Network Modeling and Control of Physical Systems, DISC

Theory of Port-Hamiltonian systems

Chapter 1: Port-Hamiltonian formulation of network models; the lumped-parameter case

A.J. van der Schaft*

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Abstract

It is shown how *port-based modeling* of lumped-parameter complex physical systems (multi-body systems, electrical circuits, electromechanical systems, ..) naturally leads to a geometrically defined class of systems, called *port-Hamiltonian systems*. These are Hamiltonian systems defined with respect to a power-conserving geometric structure capturing the basic interconnection laws, and a Hamiltonian function given by the total stored energy. The structural properties of port-Hamiltonian systems are discussed, in particular the existence of Casimir functions and its implications for stability.

1 Introduction

In this chapter we discuss how *network modeling* of lumped-parameter physical systems naturally leads to a geometrically defined class of systems, called *port-Hamiltonian systems*. This provides a unified mathematical framework for the description of physical systems stemming from different physical domains, such as mechanical, electrical, thermal, as well as mixtures of them.

Historically, the Hamiltonian approach has its roots in analytical mechanics and starts from the principle of least action, via the Euler-Lagrange equations and the Legendre transform, towards the Hamiltonian equations of motion. On the other hand, the network approach stems from electrical engineering, and constitutes a cornerstone of systems theory. While most of the *analysis* of physical systems has been performed within the Lagrangian and Hamiltonian framework, the network modelling point of view is prevailing in *modelling* and *simulation* of (complex) physical systems. The framework of port-Hamiltonian systems *combines* both points of view, by associating with the interconnection structure ("generalized junction structure" in bond graph terminology) of the network model a *geometric structure* given by a *Poisson structure*, or more generally a *Dirac structure*. The Hamiltonian dynamics

*Dept. of Applied Mathematics, University of Twente, PO Box 217, 7500 AE Enschede, The Netherlands

is then defined with respect to this Poisson (or Dirac) structure *and* the Hamiltonian given by the total stored energy, as well as the energy-dissipating elements and the ports of the system. This is discussed in Section 2 for the case of Poisson structures (no algebraic constraints), and in Section 3 for the general case of Dirac structures. Dirac structures encompass the 'canonical' structures which are classically being used in the geometrization of mechanics, since they also allow to describe the geometric structure of systems with *constraints* as arising from the interconnection of sub-systems. Furthermore, Dirac structures allow to extend the Hamiltonian description of *distributed-parameter systems* to include variable boundary conditions, leading to distributed-parameter port-Hamiltonian systems with boundary ports. This will be the topic of the third chapter.

The structural properties of lumped-parameter port-Hamiltonian systems are investigated in Section 4 through geometric tools stemming from the theory of Hamiltonian systems. It is indicated how the *interconnection* of port-Hamiltonian systems again leads to a port-Hamiltonian system, and how this may be exploited for control and design. In particular, we investigate the existence of Casimir functions for the feedback interconnection of a plant port-Hamiltonian system and a controller port-Hamiltonian system, leading to a reduced port-Hamiltonian system on invariant manifolds with *shaped* energy. We thus provide an interpretation of *passivity-based control* from an *interconnection* point of view.

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2 Finite-dimensional port-Hamiltonian systems

2.1 From the Euler-Lagrange and Hamiltonian equations to port-Hamiltonian systems

In this subsection we indicate how the classical framework of Lagrangian and Hamiltonian differential equations as originating from analytical mechanics can be extended to port-Hamiltonian systems. Let us briefly recall the standard Euler-Lagrange and Hamiltonian equations of motion. The standard *Euler-Lagrange equations* are given as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right) - \frac{\partial L}{\partial q}(q,\dot{q}) = \tau, \tag{1}$$

where $q = (q_1, \ldots, q_k)^T$ are generalized configuration coordinates for the system with k degrees of freedom, the Lagrangian L equals the difference $K - P$ between kinetic energy K and potential energy P, and $\tau = (\tau_1, \ldots, \tau_k)^T$ is the vector of generalized forces acting on the system. Furthermore, $\frac{\partial L}{\partial \dot{q}}$ denotes the column-vector of partial derivatives of $L(q,\dot{q})$ with respect to the generalized velocities $\dot{q}_1, \ldots, \dot{q}_k$, and similarly for $\frac{\partial L}{\partial q}$. In standard mechanical systems the kinetic energy K is of the form

$$K(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} \tag{2}$$

where the $k \times k$ inertia (generalized mass) matrix $M(q)$ is symmetric and positive definite for all q. In this case the vector of generalized *momenta* $p = (p_1, \ldots, p_k)^T$, defined for any

Lagrangian L as $p = \frac{\partial L}{\partial \dot{q}}$, is simply given by

$$p = M(q)\dot{q}, \tag{3}$$

and by defining the state vector $(q_1, \ldots, q_k, p_1, \ldots, p_k)^T$ the k second-order equations (1) transform into $2k$ first-order equations

$$\begin{aligned} \dot{q} &= \tfrac{\partial H}{\partial p}(q,p) \qquad (= M^{-1}(q)p) \\ \dot{p} &= -\tfrac{\partial H}{\partial q}(q,p) + \tau \end{aligned} \tag{4}$$

where

$$H(q,p) = \frac{1}{2}p^T M^{-1}(q)p + P(q) \quad (= \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)) \tag{5}$$

is the total energy of the system. The equations (4) are called the *Hamiltonian equations* of motion, and H is called the *Hamiltonian.* The following *energy balance* immediately follows from (4):

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial q}(q,p)\dot{q} + \frac{\partial^T H}{\partial p}(q,p)\dot{p} = \frac{\partial^T H}{\partial p}(q,p)\tau \quad = \quad \dot{q}^T \tau, \tag{6}$$

expressing that the increase in energy of the system is equal to the supplied work (*conservation of energy*).

If the Hamiltonian $H(q,p)$ is assumed to be the sum of a positive kinetic energy and a potential energy which is *bounded from below*, that is

$$\begin{aligned} H(q,p) \quad &= \quad \frac{1}{2}p^T M^{-1}(q)p + P(q) \\ M(q) = M^T(q) \quad &> \quad 0, \qquad \exists C > -\infty \quad \text{such that } P(q) \geq C. \end{aligned} \tag{7}$$

then it follows that (4) with inputs $u = \tau$ and outputs $y = \dot{q}$ is a *passive* (in fact, *lossless*) state space system with storage function $H(q,p) - C \geq 0$ (see e.g. [62, 20, 47] for the general theory of passive and dissipative systems). Since the energy is only defined up to a constant, we may as well as take as potential energy the function $P(q) - C \geq 0$, in which case the total energy $H(q,p)$ becomes nonnegative and thus itself is the storage function.

System (4) is an example of a *Hamiltonian system* with collocated inputs and outputs, which more generally is given in the following form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \quad , \quad (q,p) = (q_1, \ldots, q_k, p_1, \ldots, p_k) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + B(q)u, \qquad\qquad u \in \mathbb{R}^m, \\ y &= B^T(q)\frac{\partial H}{\partial p}(q,p) \qquad (= B^T(q)\dot{q}), \qquad y \in \mathbb{R}^m, \end{aligned} \tag{8}$$

Here $B(q)$ is the input force matrix, with $B(q)u$ denoting the generalized forces resulting from the control inputs $u \in \mathbb{R}^m$. The state space of (8) with local coordinates (q,p) is usually called the *phase space.* In case $m < k$ we speak of an *underactuated* system. If $m = k$ and the matrix $B(q)$ is everywhere invertible, then the Hamiltonian system is called fully actuated.

Because of the form of the output equations $y = B^T(q)\dot{q}$ we again obtain the energy balance

$$\frac{dH}{dt}(q(t), p(t)) = u^T(t)y(t) \tag{9}$$

Hence if H is non-negative (or, *bounded from below*), any Hamiltonian system (8) is a *lossless* state space system. ('Lossless' is a strong form of 'passive'; in the latter case (9) need only be satisfied with the equality sign '=' replaced by the inequality sign '$\leq$'.) For a system-theoretic treatment of the Hamiltonian systems (8), especially if the output y can be written as the time-derivative of a vector of generalized configuration coordinates, we refer to e.g. [8, 43, 44, 10, 36].

A major generalization of the class of Hamiltonian systems (8) is to consider systems which are described in local coordinates as

$$\begin{aligned} \dot{x} &= J(x)\tfrac{\partial H}{\partial x}(x) + g(x)u, \quad x \in \mathcal{X}, u \in \mathbb{R}^m \\ y &= g^T(x)\tfrac{\partial H}{\partial x}(x), \qquad\qquad y \in \mathbb{R}^m \end{aligned} \tag{10}$$

Here $J(x)$ is an $n \times n$ matrix with entries depending smoothly on x, which is assumed to be *skew-symmetric*

$$J(x) = -J^T(x), \tag{11}$$

and $x = (x_1, \ldots, x_n)$ are local coordinates for an n-dimensional state space manifold $\mathcal{X}$. Because of (11) we easily recover the energy-balance $\frac{dH}{dt}(x(t)) = u^T(t)y(t)$, showing that (10) is lossless if $H \geq 0$. We call (10) with J satisfying (11) a *port-Hamiltonian system* with *structure matrix* $J(x)$ and *Hamiltonian* H ([24, 30, 25]). Note that (8) (and hence (4)) is a particular case of (10) with $x = (q, p)$, and $J(x)$ being given by the constant skew-symmetric matrix $J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix}$, and $g(q,p) = \begin{bmatrix} 0 \\ B(q) \end{bmatrix}$.

As an important mathematical note, we remark that in many examples the structure matrix J will satisfy the "*integrability*" conditions

$$\sum_{l=1}^{n} \left[J_{lj}(x)\frac{\partial J_{ik}}{\partial x_l}(x) + J_{li}(x)\frac{\partial J_{kj}}{\partial x_l}(x) + J_{lk}(x)\frac{\partial J_{ji}}{\partial x_l}(x) \right] = 0, i, j, k = 1, \ldots, n \tag{12}$$

In this case we may find, by Darboux's theorem (see e.g. [61]) around any point x_0 where the rank of the matrix $J(x)$ is constant, local coordinates $\tilde{x} = (q, p, s) = (q_1, \ldots, q_k, p_1, \ldots, p_k, s_1, \ldots s_l)$, with $2k$ the rank of J and $n = 2k + l$, such that J in these coordinates takes the form

$$J = \begin{bmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{13}$$

The coordinates (q, p, s) are called *canonical* coordinates, and J satisfying (11) and (12) is called a *Poisson structure matrix.* In such canonical coordinates the equations (10) take the form

$$
\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p}(q,p,s) + g_q(q,p,s)u \\
\dot{p} &= -\frac{\partial H}{\partial q}(q,p,s) + g_p(q,p,s)u \\
\dot{s} &= g_s(q,p,s)u \\
y &= g_q^T(q,p,s)\frac{\partial H}{\partial q}(q,p,s) + g_p^T(q,p,s)\frac{\partial H}{\partial p}(q,p,s) + g_s^T(q,p,s)\frac{\partial H}{\partial s}(q,p,s)
\end{aligned}
\tag{14}
$$

which is, apart from the appearance of the variables s, very close to the standard Hamiltonian form (8). In particular, if $g_s = 0$, then the variables s are merely an additional set of *constant* parameters.

2.2 From port-based network modelling to port-Hamiltonian systems

In the preceeding subsection we have seen how the classical Hamiltonian equations of motion can be extended to port-Hamiltonian systems. This has been basically done by adding to the (generalized) Hamiltonian equations of motion *ports* modeling the interaction of the system with its environment.

In this subsection we take a different point of view. Indeed, port-Hamiltonian systems arise systematically from *port-based network models* of physical systems, e.g. using bond graphs. In port-based network models of complex physical systems the overall system is seen as the *interconnection* of energy-storing elements via basic interconnection (balance) laws as Newton's third law or Kirchhoff's laws, as well as power-conserving elements like transformers, kinematic pairs and ideal constraints, together with energy-dissipating elements. The basic point of departure for the theory of port-Hamiltonian systems is to formalize the basic interconnection laws together with the power-conserving elements by a *geometric structure*, and to define the Hamiltonian as the total energy stored in the system. Indeed, for the (restricted) form of port-Hamiltonian systems given in the previous subsection the structure matrix $J(x)$ and the input matrix $g(x)$ may be directly associated with the network interconnection structure, while the Hamiltonian H is just the sum of the energies of all the energy-storing elements; see the papers [30, 24, 32, 31, 51, 53, 27, 46, 59]. In particular, network models of complex physical systems formalized within the (generalized) bond graph language ([41, 7]) can be shown to immediately lead to port-Hamiltonian systems; see e.g. [19].

Example 2.1 (LCTG circuits). Consider a controlled LC-circuit (see Figure 1) consisting of two inductors with magnetic energies $H_1(\varphi_1), H_2(\varphi_2)$ (φ_1 and φ_2 being the magnetic flux linkages), and a capacitor with electric energy $H_3(Q)$ (Q being the charge). If the elements are linear then $H_1(\varphi_1) = \frac{1}{2L_1}\varphi_1^2$, $H_2(\varphi_2) = \frac{1}{2L_2}\varphi_2^2$ and $H_3(Q) = \frac{1}{2C}Q^2$. Furthermore let $V = u$ denote a voltage source. Using Kirchhoff's laws one immediately arrives at the dynamical

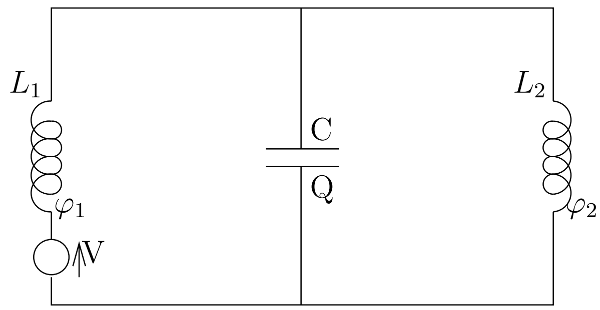

Figure 1: Controlled LC-circuit

equations

$$\begin{aligned}
\begin{bmatrix} \dot{Q} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{J} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial \varphi_1} \\ \frac{\partial H}{\partial \varphi_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u \qquad (15) \\
y &= \frac{\partial H}{\partial \varphi_1} \qquad (= \text{current through first inductor})
\end{aligned}$$

with $H(Q, \varphi_1, \varphi_2) := H_1(\varphi_1) + H_2(\varphi_2) + H_3(Q)$ the total energy. Clearly the matrix J is skew-symmetric, and since J is constant it trivially satisfies (12). In [31] it has been shown that in this way every LC-circuit with independent elements can be modelled as a port-Hamiltonian system. Furthermore, also any LCTG-circuit with independent elements can be modelled as a port-Hamiltonian system, with J determined by Kirchhoff's laws *and* the constitutive relations of the transformers T and gyrators G. □

Example 2.2 (Actuated rigid body). Consider a rigid body spinning around its center of mass in the absence of gravity. The energy variables are the three components of the body angular momentum p along the three principal axes: $p = (p_x, p_y, p_z)$, and the energy is the kinetic energy

$$H(p) = \frac{1}{2}\left(\frac{p_x^2}{I_x} + \frac{p_y^2}{I_y} + \frac{p_z^2}{I_z}\right),$$

where I_x, I_y, I_z are the principal moments of inertia. Euler's equations describing the dynamics are

$$\begin{bmatrix} \dot{p_x} \\ \dot{p_y} \\ \dot{p_z} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}}_{J(p)} \begin{bmatrix} \frac{\partial H}{\partial p_x} \\ \frac{\partial H}{\partial p_y} \\ \frac{\partial H}{\partial p_z} \end{bmatrix} + g(p)u \qquad (16)$$

It can be checked that the skew-symmetric matrix $J(p)$ satisfies (12). (In fact, $J(p)$ is the canonical Lie-Poisson structure matrix on the dual of the Lie algebra $so(3)$ corresponding to the configuration space $SO(3)$ of the rigid body.) In the scalar input case the term $g(p)u$

denotes the torque around an axis with coordinates $g = (b_x \ b_y \ b_z)^T$, with corresponding collocated output given as

$$y = b_x \frac{p_x}{I_x} + b_y \frac{p_y}{I_y} + b_z \frac{p_z}{I_z}, \tag{17}$$

which is the velocity around the same axis $(b_x \ b_y \ b_z)^T$. □

Example 2.3. A third important class of systems that naturally can be written as port-Hamiltonian systems, is constituted by mechanical systems with *kinematic constraints.* Consider as before a mechanical system with k degrees of freedom, locally described by k configuration variables $q = (q_1, \dots, q_k)$. Suppose that there are constraints on the generalized velocities $\dot{q}$, described as

$$A^T(q)\dot{q} = 0, \tag{18}$$

with $A(q)$ a $r \times k$ matrix of rank r everywhere (that is, there are r independent kinematic constraints). Classically, the constraints (18) are called *holonomic* if it is possible to find new configuration coordinates $\overline{q} = (\overline{q}_1, \dots, \overline{q}_k)$ such that the constraints are equivalently expressed as

$$\dot{\overline{q}}_{k-r+1} = \dot{\overline{q}}_{n-r+2} = \cdots = \dot{\overline{q}}_k = 0 \ , \tag{19}$$

in which case one can eliminate the configuration variables $\overline{q}_{k-r+1}, \dots, \overline{q}_k$, since the kinematic constraints (19) are equivalent to the *geometric* constraints

$$\overline{q}_{k-r+1} = c_{k-r+1}, \dots, \overline{q}_k = c_k \ , \tag{20}$$

for certain constants $c_{k-r+1}, \dots, c_k$ determined by the initial conditions. Then the system reduces to an *unconstrained* system in the remaining configuration coordinates $(\overline{q}_1, \dots, \overline{q}_{k-r})$. If it is *not* possible to find coordinates $\overline{q}$ such that (19) holds (that is, if we are not able to *integrate* the kinematic constraints as above), then the constraints are called *nonholonomic*.

The equations of motion for the mechanical system with Lagrangian $L(q, \dot{q})$ and constraints (18) are given by the Euler-Lagrange equations [35]

$$\begin{array}{rcl} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) - \dfrac{\partial L}{\partial q} & = & A(q)\lambda + B(q)u, \qquad \lambda \in \mathbb{R}^r, u \in \mathbb{R}^m \\ A^T(q)\dot{q} & = & 0 \end{array} \tag{21}$$

where $B(q)u$ are the external forces (controls) applied to the system, for some $k \times m$ matrix $B(q)$, while $A(q)\lambda$ are the *constraint forces.* The Lagrange multipliers $\lambda(t)$ are uniquely determined by the requirement that the constraints $A^T(q(t))\dot{q}(t) = 0$ have to be satisfied for all t.

Defining as before (cf. (3)) the generalized momenta the constrained Euler-Lagrange equations (21) transform into *constrained Hamiltonian equations* (compare with (8)),

$$\begin{array}{rcl} \dot{q} & = & \dfrac{\partial H}{\partial p}(q,p) \\ \dot{p} & = & -\dfrac{\partial H}{\partial q}(q,p) + A(q)\lambda + B(q)u \\ y & = & B^T(q)\dfrac{\partial H}{\partial p}(q,p) \\ 0 & = & A^T(q)\dfrac{\partial H}{\partial p}(q,p) \end{array} \tag{22}$$

with $H(q,p) = \frac{1}{2}p^T M^{-1}(q)p + P(q)$ the total energy. The *constrained* state space is therefore given as the following subset of the phase space:

$$\mathcal{X}_c = \{(q,p) \mid A^T(q)\frac{\partial H}{\partial p}(q,p) = 0\} \tag{23}$$

One way of proceeding with these equations is to *eliminate* the constraint forces, and to *reduce* the equations of motion to the constrained state space. In [50] it has been shown that this leads to a port-Hamiltonian system (10). Furthermore, the structure matrix J_c of the port-Hamiltonian system satisfies the integrability conditions (12) if and only if the constraints (18) are *holonomic*. (In fact, if the constraints are holonomic then the coordinates s as in (13) can be taken to be equal to the "integrated constraint functions" $\overline{q}_{k-r+1}, \ldots, \overline{q}_k$ of (20), and the matrix g_s as in (14) is zero.)

An alternative way of approaching the system (22) is to formalize it directly as an *implicit* port-Hamiltonian system, as will be discussed in the next Section 3.

2.3 Basic properties of port-Hamiltonian systems

As allude to above, port-Hamiltonian systems naturally arise from a *network modeling* of physical systems without dissipative elements, see our papers [24, 30, 25, 32, 31, 26, 51, 49, 53, 27, 46]. Recall that a port-Hamiltonian system is defined by a state space manifold $\mathcal{X}$ endowed with a *triple* (J, g, H). The pair $(J(x), g(x))\,, x \in \mathcal{X}$, captures the *interconnection structure* of the system, with $g(x)$ modeling in particular the *ports* of the system. This is very clear in Example 2.1, where the pair $(J(x), g(x))$ is determined by Kirchhoff's laws, the paradigmatic example of a power-conserving interconnection structure, but it naturally holds for other physical systems without dissipation as well. Independently from the interconnection structure, the function $H : \mathcal{X} \to \mathbb{R}$ defines the total stored *energy* of the system. Furthermore, port-Hamiltonian systems are intrinsically *modular* in the sense that a power-conserving interconnection of a number of port-Hamiltonian systems again defines a port-Hamiltonian system, with its overall interconnection structure determined by the interconnection structures of the composing individual systems together with their power-conserving interconnection, and the Hamiltonian just the sum of the individual Hamiltonians (see [53, 46, 11]).

As we have seen before, a basic property of port-Hamiltonian systems is the energy-balancing property $\frac{dH}{dt}(x(t)) = u^T(t)y(t)$. Physically this corresponds to the fact that the *internal* interconnection structure is power-conserving (because of skew-symmetry of $J(x)$), while u and y are the *power-variables* of the ports defined by $g(x)$, and thus $u^T y$ is the externally supplied power.

From the structure matrix $J(x)$ of a port-Hamiltonian system one can directly extract useful information about the dynamical properties of the system. Since the structure matrix is directly related to the modeling of the system (capturing the interconnection structure) this information usually has a direct physical interpretation.

A very important property which may be directly inferred from the structure matrix is the existence of dynamical invariants *independent* of the Hamiltonian H, called *Casimir functions.* Consider the set of p.d.e.'s

$$\frac{\partial^T C}{\partial x}(x)J(x) = 0, \qquad x \in \mathcal{X}, \tag{24}$$

in the unknown (smooth) function $C : \mathcal{X} \to \mathbb{R}$. If (24) has a solution C then it follows that the time-derivative of C along the port-controlled Hamiltonian system (10) satisfies

$$\begin{aligned} \frac{dC}{dt} &= \frac{\partial^T C}{\partial x}(x) J(x) \frac{\partial H}{\partial x}(x) + \frac{\partial^T C}{\partial x}(x) g(x) u \\ &= \frac{\partial^T C}{\partial x}(x) g(x) u \end{aligned} \tag{25}$$

Hence, for the input $u = 0$, or for *arbitrary* input functions if additionally $\frac{\partial^T C}{\partial x}(x) g(x) = 0$, the function $C(x)$ *remains constant* along the trajectories of the port-Hamiltonian system, *irrespective* of the precise form of the Hamiltonian H. A function $C : \mathcal{X} \to \mathbb{R}$ satisfying (24) is called a *Casimir function* (of the structure matrix $J(x)$).

The existence of non-trivial solutions C to (24) clearly assumes that rank $J(x) < \dim \mathcal{X}$, but is also related to the integrability conditions (12). In fact, if canonical coordinates (q, p, s) as in (13) have been found, then the Casimir functions are precisely all functions $C : \mathcal{X} \to \mathbb{R}$ depending *only* on the s-coordinates.

From (25) it follows that the level sets $L_C := \{x \in \mathcal{X} | C(x) = c\}, c \in \mathbb{R}$, of a Casimir function C are *invariant* sets for the autonomous Hamiltonian system $\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$. Furthermore, the dynamics $\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$ *restricted* to any level set L_C is given as the *reduced* Hamiltonian dynamics

$$\dot{x}_C = J_C(x_C) \frac{\partial H_C}{\partial x}(x_C) \tag{26}$$

with H_C and J_C the *restriction* of H, respectively J, to L_C. More generally, if $C = (C_1, \ldots, C_r)$ are independent Casimir functions, then in any set of local coordinates $(z_1, \ldots, z_l, C_1, \ldots, C_r)$ for $\mathcal{X}$ the Hamiltonian dynamics $\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$ takes the form

$$\begin{bmatrix} \dot{z} \\ \dot{C} \end{bmatrix} = \begin{bmatrix} \tilde{J}(z, C) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial z} \\ \frac{\partial H}{\partial C} \end{bmatrix},$$

leading to the *reduced* Hamiltonian dynamics

$$\dot{z} = \tilde{J}(z, C = c) \frac{\partial H}{\partial z}$$

on any multi-level set $\{x \in \mathcal{X} | \, (C_1(x), \ldots, C_r(x)) = c \in \mathbb{R}^r\}$.

The existence of Casimir functions has immediate consequences for stability analysis of (10) for $u = 0$. Indeed, if $C_1, \cdots, C_r$ are Casimirs, then by (24) not only $\frac{dH}{dt} = 0$ for $u = 0$, but

$$\frac{d}{dt}\left(H + H_a(C_1, \cdots, C_r)\right)(x(t)) = 0 \tag{27}$$

for *any* function $H_a : \mathbb{R}^r \to \mathbb{R}$. Hence, even if H is not positive definite at an equilibrium $x^* \in \mathcal{X}$, then $H + H_a(C_1, \cdots, C_r)$ may be positive definite at x^* by a proper choice of H_a, and thus may serve as a Lyapunov function. This method for stability analysis is called the *Energy-Casimir method*, see e.g. [23].

Example 2.4 (Example 2.1 continued). The quantity $\phi_1 + \phi_2$ is a Casimir function.

Example 2.5 (Example 2.2 continued). The quantity $\frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}p_z^2$ (total angular momentum) is a Casimir function.

For a further discussion of the dynamical properties of Hamiltonian systems (especially if J satisfies the integrability conditions (12)) we refer to the extensive literature on this topic, see e.g. [1, 23].

2.4 Port-Hamiltonian systems with dissipation

Energy-dissipation is included in the framework of port-Hamiltonian systems (10) by terminating some of the ports by resistive elements. Indeed, consider instead of $g(x)u$ in (10) a term

$$\begin{bmatrix} g(x) & g_R(x) \end{bmatrix} \begin{bmatrix} u \\ u_R \end{bmatrix} = g(x)u + g_R(x)u_R \tag{28}$$

and extend correspondingly the output equations $y = g^T(x)\frac{\partial H}{\partial x}(x)$ to

$$\begin{bmatrix} y \\ y_R \end{bmatrix} = \begin{bmatrix} g^T(x)\frac{\partial H}{\partial x}(x) \\ \\ g_R^T(x)\frac{\partial H}{\partial x}(x) \end{bmatrix} \tag{29}$$

Here $u_R, y_R \in \mathbb{R}^{m_r}$ denote the power variables at the ports which are terminated by static resistive elements

$$u_R = -F(y_R) \tag{30}$$

where the resistive characteristic $F : \mathbb{R}^{m_r} \to \mathbb{R}^{m_r}$ satisfies

$$y_R^T F(y_R) \geq 0, \quad y_R \in \mathbb{R}^{m_r} \tag{31}$$

(In many cases, F will be derivable from a so-called *Rayleigh dissipation function* $R : \mathbb{R}^{m_r} \to \mathbb{R}$ in the sense that $F(y_R) = \frac{\partial R}{\partial y_R}(y_R)$.) In the sequel we concentrate on port-Hamiltonian systems with ports terminated by *linear* resistive elements

$$u_R = -Sy_R \tag{32}$$

for some positive semi-definite symmetric matric $S = S^T \geq 0$. Substitution of (32) into (28) leads to a model of the form

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\,\frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\frac{\partial H}{\partial x}(x) \end{aligned} \tag{33}$$

where $R(x) := g_R(x)Sg_R^T(x)$ is a positive semi-definite symmetric matrix, depending smoothly on x. In this case the energy-balancing property (8) takes the form

$$\begin{aligned} \frac{dH}{dt}(x(t)) &= u^T(t)y(t) - \frac{\partial^T H}{\partial x}(x(t))R(x(t))\frac{\partial H}{\partial x}(x(t)) \\ &\leq u^T(t)y(t). \end{aligned} \tag{34}$$

showing that a port-Hamiltonian system is passive if the Hamiltonian H is bounded from below. We call (33) a *port-Hamiltonian system with dissipation.* Note that in this case *two* geometric structures play a role: the internal interconnection structure given by $J(x)$, and an additional resistive structure given by $R(x)$, which is determined by the port structure $g_R(x)$ and the linear constitutive relations $u_R = -Sy_R$ of the resistive elements.

Regarding Casimir functions for a port-Hamiltonian system with dissipation (33) we consider functions $C : \mathcal{X} \to \mathbb{R}$ satisfying the set of p.d.e.'s

$$\frac{\partial^T C}{\partial x}(x)\,[J(x) - R(x)] = 0, \quad x \in \mathcal{X}, \tag{35}$$

implying that the time-derivative of C along solutions of the system (33) for $u = 0$ is zero (irrespective of the Hamiltonian H). Post-multiplication of (35) by $\frac{\partial C}{\partial x}(x)$ and sunsequent transposition of the first result yields by skew-symmetry of J and symmetry of R

$$\begin{array}{rcl} \frac{\partial^T C}{\partial x}(x)\,[J(x) - R(x)]\,\frac{\partial C}{\partial x}(x) & = & 0 \\ \frac{\partial^T C}{\partial x}(x)\,[-J(x) - R(x)]\,\frac{\partial C}{\partial x}(x) & = & 0 \end{array} \tag{36}$$

which by semi-positive definiteness of R yields

$$\begin{array}{l} \frac{\partial^T C}{\partial x}(x) J(x) = 0 \\ \\ \frac{\partial^T C}{\partial x}(x) R(x) = 0 \end{array} \tag{37}$$

Thus C is a Casimir for *both* geometric structures defined, respectively, by $J(x)$ and $R(x)$.

If (37) holds for independent functions $C_1, \ldots, C_r$, then in any set of local coordinates $(z, C) = (z_1, \ldots z_l, C_1, \ldots C_r)$ the dynamics (33) for $u = 0$ takes the form

$$\left[\begin{array}{c} \dot{z} \\ \dot{C} \end{array}\right] = \left(\left[\begin{array}{cc} \tilde{J}(z, C) & 0 \\ 0 & 0 \end{array}\right] - \left[\begin{array}{cc} \tilde{R}(z, C) & 0 \\ 0 & 0 \end{array}\right] \right) \left[\begin{array}{c} \frac{\partial H}{\partial z} \\ \\ \frac{\partial H}{\partial C} \end{array}\right], \tag{38}$$

which can be restricted on any multi-level set $\{x \in \mathcal{X} | (C_1(x), \ldots, C_r(x)) \ = c \in \mathbb{R}^r\}$ to

$$\dot{z} = \left[\tilde{J}(z, C = c) - \tilde{R}(z, C = c)\right] \frac{\partial H}{\partial z}(z, C = c) \tag{39}$$

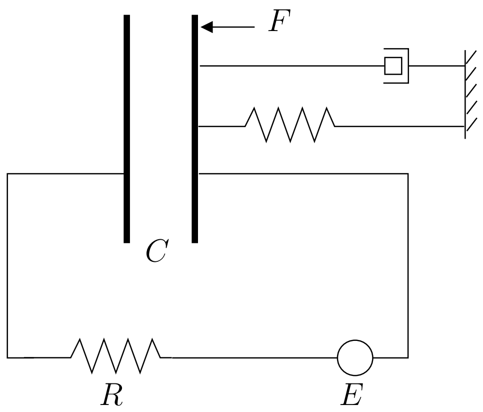

Figure 2: Capacitor microphone

Example 2.6. ([35]) Consider the capacitor microphone depicted in Figure 2. Here the capacitance $C(q)$ of the capacitor is varying as a function of the displacement q of the right plate (with mass m), which is attached to a spring (with spring constant $k > 0$) and a damper (with constant $c > 0$), and affected by a mechanical force F (air pressure arising from sound). Furthermore, E is a voltage source. The dynamical equations of motion can be written as the port-Hamiltonian system with dissipation

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{Q} \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1/R \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial Q} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} F + \begin{bmatrix} 0 \\ 0 \\ 1/R \end{bmatrix} E \tag{40}$$

$$y_1 = \frac{\partial H}{\partial p} = \dot{q}$$

$$y_2 = \frac{1}{R}\frac{\partial H}{\partial Q} = I$$

with p the momentum, R the resistance of the resistor, I the current through the voltage source, and the Hamiltonian H being the total energy

$$H(q, p, Q) = \frac{1}{2m}p^2 + \frac{1}{2}k(q - \bar{q})^2 + \frac{1}{2C(q)}Q^2, \tag{41}$$

with $\bar{q}$ denoting the equilibrium position of the spring. Note that $F\dot{q}$ is the mechanical power, and EI the electrical power applied to the system. In the application as a microphone the voltage over the resistor will be used (after amplification) as a measure for the mechanical force F.

Example 2.7. ([Ortega et al. [39]]) A permanent magnet synchronous motor can be written as a port-Hamiltonian system with dissipation (in a rotating reference, i.e. the dq frame) for the state vector

$$x = M \begin{bmatrix} i_d \\ i_q \\ \omega \end{bmatrix}, \qquad M = \begin{bmatrix} L_d & 0 & 0 \\ 0 & L_q & 0 \\ 0 & 0 & \frac{j}{n_p} \end{bmatrix} \tag{42}$$

the magnetic flux linkages and mechanical momentum (i_d, i_q being the currents, and ω the angular velocity), L_d, L_q stator inductances, j the moment of inertia, and n_p the number of pole pairs. The Hamiltonian $H(x)$ is given as $H(x) = \frac{1}{2}x^T M^{-1} x$ (total energy), while furthermore $J(x), R(x)$ and $g(x)$ are determined as

$$J(x) = \begin{bmatrix} 0 & L_0 x_3 & 0 \\ -L_0 x_3 & 0 & -\Phi_{q0} \\ 0 & \Phi_{q0} & 0 \end{bmatrix}, \quad R(x) = \begin{bmatrix} R_S & 0 & 0 \\ 0 & R_S & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; g(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{n_p} \end{bmatrix} \tag{43}$$

with R_S the stator winding resistance, Φ_{q0} a constant term due to interaction of the permanent magnet and the magnetic material in the stator, and $L_0 := L_d n_p/j$. The three inputs are the stator voltage $(v_d, v_q)^T$ and the (constant) load torque. Outputs are i_d, i_q and ω.

In some cases the interconnection structure $J(x)$ may be actually *varying*, depending on the mode of operation of the system, as exemplified by the following simple dc-to-dc power converter with a single switch. See for a further treatment of power converters in this context [13].

Example 2.8. Consider the ideal boost converter given in Figure 3.

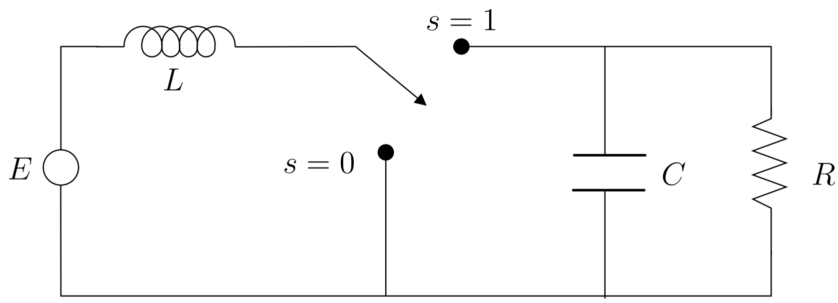

Figure 3: Ideal boost converter

The system equations are given as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \left(\begin{bmatrix} 0 & -s \\ s & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1/R \end{bmatrix} \right) \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} E \\ y &= \frac{\partial H}{\partial x_1} \end{aligned} \tag{44}$$

with x_1 the magnetic flux linkage of the inductor, x_2 the charge of the capacitor, and $H(x_1, x_2) = \frac{1}{2L}x_1^2 + \frac{1}{2C}x_2^2$ the total stored energy. The internal interconnection structure matrix J is either $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, depending on the ideal switch being in position $s = 0$ or $s = 1$.

3 Implicit port-Hamiltonian systems

From a general modeling point of view physical systems are, at least in first instance, often described as DAE's, that is, a mixed set of differential and *algebraic* equations. This stems from the fact that in network modeling the system under consideration is naturally regarded as obtained from interconnecting simpler sub-systems. These interconnections in general, give rise to algebraic constraints between the state space variables of the sub-systems; thus leading to implicit systems. While in the linear case one may argue that it is often relatively straightforward to eliminate the algebraic constraints, and thus to reduce the system to an *explicit* form, in the nonlinear case such a conversion from implicit to explicit form is usually fraught with difficulties. Indeed, if the algebraic constraints are nonlinear they need not be analytically solvable (locally or globally). More importantly perhaps, even if they are analytically solvable, then often one would prefer *not* to eliminate the algebraic constraints, because of the complicated and physically not easily interpretable expressions for the reduced system which may arise.

Therefore it is important to extend the framework of port-Hamiltonian systems, as sketched in the previous sections, to the context of *implicit systems*; that is, systems with algebraic constraints.

In order to give the definition of an implicit port-Hamiltonian system (with dissipation) we first consider the notion of a Dirac structure, formalizing the concept of a power-conserving interconnection, and generalizing the notion of a structure matrix $J(x)$ as encountered before.

3.1 Power-conserving interconnections

Let us return to the basic setting of passivity, starting with a finite-dimensional linear space and its dual, in order to define *power*. Thus, let $\mathcal{F}$ be an ℓ-dimensional linear space, and denote its dual (the space of linear functions on $\mathcal{F}$) by $\mathcal{F}^*$. The product space $\mathcal{F} \times \mathcal{F}^*$ is considered to be the space of power variables, with power defined by

$$P =< f^*|f >, \quad (f, f^*) \in \mathcal{F} \times \mathcal{F}^*, \tag{45}$$

where $< f^*|f >$ denotes the duality product, that is, the linear function $f^* \in \mathcal{F}^*$ acting on $f \in \mathcal{F}$. Often we call $\mathcal{F}$ the space of *flows* f, and $\mathcal{F}^*$ the space of *efforts* e, with the power of an element $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ denoted as $< e|f >$.

Remark 3.1. If $\mathcal{F}$ is endowed with an *inner product* structure $<,>$, then $\mathcal{F}^*$ can be naturally *identified* with $\mathcal{F}$ in such a way that $< e|f >=< e, f >,\ f \in \mathcal{F},\ e \in \mathcal{F}^* \simeq \mathcal{F}$.

Example 3.2. Let $\mathcal{F}$ be the space of generalized *velocities*, and $\mathcal{F}^*$ be the space of generalized *forces*, then $< e|f >$ is mechanical power. Similarly, let $\mathcal{F}$ be the space of *currents*, and $\mathcal{F}^*$ be the space of *voltages*, then $< e|f >$ is electrical power.

There exists on $\mathcal{F} \times \mathcal{F}^*$ a canonically defined symmetric bilinear form

$$< (f_1, e_1), (f_2, e_2) >_{\mathcal{F}\times\mathcal{F}^*} :=< e_1|f_2 > + < e_2|f_1 > \tag{46}$$

for $f_i \in \mathcal{F},\ e_i \in \mathcal{F}^*, i = 1, 2$. Now consider a linear subspace

$$S \subset \mathcal{F} \times \mathcal{F}^* \tag{47}$$

and its orthogonal complement with respect to the bilinear form $<,>_{\mathcal{F}\times\mathcal{F}^*}$ on $\mathcal{F}\times\mathcal{F}^*$, denoted as

$$S^\perp \subset \mathcal{F} \times \mathcal{F}^*. \tag{48}$$

Clearly, if S has dimension d, then the subspace $S^\perp$ has dimension $2\ell - d$. (Since dim $(\mathcal{F} \times \mathcal{F}^*) = 2\ell$, and $<,>_{\mathcal{F}\times\mathcal{F}^*}$ is a non-degenerate form.)

Definition 3.3. [9, 12, 11] A constant Dirac structure on $\mathcal{F}$ is a linear subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ such that

$$\mathcal{D} = \mathcal{D}^\perp \tag{49}$$

It immediately follows that the dimension of any Dirac structure $\mathcal{D}$ on an ℓ-dimensional linear space is equal to ℓ. Furthermore, let $(f,e) \in \mathcal{D} = \mathcal{D}^{\perp}$. Then by (46)

$$0 =< (f,e),(f,e) >_{\mathcal{F}\times\mathcal{F}^*}= 2 < e|f > . \tag{50}$$

Thus for all $(f,e) \in \mathcal{D}$ we obtain

$$< e \mid f >= 0. \tag{51}$$

Hence a Dirac structure $\mathcal{D}$ on $\mathcal{F}$ defines a power-conserving relation between the power variables $(f,e) \in \mathcal{F} \times \mathcal{F}^*$.

Remark 3.4. The condition dim $\mathcal{D} =$ dim $\mathcal{F}$ is intimately related to the usually expressed statement that a physical interconnection can *not* determine at the same time both the flow and effort (e.g. current *and* voltage, or velocity *and* force).

Constant Dirac structures admit different *matrix representations*. Here we just list a number of them, without giving proofs and algorithms to convert one representation into another, see e.g. [11].
Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$, with dim $\mathcal{F} = \ell$, be a constant Dirac structure. Then $\mathcal{D}$ can be represented as

1. (*Kernel and Image representation*, [11, 51]).

$$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* | Ff + Ee = 0\} \tag{52}$$

for $\ell \times \ell$ matrices F and E satisfying

$$\begin{array}{ll} (i) & EF^T + FE^T = 0 \\ \\ (ii) & \text{rank } [F \vdots E] = \ell \end{array} \tag{53}$$

Equivalently,

$$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* | f = E^T\lambda,\ e = F^T\lambda,\ \lambda \in \mathbb{R}^\ell\} \tag{54}$$

2. (*Constrained input-output representation*, [11]).

$$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* | f = Je + G\lambda,\ G^Te = 0\} \tag{55}$$

for an $\ell \times \ell$ skew-symmetric matrix J, and a matrix G such that ImG $=$ $\{f|(f,0) \in \mathcal{D}\}$. Furthermore, Ker$J = \{e|(0,e) \in \mathcal{D}\}$.

3. (*Hybrid input-output representation*, [6]).
Let $\mathcal{D}$ be given as in (52). Suppose rank $F = \ell^1 (\leq \ell)$. Select ℓ^1 independent columns of F, and group them into a matrix F^1. Write (possibly after permutations) $F = [F^1 \vdots F^2]$ and, correspondingly $E = [E^1 \vdots E^2]$, $f = \begin{bmatrix} f^1 \\ f^2 \end{bmatrix}, e = \begin{bmatrix} e^1 \\ e^2 \end{bmatrix}$. Then the matrix $[F^1 \vdots E^2]$ can be shown to be invertible, and

$$\mathcal{D} = \left\{ \begin{pmatrix} f^1 \\ f^2 \end{pmatrix}, \begin{pmatrix} e^1 \\ e^2 \end{pmatrix} \middle| \begin{pmatrix} f^1 \\ e^2 \end{pmatrix} = J \begin{pmatrix} e^1 \\ f^2 \end{pmatrix} \right\} \tag{56}$$

with $J := -\left[F^1 \vdots E^2\right]^{-1} \left[F^2 \vdots E^1\right]$ skew-symmetric.

4. (*Canonical coordinate representation*, [9]).
   There exist linear coordinates (q,p,r,s) for $\mathcal{F}$ such in these coordinates and dual coordinates for $\mathcal{F}^*$, $(f,e) = (f_q, f_p, f_r, f_s, e_q, e_p, e_r, e_s) \in \mathcal{D}$ if and only if

$$\begin{cases} f_q = e_p, & f_p = -e_q \\ f_r = 0, & e_s = 0 \end{cases} \tag{57}$$

Example 3.5. Kirchhoff's laws are an example of (52), taking $\mathcal{F}$ the space of currents and $\mathcal{F}^*$ the space of voltages.

Given a Dirac structure $\mathcal{D}$ on $\mathcal{F}$, the following subspaces of $\mathcal{F}$, respectively $\mathcal{F}^*$, are of importance

$$\begin{aligned} G_1 &:= \{f \in \mathcal{F} \mid \exists e \in \mathcal{F}^* \text{ s.t. } (f,e) \in \mathcal{D}\} \\ P_1 &:= \{e \in \mathcal{F}^* \mid \exists f \in \mathcal{F} \text{ s.t. } (f,e) \in \mathcal{D}\} \end{aligned} \tag{58}$$

The subspace G_1 expresses the set of admissible flows, and P_1 the set of admissible efforts. It follows from the image representation (54) that

$$\begin{aligned} G_1 &= \text{Im } E^T \\ P_1 &= \text{Im } F^T \end{aligned} \tag{59}$$

3.2 Implicit port-Hamiltonian systems

From a network modeling perspective a (lumped-parameter) physical system is naturally described by a set of (possibly multi-dimensional) *energy-storing* elements, a set of *energy-dissipating* or *resistive* elements, and a set of *ports* (by which interaction with the environment can take place), interconnected to each other by a *power-conserving interconnection*, see Figure 4.

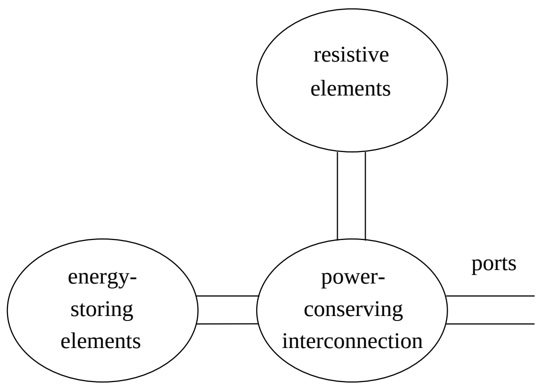

Figure 4: Implicit port-Hamiltonian system with dissipation

Here the power-conserving interconnection also includes power-conserving elements like (in the electrical domain) transformers, gyrators, or (in the mechanical domain) transformers,

kinematic pairs and kinematic constraints.
Associated with the energy-storing elements are energy-variables $x_1, \cdots, x_n$, being coordinates for some n-dimensional state space manifold $\mathcal{X}$, and a total energy $H : \mathcal{X} \to \mathbb{R}$. The power-conserving interconnection is formalized in first instance (see later on for the non-constant case) by a constant Dirac structure $\mathcal{D}$ on the finite-dimensional linear space $\mathcal{F} := \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_P$, with $\mathcal{F}_S$ denoting the space of flows f_S connected to the energy-storing elements, $\mathcal{F}_R$ denoting the space of flows f_R connected to the dissipative (resistive) elements, and $\mathcal{F}_P$ the space of external flows f_P which can be connected to the environment. Dually, we write $\mathcal{F}^* = \mathcal{F}_S^* \times \mathcal{F}_R^* \times \mathcal{F}_P^*$, with $e_S \in \mathcal{F}_S^*$ the efforts connected to the energy-storing elements, $e_R \in \mathcal{F}_R^*$ the efforts connected to the resistive elements, and $e_P \in \mathcal{F}_P^*$ the efforts to be connected to the environment of the system.

The flow variables of the energy-storing elements are given as $\dot{x}(t) = \frac{dx}{dt}(t), t \in \mathbb{R}$, and the effort variables of the energy-storing elements as $\frac{\partial H}{\partial x}(x(t))$ (implying that $< \frac{\partial H}{\partial x}(x(t)) | \dot{x}(t) >= \frac{dH}{dt}(x(t))$ is the increase in energy). In order to have a consistent sign convention for energy flow we put

$$\begin{aligned} f_S &= -\dot{x} \\ e_S &= \frac{\partial H}{\partial x}(x) \end{aligned} \tag{60}$$

Similarly, restricting to *linear* resistive elements as in (32), the flow and effort variables connected to the resistive elements are related as

$$f_R = -Re_R \tag{61}$$

for some matrix $R = R^T \geq 0$.

Substitution of (60) and (61) into the Dirac structure $\mathcal{D}$ leads to the following geometric description of the dynamics

$$(f_S = -\dot{x}, f_R = -Re_R, f_P, e_S = \frac{\partial H}{\partial x}(x), e_R, e_P) \in \mathcal{D} \tag{62}$$

We call (62) an *implicit port-Hamiltonian system* (with dissipation), defined with respect to the constant Dirac structure $\mathcal{D}$, the Hamiltonian H and the resistive structure R.

An equational representation of an implicit port-Hamiltonian system is obtained by taking a matrix representation of the Dirac structure $\mathcal{D}$ as discussed in the previous subsection. For example, in kernel representation the Dirac structure on $\mathcal{F} = \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_P$ may be given as

$$\begin{aligned} \mathcal{D} = \{ & (f_S, f_R, f_P, e_S, e_R, e_P) \mid \\ & F_S f_S + E_S e_S + F_R f_R + E_R e_R + F_P f_P + E_P e_P = 0 \} \end{aligned} \tag{63}$$

for certain matrices $F_S, E_S, F_R, E_R, F_P, E_P$ satisfying

$$\begin{aligned} (i) \quad & E_S F_S^T + F_S E_S^T + E_R F_R^T + F_R E_R^T + E_P F_P^T + F_P E_P^T = 0 \\ (ii) \quad & \text{rank} \left[F_S \vdots F_R \vdots F_P \vdots E_S \vdots E_R \vdots E_P \right] = \dim \mathcal{F} \end{aligned} \tag{64}$$

Then substitution of (60) and (61) into (63) yields the following set of differential-algebraic equations for the implicit port-Hamiltonian system

$$F_S \dot{x}(t) = E_S \frac{\partial H}{\partial x}(x(t)) - F_R R e_R + E_R e_R + F_P f_P + E_P e_P, \tag{65}$$

Different representations of the Dirac structure $\mathcal{D}$ lead to different representations of the implicit port-Hamiltonian system, and this freedom may be exploited for simulation and analysis.

Actually, for many purposes this definition of port-Hamiltonian system is not general enough, since often the Dirac structure is not constant, but *modulated* by the state variables x. In this case the matrices $F_S, E_S, F_R, E_R, F_P, E_P$ in the kernel representation depend (smoothly) on x, leading to the implicit port-Hamiltonian system

$$\begin{array}{l} F_S(x(t))\dot{x}(t) = E_S(x(t))\frac{\partial H}{\partial x}(x(t)) - F_R(x(t))Re_R(t) \\ \\ +E_R(x(t))e_R(t) + F_P(x(t))f_P(t) + E_P(x(t))e_P(t), \quad t \in \mathbb{R} \end{array} \tag{66}$$

with

$$\begin{array}{l} E_S(x)F_S^T(x) + F_S(x)E_S^T(x) + E_R(x)F_R^T(x) + F_R(x)E_R^T(x) \\ \\ +\ E_P(x)F_P^T(x) + F_P(x)E_P^T(x) = 0, \quad \forall x \in \mathcal{X} \\ \\ \text{rank } \left[F_S(x) \vdots F_R(x) \vdots F_P(x) \vdots E_S(x) \vdots E_R(x) \vdots E_P(x) \right] = \dim \mathcal{F} \end{array} \tag{67}$$

Remark 3.6. Strictly speaking the flow and effort variables $\dot{x}(t) = -f_S(t)$, respectively $\frac{\partial H}{\partial x}(x(t)) = e_S(t)$, are not living in a constant linear space $\mathcal{F}_S$, respectively $\mathcal{F}_S^*$, but instead in the tangent spaces $T_{x(t)}\mathcal{X}$, respectively co-tangent spaces $T^*_{x(t)}\mathcal{X}$, to the state space manifold $\mathcal{X}$. This is formalized in the definition of a *non-constant Dirac structure on a manifold*; see the references [9, 12, 11, 47].

It can be checked that the definition of a port-Hamiltonian system as given in (33) is a special case of (66), see [47]. By the power-conservation property of a Dirac structure (cf. (51)) it follows directly that any implicit port-Hamiltonian system satisfies the energy-balance

$$\begin{array}{rcl} \frac{dH}{dt}(x(t)) & = & < \frac{\partial H}{\partial x}(x(t)) | \dot{x}(t) >= \\ \\ & = & -e_R^T(t)Re_R(t) + e_P^T(t)f_P(t), \end{array} \tag{68}$$

as was the case for an (explicit) port-Hamiltonian system (33).

The *algebraic constraints* that are present in the implicit system (66) are expressed by the subspace P_1, and the Hamiltonian H. In fact, since the Dirac structure $\mathcal{D}$ is modulated by the x-variables, also the subspace P_1 is modulated by the x-variables, and thus the effort variables e_S, e_R and e_P necessarily satisfy

$$(e_S, e_R, e_P) \in P_1(x), \quad x \in \mathcal{X}, \tag{69}$$

or, because of (59),

$$e_S \in \text{ Im } F_S^T(x), e_R \in \text{ Im } F_R^T(x), e_P \in \text{ Im } F_P^T(x). \tag{70}$$

The second and third inclusions entail the expression of e_R and e_P in terms of the other variables, while the first inclusion determines, since $e_S = \frac{\partial H}{\partial x}(x)$, the following algebraic constraints on the state variables

$$\frac{\partial H}{\partial x}(x) \in \text{ Im } F_S^T(x). \tag{71}$$

Remark 3.7. Under certain non-degeneracy conditions the elimination of the algebraic constraints (71) for an implicit port-Hamiltonian system (62) can be shown to result in an *explicit* port-Hamiltonian system.

The *Casimir functions* $C : \mathcal{X} \to \mathbb{R}$ of the implicit system (66) are determined by the subspace $G_1(x)$. Indeed, necessarily $(f_S, f_R, f_P) \in G_1(x)$, and thus by (59)

$$f_S \in \text{ Im } E_S^T(x), f_R \in \text{ Im } E_R^T(x), f_P \in \text{ Im } E_P^T(x). \tag{72}$$

Since $f_S = \dot{x}(t)$, the first inclusion yields the *flow constraints*

$$\dot{x}(t) \in \text{ Im } E_S^T(x(t)), \quad t \in \mathbb{R}. \tag{73}$$

Thus $C : \mathcal{X} \to \mathbb{R}$ is a Casimir function if $\frac{dC}{dt}(x(t)) = \frac{\partial^T C}{\partial x}(x(t))\dot{x}(t) = 0$ for all $\dot{x}(t) \in \text{ Im } E_S^T(x(t))$. Hence $C : \mathcal{X} \to \mathbb{R}$ is a Casimir of the implicit port-Hamiltonian system (62) if it satisfies the set of p.d.e.'s

$$\frac{\partial C}{\partial x}(x) \in \text{ Ker } E_S(x) \tag{74}$$

Remark 3.8. Note that $C : \mathcal{X} \to \mathbb{R}$ satisfying (74) is a Casimir function of (62) in a *strong sense*: it is a dynamical invariant $\left(\frac{dC}{dt}(x(t)) = 0\right)$ for *every* port behavior and every resistive relation (61).

Example 3.9. [11, 52] The constrained Hamiltonian equations (22) can be viewed as an implicit port-Hamiltonian system, with respect to the Dirac structure $\mathcal{D}$, given in constrained input-output representation (55) by

$$\begin{aligned} \mathcal{D} = \{(f_S, f_P, e_S, e_P) | 0 = A^T(q)e_S,\ e_P = B^T(q)e_S, \\ -f_S = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} e_S + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_P,\ \lambda \in \mathbb{R}^r\} \end{aligned} \tag{75}$$

In this case, the algebraic constraints on the state variables (q, p) are given as

$$0 = A^T(q)\frac{\partial H}{\partial p}(q, p) \tag{76}$$

while the Casimir functions C are determined by the equations

$$\frac{\partial^T C}{\partial q}(q)\dot{q} = 0, \qquad \text{for all } \dot{q} \text{ satisfying } A^T(q)\dot{q} = 0. \tag{77}$$

Hence, finding Casimir functions amounts to *integrating* the kinematic constraints $A^T(q)\dot{q} = 0$. In particular, if the kinematic constraints are *holonomic*, and thus can be expressed as (19), then $\bar{q}_{k-r+1}, \cdots, \bar{q}_k$ generate all the Casimir functions. □

Remark 3.10. For a proper notion of *integrability* of non-constant Dirac structures, generalizing the integrability conditions (12) of the structure matrix $J(x)$, we refer e.g. to [11].

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