Symplectic Hamiltonian Formulation of Transmission Line Systems with Boundary Energy Flow

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Abstract: The classical Lagrangian and Hamiltonian formulation of an electrical transmission line is reviewed and extended to allow for varying boundary conditions. This extension is based on the definition of an infinite-dimensional analogue of the affine Lagrangian and Hamiltonian input-output systems formulation. However, the framework is limited to a line that is terminated on both ends by independent voltage sources. Additionally, the inclusion of the usual line resistance and shunt conductance via a Rayleigh dissipation functional is nontrivial. To overcome these problems, a family of alternative Lagrangian functionals is proposed. The method is inspired by a (not so well-known) concept from network theory called 'the traditor'.

Keywords: Distributed-Parameter Systems, Hamiltonian Equations, Lagrangian Equations, Transmission Line, Traditor.

1 Introduction

It is well-known that the Lagrangian and Hamiltonian formalism from classical mechanics can be extended to describe a diverse range of lumped- and distributed-parameter physical systems. A typical example of such extension is the Lagrangian and Hamiltonian formu-

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lation of the wave propagation in an electrical transmission line. In mathematical terms, a transmission line is described by a system of partial differential equations of the form[1]

$$LI_t(z,t) + V_z(z,t) = 0 \tag{1}$$

$$CV_t(z,t) + I_z(z,t) = 0, \tag{2}$$

where $I(z,t)$ and $V(z,t)$ denote the current and voltage propagations within the spatial domain $Z = [0,1]$. The constants L and C represent the distributed inductance and shunt capacitance, respectively [9]. Apart from its strong pedagogical value in explaining abstract ideas associated with field theory, transmission lines appear in many applications and are used to interconnect various subsystems that exchange energy among each other. Hence, from an interconnection and control point of view, it is essential to be able to describe a transmission line with varying boundary conditions.

In extending the classical Lagrangian and Hamiltonian theory a fundamental difficulty arises in the treatment of boundary conditions. Indeed, the literature seems to be mainly focused on transmission lines with infinite spatial domain, i.e., having infinite length, or having open ports such that the energy exchange through the boundary is zero, e.g., [8] and [10]. The main problem is that for non-zero boundary conditions the spatial differential operator $(\partial/\partial z)$ is not skew-symmetric anymore (since after integration by parts the remaining boundary terms are not zero). On the other hand, in the context of Hamiltonian systems, these difficulties can be avoided by invoking the notion of an infinite-dimensional Dirac structure [12]. This, in turn, has led to a class of Hamiltonian boundary control systems—called infinite-dimensional port-Hamiltonian systems—that generalize the classical (symplectic and Poisson) formulations and allow for non-zero energy flow at the boundary in a mathematically sound way. However, there is no (direct) variational principle (and its associated Lagrangian equations of motion) involved in this description.

The main contributions of this paper can be summarized as follows. First, the classical Lagrangian and Hamiltonian approach is extended by invoking an infinite-dimensional analogue of the affine Lagrangian and Hamiltonian control systems formulation, as originally introduced in [4] (see also [11] for a summary and further developments on the topic). It will turn out that the inclusion of the boundary port variables via so-called *interaction* Lagrangians and Hamiltonians provides a solution to the boundary energy flow problem. The associated Hamiltonian equations of motion remain symplectic in form, while the *internal* Hamiltonian still coincides with the total stored energy in the transmission line. Secondly, since the generalized coordinates in the classical Lagrangian formulation of a transmission line are usually associated with the distributed charges, the corresponding equation of motion yields a homogeneous wave equation in terms of a charge wave. As this method is essentially based on the infinite-dimensional analogue of a loop-current analysis, only one of the two transmission line (or telegrapher's) equations (i.e., the voltage balance equation) is described, whereas the other (i.e., the current balance equation) is hidden as a constraint. An additional complication is that the inclusion of the usual line resistance and shunt conductance is far from trivial, if not, impossible—especially in the nonlinear case. A solution to these problems is presented that invokes the (to our knowledge) not so well-

[1]The subscript notation $(\bullet)_u$ denotes partial differentiation with respect to u. When clear from the context, the explicit time- and spatial dependence of the variables will be omitted. Furthermore, for ease of presentation, all variables are assumed to be null at $t \leq 0$.

known network-theoretic concept called *the traditor*, proposed by Duinker [6] in the late fifties as part of his development of a complete set of basic network elements. Although we will start the analysis from the foremost simplest version of the traditor, namely the ideal transformer, the concept in itself will lead to a rather novel family of alternative Lagrangian variational principles and associated (symplectic) Hamiltonians.

2 Classical Lagrangian and Hamiltonian Formulation

As shown in [10], denoting the *integrated charge density*

$$Q(z,t) = \int I(z,t)\,\mathrm{d}t, \tag{3}$$

as the generalized displacement and the *current density* $Q_t(z,t)$ $(= I(z,t))$ as the generalized velocity, the transmission line equations (1)–(2) can be associated with a Lagrangian functional of the form

$$\mathscr{L}(Q,Q_t) = \int_Z \overline{\mathscr{L}}(Q_t,Q_z)\,\mathrm{d}z, \tag{4}$$

with Lagrangian density $\overline{\mathscr{L}}(Q_t,Q_z) = \frac{1}{2}\left(LQ_t^2 - C^{-1}Q_z^2\right)$. Indeed, invoking Hamilton's principle

$$\delta\int_T \mathscr{L}(Q,Q_t)\,\mathrm{d}t = \int_T\int_Z \left(-LQ_{tt} + \frac{Q_{zz}}{C}\right)\delta Q\,\mathrm{d}z\mathrm{d}t - \int_T \frac{Q_z}{C}\delta Q\Big|_{z=0}^{z=1}\mathrm{d}t = 0, \tag{5}$$

with $T = [t_0,t_1]$, for $t_1 \geq t_0$, and imposing the boundary condition $-C^{-1}Q_z|_{\partial Z} = V|_{\partial Z} = 0$, yields a homogeneous wave equation for a lossless transmission line in terms of a charge wave

$$Q_{tt} - \frac{Q_{zz}}{LC} = 0. \tag{6}$$

The Hamiltonian counterpart is obtained by introducing the conjugate momentum $\Pi = \delta_{Q_t}\mathscr{L}(Q,Q_t) = LQ_t$, where $\delta_{(\bullet)}$ denotes the functional derivative [1, 5], and completing the Legendre transformation $\mathscr{H}(Q,\Pi)$ of $\mathscr{L}(Q,Q_t)$ as

$$\mathscr{H}(Q,\Pi) = \int_Z \frac{1}{2}\left(\frac{\Pi^2}{L} + \frac{Q_z^2}{C}\right)\mathrm{d}z. \tag{7}$$

Hence, imposing the same boundary condition on Q_z as before, the Hamiltonian equations of motion read

$$\begin{pmatrix} Q_t \\ \Pi_t \end{pmatrix} = \mathbf{F}\begin{pmatrix} \delta_Q\mathscr{H}(Q,\Pi) \\ \delta_\Pi\mathscr{H}(Q,\Pi) \end{pmatrix} = \begin{pmatrix} \dfrac{\Pi}{L} \\ \dfrac{Q_{zz}}{C} \end{pmatrix}, \tag{8}$$

with the symplectic structure matrix

$$\mathbf{F} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9}$$

3 Classical Formulation Revisited

In the previous section it is observed that the classical formulation of the transmission line is insufficient in the distinctive case of nonzero boundary energy flow. As will be shown next, this problem can be circumvented introducing an infinite-dimensional analogue of the affine input-output Lagrangian description [4, 11].

Consider again the classical Lagrangian formulation outlined in the previous section and define the triple integrated charge vector

$$[Q] = \begin{pmatrix} Q \\ Q^0 \\ Q^1 \end{pmatrix}.$$

Denoting

$$\mathscr{L}^{\text{int}}(Q, Q_t) = \int_Z \underbrace{\frac{1}{2}\left(LQ_t^2 - \frac{Q_z^2}{C}\right)}_{\overline{\mathscr{L}}^{\text{int}}(Q_z, Q_t)} \mathrm{d}z \tag{10}$$

as the *internal Lagrangian*, and introducing the *interaction Lagrangians* $\mathscr{L}^0(Q^0, e^0)$ and $\mathscr{L}^1(Q^1, e^1)$, where e^0 and e^1 are independent external (control) variables at the boundary. This results in a *boundary control Lagrangian* functional of the form

$$\mathscr{L}[Q, Q_t, e] = \mathscr{L}^{\text{int}}(Q, Q_t) + \mathscr{L}^0(Q^0, e^0) + \mathscr{L}^1(Q^1, e^1). \tag{11}$$

Invoking Hamilton's principle, we now obtain

$$\begin{aligned}\delta \int_T \mathscr{L}[Q, Q_t, e]\,\mathrm{d}t &= \int_T \int_Z \left(-LQ_{tt} + \frac{Q_{zz}}{C}\right) \delta Q\,\mathrm{d}z\mathrm{d}t \\ &\quad + \int_T \left\{ \left(\mathscr{L}_Q^0 + \frac{Q_z}{C}\right) \delta Q\Big|_{z=0} + \left(\mathscr{L}_Q^1 - \frac{Q_z}{C}\right) \delta Q\Big|_{z=1} \right\} \mathrm{d}t = 0.\end{aligned}$$

In order to restore to the original equation of motion (6), and since $C^{-1}Q_z|_{\partial Z} = -V|_{\partial Z}$, we select $\mathscr{L}^0 = +Q^0E^0$ and $\mathscr{L}^1 = -Q^1E^1$, where E^0 and E^1 denote independent external voltage sources. This establishes a well-posed and mathematically sound variational principle. Moreover, the latter observations directly suggest the definition of a Lagrangian boundary control system of the form

$$\left(\delta_{[Q_t]}\mathscr{L}[Q, Q_t, e]\right)_t - \delta_{[Q]}\mathscr{L}[Q, Q_t, e] = 0, \tag{12}$$

where $\delta_{[\bullet]}\mathscr{L}$ is the extended functional derivative [3]

$$\delta_{[\bullet]}\mathscr{L} = \begin{pmatrix} \overline{\mathscr{L}}_{(\bullet)} - \left(\overline{\mathscr{L}}_{(\bullet)_z}\right)_z \\ \left(\mathscr{L}^0_{(\bullet)} - \overline{\mathscr{L}}_{(\bullet)_z}\right)\Big|_{z=0} \\ \left(\mathscr{L}^1_{(\bullet)} + \overline{\mathscr{L}}_{(\bullet)_z}\right)\Big|_{z=1} \end{pmatrix}. \tag{13}$$

(Recall that $\overline{\mathscr{L}}$ represents the Lagrangian density.)

In passing on to the Hamiltonian formulation, one is tempted to define the generalized momentum triple vector

$$[\Pi] = \delta_{[Q_t]}\mathscr{L}[Q, Q_t, e]. \tag{14}$$

However, since the interaction-boundary Lagrangians $\mathscr{L}^0$ and $\mathscr{L}^1$ are functions of the boundary charges only, the boundary momenta of $[\Pi]$, i.e., Π^0 and Π^1, will vanish identically which implies that the Lagrangian (11) belongs to the class of so-called singular Lagrangians. For that reason we consider the partial Legendre transformation, i.e.,

$$\mathscr{H}[Q,\Pi,e] = \mathscr{H}^{\text{int}}(Q,\Pi) + \mathscr{H}^0(Q^0,e^0) + \mathscr{H}^1(Q^1,e^1), \tag{15}$$

where $\mathscr{H}^{\text{int}}(Q,\Pi)$ is the Legendre transformation of $\mathscr{L}^{\text{int}}(Q,Q_t)$, while the interaction-boundary terms $\mathscr{H}^0(Q^0,e^0) = -\mathscr{L}^0(Q^0,e^0)$ and $\mathscr{H}^1(Q^1,e^1) = -\mathscr{L}^1(Q^1,e^1)$. Indeed, differentiating the internal Hamiltonian

$$\mathscr{H}^{\text{int}}(Q,\Pi) = \int_{\mathbb{Z}} \frac{1}{2}\left(\frac{\Pi^2}{L} + \frac{Q_z^2}{C}\right) \mathrm{d}z, \tag{16}$$

we now obtain the energy flow balance

$$\dot{\mathscr{H}}^{\text{int}}(Q,\Pi) = \frac{Q_z}{C} Q_t \bigg|_{z=0}^{z=1} = -I^1E^1 + I^0E^0, \tag{17}$$

which precisely coincides with the power-balance obtained in the infinite-dimensional port-Hamiltonian framework proposed in [12].

Finally, in a similar fashion as in the case of finite dimensional systems [11], we can define a set of *natural outputs* for the transmission line system as follows:

$$\begin{aligned} y^0 &= -\mathscr{H}_{e^0}[Q,\Pi,e] \\ y^1 &= -\mathscr{H}_{e^1}[Q,\Pi,e]. \end{aligned}$$

The selection of $e^0 = E^0$ and $e^1 = E^1$ is tantamount to terminating the transmission line by independent voltage sources (the controls), resulting in $y^0 = Q^0$ and $y^1 = -Q^1$ as the natural outputs. We refer to this particular (causality) configuration as a *voltage/voltage-controlled (VV)* transmission line system.

4 A Novel Variational Boundary Control Principle

In the previous section we have accommodated the classical Lagrangian and Hamiltonian formulation to include the practically relevant situation of nonzero boundary energy flow. Moreover, the associated Hamiltonian equations of motion remain symplectic in form, while the *internal* Hamiltonian still coincides with the total stored energy in the transmission line. However, since the generalized coordinates in the classical Lagrangian formulation of a transmission line are associated with the distributed charges, the corresponding equation of motion yields a homogeneous wave equation in terms of a charge wave. As this method is essentially based on the infinite-dimensional analogue of a loop-current analysis (note that $Q_t = I$), only one of the two transmission line equations (i.e., the voltage balance equation (1)) is described, whereas the other (i.e., the current balance equation (2)) is hidden as a constraint. Hence, the boundary energy flow is restricted to a voltage/voltage controlled configuration. Of course, the current balance (and associated current/current controlled (CC) configuration) can be obtained from the introduction of an *integrated flux density*

$$P(z,t) = \int V(z,t)\,\mathrm{d}t, \tag{18}$$

and the definition of a *co*-Lagrangian, i.e., the dual of (11). An additional complication is that the inclusion of the usual line resistance and shunt conductance is far from trivial, if not, impossible—especially in the nonlinear case.

4.1 The Traditor

A solution to these problems is presented that invokes the (to our knowledge) not so well-known network-theoretic concept called *the traditor*, proposed by Duinker in the late fifties as part of his development of a complete set of basic *lumped-parameter* network elements [6]. A general n-th order traditor is defined as an n-port element with Lagrangian function $\mathscr{S} = f(x_1,\ldots,x_n)\dot{x}_n$. It is a *non-energic* element since it is characterized by the fact that it neither stores nor dissipates energy [2]. This means that at any instant the total power delivered to a traditor is equal to zero, which is also evident from the fact that the associated Hamiltonian, say $\mathscr{S}^*$, equals

$$\mathscr{S}^* = \dot{x}_n \mathscr{S}_{\dot{x}_n} - \mathscr{S} = 0, \tag{19}$$

and thus $\dot{\mathscr{S}}^* = 0$.

Traditors are defined in various degrees. The simplest examples of a traditor are an open- and short-circuited branch. These two situations are classified as first-degree traditors. Traditors of the second-degree are the ideal transformer and the gyrator. However, in later works, Duinker allocated the name traditor specifically to traditors of the third-degree since these are the simplest to be actually nonlinear and, in addition to the gyrator, synthesize the lower- and higher-degree traditors. The interested reader is referred to [6] and [7] for further details.

The concept of the traditor, though in its foremost simplest form, can easily be carried over to the distributed-parameter domain and will be used to derive a novel class of variational principles that lead to both the transmission line equations simultaneously.

4.2 Alternative Lagrangian Functionals

Considering both the integrated charge Q and flux P simultaneously, we propose instead of (10) an internal Lagrangian

$$\mathscr{L}^{\mathrm{int}}(\bullet, Q_t, P_t) = \int_{\mathbb{Z}} \frac{1}{2}\left(LQ_t^2 + CP_t^2\right)\mathrm{d}z + \mathscr{S}(\bullet), \tag{20}$$

where $\mathscr{S}(\bullet)$ is selected from one of the following functionals

$$\mathscr{S}(\bullet) = \begin{cases} \int_{\mathbb{Z}} P_t Q_z\,\mathrm{d}z \\ \int_{\mathbb{Z}} Q_t P_z\,\mathrm{d}z, \end{cases} \tag{21}$$

each giving rise to a different set of boundary conditions. In a similar fashion as before, we define the boundary control Lagrangian

$$\mathscr{L}[\mathbf{q},\mathbf{q}_t,\mathbf{e}] = \mathscr{L}^{\mathrm{int}}(\mathbf{q},\mathbf{q}_t) + \mathscr{L}^0(\bullet,\mathbf{e}^0) + \mathscr{L}^1(\bullet,\mathbf{e}^1), \tag{22}$$

yielding the following Lagrangian boundary control system:

$$\left(\delta_{\mathbf{q}_t}\mathscr{L}[\mathbf{q},\mathbf{q}_t,\mathbf{e}]\right)_t - \delta_{\mathbf{q}}\mathscr{L}[\mathbf{q},\mathbf{q}_t,\mathbf{e}] = \mathbf{0}, \tag{23}$$

where $\mathbf{q} = \mathrm{col}(Q,P)$ and $\mathbf{q}_t = \mathrm{col}(Q_t,P_t)$. Hence, starting for example from

$$\mathscr{L}^{\mathrm{int}}(Q,Q_t,P_t) = \int_{\mathbb{Z}} \frac{1}{2}\left(LQ_t^2 + CP_t^2\right)\mathrm{d}z + \int_{\mathbb{Z}} P_t Q_z \mathrm{d}z, \tag{24}$$

and leaving the interaction-boundary Lagrangians unprescribed for the moment, yields the Lagrangian boundary control system:

$$\begin{pmatrix} LQ_{tt} \\ CP_{tt} + Q_{zt} \\ \left(\mathscr{L}^0_{Q_t^0}\right)_t \\ \left(\mathscr{L}^0_{P_t^0}\right)_t \\ \left(\mathscr{L}^1_{Q_t^1}\right)_t \\ \left(\mathscr{L}^1_{P_t^1}\right)_t \end{pmatrix} - \begin{pmatrix} -P_{tz} \\ 0 \\ \mathscr{L}^0_{Q^0} - P_t^0 \\ \mathscr{L}^0_{P^0} \\ \mathscr{L}^1_{Q^1} + P_t^1 \\ \mathscr{L}^1_{P^1} \end{pmatrix} = \mathbf{0}. \tag{25}$$

First note that, after substitution of $Q_t = I$ and $P_t = V$, the first two equations precisely coincide with (1) and (2). Furthermore, returning to a transmission line that is terminated on both ends by independent voltage sources (VV), $\mathbf{e}^0 = \mathrm{col}(E^0,0)$ and $\mathbf{e}^1 = \mathrm{col}(E^1,0)$, it is readily found that in this case $\mathscr{L}^0 = E^0Q^0$ and $\mathscr{L}^1 = -E^1Q^1$.

On the other hand, if the line is terminated by current sources (CC) we need to select either $\overline{\mathscr{S}}(P_z,Q_t) = Q_tP_z$, or admit for interaction Lagrangians that also depend on the generalized velocities at the boundary, i.e., $\mathscr{L}^0 = \mathscr{L}^0(\mathbf{q}^0,\mathbf{q}_t^0,\mathbf{e}^0)$ and $\mathscr{L}^1 = \mathscr{L}^1(\mathbf{q}^1,\mathbf{q}_t^1,\mathbf{e}^1)$. Indeed, if in the present setting the voltage sources are replaced by current sources $\mathbf{e}^0 = \mathrm{col}(0,J^0)$ and $\mathbf{e}^1 = \mathrm{col}(0,J^1)$, then the interaction Lagrangians need to be set as $\mathscr{L}^0 = (J^0 - Q_t^0)P^0$ and $\mathscr{L}^1 = -(J^1 - Q_t^1)P^1$. This also allows for combinations, i.e., voltage/current (VC) and current/voltage (CV) causalities, providing the four different boundary configurations summarized in Table 1. However, as will be demonstrated next, the corresponding Hamiltonian control system formulation does not allow for the mixed cases.

4.3 Symplectic Hamiltonian Boundary Control System

In passing on to the Hamiltonian formulation, we now define the conjugate momenta

$$[\mathbf{p}] = \delta_{[\mathbf{q}_t]}\mathscr{L}[\mathbf{q},\mathbf{q}_t,\mathbf{e}], \tag{26}$$

where $\mathbf{q} = \mathrm{col}(\Pi,\Gamma)$, and find the Legendre transformation

$$\mathscr{H}[\mathbf{q},\mathbf{p},\mathbf{e}] = \int_Z \mathbf{p}\cdot\mathbf{q}_t\,\mathrm{d}z - \mathscr{L}[\mathbf{q},\mathbf{q}_t,\mathbf{e}]. \tag{27}$$

Table 1. *Causality configurations of a terminated transmission line for the boundary control system (23), starting from either $\overline{\mathscr{S}} = Q_t P_z$ or $\overline{\mathscr{S}} = P_t Q_z$.*

	e_1^0	e_2^0	e_1^1	e_2^1	$\mathscr{L}^0$ ($\overline{\mathscr{S}} = Q_t P_z$)	$\mathscr{L}^1$ ($\overline{\mathscr{S}} = Q_t P_z$)	$\mathscr{L}^0$ ($\overline{\mathscr{S}} = P_t Q_z$)	$\mathscr{L}^1$ ($\overline{\mathscr{S}} = P_t Q_z$)
VV	E^0	0	E^1	0	$(E^0 - P_t^0)Q^0$	$-(E^1 - P_t^1)Q^1$	$E^0 Q^0$	$-E^1 Q^1$
VC	E^0	0	0	J^1	$(E^0 - P_t^0)Q^0$	$-J^1 P^1$	$E^0 Q^0$	$-(J^1 - Q_t^1)P^1$
CV	0	J^0	E^1	0	$J^0 P^0$	$-(E^1 - P_t^1)Q^1$	$(J^0 - Q_t^0)P^0$	$-E^1 Q^1$
CC	0	J^0	0	J^1	$J^0 P^0$	$-J^1 P^1$	$(J^0 - Q_t^0)P^0$	$-(J^1 - Q_t^1)P^1$

If the interaction terms $\mathscr{L}^0$ and $\mathscr{L}^1$ in (22) depend only on the generalized displacements then they do not contribute any conjugate boundary momenta to the Legendre transformation. In that case, the interaction Hamiltonians follow verbatim from the respective $\mathscr{L}^0$ and $\mathscr{L}^1$ defined in Table 1. For example, insisting that $\mathscr{L}^0 = \mathscr{L}^0(\mathbf{q}^0, \mathbf{e}^0)$ and $\mathscr{L}^1 = \mathscr{L}^0(\mathbf{q}^1, \mathbf{e}^1)$, the VV configuration suggests to start with (24). This yields a boundary control Hamiltonian of the form

$$\mathscr{H}[Q,\Pi,\Gamma,E^0,E^1] = \underbrace{\int_Z \frac{1}{2L}\Pi^2\,\mathrm{d}z + \int_Z \frac{1}{2C}(\Gamma - Q_z)^2\,\mathrm{d}z}_{\mathscr{H}^{\mathrm{int}}(Q,\Pi,\Gamma)} - E^0 Q^0 + E^1 Q^1, \tag{28}$$

together with the boundary conditions

$$\begin{aligned} -E^0 + \frac{1}{C}(\Gamma - Q_z)\Big|_{z=0} &= 0 \\ E^1 - \frac{1}{C}(\Gamma - Q_z)\Big|_{z=1} &= 0. \end{aligned}$$

Note that the generalized momenta are given by $\Pi = LQ_t$ and $\Gamma = CP_t + Q_z$, which directly shows that the internal Hamiltonian coincides with the internally stored energy, i.e., $\frac{1}{2L}\Pi^2 = \frac{1}{2}LQ_t^2$ and $\frac{1}{2C}(\Gamma - Q_z)^2 = \frac{1}{2}CP_t^2$, where we recall that $Q_t = I$ and $P_t = V$, respectively. Furthermore, the terms $\frac{1}{C}(\Gamma - Q_z)|_{z=0} = P_t^0$ and $\frac{1}{C}(\Gamma - Q_z)|_{z=1} = P_t^1$ coincide with the voltages at the ports.

Similarly, variation of the total Hamiltonian associated with $\overline{\mathscr{S}} = Q_t P_z$ given by

$$\mathscr{H}[P,\Pi,\Gamma,J^0,J^1] = \underbrace{\int_Z \frac{1}{2L}(\Pi - P_z)^2\,\mathrm{d}z + \int_Z \frac{1}{2C}\Gamma^2\mathrm{d}z}_{\mathscr{H}^{\mathrm{int}}(P,\Pi,\Gamma)} + J^0 P^0 - J^1 P^1, \tag{29}$$

yields the boundary conditions

$$\begin{aligned} J^0 - \frac{1}{L}(\Pi - P_z)\Big|_{z=0} &= 0 \\ -J^1 + \frac{1}{L}(\Pi - P_z)\Big|_{z=1} &= 0. \end{aligned}$$

In this case the momenta are $\Pi = LQ_t + P_z$ and $\Gamma = CP_t$, so that $\frac{1}{L}(\Pi - P_z)\big|_{z=0} = Q_t^0$ and $\frac{1}{L}(\Pi - P_z)\big|_{z=1} = Q_t^1$ coincide with the currents at the ports, and the internal Hamiltonian again equals the internally stored energy.

These descriptions can also be merged into a boundary control formulation by introducing a skew-symmetric matrix $[\mathbf{F}] = -[\mathbf{F}]^T$ such that

$$[\mathbf{F}]^T[\mathbf{x}_t] = \delta_{[\mathbf{x}]}\mathscr{H}[\mathbf{x},\mathbf{e}], \tag{30}$$

where $\mathbf{x} = \mathrm{col}(\mathbf{q},\mathbf{p})$,

$$[\mathbf{F}] = \begin{pmatrix} \mathbf{F} & & \\ & \mathbf{F}^0 & \\ & & \mathbf{F}^1 \end{pmatrix}, \text{ with } \mathbf{F} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{2\times 2} \\ -\mathbf{I}_{2\times 2} & \mathbf{0} \end{pmatrix}, \tag{31}$$

and $\mathbf{F}^0 = \mathbf{F}^1 = \mathbf{0}$.

On the other hand, for a mixed causality like the VC configuration, we have for $\overline{\mathscr{S}} = P_t Q_z$ that $\mathscr{L}^0 = -E^0Q^0$, but $\mathscr{L}^1 = -(J^1 - Q_t^1)P^1$. As a result, the latter interaction term now contributes a conjugate boundary momentum

$$[\mathbf{p}] = \delta_{[\mathbf{q}_t]}\mathscr{L}[\mathbf{q},\mathbf{q}_t,\mathbf{e}] = \left(\begin{array}{c} LQ_t \\ CP_t + Q_z \\ \hline 0 \\ 0 \\ \hline P^1 \\ 0 \end{array}\right), \tag{32}$$

yielding that the Legendre transformation

$$\begin{aligned} \mathscr{H}[Q,P,\Pi,\Gamma,E^0,J^1] &= \mathscr{H}^{\mathrm{int}}(Q,\Pi,\Gamma) + P^1\frac{\Pi^1}{L} - \left\{E^0Q^0 - \left(J^1 - \frac{\Pi^1}{L}\right)P^1\right\} \\ &= \mathscr{H}^{\mathrm{int}}(Q,\Pi,\Gamma) - E^0Q^0 + J^1P^1. \end{aligned}$$

Although the latter functional seems a valid Hamiltonian, its functional derivative is only well-defined if the associated boundary terms satisfy a rather unpractical condition. Indeed, since $\Pi^1 = P^1$ we find in terms of the Hamiltonian boundary control system formulation (30) that $\mathbf{F}^0 = \mathbf{0}$, but

$$\mathbf{F}^1 = \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{33}$$

This results in the following set of boundary conditions:

$$\begin{aligned} -E^0 + \frac{1}{C}(\Gamma - Q_z)\Big|_{z=0} &= 0 \\ -\Pi_t^1 &= -\frac{1}{C}(\Gamma - Q_z)\Big|_{z=1} \ \left(= -P_t^1\right) \\ J^1 &= 0 \\ Q_t^1 &= 0. \end{aligned}$$

Clearly, since the boundary conditions need to satisfy $J^1 = Q_t^1 = 0$ there can not be any energy flow through the boundary at $z = 1$. A similar discussion holds for the CV configuration.

5 On the Role of Dissipation

So far we have considered only the description of a lossless transmission line. Let us next turn to the case of a lossy line. The transmission equations (1) and (2) then take the form

$$LI_t + V_z = -RI \tag{34}$$
$$CV_t + I_z = -GV, \tag{35}$$

where R and G are the distributed resistance and shunt conductance, respectively.

In the context of the Lagrangian formalism, a standard approach is to include dissipative effects by introducing a *Rayleigh dissipation* or *content* function(al). However, in the classical approach, where we have started from an integrated charge description, the corresponding equation of motion only constitutes the lines voltage balance (6). This means that we can only include the transversal dissipation effects by considering a Rayleigh dissipation functional of the form

$$\mathscr{R}(Q_t) = \int_Z \frac{1}{2} R Q_t^2 \,\mathrm{d}z, \tag{36}$$

The corresponding equation of motion (6) extends to

$$LQ_{tt} - \frac{Q_{zz}}{C} = -\delta_{Q_t}\mathscr{R}(Q_t) = -RQ_t, \tag{37}$$

which in turn coincides with (34).

On the other hand, invoking a co-Lagrangian description starting from an integrated flux coordinate suggests the introduction of a *co*-Rayleigh dissipation (or *co*-content) functional

$$\mathscr{R}^*(P_t) = \int_Z \frac{1}{2} G P_t^2 \,\mathrm{d}z. \tag{38}$$

This yields the current balance (35) in terms of the integrated flux:

$$CP_{tt} - \frac{P_{zz}}{L} = -\delta_{P_t}\mathscr{R}^*(P_t) = -GP_t. \tag{39}$$

Now, invoking the theory presented in the previous sections, these two separate results can be derived from a single Lagrangian. This means that starting, for example, from an internal Lagrangian of the form (24), we obtain for a lossy line

$$LQ_{tt} + P_{tz} = -\delta_{Q_t}\mathscr{R}^{\mathrm{int}}(Q_t, P_t) = -RQ_t \tag{40}$$
$$CP_{tt} + Q_{zt} = -\delta_{P_t}\mathscr{R}^{\mathrm{int}}(Q_t, P_t) = -GP_t, \tag{41}$$

where $\mathscr{R}^{\mathrm{int}}(Q_t, P_t)$ is the total *internal Rayleigh dissipation functional* defined by

$$\mathscr{R}^{\mathrm{int}}(Q_t, P_t) = \int_Z \frac{1}{2}\left(RQ_t^2 + GP_t^2\right) \mathrm{d}z. \tag{42}$$

In a similar fashion as before, it is also possible to define an *interaction Rayleigh dissipation functional* to include resistances or conductances that appear at the ports.

6 Final Remarks

In this paper the classical Lagrangian and Hamiltonian formulation of a uniform transmission line is accommodated to account for nonzero boundary energy flow. The framework is, however, limited to a line that is terminated on both ends by independent voltage sources. This has motivated the search for alternative Lagrangian variational principles that yield both the transmission line equations. The approach is inspired and motivated by an infinite-dimensional generalization of a network-theoretic concept called the traditor. Although the new Lagrangian formulation allows for mixed boundary conditions, the associated Hamiltonian formulation only allows both ends to be terminated either by independent voltage sources or current sources, but not both.

Additionally, the present setting allows energy dissipation to be included in a clear and transparent manner introducing the usual Rayleigh dissipation function(al). As shown in [10], an alternative way to account for losses is to consider a modified version of Hamilton's principle using a weighted Lagrangian density with a time-dependent exponential factor. However, apart from the fact that one runs into problems when dealing with nonlinear resistances and/or shunt conductances, the associated Hamiltonian does not have the interpretation of the total stored energy in the system.

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