Lecture 2: Controllability of nonlinear systems

Nonlinear Dynamical Control Systems, Chapter 3

See www.math.rug.nl/˜arjan (under **teaching**) for info on course schedule and homework sets.

Take-Home Exam I on homepage on March 16.

Recall: **Kinematic model of the unicycle**

$$\begin{aligned}\dot{x}_1 &= u_1 \cos x_3 \\ \dot{x}_2 &= u_1 \sin x_3 \\ \dot{x}_3 &= u_2\end{aligned}$$

written as a system with two input vector fields and zero drift vector field

$$\dot{x} = \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

The Lie bracket of the two input vector fields is given as

$$-\begin{bmatrix} 0 & 0 & -\sin x_3 \\ 0 & 0 & \cos x_3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin x_3 \\ -\cos x_3 \\ 0 \end{bmatrix}$$

which is a vector field that is **independent** from the two input vector fields.
Claim: This new independent direction guarantees controllability of the unicycle system.

Interpretation of the Lie bracket:

Proposition 1 *Let X, Y be two vector fields such that*

$$[X, Y] = 0$$

Then the solution flows of the vector fields are **commuting**.
In fact, we may find local coordinates $x_1, \ldots, x_n$ such that

$$X = \frac{\partial}{\partial x_1}, \quad Y = \frac{\partial}{\partial x_2}$$

Thus, the Lie bracket $[X, Y]$ characterizes the amount of **non-commutativity** of the vector fields X, Y.

In fact, let the control strategy $u = \mathrm{col}(u_1, u_2)$ be defined by

$$u(t) = \begin{cases} (1,0), & t \in [0,\varepsilon),\ \ \varepsilon > 0 \\ (0,1), & t \in [\varepsilon, 2\varepsilon) \\ (-1,0), & t \in [2\varepsilon, 3\varepsilon) \\ (0,-1), & t \in [3\varepsilon, 4\varepsilon), \end{cases}$$

Then the motion of the system is described by

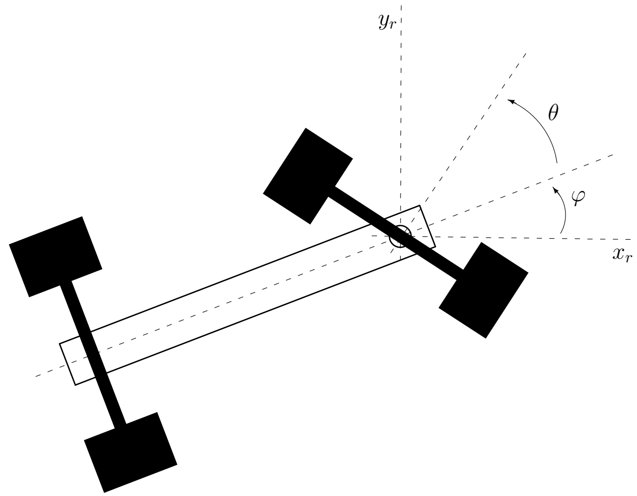
$$x(4\varepsilon) = x_0 + \varepsilon^2 [g_1, g_2](x_0) + \mathcal{O}(\varepsilon^3).$$

which indicates controllability, since $[g_1, g_2]$ is everywhere independent from g_1, g_2.

This formula holds in general.

This is enough for systems with two inputs and three state variables, but what can we do if the dimension of the state is > 3?

Answer: consider **higher-order** Lie brackets.



Example 2 *Consider the cart with fixed rear axis*

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \varphi \\ \theta \end{bmatrix} = \begin{bmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2$$

with u_1 the driving input, and u_2 the steering input.

Define

$$g_1(x) = \underbrace{\begin{bmatrix} \cos(x_3 + x_4) \\ \sin(x_3 + x_4) \\ \sin(x_4) \\ 0 \end{bmatrix}}_{\text{Drive}}, \;\; g_2(x) = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{Steer}}.$$

Compute

$$
\begin{aligned}
[\mathrm{Steer}, \mathrm{Drive}] &= \frac{\partial g_1}{\partial x} g_2 - \frac{\partial g_2}{\partial x} g_1 \\
&= \begin{bmatrix} 0 & 0 & -\sin(x_3+x_4) & -\sin(x_3+x_4) \\ 0 & 0 & \cos(x_3+x_4) & \cos(x_3+x_4) \\ 0 & 0 & 0 & \cos(x_4) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - 0 \\
&= \begin{bmatrix} -\sin(x_3+x_4) \\ \cos(x_3+x_4) \\ \cos(x_4) \\ 0 \end{bmatrix} =: \mathrm{Wriggle}.
\end{aligned}
$$

Another independent direction is obtained by the third-order Lie bracket

$$[\mathrm{Wriggle}, \mathrm{Drive}] = \begin{bmatrix} -\sin(x_3) \\ \cos(x_3) \\ 0 \\ 0 \end{bmatrix} =: \mathrm{Slide}.$$

This shows that you can manoeuver your car into any parking lot by applying controls corresponding to the ‘Slide’ direction, i.e., by applying the control sequence {Wriggle, Drive, $-$Wriggle, $-$Drive}.

What to do with the drift vector field ?

The system

$$\dot{x} = f(x) + g(x)u$$

can be considered as a special case of

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2,$$

with $u_1 = 1$. This means that care has to be taken with respect to brackets involving f:

$$[f, g], [g, [f, g]], [f, [f, g]], \ldots$$

Example 3 *Consider the system on* $\mathbb{R}^2$

$$\begin{aligned}\dot{x}_1 &= x_2^2\\ \dot{x}_2 &= u.\end{aligned}$$

Compute the Lie brackets of the vector fields

$$f(x) = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

yielding

$$[f,g](x) = \begin{bmatrix} -2x_2 \\ 0 \end{bmatrix}, \quad [[f,g],g](x) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Clearly, we have obtained two independent directions. However, since $x_2^2 \geq 0$*, the* x_1*-coordinate is always non-decreasing. Hence, the system is not really controllable.*

A weaker form of controllability: **local accessibility**

Let $\mathbb{V}$ be a neighborhood of x_0, then $\mathcal{R}^{\mathbb{V}}(x_0, t_1)$ denotes the reachable set from x_0 at time $t_1 \geq 0$, following the trajectories which remain in the neighborhood $\mathbb{V}$ of x_0 for $t \leq t_1$, i.e., all points x_1 for which there exists an input $u(\cdot)$ such that the evolution of the system for $x(0) = x_0$ satisfies $x(t) \in \mathbb{V}$, $0 \leq t \leq t_1$, and $x(t_1) = x_1$. Furthermore, let

$$\mathcal{R}^{\mathbb{V}}_{t_1}(x_0) = \bigcup_{\tau \leq t_1} \mathcal{R}^{\mathbb{V}}(x_0, \tau).$$

Definition 4 (Local accessibility) A system is said to be **locally accessible** from x_0 if $\mathcal{R}^{\mathbb{V}}_{t_1}(x_0)$ contains a *non-empty open subset* of $\mathcal{X}$ for all non-empty neighborhoods $\mathbb{V}$ of x_0 and all $t_1 > 0$. If the latter holds for all $x_0 \in \mathcal{X}$ then **the system is called locally accessible**.

Definition 5 (Accessibility algebra) Consider the system

$$\dot{x} = f(x) + g_1(x)u_1 + \cdots + g_m(x)u_m$$

The **accessibility algebra** $\mathcal{C}$ are the linear combinations of repeated Lie brackets of the form

$$[X_k, [X_{k-1}, [\cdots, [X_2, X_1] \cdots]]], \; k = 1, 2, \ldots,$$

where X_i, is a vector field in the set $\{f, g_1, \ldots, g_m\}$.

This linear space is a **Lie algebra** under the Lie bracket.

Definition 6 *The* **accessibility distribution** C *is the* **distribution** *generated by the accessibility algebra* $\mathcal{C}$*:*

$$C(x) = \operatorname{span}\{X(x) \mid X \text{ vector field in } \mathcal{C}\}, \; x \in \mathcal{X}$$

Intermezzo: **Distributions on manifolds**

A distribution D on a manifold $\mathcal{X}$ is specified by a subspace

$$D(x) \subset T_x\mathcal{X}$$

for all $x \in \mathcal{X}$.

Let $X_1, X_2, \ldots, X_k$ be vector fields on $\mathcal{X}$. Then

$$D(x) = \text{span}(X_1(x), X_2(x), \ldots, X_k(x)).$$

defines a distribution.

A distribution D is called **involutive** if, whenever $f, g \in D$, also $[f, g] \in D$.

The distribution D is called **constant-dimensional** whenever the dimension of $D(x)$ is constant.

Example 7 Let $\mathcal{X} = \mathbb{R}^3$ and $D = \mathrm{span}(f_1, f_2)$, where

$$f_1(x) = \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}.$$

Since f_1 and f_2 are linearly independent, we have that $\dim(D(x)) = 2$, for all x. Furthermore, we have

$$[f_1, f_2](x) = \frac{\partial f_2}{\partial x}(x) f_1(x) - \frac{\partial f_1}{\partial x}(x) f_2(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$[f_1, f_2] \in D$ if and only if $\mathrm{rank}(f_1(x), f_2(x), [f_1, f_2](x)) = 2$, for all x.
However,

$$\mathrm{rank}(f_1(x), f_2(x), [f_1, f_2](x)) = \mathrm{rank}\begin{bmatrix} 2x_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & 1 \end{bmatrix} = 3,$$

for all x. Hence, D is not involutive.

Let D be a nonsingular distribution on $\mathcal{X}$, generated by the independent vector fields $f_1, \ldots, f_r$. Then D is said to be **integrable** if for each $x_0 \in \mathcal{X}$, there exists a neighborhood N of x_0 and $n-r$ real-valued **independent** functions $h_1(x), \ldots, h_{n-r}(x)$ defined on N, such that $h_1(x), \ldots, h_{n-r}(x)$ satisfy the partial differential equations

$$\frac{\partial h_j}{\partial x}(x) f_i(x) = 0, \qquad (1)$$

for all indices $i = 1, \ldots, r$, $j = 1, \ldots, n-r$.

Frobenius' theorem

A constant-dimensional distribution is *integrable if and only if it is involutive*.

The **necessity** of involutivity for complete integrability is easily seen. Indeed, suppose that (1) is satisfied. This is the same as

$$L_{f_i} h_j = 0$$

It follows that

$$L_{[f_i,f_k]}h_j = L_{f_i}L_{f_k}h_j - L_{f_k}L_{f_i}h_j = 0$$

Since the functions $h_1(x), \ldots, h_{n-r}(x)$ are independent, this implies that the Lie brackets $[f_i, f_k]$ are (pointwise) linear combinations of the vector fields $f_1, \ldots, f_r$, and are thus contained in the distribution D.

A geometric description of Frobenius' theorem is as follows. Let the independent functions $h_1(x), \ldots, h_{n-r}(x)$ satisfy (1). Then their level sets, i.e., all sets of the form

$$\{x \mid h_1(x) = c_1, \ldots, h_{n-r}(x) = c_{n-r}\}$$

for arbitrary constants $c_1, \ldots, c_{n-r}$, are well-defined r-dimensional submanifolds of $\mathcal{X}$, to which all the vector fields $f_1, \ldots, f_r$ are *tangent*, and, as a consequence, also all their Lie brackets are tangent.

Example 8 Consider the following set of partial differential equations

$$\begin{aligned} 0 &= x_1 \frac{\partial \phi}{\partial x_1} + x_2 \frac{\partial \phi}{\partial x_2} + x_3 \frac{\partial \phi}{\partial x_3} \\ 0 &= \frac{\partial \phi}{\partial x_3} \end{aligned}$$

Define the vector fields

$$f_1(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \; f_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It is checked that $D := \mathrm{span}(f_1, f_2)$ has constant dimension $= 2$ on the set $\mathcal{X} = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \neq 0\}$ (that is, $\mathbb{R}^3$ excluding the x_3-axis), and is involutive. Thus, by Frobenius' theorem, D is integrable.

Consequently, for each $x_0 \in \mathcal{X}$, there exists a neighborhood N of x_0 and a real-valued function $\phi(x)$ with $d\phi(x) \neq 0$ that satisfies the given set of partial differential equations. In fact, $\phi(x) = \ln x_1 - \ln x_2$ is a (global) solution.
Note that the solution is not unique. In particular, $\phi(x) = \tan^{-1} \frac{x_2}{x_1}$ is also a global solution.

By construction, the accessibility distribution C is **involutive**.

Theorem 9 (Local accessibility) *A sufficient condition for the system to be locally accessible from $x \in \mathcal{X}$ is*

$$\dim C(x) = n \tag{2}$$

If this holds for all $x \in \mathcal{X}$ then the system is locally accessible. Conversely, if the system is locally accessible then (2) holds for all x in an open and dense subset of $\mathcal{X}$.

We call (2) the *accessibility rank condition* at x.

Key idea of the proof

Consider the system

$$\dot{x} = f(x) + g_1(x)u_1 + \dots g_m(x)u_m$$

and its generated system vector fields

$$\mathcal{F} = \{X \mid \exists u_1, \dots, u_m \text{ such that } X(x) = f(x) + g_1(x)u_1 + \dots g_m(x)u_m\}$$

Then for every $k \leq n$ there exists a submanifold N_k around x_0 of dimension k given as

$$N_k = \{x \mid x = X_k^{t_k} \circ X_{k-1}^{t_{k-1}} \circ \dots \circ X_1^{t_1}(x_0),\, 0 \leq \sigma_i < t_i < \tau_i\}$$

with $X_i \in \mathcal{F}$. Indeed, suppose for a certain $k < n$ we can**not** construct N_{k+1}. This means that all system vector fields $X \in \mathcal{F}$ are tangent to N_k, and hence all vector fields $f, g_1, \dots, g_m$. This also means that all **Lie brackets** of these vector fields are tangent to N_k, and thus $\dim C(x) \leq k$, which is a contradiction.

If there is no drift vector field then we obtain real controllability:

Theorem 10 *Consider*

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2 + \ldots + g_m(x)u_m$$

If $\dim C(x) = n$ *for all* $x \in \mathcal{X}$ *then the system is controllable.*

Consider the map

$$(t_1, \ldots, t_n) \to X_n^{t_n} \circ X_{n-1}^{t_{n-1}} \circ \cdots \circ X_1^{t_1}(x_0), \quad 0 \leq \sigma_i < t_i < \tau_i$$

having image N_n, which is an n-dimensional open part of $\mathcal{X}$.
Now let $s_1, \ldots, s_n$ be such that $0 \leq \sigma_i < s_i < \tau_i$. Then the map

$$(t_1, \ldots, t_n) \to (-X_1)^{s_1} \circ (-X_2)^{s_2} \circ \cdots \circ (-X_n)^{s_n} \circ X_n^{t_n} \circ X_{n-1}^{t_{n-1}} \circ \cdots \circ X_1^{t_1}(x_0), 0 \leq \sigma_i < t_i < \tau$$

has an image which is an open neighborhood of x_0. Thus the reachable set $R(x_0)$ from x_0 contains an open neighborhood of x_0.

Suppose now that the reachable set is smaller than $\mathcal{X}$. Then take any point on the boundary of the reachable set $R(x_0)$. Then the set of points reachable from this point is again open. Contradiction.

Thus the unicycle and the cart are indeed controllable.

Note that the actual **construction** of the input functions which steers the system from x_0 to x_1 has not been addressed.

Sometimes local accessibility is heavily depending on the flow of the drift vector field; consider for example the system

$$\begin{aligned}\dot{x}_1 &= 1\\ \dot{x}_2 &= u\end{aligned}$$

This system is locally accessible, but of course very far from controllability. In order to improve the situation we look at a

stronger form of accessibility: **local strong accessibility**

A system is **locally strongly accessible** from x_0 if for any neighborhood $\mathbb{V}$ of x_0 the set $\mathcal{R}^{\mathbb{V}}(x_0, t_1)$ contains a non-empty set for any $t_1 > 0$ sufficiently small. If the latter holds for all $x \in \mathcal{X}$ then the system is called locally strongly accessible.
(The example given above is **not** locally strongly accessible.)

Define $\mathcal{C}_0$ as the smallest algebra which contains $g_1, \ldots, g_m$ and satisfies $[f, w] \in \mathcal{C}_0$ for all $w \in \mathcal{C}_0$. Define the corresponding involutive distribution

$$C_0(x) := \mathrm{span}\{X(x) \mid X \text{ vector field in } \mathcal{C}_0\}.$$

We refer to $\mathcal{C}_0$ and C_0 as the **strong accessibility algebra** and the **strong accessibility distribution**, respectively.

Notice that the strong accessibility algebra $\mathcal{C}_0$ does **not** contain the drift vector field f).

Theorem 11 (Strong accessibility) *A sufficient condition for the system to be locally strongly accessible from x is*

$$\dim C_0(x) = n$$

Furthermore, the system is locally strongly accessible if this holds for all x. Conversely, if the system is locally strongly accessible then it holds for all x in an open and dense subset of $\mathcal{X}$.

The system given before:

$$\begin{aligned} \dot{x}_1 &= x_2^2 \\ \dot{x}_2 &= u. \end{aligned}$$

is not only locally accessible, but also locally strongly accessible, since $g(x)$ and $[[f, g], g](x)$ are everywhere independent.

Let us apply the theory developed above to a linear system

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i, \;\; x \in \mathbb{R}^n,$$

where $b_1, \ldots, b_m$ are the columns of the input matrix B.
Clearly, the Lie brackets of the constant input vector fields given by the input vectors $b_1, \ldots, b_m$ are all zero, i.e.,

$$[b_i, b_j] = 0, \;\text{ for all } i, j = 1, \ldots, m.$$

Furthermore, the Lie bracket of the linear drift vector field Ax with an input vector field b_i yields the constant vector field

$$[Ax, b_i] = -Ab_i.$$

The Lie brackets of Ab_i with Ab_j or b_j are again all zero, while

$$[Ax, -Ab_i] = A^2 b_i.$$

Hence we conclude that $\mathcal{C}$ is spanned by all constant vector fields b_i, Ab_i, $A^2b_i, \ldots,$ $i \in m$, together with the linear drift vector field Ax, i.e.,

$$\mathcal{C} = \{Ax, b_i, Ab_i, A^2b_i \ldots, A^{n-1}b_i,\ i = 1, \ldots, m\}.$$

while

$$\mathcal{C}_0 = \text{ columns of } (B, AB, A^2B \ldots, A^{n-1}B)$$

We see that for linear systems the rank condition for strong accessibility coincides with the Kalman rank condition for controllability. Hence, if we would not have known anything special about linear systems, then at least a linear system which satisfies the Kalman rank condition is **locally strongly accessible**.

Example 12 (Actuated rotating rigid body)

Consider

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} A_1\omega_2\omega_3 \\ A_2\omega_3\omega_1 \\ A_3\omega_1\omega_2 \end{bmatrix} + \begin{bmatrix} \alpha_1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \alpha_2 \\ 0 \end{bmatrix} u_2$$

with $\alpha_1 \neq 0, \alpha_2 \neq 0$. Here the constants A_1, A_2, A_3 are determined by the **moments of inertia** a_1, a_2, a_3. Compute

$$[g_1, f](\omega) = \begin{bmatrix} 0 \\ \alpha_1 A_2 \omega_3 \\ \alpha_1 A_3 \omega_2 \end{bmatrix}$$

$$[g_2, f](\omega) = \begin{bmatrix} \alpha_2 A_1 \omega_3 \\ 0 \\ \alpha_2 A_3 \omega_1 \end{bmatrix}$$

On the other hand

$$[g_2, [g_1, f]] = \begin{bmatrix} 0 \\ 0 \\ \alpha_1 \alpha_2 A_3 \end{bmatrix}$$

Thus the system is locally strongly accessible if

$$A_3 \neq 0$$

which is equivalent to $a_1 \neq a_2$.
In fact, this is the **if and only if** condition. Indeed, if $A_3 = 0$ then $\dot{\omega}_3 = 0$, showing that the system is not locally strongly accessible.

Remark 13 *Due to the specific properties of the drift vector field, i.c. Poisson stability, it can be shown that the system is in fact* **controllable** *if and only if the two first moments of inertia* a_1 *and* a_2 *are different.)*