], [?], [?] for related work.

In this paper we concentrate on model reduction of port-Hamiltonian systems using rational Krylov methods. We show that rational Krylov methods may be applied to port-Hamiltonian systems in such a way as to not only match moments of the transfer functions at specific points in the complex plane but also preserve the port-Hamiltonian structure and passivity. We introduce a related algorithm for  $\mathcal{H}_2$ -optimal structure-preserving model reduction of port-Hamiltonian systems.

## II. INTERPOLATORY MODEL REDUCTION

Consider a linear, time invariant (LTI) single-input/single-output (SISO) continuous-time system  $G(s)$  described via

$$G(s) : \begin{cases} \mathbf{E}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u, \\ y &= \mathbf{c}\mathbf{x}, \end{cases} \quad (1)$$

where  $\mathbf{A}, \mathbf{E} \in \mathbb{R}^{n \times n}$  and  $\mathbf{b}, \mathbf{c}^T \in \mathbb{R}^n$ .  $\mathbf{x}(t) \in \mathbb{R}^n$  is the *state*,  $u(t) \in \mathbb{R}$  is the *input*, and  $y(t) \in \mathbb{R}$  is the *output* of  $G(s)$ . The transfer function is given by  $G(s) = \mathbf{c}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$ . With usual abuse of notation, both the underlying dynamical system and its transfer function will be denoted by  $G(s)$ .

In many applications; see, for example, [1], [11], the system dimension  $n$  is quite large, making computation infeasible due to memory, time limitations, and numerical ill-conditioning. The goal of model reduction is to produce a much smaller order system  $G_r(s)$  with state space form

$$G_r(s) : \begin{cases} \mathbf{E}_r \dot{\mathbf{x}}_r(t) &= \mathbf{A}_r \mathbf{x}_r(t) + \mathbf{b}_r u(t) \\ y_r(t) &= \mathbf{c}_r \mathbf{x}_r(t), \end{cases} \quad (2)$$

where  $\mathbf{A}_r, \mathbf{E}_r \in \mathbb{R}^{r \times r}$ ,  $\mathbf{b}_r, \mathbf{c}_r^T \in \mathbb{R}^r$  with  $r \ll n$  such that  $y_r(t)$  approximates  $y(t)$  well in some norm. Here we use an  $\mathcal{H}_2$  performance measure, which is introduced in Section V.

We will construct reduced order models  $G_r(s)$  via projection. That is, we construct matrices  $\mathbf{V}_r, \mathbf{W}_r \in \mathbb{R}^{n \times r}$  such that the reduced order  $G_r(s)$  in (2) is then obtained using

$$\mathbf{A}_r = \mathbf{W}_r^T \mathbf{A} \mathbf{V}_r, \quad \mathbf{E}_r = \mathbf{W}_r^T \mathbf{E} \mathbf{V}_r, \quad \mathbf{b}_r = \mathbf{W}_r^T \mathbf{b}, \quad \mathbf{c}_r = \mathbf{c} \mathbf{V}_r. \quad (3)$$

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### A. Moment matching and Krylov-based model reduction

Given the full order model as in (1), the moment matching problem is to find a reduced system as in (2) so that  $G_r(s)$  interpolates  $G(s)$  (and perhaps some of its derivatives as well) at selected interpolation points in the complex plane — the *shifts*,  $s_k$ . Simple interpolation will suffice (see the discussion of Section V); hence we seek  $\mathbf{A}_r$ ,  $\mathbf{E}_r$ ,  $\mathbf{b}_r$ , and  $\mathbf{c}_r$  so that for  $k = 1, \dots, r$ :  $G_r(s_k) = G(s_k)$ , or equivalently,  $\mathbf{c}(s_k \mathbf{E} - \mathbf{A})^{-1} \mathbf{b} = \mathbf{c}_r(s_k \mathbf{E}_r - \mathbf{A}_r)^{-1} \mathbf{b}_r$ .

The quantity  $\mathbf{c}(s_k \mathbf{E} - \mathbf{A})^{-(j+1)} \mathbf{b}$  is called the  $j^{\text{th}}$  moment of  $G(s)$  at  $s_k$ . Moment matching for finite  $s_k \in \mathbb{C}$ , becomes *rational interpolation*, see for example [2]. Solution of the rational interpolation problem via projection was introduced by Skelton *et. al.* in [4], [23], [24]. Grimme [6] showed how to construct the required projection using the numerically efficient rational Krylov methods of Ruhe [16]. Gugercin *et al.* [7] presented a concise proof of rational interpolation via Krylov projection. Theorem 1 below shows how to solve by projection the rational interpolation problem stated above: find  $\mathbf{A}_r$ ,  $\mathbf{E}_r$ ,  $\mathbf{b}_r$ , and  $\mathbf{c}_r$  so that  $G_r(s_k) = G(s_k)$ . More general cases are discussed in [6], [4], [23], [24].

*Theorem 1:* Given  $G(s) = \mathbf{c}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}$  and a set of distinct shifts  $\{s_k\}_{k=1}^r$ , construct  $\mathbf{V}_r$  such that

$$\mathbf{V}_r = [(s_1 \mathbf{E} - \mathbf{A})^{-1} \mathbf{b}, \dots, (s_r \mathbf{E} - \mathbf{A})^{-1} \mathbf{b}] \quad (4)$$

Then for any  $\mathbf{W}_r \in \mathbb{C}^{n \times r}$ , the reduced order system  $G_r(s) = \mathbf{c}_r(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r$  defined by  $\mathbf{A}_r = \mathbf{W}_r^T \mathbf{A} \mathbf{V}_r$ ,  $\mathbf{E}_r = \mathbf{W}_r^T \mathbf{E} \mathbf{V}_r$ ,  $\mathbf{b}_r = \mathbf{W}_r^T \mathbf{b}$ ,  $\mathbf{c}_r = \mathbf{c} \mathbf{V}_r$  satisfies  $G_r(s_k) = G(s_k)$  for  $k = 1, \dots, r$ , provided that  $s_i \mathbf{E} - \mathbf{A}$  and  $s_i \mathbf{E}_r - \mathbf{A}_r$  are invertible for  $i = 1, \dots, r$ .

*Remark 1:* The interpolation property still holds if  $\mathbf{V}_r$  above is replaced with any matrix  $\widehat{\mathbf{V}}_r$  having the same range as  $\mathbf{V}_r$ :  $\widehat{\mathbf{V}}_r = \mathbf{V}_r \mathbf{L}$  for invertible  $\mathbf{L} \in \mathbb{R}^{r \times r}$ . This is because the basis change  $\mathbf{L}$  taking  $\mathbf{V}_r$  to  $\widehat{\mathbf{V}}_r$  is simply a state space transformation for the reduced model. This property guarantees that so long as the shift set is closed under conjugation (as always done in practice),  $\widehat{\mathbf{V}}_r$  can always be chosen to be real and hence the reduced order quantities are guaranteed to be real as well.

### III. LINEAR PORT-HAMILTONIAN SYSTEMS

In the absence of algebraic constraints, linear port-Hamiltonian systems take the form ([19], [15], [14]):

$$\begin{cases} \dot{\mathbf{x}} &= (\mathbf{J} - \mathbf{R})\mathbf{Q}\mathbf{x} + \mathbf{b}u \\ y &= \mathbf{b}^T \mathbf{Q}\mathbf{x} \end{cases} \quad (5)$$

where  $\mathbf{Q} = \mathbf{Q}^T$  is the *energy* matrix associated with the *total energy* (Hamiltonian),  $H(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T \mathbf{Q}\mathbf{x}$ .  $\mathbf{R} = \mathbf{R}^T \geq 0$  is the *dissipation* matrix and both  $\mathbf{J} = -\mathbf{J}^T$  and  $\mathbf{b}$  specify the *interconnection* structure. Note that the system (5) has the form (1) with  $\mathbf{E} = \mathbf{I}$ ,  $\mathbf{A} = (\mathbf{J} - \mathbf{R})\mathbf{Q}$ , and  $\mathbf{c} = \mathbf{b}^T \mathbf{Q}$ .

*Theorem 2:* A port-Hamiltonian system (5) is *passive* if the energy matrix  $\mathbf{Q}$  is positive semidefinite.

*Proof:* Follows from passivity theory, see [20], [19]. ■

The state variables  $\mathbf{x} \in \mathbb{R}^n$  are called *energy* variables, since the total energy  $H(\mathbf{x})$  is expressed with respect to  $\mathbf{x}$ .

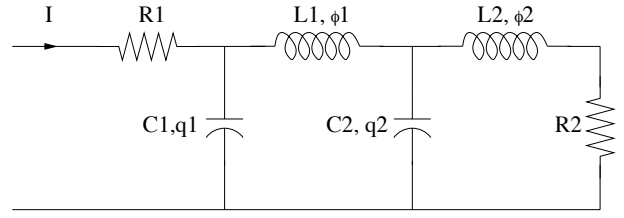


Fig. 1. Ladder Network

The variables  $u, y \in \mathbb{R}$  are called *power* variables, since their product  $u \cdot y$  is the power supplied to the system.

*Example 1:* Consider the Ladder Network in Fig. 1, with  $C_1, C_2, L_1, L_2, R_1, R_2$  representing (linear) capacitances, inductances and resistances as shown. The port-Hamiltonian representation of this physical system will be of the form (5) with the corresponding matrices

$$\begin{aligned} \mathbf{J} &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ \mathbf{R} &= \text{diag}\{0, 0, 0, R_2\}, \\ \mathbf{Q} &= \text{diag}\{C_1^{-1}, L_1^{-1}, C_2^{-1}, L_2^{-1}\}, \\ \mathbf{b} &= [1 \ 0 \ 0 \ 0]^T. \end{aligned} \quad (6)$$

The  $\mathbf{A}$  matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & -L_1^{-1} & 0 & 0 \\ C_1^{-1} & 0 & -C_2^{-1} & 0 \\ 0 & L_1^{-1} & 0 & -L_2^{-1} \\ 0 & 0 & C_2^{-1} & -R_2 L_2^{-1} \end{bmatrix}$$

and the state space vector  $\mathbf{x}$  is given as

$$\mathbf{x}^T = [q_1 \ \phi_1 \ q_2 \ \phi_2]$$

with  $q_1, q_2$  the charges on the capacitors  $C_1, C_2$  and  $\phi_1, \phi_2$  the fluxes through the inductors  $L_1, L_2$  respectively. The input of the system  $u$  is given by the current  $I$  from the external current source; the output  $y$  is the voltage over the first capacitor.

The system matrices  $\mathbf{A}$  and  $\mathbf{b}$  follow directly from writing the linear input-state differential equation for this system.  $\mathbf{Q}$  comes from the Hamiltonian  $H(\mathbf{x}) = \mathbf{x}^T \mathbf{Q}\mathbf{x}$  (known from physics). With  $\mathbf{A}$  and  $\mathbf{Q}$  it is easy to derive a dissipation matrix  $\mathbf{R}$  and a structure matrix  $\mathbf{J}$  so that  $\mathbf{A} = (\mathbf{J} - \mathbf{R})\mathbf{Q}$ . The port-Hamiltonian output matrix  $\mathbf{c}$  is given as  $\mathbf{b}^T \mathbf{Q}$ .

In order to proceed we recall from [15], [14] that a port-Hamiltonian system (5) in *co-energy coordinates* takes the following form

$$\begin{cases} \dot{\mathbf{e}} &= \mathbf{Q}(\mathbf{J} - \mathbf{R})\mathbf{e} + \mathbf{Q}\mathbf{b}u, \\ y &= \mathbf{b}^T \mathbf{e}, \end{cases} \quad (7)$$

The coordinate transformation [15] between energy  $\mathbf{x}$  and co-energy  $\mathbf{e}$  coordinates is given by the energy matrix  $\mathbf{Q}$

$$\mathbf{e} = \mathbf{Q}\mathbf{x}. \quad (8)$$

*Example 2:* (continued). The state space vector  $\mathbf{e}$  for the Ladder Network of Example 1 in co-energy coordinates is

$$\mathbf{e}^T = [ V_1 \quad I_1 \quad V_2 \quad I_2 ]$$

with  $V_1, V_2$  the voltages on the capacitors  $C_1, C_2$  and  $I_1, I_2$  the currents through the inductors  $L_1, L_2$  respectively.

#### IV. REDUCTION OF PORT-HAMILTONIAN SYSTEMS BY RATIONAL KRYLOV METHODS

In this section we show how to use *rational Krylov methods* in order to construct reduced order models which would not only interpolate the original system but also preserve the original port-Hamiltonian structure of the full order systems. The key observation is that even though simple application of Theorem 1 to the original port-Hamiltonian state space structure in (5) would satisfy interpolation conditions, the reduced-model would *no longer* be in the port-Hamiltonian form; i.e. the structure and properties such as passivity will be lost. The loss of structure follows from the fact that when Theorem 1 is directly applied to (5), the reduced system matrix will have the form  $\mathbf{W}_r^T(\mathbf{J} - \mathbf{R})\mathbf{Q}\mathbf{V}_r$ . Since this reduced form can be no longer represented as  $(\mathbf{J}_r - \mathbf{R}_r)\mathbf{Q}_r$  where  $\mathbf{J}_r$  is skew-symmetric, and  $\mathbf{R}_r$  and  $\mathbf{Q}_r$  are symmetric positive semi-definite. Therefore, in order to preserve the structure as well as to guarantee interpolation, one would need to perform Krylov-based model reduction in such a way that the reduced order quantities will satisfy  $\mathbf{J}_r = \widehat{\mathbf{V}}_r^T \mathbf{J} \widehat{\mathbf{V}}_r$ ,  $\mathbf{R}_r = \widehat{\mathbf{V}}_r^T \mathbf{R} \widehat{\mathbf{V}}_r$ , and  $\mathbf{Q}_r = \widehat{\mathbf{V}}_r^T \mathbf{Q} \widehat{\mathbf{V}}_r$ . Clearly, these reduced order matrices preserve original structure. In what follows we show how to achieve this goal; i.e. forcing interpolation and preserving structure simultaneously.

##### A. Model reduction in original energy coordinates

The key component for obtaining a structure-preserving Krylov-based model reduction is the freedom in choosing the matrix  $\mathbf{W}_r$  once  $\mathbf{V}_r$  is chosen appropriately as discussed in Theorem 1. We present our first main results that exploits this property:

*Theorem 3:* Consider the full order port-Hamiltonian system (5). Let  $s_1, \dots, s_r$  be  $r$  distinct interpolation points. Construct  $\mathbf{V}_r$  as in (4) using  $\mathbf{A} = (\mathbf{J} - \mathbf{R})\mathbf{Q}$  and  $\mathbf{E} = \mathbf{I}$ , i.e.

$$\mathbf{V}_r = [(s_1\mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1}\mathbf{b}, \dots, (s_r\mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1}\mathbf{b}]. \quad (9)$$

Let  $\widehat{\mathbf{V}}_r = \mathbf{V}_r \mathbf{L}^{-1}$  where  $\mathbf{L}$  is the Cholesky factor of  $\mathbf{V}_r^T \mathbf{Q} \mathbf{V}_r$ , i.e.  $\mathbf{V}_r^T \mathbf{Q} \mathbf{V}_r = \mathbf{L}^T \mathbf{L}$ . Choose  $\widehat{\mathbf{W}}_r = \mathbf{Q} \widehat{\mathbf{V}}_r$ . Define

$$\begin{aligned} \mathbf{J}_r &= \widehat{\mathbf{W}}_r^T \mathbf{J} \mathbf{Q} \widehat{\mathbf{V}}_r = \widehat{\mathbf{V}}_r^T \mathbf{Q} \mathbf{J} \mathbf{Q} \widehat{\mathbf{V}}_r, \\ \mathbf{R}_r &= \widehat{\mathbf{W}}_r^T \mathbf{R} \mathbf{Q} \widehat{\mathbf{V}}_r = \widehat{\mathbf{V}}_r^T \mathbf{Q} \mathbf{R} \mathbf{Q} \widehat{\mathbf{V}}_r, \\ \mathbf{Q}_r &= \widehat{\mathbf{W}}_r^T \widehat{\mathbf{V}}_r = \mathbf{I}_r \\ \mathbf{b}_r &= \widehat{\mathbf{W}}_r^T \mathbf{b} = \widehat{\mathbf{V}}_r^T \mathbf{Q} \mathbf{b}. \end{aligned} \quad (10)$$

Then, the reduced order model

$$\begin{cases} \dot{\mathbf{x}}_r &= (\mathbf{J}_r - \mathbf{R}_r)\mathbf{Q}_r \mathbf{x}_r + \mathbf{b}_r u \\ y_r &= \mathbf{b}_r^T \mathbf{x}_r \end{cases} \quad (11)$$

is port-Hamiltonian and passive. Furthermore the reduced order port-Hamiltonian system (11) interpolates the original system (5) at  $s_1, \dots, s_r$ .

*Proof:* Note that by construction  $\mathbf{E}_r = \widehat{\mathbf{W}}_r^T \mathbf{E} \widehat{\mathbf{V}}_r = \mathbf{I}_r$ . Also,

$$\begin{aligned} \mathbf{A}_r &= \widehat{\mathbf{W}}_r^T \mathbf{A} \widehat{\mathbf{V}}_r = \widehat{\mathbf{V}}_r^T \mathbf{Q} (\mathbf{J} - \mathbf{R}) \mathbf{Q} \widehat{\mathbf{V}}_r \\ &= (\mathbf{J}_r - \mathbf{R}_r)\mathbf{Q}_r \end{aligned}$$

where  $\mathbf{J}_r$ ,  $\mathbf{R}_r$  and  $\mathbf{Q}_r$  are as defined in (10). Similarly,  $\mathbf{b}_r = \widehat{\mathbf{W}}_r^T \mathbf{b} = \widehat{\mathbf{V}}_r^T \mathbf{Q} \mathbf{b}$  and  $\mathbf{c}_r = \mathbf{b}^T \mathbf{Q} \widehat{\mathbf{V}}_r = \mathbf{b}_r^T$ . Hence, the reduced order model has the state space form (11). Moreover, by construction  $\mathbf{J}_r$  is skew-symmetric,  $\mathbf{R}_r$  is symmetric and positive semidefinite and  $\mathbf{Q}_r$  is symmetric positive definite. Therefore, the reduced-model (11) is in the form of (5), and hence is a reduced port-Hamiltonian system. The interpolation property is a direct consequence of Theorem 1. ■

##### B. Model reduction in scaled energy coordinates

In this section, we will introduce Krylov-based model reduction of port-Hamiltonian system in the scaled energy coordinates. Unlike the model reduction in the original coordinates (5), the new coordinate system will allow choosing  $\mathbf{W}_r = \mathbf{V}_r$  in the reduction step while preserving port-Hamiltonian structure and guaranteeing interpolation.

Consider a full order port-Hamiltonian system (5) with  $\mathbf{Q} > 0$ . Then there exists a coordinate transformation  $\mathbf{S}$  such that in the new coordinates  $\tilde{\mathbf{x}} = \mathbf{S}^{-1}\mathbf{x}$ , we obtain

$$\tilde{\mathbf{Q}} = \mathbf{S}^T \mathbf{Q} \mathbf{S} = \mathbf{I}. \quad (12)$$

By defining the system matrices

$$\tilde{\mathbf{J}} = \mathbf{S}^{-1} \mathbf{J} \mathbf{S}^{-T}, \quad \tilde{\mathbf{R}} = \mathbf{S}^{-1} \mathbf{R} \mathbf{S}^{-T}, \quad \tilde{\mathbf{b}} = \mathbf{S}^{-1} \mathbf{b}, \quad (13)$$

we obtain the port-Hamiltonian system (5) in the new coordinate system:

$$\begin{cases} \dot{\tilde{\mathbf{x}}} &= (\tilde{\mathbf{J}} - \tilde{\mathbf{R}})\tilde{\mathbf{x}} + \tilde{\mathbf{b}} u, \\ y &= \tilde{\mathbf{b}}^T \tilde{\mathbf{x}}. \end{cases} \quad (14)$$

Note that there are many ways to compute the coordinate transformation  $\mathbf{S}$  in (12), among them being the computationally efficient Cholesky factorisation of the matrix  $\mathbf{Q}$ .

*Theorem 4:* Consider the full order port-Hamiltonian system (5). Let  $s_1, \dots, s_r$  be  $r$  distinct interpolation points. Transform (5) into energy coordinates (14) using (13) and construct  $\mathbf{U}_r$  as in Theorem 1 using  $\mathbf{A} = \tilde{\mathbf{J}} - \tilde{\mathbf{R}}$ ,  $\mathbf{E} = \mathbf{I}$  and  $\mathbf{b} = \tilde{\mathbf{b}}$ , i.e.

$$\mathbf{U}_r = [(s_1\mathbf{I} - (\tilde{\mathbf{J}} - \tilde{\mathbf{R}}))^{-1}\tilde{\mathbf{b}}, \dots, (s_r\mathbf{I} - (\tilde{\mathbf{J}} - \tilde{\mathbf{R}}))^{-1}\tilde{\mathbf{b}}] \quad (15)$$

Let  $\widehat{\mathbf{U}}_r$  be an orthogonal basis for the columns of  $\mathbf{U}_r$ . Define

$$\tilde{\mathbf{J}}_r = \widehat{\mathbf{U}}_r^T \tilde{\mathbf{J}} \widehat{\mathbf{U}}_r, \quad \tilde{\mathbf{R}}_r = \widehat{\mathbf{U}}_r^T \tilde{\mathbf{R}} \widehat{\mathbf{U}}_r, \quad \tilde{\mathbf{b}}_r = \widehat{\mathbf{U}}_r^T \tilde{\mathbf{b}}. \quad (16)$$

Then, the reduced order model

$$\begin{cases} \dot{\tilde{\mathbf{x}}}_r &= (\tilde{\mathbf{J}}_r - \tilde{\mathbf{R}}_r)\tilde{\mathbf{x}}_r + \tilde{\mathbf{b}}_r u \\ y_r &= \tilde{\mathbf{b}}_r^T \tilde{\mathbf{x}}_r \end{cases} \quad (17)$$

is port-Hamiltonian and passive. Furthermore the reduced order port-Hamiltonian system (17) interpolates the original system (5) at  $s_1, \dots, s_r$ .

*Proof:* Note that by construction  $\mathbf{E}_r = \widehat{\mathbf{U}}_r^T \mathbf{E} \widehat{\mathbf{U}}_r = \mathbf{I}_r$ . Moreover,  $\mathbf{A}_r = \widehat{\mathbf{U}}_r^T \mathbf{A} \widehat{\mathbf{U}}_r = \widehat{\mathbf{U}}_r^T (\widetilde{\mathbf{J}} - \widetilde{\mathbf{R}}) \widehat{\mathbf{U}}_r = \widetilde{\mathbf{J}}_r - \widetilde{\mathbf{R}}_r$ . Clearly  $\widetilde{\mathbf{J}}_r$  is skew-symmetric and  $\widetilde{\mathbf{R}}_r$  is symmetric and positive semidefinite. Hence, the reduced order model (17) is of the same form as (14); and hence is port-Hamiltonian and passive. Interpolation property is direct consequence of Theorem 1. ■

We have thus introduced two ways of Krylov-based structure-preserving model reduction of port-Hamiltonian systems; one in the original energy coordinates, one in the scaled energy coordinates. The next result connects these two frameworks.

*Theorem 5:* Consider the full order port-Hamiltonian system (5). Let  $s_1, \dots, s_r$  be  $r$  distinct interpolation points. Then, the reduced order model of Theorem 3 and Theorem 4 are equivalent.

*Proof:* As noted before, model reduction using  $\mathbf{V}_r$  and  $\mathbf{W}_r$  is equivalent to model reduction using  $\mathbf{V}_r \mathbf{T}$  and  $\mathbf{W}_r \mathbf{Z}$  where  $\mathbf{T}$  and  $\mathbf{Z}$  are invertible matrices. Therefore, it is enough to show the equivalence of the reduced order models of Theorems 3 and 4 without the orthogonalization of the reducing matrices in each case. For Theorem 3, we will have  $\mathbf{E}_r = \mathbf{V}_r^T \mathbf{Q} \mathbf{V}_r$ ,  $\mathbf{A}_r = \mathbf{V}_r^T \mathbf{Q} (\mathbf{J} - \mathbf{R}) \mathbf{Q} \mathbf{V}_r$ ,  $\mathbf{b}_r = \mathbf{V}_r^T \mathbf{Q} \mathbf{b}$  and  $\mathbf{c}_r = \mathbf{b}^T \mathbf{Q} \mathbf{V}_r$ ; leading to the transfer function

$$G_r(s) = \mathbf{b}^T \mathbf{Q} \mathbf{V}_r \left( s \mathbf{V}_r^T \mathbf{Q} \mathbf{V}_r - \mathbf{V}_r^T \mathbf{Q} (\mathbf{J} - \mathbf{R}) \mathbf{Q} \mathbf{V}_r \right)^{-1} \mathbf{V}_r^T \mathbf{Q} \mathbf{b}$$

In the case of Theorem 4, we obtain  $\mathbf{E}_r = \mathbf{U}_r^T \mathbf{U}_r$ ,  $\mathbf{A}_r = \mathbf{U}_r^T (\widetilde{\mathbf{J}} - \widetilde{\mathbf{R}}) \mathbf{U}_r$ ,  $\mathbf{b}_r = \mathbf{U}_r^T \widetilde{\mathbf{b}}$  and  $\mathbf{c}_r = \widetilde{\mathbf{b}}^T \mathbf{U}_r$ ; leading to the reduced order transfer function

$$\widetilde{G}_r(s) = \widetilde{\mathbf{b}}^T \mathbf{U}_r \left( s \mathbf{U}_r^T \mathbf{U}_r - \mathbf{U}_r^T (\widetilde{\mathbf{J}} - \widetilde{\mathbf{R}}) \mathbf{U}_r \right)^{-1} \mathbf{U}_r^T \widetilde{\mathbf{b}} \quad (18)$$

We use the definitions of  $\widetilde{\mathbf{J}}, \widetilde{\mathbf{R}}$  and  $\widetilde{\mathbf{b}}$  in (13) with the definition of  $\mathbf{U}_r$  in (15) and obtain  $\mathbf{U}_r = \mathbf{S}^{-1} \mathbf{V}_r$  where  $\mathbf{V}_r$  is as defined in (9). Then, plugging  $\mathbf{U}_r = \mathbf{S}^{-1} \mathbf{V}_r$  together with the definitions of  $\widetilde{\mathbf{J}}, \widetilde{\mathbf{R}}$  and  $\widetilde{\mathbf{b}}$  from (13) into  $\widetilde{G}_r(s)$  in (18), one obtains the desired result that  $\widetilde{G}_r(s) = G_r(s)$ . ■

*Remark 2:* Theorem 5 proves the equivalence of two model reduction approaches in original energy coordinates and in scaled energy coordinates. Model reduction in original coordinates is usually numerically more efficient since there is then no need for a full order state space transformation. Moreover, such transformations are usually ill-conditioned. In all our examples, we employ the method of Section IV-A.

### C. Scaled co-energy coordinates

Given the full order port-Hamiltonian system (7) in the co-energy coordinates, construct a state transformation  $\mathbf{T}$  such that in the new coordinates  $\widehat{\mathbf{e}} = \mathbf{T}^{-1} \mathbf{e}$ , the energy matrix  $\widehat{\mathbf{Q}}$  is an identity matrix, i.e.  $\widehat{\mathbf{Q}} = \mathbf{T}^{-1} \mathbf{Q} \mathbf{T}^{-T} = \mathbf{I}$ , and consequently the state-dynamics have the form

$$\begin{cases} \dot{\widehat{\mathbf{e}}} &= (\widehat{\mathbf{J}} - \widehat{\mathbf{R}}) \widehat{\mathbf{e}} + \widehat{\mathbf{b}} u, \\ y &= \widehat{\mathbf{b}}^T \widehat{\mathbf{e}}, \end{cases} \quad (19)$$

$$\text{where } \widehat{\mathbf{J}} = \mathbf{T}^T \mathbf{J} \mathbf{T}, \quad \widehat{\mathbf{R}} = \mathbf{T}^T \mathbf{R} \mathbf{T}, \quad \text{and } \widehat{\mathbf{b}} = \mathbf{T}^T \mathbf{b}. \quad (20)$$

These are called scaled co-energy coordinates.

*Theorem 6:* Consider the full order port-Hamiltonian system (19). Let  $s_1, \dots, s_r$  be  $r$  distinct interpolation points. Construct an orthogonal matrix  $\mathbf{V}_r$  as as in Theorem 1 and Remark 1 using  $\mathbf{A} = \widehat{\mathbf{J}} - \widehat{\mathbf{R}}$ ,  $\mathbf{b} = \widehat{\mathbf{b}}$  where  $\widehat{\mathbf{J}}, \widehat{\mathbf{R}}$  and  $\widehat{\mathbf{b}}$  are as defined in (20). Then the reduced order system

$$\begin{cases} \dot{\widehat{\mathbf{e}}}_r &= (\widehat{\mathbf{J}}_r - \widehat{\mathbf{R}}_r) \widehat{\mathbf{e}}_r + \widehat{\mathbf{b}}_r u, \\ y_r &= \widehat{\mathbf{c}}_r \widehat{\mathbf{e}}_r \end{cases} \quad (21)$$

$$\text{with } \widehat{\mathbf{J}}_r = \mathbf{V}_r^T \widehat{\mathbf{J}} \mathbf{V}_r, \quad \widehat{\mathbf{R}}_r = \mathbf{V}_r^T \widehat{\mathbf{R}} \mathbf{V}_r, \\ \widehat{\mathbf{Q}} = \mathbf{V}_r^T \widehat{\mathbf{Q}} \mathbf{V}_r = \mathbf{I}_r, \quad \widehat{\mathbf{b}}_r = \mathbf{V}_r^T \widehat{\mathbf{b}}, \quad \widehat{\mathbf{c}}_r = \widehat{\mathbf{b}}_r^T,$$

is a port-Hamiltonian system; hence is also passive. The reduced order port-Hamiltonian system (21) interpolates the original system (19) at  $s_1, \dots, s_r$ .

*Proof:* Similar to that of Theorem 4; hence omitted. ■

*Remark 3:* Structure-preserving interpolatory model reduction for port-Hamiltonian system in co-energy coordinates can also be performed without the scaling of the co-energy coordinates by representing (7) equivalently in the descriptor form as

$$\begin{cases} \mathbf{Q}^{-1} \dot{\mathbf{e}} &= (\mathbf{J} - \mathbf{R}) \mathbf{e} + \mathbf{b} u, \\ y &= \mathbf{b}^T \mathbf{e}, \end{cases} \quad (22)$$

Then the reduced order system takes the form

$$\begin{cases} \mathbf{V}_r^T \mathbf{Q}^{-1} \mathbf{V}_r \dot{\mathbf{e}}_r &= \mathbf{V}_r^T (\mathbf{J} - \mathbf{R}) \mathbf{V}_r \mathbf{e}_r + \mathbf{V}_r^T \mathbf{b} u, \\ y_r &= \mathbf{b}^T \mathbf{V}_r \mathbf{e}_r \end{cases} \quad (23)$$

has the same properties as the one in Theorem 6. Moreover, one can construct the projection map  $\mathbf{V}_r$  in (23) such that  $\mathbf{V}_r^T \mathbf{Q}^{-1} \mathbf{V}_r = \mathbf{I}_r$ . We omit details here for conciseness but will include in them in the full paper.

*Theorem 7:* Consider the port-Hamiltonian system (5). Let  $s_1, \dots, s_r$  be  $r$  distinct interpolation points. Then, the reduced models of Theorem 3 and Theorem 6 are equivalent.

*Proof:* Similar to that of Theorem 5; hence omitted. ■

## V. INTERPOLATION-BASED OPTIMAL $\mathcal{H}_2$ APPROXIMATION OF PORT-HAMILTONIAN SYSTEMS

The main prior disadvantages of interpolatory model reduction have been the *ad hoc* selection of interpolation points and the lack of a guarantee on global  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  performance of the resulting reduced model. Recently, Gugercin *et al.* [7] have shown that an optimal shift selection strategy exists for the optimal  $\mathcal{H}_2$  approximation problem, and introduced an Iterative Rational Krylov Algorithm (**IRKA**) for model reduction that exploits it. Besides providing high quality reduced models, **IRKA** have also proved to be numerically effective and have been successfully applied to tackle the  $\mathcal{H}_2$ -optimal approximation problem for systems of order  $n > 160,000$ , see [10], [17].

### A. Optimal $\mathcal{H}_2$ Approximation

Given an  $n^{\text{th}}$  order SISO dynamical system  $G(s)$ , the goal of optimal  $\mathcal{H}_2$  approximation is to find a stable  $r^{\text{th}}$  order reduced system  $G_r(s)$  with  $r < n$ , such that  $G_r(s)$  minimizes the  $\mathcal{H}_2$  error, i.e.

$$G_r(s) = \arg \min_{\deg(\widehat{G})=r} \left\| G(s) - \widehat{G}(s) \right\|_{\mathcal{H}_2}. \quad (24)$$

where

$$\|G\|_{\mathcal{H}_2} := \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} |G(j\omega)|^2 d\omega \right)^{1/2}.$$

Many researchers have worked on this problem; see [22], [18], [3], [8], [13], [9], [21], [12], [25]. The main drawback of most methods is that they require solution of large-scale Lyapunov equations (possibly many of them) and dense matrix operations such as inversion. Such approaches rapidly become intractable in large-scale settings as the dimension increases. Obtaining the global minimum is a hard task, so the usual approach is to find a reduced order model satisfying (local) first-order conditions for (24):

*Theorem 8:* [13] Let  $G_r(s)$  solve the optimal  $\mathcal{H}_2$  problem and let  $\hat{\lambda}_k$  denote the poles of  $G_r(s)$ . Assume (for simplicity) that the poles  $\hat{\lambda}_k$  are simple. Then, the first-order necessary conditions for  $\mathcal{H}_2$  optimality are

$$G_r(-\hat{\lambda}_k) = G(-\hat{\lambda}_k) \quad \text{and} \quad (25)$$

$$G_r'(-\hat{\lambda}_k) = G'(-\hat{\lambda}_k), \quad \text{for } k = 1, \dots, r. \quad (26)$$

The work by Gugercin *et al.* [7] provides a new and simple proof of these necessary conditions for  $\mathcal{H}_2$ -optimality, and also shows equivalence with a variety of other (apparently distinct) necessary conditions for  $\mathcal{H}_2$ -optimality.

Theorem 8 states that  $G_r(s)$  has to *interpolate*  $G(s)$  and its first derivative at the mirror images of the reduced order poles. Thus, first-order conditions are given in the framework of interpolation. The method of Gugercin *et al.* [7] produces a reduced order model  $G_r(s)$  satisfying the interpolation-based first-order necessary conditions of (25)-(26) exploiting the connection between the Krylov-based reduction and interpolation. However note that the optimal interpolation points depend on the final reduced model and are not known *a priori*. Therefore, [7] uses rational Krylov steps to iteratively correct the reduced order model  $G_r(s)$  so that the next (corrected) reduced order model interpolates the full order model at mirror images of  $\hat{\lambda}_i$  from the previous reduced order model. This continues until the poles from consecutive reduced order models stagnate. For details of **IRKA**, see the original source [7].

### B. $\mathcal{H}_2$ model reduction for port-Hamiltonian Systems

Direct application of **IRKA** in port-Hamiltonian settings will destroy the port-Hamiltonian structure. Therefore, in the port-Hamiltonian setting, we offer below a modified version of **IRKA** which enforces only (25) yet preserves the port-Hamiltonian structure.

*Algorithm 1: An  $\mathcal{H}_2$ -based Iterative Rational Krylov Algorithm for port-Hamiltonian systems  $G(s)$  as in (5):*

- 1) Make an initial shift selection  $s_i$  for  $i = 1, \dots, r$
- 2)  $\mathbf{V}_r = [(s_1\mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1}\mathbf{b}, \dots, (s_r\mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1}\mathbf{b}]$ .
- 3) Let  $\hat{\mathbf{V}}_r = \mathbf{V}_r\mathbf{L}^{-1}$  with  $\mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r = \mathbf{L}^T\mathbf{L}$ .
- 4) Define  $\hat{\mathbf{W}}_r = \mathbf{Q}\hat{\mathbf{V}}_r$ .
- 5) while (not converged)
  - a)  $\mathbf{J}_r = \hat{\mathbf{W}}_r^T\mathbf{J}\mathbf{Q}\hat{\mathbf{V}}_r$ ,  $\mathbf{R}_r = \hat{\mathbf{W}}_r^T\mathbf{R}\mathbf{Q}\hat{\mathbf{V}}_r$ , and  $\mathbf{Q}_r = \mathbf{I}_r$
  - b)  $\mathbf{A}_r = (\mathbf{J}_r - \mathbf{R}_r)\mathbf{Q}_r$
  - c)  $s_i \leftarrow -\lambda_i(\mathbf{A}_r)$  for  $i = 1, \dots, r$
  - d)  $\mathbf{V}_r = [(s_1\mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1}\mathbf{b}, \dots, (s_r\mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1}\mathbf{b}]$ .

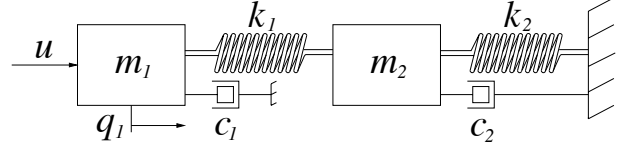


Fig. 2. Mass-Spring-Damper system

e) Let  $\hat{\mathbf{V}}_r = \mathbf{V}_r\mathbf{L}^{-1}$  with  $\mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r = \mathbf{L}^T\mathbf{L}$ .

f) Define  $\hat{\mathbf{W}}_r = \mathbf{Q}\hat{\mathbf{V}}_r$ .

6)  $\mathbf{J}_r = \hat{\mathbf{W}}_r^T\mathbf{J}\mathbf{Q}\hat{\mathbf{V}}_r$ ,  $\mathbf{R}_r = \hat{\mathbf{W}}_r^T\mathbf{R}\mathbf{Q}\hat{\mathbf{V}}_r$ ,  
 $\mathbf{b}_r = \hat{\mathbf{W}}_r^T\mathbf{b}$ ,  $\mathbf{c}_r = \mathbf{b}_r^T$ ,  $\mathbf{Q}_r = \mathbf{I}_r$ .

Step 5-c) in Algorithm 1 corresponds to reflecting the reduced order system poles as the next interpolation points. Hence, upon convergence, the desired interpolation conditions are satisfied due to the way  $\mathbf{V}_r$  is constructed. i.e.  $G_r(s)$  interpolates  $G(s)$  at the mirror images of the poles of  $G_r(s)$  but interpolation of first-derivatives does not hold. This is the price one may pay to preserve the structure. *It is not possible to force (25) and (26) while preserving port-Hamiltonian structure in general. This will only be possible if the skew-symmetric matrix  $\mathbf{J}$  were the zero matrix, since in that case the problem becomes symmetric.* Even so, the outcome of Algorithm 1 is optimal in a restricted sense.

*Theorem 9:* Given a port-Hamiltonian system  $G(s)$ , let the reduced model  $G_r(s)$  be obtained by Algorithm 1. Then,  $G_r(s)$  is also port-Hamiltonian, and passive. Also, let  $\hat{\lambda}_1, \dots, \hat{\lambda}_r$  denote the poles of  $G_r(s)$ .  $G_r(s)$  interpolates  $G(s)$  at  $-\hat{\lambda}_k$ , for  $k = 1, \dots, r$ , and minimizes the  $\mathcal{H}_2$  error  $\|G - \tilde{G}\|_{\mathcal{H}_2}$  among all  $r^{\text{th}}$  order reduced models  $\tilde{G}(s)$  having the same poles  $\hat{\lambda}_k$ .

*Proof:* The interpolation property, preservation of port-Hamiltonian structure and passivity are all a direct consequence of Theorem 3. Optimality in the  $\mathcal{H}_2$  sense is an extension of Gaier's result [5] to continuous time. ■

*Remark 4:* Theorem 9 states that among all  $r^{\text{th}}$  order reduced models having the same poles, Algorithm 1 yields the best optimal reduced order model in the  $\mathcal{H}_2$  norm.

*Remark 5:* Convergence of the original **IRKA** method and effective initialization strategies have been investigated in [7]. **IRKA** exhibits fast convergence in most cases regardless of initialization. This seems to be true for Algorithm 1 as well. Details of convergence behavior and different initialization strategies will be discussed in a later paper.

## VI. NUMERICAL EXAMPLES

### A. Mass-Spring-Damper system

Consider a Mass-Spring-Damper system as shown in Fig. 2 with masses  $m_i$ , spring constants  $k_i$  and damping constants  $c_i \geq 0$ , for  $i = 1, \dots, n/2$ .  $p_i$  and  $q_i$  is the momentum and displacement of the mass  $m_i$ , respectively. The external force acting on the first mass,  $m_1$ , is the input  $u$ , while its velocity is the output,  $y$ . State variables are defined in the following way: for  $i = 1, \dots, n/2$ ,  $x_{2i-1} = q_i$  and  $x_{2i} = p_i$ .

A minimal realization of this system for order  $n = 6$  (corresponding to three masses, springs, and dampers) is

$$\mathbf{b} = [0 \ 1 \ 0 \ 0 \ 0 \ 0]^T, \quad \mathbf{c} = \begin{bmatrix} 0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{J} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_3 \end{bmatrix}, \quad \text{and}$$

$$\mathbf{Q} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 & 0 & 0 \\ 0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 + k_2 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} & 0 & 0 \\ 0 & 0 & -k_2 & 0 & k_2 + k_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{m_3} \end{bmatrix}.$$

Then the  $\mathbf{A}$  matrix of this model is given as

$$\mathbf{A} = (\mathbf{J} - \mathbf{R})\mathbf{Q} = \begin{bmatrix} 0 & \frac{1}{m_1} & 0 & 0 & 0 & 0 \\ -k_1 & -\frac{c_1}{m_1} & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} & 0 & 0 \\ k_1 & 0 & -k_1 - k_2 & -\frac{c_2}{m_2} & k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{m_3} \\ 0 & 0 & k_2 & 0 & -k_2 - k_3 & -\frac{c_3}{m_3} \end{bmatrix}.$$

The system matrices are obtained in the same way as explained in Example 1. Adding another mass with a spring and a damper would increase the dimension of the system by two. In that case the main diagonal of the matrix  $\mathbf{A}$  will obtain 0 in the  $(n-1, n-1)$  position and  $-c_{n/2}/m_{n/2}$  in the  $(n, n)$  position. The super diagonal of  $\mathbf{A}$  will have  $k_{n/2-1}$  in the  $(n-2, n-1)$  position and  $1/m_{n/2}$  in the  $(n-1, n)$  position. The sub-diagonal of  $\mathbf{A}$  will obtain 0 in the  $(n-1, n-2)$  position and  $-k_{n/2-1} - k_{n/2}$  in the  $(n, n-1)$  position. Additionally  $\mathbf{A}$  will have  $k_{n/2-1}$  in the  $(n, n-3)$  position.

We considered a 100-dimensional Mass-Spring-Damper system with  $m_i = 2$ ,  $k_i = 1$ , and  $c_i = 1$ . We reduce the order using two approaches: the effort-constraint method in balanced coordinates of [15], [14] with the reduced order port-Hamiltonian model

$$\begin{cases} \dot{\mathbf{x}}_1^b = (\mathbf{J}_{11}^b - \mathbf{R}_{11}^b)\bar{\mathbf{Q}}^b \mathbf{x}_1^b + \mathbf{b}_1^b u, \\ y_r = (\mathbf{b}_1^b)^T \bar{\mathbf{Q}}^b \mathbf{x}_1^b, \end{cases} \quad (27)$$

where  $\bar{\mathbf{Q}}^b = \mathbf{Q}_{11}^b - \mathbf{Q}_{12}^b(\mathbf{Q}_{22}^b)^{-1}\mathbf{Q}_{21}^b$  is the Schur complement of the energy matrix  $\mathbf{Q}^b$  in the balanced coordinates

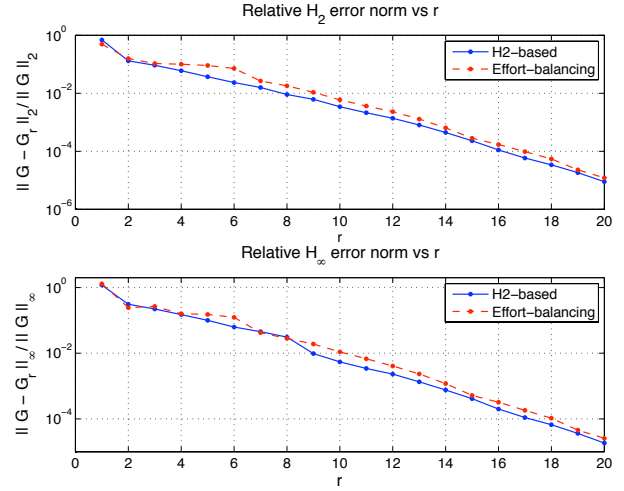


Fig. 3. Evolution of the relative  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  norms

(denoted by the superscript  $b$ ), and the  $\mathcal{H}_2$ -based interpolation method described in Algorithm 1. Using each method, we produce reduced models of order  $r = 1, 2, \dots, 20$ . For our proposed interpolation algorithm, initial shifts were simply chosen as logarithmically spaced points between  $10^{-4}$  and  $10^{-1}$ . The resulting relative  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  error norms for each order  $r$  is illustrated in Figure 3. With respect to the  $\mathcal{H}_2$  norm, our proposed interpolation method performs better for each reduction order except for  $r = 1$ . A similar observation can be made for the reduction errors relative to the  $\mathcal{H}_\infty$  norm as well with the exception of  $r = 2, 7$ , and 8. We note that our proposed method achieves this superior performance with less computational effort as well, since no dense matrix operations are needed, in contrast to what is required for system balancing within the effort-constraint approach — typically Schur decompositions are involved in solving the associated Lyapunov equations. We also note that the performance of our proposed interpolation approach may be further improved with better initialization choices as proposed in [7]. This will be the focus of future work. Finally, recall that the effort-constraint method used here is applied in the regular balanced basis. For this example, it appears that the usual balanced coordinates may not be the best coordinates to perform the effort-constraint method.

For the special case of  $r = 10$ , the Amplitude Bode plots of the full and reduced models, and that of the error models are given in Figure 4. As the top plot illustrates, both reduced models perform well. And it is clear from the bottom error plot that the  $\mathcal{H}_\infty$  error norm is slightly larger for the effort-constraint method in balanced coordinates.

## B. Ladder Network

We consider a 100-dimensional Ladder Network with a similar structure as in Example 1. The  $\mathbf{J}, \mathbf{R}, \mathbf{Q}, \mathbf{b}$  matrices will be of the same structure as those of 4-dimensional system as given in (6).

For this example, we compare the proposed approach of Algorithm 1 with *regular balanced truncation*. Balanced

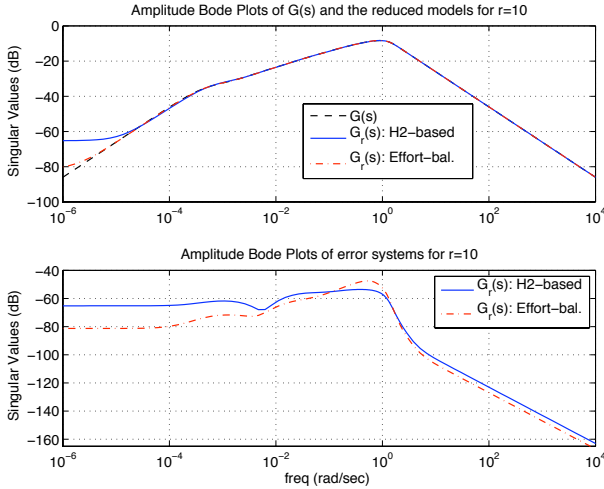


Fig. 4. Amplitude Bode plots for  $r = 10$

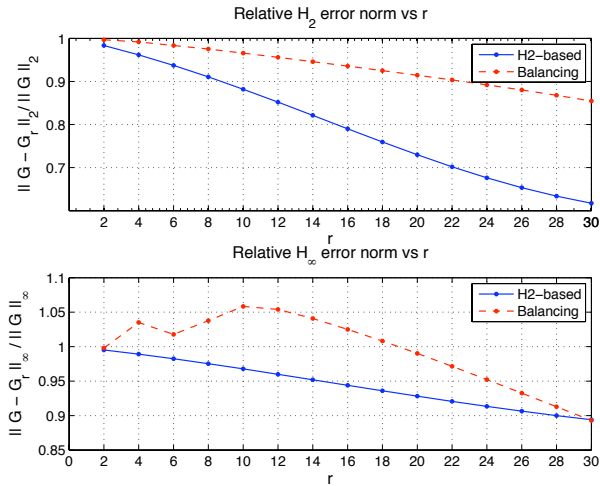


Fig. 5. Evolution of the relative  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  norms

truncation is well known to yield small  $\mathcal{H}_\infty$  and  $\mathcal{H}_2$  error norms. We note that balanced truncation does not preserve the port-Hamiltonian structure. However, to illustrate the effectiveness of our proposed method, we present the comparison with balanced truncation and show that our approach performs as well as even regular balanced truncation which is not constrained to preserve the structure.

We reduce order ranging from  $r = 2$  to  $r = 30$  in increments of 2. Figure 5 shows the resulting relative  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  errors for each  $r$ . Our  $\mathcal{H}_2$ -based interpolation method outperforms balanced truncation by a large margin with respect to  $\mathcal{H}_2$  error for all observed values of  $r$ . Similar behavior is observed for the  $\mathcal{H}_\infty$  error – although for  $r = 30$ , balanced truncation performs as well as our proposed interpolation method. Note that balanced truncation does not preserve structure, and also requires solving two large-scale Lyapunov equations. For this example, our structure-preserving numerically effective  $\mathcal{H}_2$ -based interpolation method clearly outperforms balanced truncation.

One might expect this result based on the decay of the

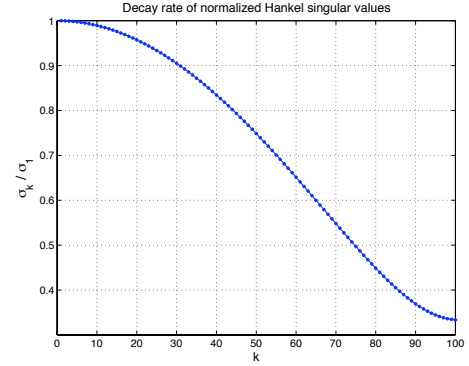


Fig. 6. Normalized Hankel singular values of the Ladder Network

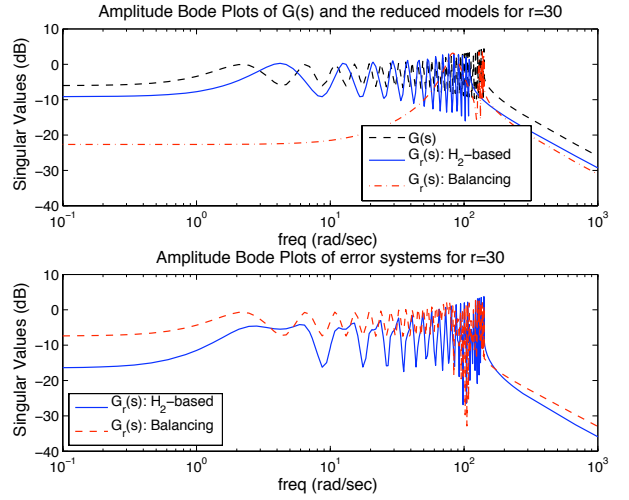


Fig. 7. Amplitude Bode plots for  $r = 30$

(normalized) Hankel singular values of the full order model depicted in Figure 6. The Hankel singular values show a very slow decay, a situation in which balanced truncation is known to show poor performance. We anticipate that our proposed  $\mathcal{H}_2$ -based method will prove even more effective in cases of slowly decaying Hankel singular values.

Figure 7 displays Amplitude Bode plots of the full and reduced models for  $r = 30$ . As the figure clearly illustrates, the reduced order model from balanced truncation does a very poor job of matching the Bode plot.

## VII. CONCLUSIONS AND FUTURE WORK

We have shown how to employ interpolatory model reduction for port-Hamiltonian systems so that the reduced system both interpolates the original model and also preserves port-Hamiltonian structure; the reduced order model is guaranteed to be passive as well. We proposed an algorithm which chooses interpolation points that satisfy a subset of the (unstructured)  $\mathcal{H}_2$  optimal necessary conditions. Two numerical examples illustrate the effectiveness of the new method.

The reduced model produced by Algorithm 1 satisfies  $r$  of the  $2r$  necessary conditions for (unstructured)  $\mathcal{H}_2$  optimality. We will investigate analogous first-order optimality conditions for a constrained optimal- $\mathcal{H}_2$  problem where

the port-Hamiltonian structure appears as the constraint. Developments for multi-input/multi-output port-Hamiltonian systems will appear in a separate work.

#### ACKNOWLEDGMENTS

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