

Network Modeling and Control of Physical Systems,
DISC

Theory of Port-Hamiltonian systems

Chapter 3: Compositional modelling of
distributed-parameter systems

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1 Introduction

The Hamiltonian formulation of distributed-parameter systems has been a challenging research area for quite some time. (A nice introduction, especially with respect to systems stemming from fluid dynamics, can be found in [26], where also a historical account is provided.) The identification of the underlying Hamiltonian structure of sets of p.d.e.'s has been instrumental in proving all sorts of results on integrability, the existence of soliton solutions, stability, reduction, etc., and in *unifying* existing results, see e.g. [11], [24], [18], [17], [25], [14].

Recently, there has been also a surge of interest in the *design* and *control* of nonlinear distributed-parameter systems, motivated by various applications. At the same time, it is well-known from *finite-dimensional* nonlinear control systems [35], [32], [6], [21], [28], [27], [34] a Hamiltonian formulation is helpful in the control design, and the same is to be expected in the distributed-parameter case. However, in extending the theory as for instance exposed in [26] to distributed-parameter *control* systems a fundamental difficulty arises in the treatment of *boundary conditions*. Indeed, the treatment of infinite-dimensional Hamiltonian systems in the literature is mostly focussed on systems with infinite spatial domain, where the variables go to zero for the spatial variables tending to infinity, or on systems with boundary

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conditions such that the energy exchange through the boundary is *zero*. On the other hand, from a control and interconnection point of view it is quite essential to be able describe a distributed-parameter system with varying boundary conditions inducing *energy exchange through the boundary*, since in many applications the interaction with the environment (e.g. actuation or measurement) will actually take place through the boundary of the system. Clear examples are the telegraph equations (describing the dynamics of a transmission line), where the boundary of the system is described by the behavior of the voltages and currents at both ends of the transmission line, or a vibrating string (or, more generally, a flexible beam), where it is natural to consider the evolution of the forces and velocities at the ends of the string. Furthermore, in both examples it is obvious that in general the boundary exchange of power (voltage times current in the transmission line example, and force times velocity for the vibrating string) will be non-zero, and that in fact one would like to consider the voltages and currents or forces and velocities as additional *boundary variables* of the system, which can be interconnected to other systems. Also for numerical integration and *simulation* of complex distributed-parameter systems it is essential to be able to describe the complex system as the interconnection or coupling of its subsystems via their boundary variables; for example in the case of coupled fluid-solid dynamics.

From a mathematical point of view, it is not obvious how to incorporate non-zero energy flow through the boundary in the existing Hamiltonian framework for distributed-parameter systems. The problem is already illustrated by the Hamiltonian formulation of e.g. the Korteweg-de Vries equation (see e.g. [26]). Here for zero boundary conditions a *Poisson bracket* can be formulated with the use of the differential operator $\frac{d}{dx}$, since by integration by parts this operator is obviously skew-symmetric. However, for boundary conditions corresponding to non-zero energy flow the differential operator is not skew-symmetric anymore (since after integrating by parts the remainders are not zero).

In [37], see also [20], we proposed a framework to overcome this fundamental problem by using the notion of a *Dirac structure*. Dirac structures were originally introduced in [5],[7] as a geometric structure generalizing both *symplectic* and *Poisson* structures. Later on (see e.g. [35], [6], [19], [2]) it was realized that in the finite-dimensional case Dirac structures can be naturally employed to formalize Hamiltonian systems with *constraints* as *implicit* Hamiltonian systems. It turns out that in order to allow the inclusion of boundary variables in distributed-parameter systems the concept of Dirac structure again provides the right type of generalization with respect to the existing framework using Poisson structures.

The Dirac structure for distributed-parameter systems employed in this paper has a specific form by being defined on certain spaces of differential forms on the spatial domain of the system and its boundary, and making use of Stokes' theorem. Its construction emphasizes the geometrical content of the physical variables involved, by identifying them as differential k -forms, for appropriate k . This interpretation is rather well-known (see e.g. [12]) in the case of Maxwell's equations (and actually directly follows from Faraday's law and Ampère's law), but seems less well-known for the telegraph equations and the description of the Euler's equations for an ideal isentropic fluid.

From the systems and control point of view the approach taken in this paper can be seen as providing the extension of the port-Hamiltonian framework established for lumped-parameter systems in [35], [6], [27], [33], [35], [34], [3] to the distributed-parameter case. In the lumped-parameter case this Hamiltonian framework has been successfully employed in the consistent (modular) modeling and simulation of complex *interconnected* lumped-parameter physical systems, including (actuated) multi-body systems with kinematic constraints and electro-mechanical systems [35], [19], [6], [34], and in the design and *control* of such systems, exploiting the Hamiltonian and passivity structure in a crucial way [32], [21], [28], [27], [34]. Similar developments can be pursued in the distributed-parameter case; see already [30], [36] for developments in this direction.

The present paper is organized as follows. In Section 2 we give a general introduction to systems of conservation laws, together with the closure equations relating the conserved quantities to the flux variables. Furthermore, we show how this leads to infinite-dimensional power-continuous interconnection structures and the definition of Hamiltonian functions for energy storage. After this general introduction the main mathematical framework is given in Section 3 and 4, following [37]. In Section 3 it is shown how the notion of a power-continuous interconnection structure as discussed before can be formalized using the geometric concept of a Dirac structure, and in particular the Stokes-Dirac structure. In Section 4 it is shown how this leads to the Hamiltonian formulation of distributed-parameter systems with boundary energy flow, generalizing the notion of finite-dimensional port-Hamiltonian systems. In Section 5 (again following [37]) this is applied to Maxwell's equations on a bounded domain (Subsection 5.1), the telegraph equations for an ideal transmission line (Subsection 5.2), and the vibrating string (Subsection 5.3). Furthermore, by modifying the Stokes-Dirac structure with an additional term corresponding to three-dimensional convection, Euler's equations for an ideal isentropic fluid are studied in Section 6. Section 7 treats the basic notions of Casimir functions determined by the Stokes-Dirac structure. This can be seen as a starting point for control by interconnection of distributed-parameter port-Hamiltonian systems. Finally, Section 8 contains the conclusions.

2 Systems of two physical domains in canonical interaction

The aim of this section is to introduce a class of infinite-dimensional physical systems and to show how they can be represented as port-Hamiltonian systems defined with respect to a special type of infinite-dimensional Dirac structure, called Stokes-Dirac structure. This will be done by formulating the distributed-parameter system as a system of conservation laws [10] [31], each describing the balance equation associated with some conserved physical quantity, coupled with a set of closure equations. These balance laws will define the Stokes-Dirac structure, while the closure equations will turn out to be equivalent with the definition of the Hamiltonian of the system.

2.1 Conservation laws, interdomain coupling and boundary energy flows: motivational examples

In this paragraph we shall introduce the main concepts of conservation law, interdomain coupling and boundary energy flow by means of three simple and classical examples of distributed-parameter systems.

The first example is the simplest one, and consists of only one conservation law on a one-dimensional spatial domain. With the aid of this simple example we shall introduce the notions of conservation law, balance equation, variational derivative, finally leading to the definition of a port-Hamiltonian system.

Example 2.1 (The inviscid Burger's equation). The *viscous Burger's equation* is a scalar parabolic equation which represents the simplest model for a fluid flow (often used as a numerical test for the asymptotic theory of the Navier-Stokes equations) [31]. It is defined on a one-dimensional spatial domain (an interval) $Z = [a, b] \subset \mathbb{R}$, while its state variable is $\alpha(z, t)$ $z \in Z$, $t \in I$, where I is an interval of \mathbb{R} satisfying the partial differential equation

$$\frac{\partial \alpha}{\partial t} + \alpha \frac{\partial \alpha}{\partial z} - \nu \frac{\partial^2 \alpha}{\partial z^2} = 0 \quad (1)$$

In the following we shall consider the *inviscid* Burger's equations (corresponding to the case $\nu = 0$), which may be alternatively expressed by the following *conservation law*:

$$\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial z} \beta = 0 \quad (2)$$

where the state variable $\alpha(z, t)$ is called the *conserved quantity* and the function $\beta(z, t)$ is called *the flux variable* and is given by $\beta = \frac{\alpha^2}{2}$. Indeed, integrating the partial differential equation (2) for $\nu = 0$ on the interval Z , one obtains the following *balance equation*:

$$\frac{d}{dt} \int_a^b \alpha dz = \beta(a) - \beta(b) \quad (3)$$

Furthermore, according to the framework of Irreversible Thermodynamics [29], one may express the flux β as a function of the *generating force* which is the *variational derivative* (or, functional derivative,) of some generating functional $H(\alpha)$ of the state variable. This variational derivative plays the same role as the gradient of a function when considering functionals instead of functions. The variational derivative $\frac{\delta H}{\delta \alpha}$ of the functional $H(\alpha)$ is uniquely defined by the requirement:

$$H(\alpha + \epsilon \eta) = H(\alpha) + \epsilon \int_a^b \frac{\delta H}{\delta \alpha} \eta dz + O(\epsilon^2) \quad (4)$$

for any $\epsilon \in \mathbb{R}$ and any smooth function $\eta(z, t)$ such that $\alpha + \epsilon \eta$ satisfies the same boundary conditions as α [26]. For the inviscid Burger's equation it is easy to see that $\beta = \frac{\alpha^2}{2}$ can be expressed as $\beta = \frac{\delta H}{\delta \alpha}$, where

$$H(\alpha) = \int_a^b \frac{\alpha^3}{6} dz \quad (5)$$

Hence the inviscid Burger's equation may be also expressed as

$$\frac{\partial \alpha}{\partial t} = -\frac{\partial}{\partial z} \frac{\delta H}{\delta \alpha} \quad (6)$$

This defines an infinite-dimensional Hamiltonian system [26] with respect to the skew-symmetric operator $\frac{\partial}{\partial z}$ (defined on the functions with support strictly contained in the interval Z).

From this formulation one immediately derives that the Hamiltonian $H(\alpha)$ is *another* conserved quantity. Indeed, by integration by parts

$$\frac{d}{dt} H = \int_a^b \frac{\delta H}{\delta \alpha} \cdot \frac{\partial \alpha}{\partial t} dz = \int_a^b \frac{\delta H}{\delta \alpha} \cdot -\frac{\partial}{\partial z} \frac{\delta H}{\delta \alpha} dz = (\beta^2(a) - \beta^2(b)) \quad (7)$$

Here it is worth to notice that the time variation of the Hamiltonian functional is a *quadratic function of the flux variables* evaluated at the boundaries of the spatial domain Z .

The second example, the *p-system*, is a classical example that we shall use in order to introduce the concept of an infinite-dimensional *port-Hamiltonian system*. It corresponds to the case of two physical domains in interaction and consists of a system of *two* conservations laws.

Example 2.2 (The p-system). The p-system is a model for a *1-dimensional isentropic gas dynamics in Lagrangian coordinates*. The independent variable z belong to an interval $Z \subset \mathbb{R}$, It is defined with the following variables: the specific volume $v(z, t) \in \mathbb{R}^+$, the velocity $u(z, t)$ and the pressure functional $p(v)$ (which is for instance in the case of a polytropic isentropic ideal gas given by $p(v) = A v^{-\gamma}$ where $\gamma \geq 1$). The *p-system* is then defined by the following system of partial differential equations:

$$\begin{aligned} \frac{\partial v}{\partial t} - \frac{\partial u}{\partial z} &= 0 \\ \frac{\partial u}{\partial t} + \frac{\partial p(v)}{\partial z} &= 0 \end{aligned} \quad (8)$$

representing the conservation of mass and of momentum. By defining the state vector as: $\alpha(z, t) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} v \\ u \end{pmatrix}$ and the vector valued flux $\beta(z, t) = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} -u \\ p(v) \end{pmatrix}$ the p-system is rewritten as

$$\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial z} \beta = 0 \quad (9)$$

Again, according to the framework of Irreversible Thermodynamics, the flux variables may be written as functions of the variational derivatives of some generating functionals. Consider the functional $H(\alpha) = \int_a^b \mathcal{H}(v, u) dz$ where $\mathcal{H}(v, u)$ denotes the energy density, which is given as the sum of the internal energy and the kinetic energy densities

$$\mathcal{H}(v, u) = \mathcal{U}(v) + \frac{u^2}{2}, \quad (10)$$

where $-\mathcal{U}(v)$ is a primitive function of the pressure. Note that the expression of the kinetic energy does not depend on the mass density which is assumed to be constant and for simplicity is set equal to 1. Hence no difference is made between the velocity and the momentum. The vector of fluxes β may now be expressed in term of the generating forces as follows

$$\beta = \begin{pmatrix} -\frac{\delta H}{\delta u} \\ -\frac{\delta H}{\delta v} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta v} \\ \frac{\delta H}{\delta u} \end{pmatrix} \quad (11)$$

The anti-diagonal matrix represents the canonical coupling between two physical domains: the kinetic and the potential (internal) domain (for lumped parameter systems this is discussed e.g. in [4]). The variational derivative of the total energy with respect to the state variable of one domain generates the flux variable for the other domain.

Combining the equations (9) and (11), the p-system may thus be written as the following Hamiltonian system:

$$\frac{\partial \alpha}{\partial t} = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \alpha_1} \\ \frac{\delta H}{\delta \alpha_2} \end{pmatrix} \quad (12)$$

From the Hamiltonian form of the system and using again integration by parts, one may derive that the total energy obeys the following *power balance equation*:

$$\frac{d}{dt} H = \beta_1(a) \beta_2(a) - \beta_1(b) \beta_2(b) \quad (13)$$

Notice again that the right-hand side of this power-balance equation is a quadratic function of the fluxes at the boundary of the spatial domain.

Remark 2.3. It is important to note that any non-linear wave equation:

$$\frac{\partial^2 g}{\partial t^2} - \frac{\partial}{\partial z} \left(\sigma \left(\frac{\partial g}{\partial z} \right) \right) = 0$$

may be expressed as a p-system using the change of variables $u = \frac{\partial g}{\partial t}$, $v = \frac{\partial g}{\partial z}$ and $p(v) = -\sigma(v)$.

The last example is the *vibrating string*. Actually it is again a system of two conservation laws representing the canonical interdomain coupling between the kinetic energy and the elastic potential energy. However in this example, unlike the p-system, the *classical* choice of the state variables leads to express the total energy as a function of some of the *spatial derivatives* of the state variables. We shall analyze how the dynamic equations and the power balance are expressed in this case and we shall subsequently draw some conclusions on the choice of the state variables.

Example 2.4 (Vibrating string). Consider an elastic string subject to traction forces at its ends. The spatial variable z belongs to the interval $Z = [a, b] \subset \mathbb{R}$. Denote by $u(t, z)$ the displacement of the string and the velocity by $v(z, t) = \frac{\partial u}{\partial t}$.

Using the vector of state variables $x(z, t) = (u, v)^T$, the dynamics of the vibrating string is described by the system of partial differential equations

$$\frac{\partial x}{\partial t} = \begin{pmatrix} v \\ \frac{1}{\mu} \frac{\partial}{\partial z} \left(T \frac{\partial u}{\partial z} \right) \end{pmatrix} \quad (14)$$

where the first equation is simply the definition of the velocity and the second one is Newton's second law.

The time variation of the state may be expressed as a function of the variational derivative of the total energy as in the preceding examples. Indeed, define the total energy as $H(x) = U(u) + K(v)$, where U denotes the elastic potential energy and K the kinetic energy of the string. The elastic potential energy is given as a function of the *strain* $\epsilon(t, z) = \frac{\partial u}{\partial z}$

$$U(u) = \int_a^b \frac{1}{2} T \left(\frac{\partial u}{\partial z} \right)^2 dz \quad (15)$$

with T the elasticity modulus. The kinetic energy K is the following function of the velocity $v(z, t) = \frac{\partial u}{\partial t}$

$$K(v) = \int_a^b \frac{1}{2} \mu v(z, t)^2 dz \quad (16)$$

Thus the total system (14) may be expressed as

$$\frac{\partial x}{\partial t} = \begin{pmatrix} 0 & \frac{1}{\mu} \\ -\frac{1}{\mu} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} \quad (17)$$

where according to the definition of the variational derivative given in (4) one obtains

$$\frac{\delta H}{\delta u} = \frac{\delta U}{\delta u} = -\frac{\partial}{\partial z} \left(T \frac{\partial u}{\partial z} \right) \quad (18)$$

which is the elastic force and

$$\frac{\delta H}{\delta v} = \frac{\delta K}{\delta v} = \mu v \quad (19)$$

which is the momentum.

In the formulation of equation (17) there appears again an anti-diagonal skew-symmetric matrix which corresponds to the expression of a canonical interdomain coupling between the elastic energy domain and the kinetic energy domain. However the system is *not* expressed as a system of conservation laws since the rate of change of the state variables is a linear combination of the variational derivatives directly (and not of their spatial derivatives). Instead of being a simplification, this reveals a drawback for the case that there is energy flow through the boundary of the spatial domain. Indeed in this case, the *variational derivative has to be completed by a boundary term* since the Hamiltonian functional depends on the *spatial derivatives of the state*. For the elastic potential energy this becomes (integration by parts)

$$U(u + \epsilon \eta) = U(u) - \epsilon \int_a^b \frac{\partial}{\partial z} \left(T \frac{\partial u}{\partial z} \right) \eta dz + \epsilon \left[\eta \left(T \frac{\partial u}{\partial z} \right) \right]_a^b + O(\epsilon^2) \quad (20)$$

On the other hand, writing the system (14) as a second order equation yields the wave equation

$$\mu \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z} \left(T \frac{\partial u}{\partial z} \right) \quad (21)$$

which according to Remark 2.3 may be alternatively expressed as a p-system.

In the sequel we shall formulate the vibrating string as a system of two conservation laws, which is however slightly different from the p-system formulated before. It differs from the p-system by the choice of the state variables in such a way that, first, the mass density may depend on the spatial variable z (which is not the case in the Hamiltonian density function defined in equation (10)), and secondly, that the variational derivatives of the total energy equal the co-energy variables.

Indeed, we take as vector of state variables

$$\alpha(z, t) = \begin{pmatrix} \epsilon \\ p \end{pmatrix} \quad (22)$$

where ϵ denotes the *strain* $\alpha_1 = \epsilon = \frac{\partial u}{\partial z}$ and p denotes the *momentum* $\alpha_2 = p = \mu v$. Recall that in these variables the total energy is written as

$$H_0 = \int_a^b \frac{1}{2} \left(T \alpha_1^2 + \frac{1}{\mu} \alpha_2^2 \right) dz \quad (23)$$

Notice that the energy functional now only depends on the state variables and *not* on their spatial derivatives. Furthermore, one may define the flux variables to be the *stress* $\beta_1 = \frac{\delta H_0}{\delta \alpha_1} = T \alpha_1$ and the *velocity* $\beta_2 = \frac{\delta H_0}{\delta \alpha_2} = \frac{\alpha_2}{\mu}$. In matrix notation, the fluxes are expressed as a function of the generating forces $\frac{\delta H_0}{\delta \alpha}$ by:

$$\beta = \begin{pmatrix} -\frac{\partial H_0}{\partial \epsilon} \\ -\frac{\partial H_0}{\partial p} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H_0}{\delta \alpha_1} \\ \frac{\delta H_0}{\delta \alpha_2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\delta H_0}{\delta \alpha} \quad (24)$$

Thus the model of the vibrating string may be expressed by the system of two conservation laws (as for the p-system):

$$\frac{\partial \alpha}{\partial t} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \frac{\delta H_0}{\delta \alpha} \quad (25)$$

which satisfies also the power balance equation (13).

2.2 Systems of two conservation laws in canonical interaction

In this section we shall consider the general class of distributed-parameter systems consisting of two conservation laws with the canonical coupling presented as in the above examples of the p-system and the vibrating string. In the first part, for 1-dimensional spatial domains, we shall introduce the concept of *interconnection structure* and *port variables* which are fundamental to the definition of port-Hamiltonian systems. On this case we shall also introduce the notion of differential forms. In the

second part we shall give the definition of systems of two conservation laws defined on n -dimensional spatial domains. We do not use the usual vector calculus formulation but express the systems in terms of differential forms [1] [16]. This leads to concise, coordinate independent formulations and unifies the notations for the various physical domains.

2.2.1 Interconnection structure, boundary energy flows and port-based formulation for 1-D spatial domains

Interconnection structure and power continuity

Let us consider the systems of two conservation laws arising from the modelling of two physical domains in canonical interaction as have been presented for the vibrating string and the p-system:

$$\frac{\partial \alpha}{\partial t} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} \frac{\delta H_0}{\delta \alpha} \quad (26)$$

where $\alpha = (\alpha_1(z, t), \alpha_2(z, t))^T$. Let us now define an interconnection structure for this system in the sense of network [13] [4] or port-based modelling [23] [35]. Define the vector of *flow variables* to be the time variation of the state and denote it by:

$$f = \frac{\partial \alpha}{\partial t} \quad (27)$$

Define the vector of *effort variables* e to be the vector of the generating forces given as

$$e = \frac{dH_0}{d\alpha} \quad (28)$$

The flow and effort variables are *power-conjugated* since their product is the time-variation of the total energy:

$$\frac{d}{dt} H_0 = \int_a^b \left(\frac{\partial \mathcal{H}_0}{\partial \alpha_1} \frac{\delta \alpha_1}{\delta t} + \frac{\partial \mathcal{H}_0}{\partial \alpha_2} \frac{\delta \alpha_2}{\delta t} \right) dz = \int_a^b (e_1 f_1 + e_2 f_2) dz \quad (29)$$

where \mathcal{H}_0 denotes the density corresponding to H_0 . Considering the right-hand side of the power balance equation (13) it is clear that the energy exchange of the system with its environment is determined by the flux variables restricted to the boundary of the domain. Therefore let us define two external boundary variables as follows:

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} e_2 \\ e_1 \end{pmatrix} = \begin{pmatrix} \frac{\delta H_0}{\delta \alpha_2} \\ \frac{\delta H_0}{\delta \alpha_1} \end{pmatrix} = \begin{pmatrix} v \\ \sigma \end{pmatrix} \quad (30)$$

These boundary variables are also power-conjugated as their product $\beta_1 \beta_2 = e_b f_b = \sigma v$ equals the right-hand side of the power balance equation (13). Considering the four power-conjugated variables $f_1, f_2, f_\partial, e_1, e_2, e_\partial$, the power balance equation (13) implies that their product is zero:

$$\int_a^b (e_1 f_1 + e_2 f_2) dz + e_\partial(b) f_\partial(b) - e_\partial(a) f_\partial(a) = 0 \quad (31)$$

This bilinear product between the power-conjugated variables is analogous to the product between the circuit variables expressing the *power continuity* relation in circuits and network models [13] [4]. Such products (or *pairings* are also central in the definition of implicit Hamiltonian systems [5] [7] and port-Hamiltonian systems in finite dimensions [35] [19]. In the forthcoming sections we shall show that this product will play the same role for infinite-dimensional port-Hamiltonian systems [20] [37].

The *interconnection structure* underlying the system (26) (analogous to Kirchhoff's laws for circuits) may now be summarized by (30) together with

$$f = \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{pmatrix} e \quad (32)$$

Introduction to differential forms

Let us now introduce for the case of the 1-dimensional spatial domain the use of differential forms in the formulation of systems of conservation laws. Until now we have simply considered the state variables α and the flux variables β as functions on the space-time domain $Z \times I$. However considering the balance equation (3)

$$\frac{d}{dt} \int_a^b \alpha dz = \beta(a) - \beta(b)$$

associated with the conservation law (2) it becomes clear that they are of different nature. The state variables α correspond to conserved quantities through integration, while the flux variables β correspond to functions which can be evaluated at any point (for instance at the boundary points of the spatial domain). This distinction may be expressed by representing the variables as *differential forms*. For the case of one-dimensional spatial domains considered in this paragraph, the state variables are identified with differential forms of degree 1, which can be integrated along one-dimensional curves. The flux variables, on the other hand, are identified with differential forms of degree 0, that means functions evaluated at points of the spatial domain. The reader is referred to the following textbooks [1] [12] [16] for an exhaustive definition of differential forms that we shall use systematically in the rest of the paper.

2.2.2 Interconnection structure, boundary energy flows and port-based formulation for n-dimensional spatial domains

Systems of two conservation laws with canonical interdomain coupling

In this paragraph we shall give the general definition of the class of systems of conservation laws that we shall consider in the forthcoming sections. We first recall the expression of systems of conservation laws defined on n -dimensional spatial domains, and secondly generalize the systems of two conservation laws with canonical interdomain coupling as defined in the previous section 2.2 to the n -dimensional spatial domain.

Define the spatial domain of the considered distributed-parameter system as $Z \in \mathbb{R}^n$ being an n -dimensional smooth manifold with smooth $(n - 1)$ -dimensional boundary ∂Z . Denote by $\Omega^k(Z)$ the vector space of (differential) k -forms on Z (respectively by $\Omega^k(\partial Z)$ the vector space of k -forms on ∂Z). Denote furthermore $\Omega = \bigoplus_{k \geq 0} \Omega^k(Z)$ the algebra of differential forms over Z and recall that it is endowed with an exterior product \wedge and an exterior derivation d [1] [16].

Definition 2.5. A *system of conservation laws* is defined by a set of *conserved quantities* $\alpha_i \in \Omega^{k_i}(Z)$, $i \in \{1, \dots, N\}$ where $N \in \mathbb{N}$, $k_i \in \mathbb{N}$, defining the state space $\mathcal{X} = \bigotimes_{i=1, \dots, N} \Omega^{k_i}(Z)$. They satisfy a set of *conservation laws*

$$\frac{\partial \alpha_i}{\partial t} + d\beta_i = g_i \quad (33)$$

where $\beta_i \in \Omega^{k_i-1}(Z)$ denote the set of *fluxes* and $g_i \in \Omega^{k_i}(Z)$ denote the set of *distributed interaction forms*. Finally, the fluxes β_i are defined by the *closure equations*

$$\beta_i = J(\alpha_i, z), \quad i = 1, \dots, N \quad (34)$$

The integral form of the conservation laws yield the following *balance equations*

$$\frac{d}{dt} \int_Z \alpha_i + \int_{\partial Z} \beta_i = \int_Z g_i \quad (35)$$

Remark 2.6. A common case is that the conserved quantities are 3-forms, that is, the balance equation is evaluated on volumes of the 3-dimensional space. Then, in vector calculus notation, the conserved quantities may be identified with vectors u_i on Z , the interaction terms g_i may also be considered as vectors, and the fluxes may be identified with vectors q_i . In this case the system of conservation laws takes the more familiar form:

$$\frac{\partial u_i}{\partial t}(z, t) + \operatorname{div}_z q_i = g_i, \quad i = 1, \dots, n \quad (36)$$

However, systems of conservation laws may correspond to differential forms of any degree. Maxwell's equations provide a classical example where the conserved quantities are actually differential forms of degree 3 [12].

In the sequel, as in the case of the 1-dimensional spatial domain, we shall consider a particular class of systems of conservation laws where the fluxes, determined by the closure equations, are (linear) functions of the derivatives of some generating function. One may note again that this is in agreement with the general assumptions of irreversible thermodynamics [29] where the flux variables are (eventually nonlinear) functions of the generating forces, being the derivative of some generating functional. More precisely, we shall consider closure equations arising from the description of the canonical interaction of two physical domains (for instance the kinetic and elastic energy in the case of the vibrating string, or the electric and magnetic energy for electromagnetic fields) [20].

First recall the general definition of the *variational derivative* of a functional $H(\alpha)$ with respect to the differential form $\alpha \in \Omega^p(Z)$ (generalizing the definition given before).

Definition 2.7. Consider a density function $\mathcal{H} : \Omega^p(Z) \times Z \rightarrow \Omega^n(Z)$ where $p \in \{1, \dots, n\}$, and denote by $H := \int_Z \mathcal{H} \in \mathbb{R}$ the associated functional. Then the uniquely defined differential form $\frac{\delta H}{\delta \alpha} \in \Omega^{n-p}(Z)$ which satisfies for all $\Delta \alpha \in \Omega^p(Z)$ and $\varepsilon \in \mathbb{R}$:

$$H(\alpha + \varepsilon \Delta \alpha) = \int_Z \mathcal{H}(\alpha + \varepsilon \Delta \alpha) = \int_Z \mathcal{H}(\alpha) + \varepsilon \int_Z \left[\frac{\delta H}{\delta \alpha} \wedge \Delta \alpha \right] + O(\varepsilon^2)$$

is called the *variational derivative* of H with respect to $\alpha \in \Omega^p(Z)$.

Now we define the generalization of the systems presented in the section 2.2 to spatial domains of arbitrary dimension.

Definition 2.8. *Systems of two conservation laws with canonical interdomain coupling* are systems of two conservation laws involving a pair of conserved quantities $\alpha_p \in \Omega^p(Z)$ and $\alpha_q \in \Omega^q(Z)$, differential forms on the n -dimensional spatial domain Z of degree p and q respectively, where the integers p and q satisfy $p + q = n + 1$. The closure equations generated by a *Hamiltonian density function* $\mathcal{H} : \Omega^p(Z) \times \Omega^q(Z) \times Z \rightarrow \Omega^n(Z)$ resulting in the total Hamiltonian $H := \int_Z \mathcal{H} \in \mathbb{R}$ are given by:

$$\begin{pmatrix} \beta_p \\ \beta_q \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & (-1)^r \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \alpha_p} \\ \frac{\delta H}{\delta \alpha_q} \end{pmatrix} \quad (37)$$

where $r = pq + 1$, $\varepsilon \in \{-1, +1\}$ depending on the sign convention of the considered physical domain.

Remark 2.9. The total Hamiltonian $H(\alpha_q, \alpha_p)$ corresponds to the energy function of a physical system, the state variables α_i are called the *energy variables* and the variational derivatives $\frac{\delta H}{\delta \alpha_i}$ are called the *co-energy variables*.

Boundary port variables and the power continuity relation

In the same way as for systems defined on 1-dimensional spatial domains, one may define for n - spatial domains pairs of power conjugated variables. Define the *flow variables* to be the time-variation of the state denoted by

$$\begin{pmatrix} f_p \\ f_q \end{pmatrix} = \begin{pmatrix} \frac{\partial \alpha_p}{\partial t} \\ \frac{\partial \alpha_q}{\partial t} \end{pmatrix} \quad (38)$$

Furthermore, define the vector of *effort variables* to be the vector of the generating forces denoted by

$$\begin{pmatrix} e_p \\ e_q \end{pmatrix} = \begin{pmatrix} \frac{\delta H}{\delta \alpha_p} \\ \frac{\delta H}{\delta \alpha_q} \end{pmatrix} \quad (39)$$

The flow and effort variables are *power-conjugated* as their product is the time-variation of the Hamiltonian function:

$$\frac{dH}{dt} = \int_Z \left(\frac{\delta H}{\delta \alpha_p} \wedge \frac{\partial \alpha_p}{\partial t} + \frac{\delta H}{\delta \alpha_q} \wedge \frac{\partial \alpha_q}{\partial t} \right) = \int_Z (e_p \wedge f_p + e_q \wedge f_q) \quad (40)$$

Using the conservation laws (36), the closure relations (37) and the properties of the exterior derivative and Stokes' theorem, one may write the time-variation of the Hamiltonian as

$$\begin{aligned}
\frac{dH}{dt} &= \int_Z (\varepsilon \beta_q \wedge (-d\beta_p) + (-1)^r \beta_p \wedge \varepsilon(-d\beta_q)) \\
&= -\varepsilon \int_Z (\beta_q \wedge d\beta_p + (-1)^{p+q+1} (-1)^{(p-1)q} \beta_q \wedge d\beta_p) \\
&= -\varepsilon \int_Z (\beta_q \wedge d\beta_p + (-1)^q \beta_q \wedge d\beta_p) \\
&= -\varepsilon \int_{\partial Z} \beta_q \wedge \beta_p
\end{aligned} \tag{41}$$

Finally we define *flow and effort variables on the boundary* of the system as the *restriction* of the flux variables to the boundary ∂Z of the domain Z :

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} \beta_q|_{\partial Z} \\ \beta_p|_{\partial Z} \end{pmatrix} \tag{42}$$

They are also power conjugated variables as their product defined in (42) is the time variation of the Hamiltonian functional (the total energy of the physical system).

On the total space of power-conjugated variables, the differential forms (f_p, e_p) and (f_q, e_q) on the domain Z and the differential forms (f_∂, e_∂) defined on the boundary ∂Z , one may define an *interconnection structure*, underlying the system of two conservation laws with canonical interdomain coupling of Definition 2.8. This interconnection structure is defined by the equation (42) together with (combining the conservation laws (36) with the closure equation (37))

$$\begin{pmatrix} f_q \\ f_p \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & (-1)^r & d \\ d & 0 & \end{pmatrix} \begin{pmatrix} e_q \\ e_p \end{pmatrix} \tag{43}$$

This interconnection is power-continuous in the sense that the power-conjugated variables related by (42) and (43) satisfy the *power continuity relation*:

$$\int_Z (e_p \wedge f_p + e_q \wedge f_q) + \varepsilon \int_{\partial Z} f_\partial \wedge e_\partial = 0 \tag{44}$$

This expression is the straightforward consequence of the two expressions of the variation of the Hamiltonian H in (40) (41).

In the next sections 3 and 4 we shall show how the above power-continuous interconnection structure can be formalized as geometric structure, called Dirac structure, and how this leads to the definition of infinite-dimensional Hamiltonian systems with energy flows at the boundary of their spatial domain, called port-Hamiltonian systems.

3 Stokes-Dirac structures

3.1 Dirac structures

The notion of a Dirac structure was originally introduced in [5], [7] as a geometric structure generalizing both *symplectic* and *Poisson* structures. In e.g. [35], [19],

[33], [2], [6], [34], [3], it was employed as the geometrical notion formalizing general *power-conserving interconnections*, thereby allowing the Hamiltonian formulation of interconnected and constrained mechanical and electrical systems.

A definition of Dirac structures (which is actually slightly more general than the one in [5], [7]) can be given as follows. Let \mathcal{F} and \mathcal{E} be linear spaces, equipped with a pairing, that is, a bilinear operation

$$\mathcal{F} \times \mathcal{E} \rightarrow L \quad (45)$$

with L a linear space. The pairing will be denoted by $\langle e|f \rangle \in L$, $f \in \mathcal{F}$, $e \in \mathcal{E}$. By symmetrizing the pairing we obtain a symmetric bilinear form \ll, \gg on $\mathcal{F} \times \mathcal{E}$, with values in L , defined as

$$\ll (f_1, e_1), (f_2, e_2) \gg := \langle e_1|f_2 \rangle + \langle e_2|f_1 \rangle, \quad (f_i, e_i) \in \mathcal{F} \times \mathcal{E} \quad (46)$$

Definition 3.1. Let \mathcal{F} and \mathcal{E} be linear spaces with a pairing $\langle | \rangle$. A Dirac structure is a linear subspace $D \subset \mathcal{F} \times \mathcal{E}$ such that $D = D^\perp$, with \perp denoting the orthogonal complement with respect to the bilinear form \ll, \gg .

Example 3.2. Let \mathcal{F} be a linear space over \mathbb{R} . Let \mathcal{E} be given as \mathcal{F}^* (the space of linear functionals on \mathcal{F}), with pairing $\langle | \rangle$ the duality product $\langle e|f \rangle \in \mathbb{R}$.

- (a) Let $J : \mathcal{E} \rightarrow \mathcal{F}$ be a skew-symmetric map. Then graph $J \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure.
- (b) Let $\omega : \mathcal{F} \rightarrow \mathcal{E}$ be a skew-symmetric map. Then graph $\omega \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure.
- (c) Let $V \subset \mathcal{F}$ be a finite-dimensional linear subspace. Then $V \times V^{orth} \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure, where $V^{orth} \subset \mathcal{E}$ is the annihilating subspace of V . The same holds if \mathcal{F} is a topological vectorspace, \mathcal{E} is the space of linear continuous functionals on \mathcal{F} , and V is a *closed* subspace of \mathcal{F} .

Example 3.3. Let M be a finite-dimensional manifold. Let $\mathcal{F} = V(M)$ denote the Lie algebra of smooth vector fields on M , and let $\mathcal{E} = \Omega^1(M)$ be the linear space of smooth 1-forms on M . Consider the usual pairing $\langle \alpha|X \rangle = i_X \alpha$ between 1-forms α and vectorfields X ; implying that L is the linear space of smooth functions on M .

- (a) Let J be a Poisson structure on M , defining a skew-symmetric mapping $J : \Omega^1(M) \rightarrow V(M)$. Then graph $J \subset V(M) \times \Omega^1(M)$ is a Dirac structure.
- (b) Let ω be a (pre-)symplectic structure on M , defining a skew-symmetric mapping $\omega : V(M) \rightarrow \Omega^1(M)$. Then graph $\omega \subset V(M) \times \Omega^1(M)$ is a Dirac structure.
- (c) Let V be a constant-dimensional distribution on M , and let $annV$ be its annihilating co-distribution. Then $V \times annV$ is a Dirac structure.

Remark 3.4. Usually in Example 3.3 an additional integrability condition is imposed on the Dirac structure, cf. [5], [7]. In part (a) this condition is equivalent to the *Jacobi-identity* for the Poisson structure; in part (b) it is equivalent to the *closedness* of the presymplectic structure, while in part (c) it is equivalent to the *involutivity* of the distribution D . Integrability is equivalent to the existence of canonical coordinates, cf. [5], [7], [6]. Various formulations of integrability of Dirac structures and their implications have been worked out in [6]. For the developments of the current paper the notion of integrability is not crucial; see however the comment in the Conclusions.

From the defining property $D = D^\perp$ of a Dirac structure it directly follows that for any $(f, e) \in D$

$$0 = \ll (f, e), (f, e) \gg = 2 \langle e|f \rangle \quad (47)$$

Thus if (f, e) is a pair of *power variables* (e.g., currents and voltages in an electric circuit context, or forces and velocities in a mechanical context), then the condition $(f, e) \in D$ implies *power-conservation* $\langle e|f \rangle = 0$ (as do Kirchhoff's laws or Newton's third law). This is the starting point for the geometric formulation of general power-conserving interconnections in physical systems by Dirac structures as alluded to above.

3.2 Stokes-Dirac structures

In this subsection we treat the underlying geometric framework for the Hamiltonian formulation of distributed-parameter systems on a bounded spatial domain, with non-zero energy flow through the boundary. The key concept is the introduction of a special type of Dirac structure on suitable spaces of differential forms on the spatial domain and its boundary, making use of Stokes' theorem. A preliminary treatment of this Dirac structure has been given in [20], [22].

Throughout, let Z be an n -dimensional smooth manifold with smooth $(n - 1)$ -dimensional boundary ∂Z , representing the space of *spatial variables*.

Denote by $\Omega^k(Z)$, $k = 0, 1, \dots, n$, the space of exterior k -forms on Z , and by $\Omega^k(\partial Z)$, $k = 0, 1, \dots, n - 1$, the space of k -forms on ∂Z . (Note that $\Omega^0(Z)$, respectively $\Omega^0(\partial Z)$, is the space of smooth functions on Z , respectively ∂Z .) Clearly, $\Omega^k(Z)$ and $\Omega^k(\partial Z)$ are (infinite-dimensional) linear spaces (over \mathbb{R}). Furthermore, there is a natural pairing between $\Omega^k(Z)$ and $\Omega^{n-k}(Z)$ given by

$$\langle \beta|\alpha \rangle := \int_Z \beta \wedge \alpha \quad (\in \mathbb{R}) \quad (48)$$

with $\alpha \in \Omega^k(Z)$, $\beta \in \Omega^{n-k}(Z)$, where \wedge is the usual wedge product of differential forms yielding the n -form $\beta \wedge \alpha$. In fact, the pairing (48) is *non-degenerate* in the sense that if $\langle \beta|\alpha \rangle = 0$ for all α , respectively for all β , then $\beta = 0$, respectively $\alpha = 0$.

Similarly, there is a pairing between $\Omega^k(\partial Z)$ and $\Omega^{n-1-k}(\partial Z)$ given by

$$\langle \beta|\alpha \rangle := \int_{\partial Z} \beta \wedge \alpha \quad (49)$$

with $\alpha \in \Omega^k(\partial Z), \beta \in \Omega^{n-1-k}(\partial Z)$. Now let us define the linear space

$$\mathcal{F}_{p,q} := \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z), \quad (50)$$

for any pair p, q of positive integers satisfying

$$p + q = n + 1, \quad (51)$$

and correspondingly let us define

$$\mathcal{E}_{p,q} := \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times \Omega^{n-q}(\partial Z). \quad (52)$$

Then the pairing (48) and (49) yields a (non-degenerate) pairing between $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$ (note that by (51) $(n-p) + (n-q) = n-1$). As before (see (46)), symmetrization of this pairing yields the following bilinear form on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ with values in \mathbb{R} :

$$\begin{aligned} \ll (f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1), (f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2) \gg := \\ \int_Z [e_p^1 \wedge f_p^2 + e_q^1 \wedge f_q^2 + e_b^2 \wedge f_p^1 + e_q^2 \wedge f_q^1] + \int_{\partial Z} [e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1] \end{aligned} \quad (53)$$

where for $i = 1, 2$

$$\begin{aligned} f_p^i &\in \Omega^p(Z), \quad f_q^i \in \Omega^q(Z) \\ e_p^i &\in \Omega^{n-p}(Z), \quad e_q^i \in \Omega^{n-q}(Z) \\ f_b^i &\in \Omega^{n-p}(\partial Z), \quad e_b^i \in \Omega^{n-q}(\partial Z) \end{aligned} \quad (54)$$

The spaces of differential forms $\Omega^p(Z)$ and $\Omega^q(Z)$ will represent the energy variables of two different physical energy domains interacting with each other, while $\Omega^{n-p}(\partial Z)$ and $\Omega^{n-q}(\partial Z)$ will denote the boundary variables whose (wedge) product represents the boundary energy flow. For example, in Maxwell's equations (Example 3.1) we will have $n = 3$ and $p = q = 2$; with $\Omega^p(Z) = \Omega^2(Z)$, respectively $\Omega^q(Z) = \Omega^2(Z)$, being the space of electric field inductions, respectively magnetic field inductions, and $\Omega^{n-p}(\partial Z) = \Omega^1(\partial Z)$ denoting the electric and magnetic field intensities at the boundary, with product the Poynting vector.

Theorem 3.5. *Consider $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$ given in (50), (52) with p, q satisfying (51), and bilinear form \ll, \gg given by (53). Define the following linear subspace D of $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$*

$$\begin{aligned} D &= \{(f_p, f_q, f_b, e_p, e_q, e_b) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \mid \\ &\quad \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \\ &\quad \left. \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_p|_{\partial Z} \\ e_q|_{\partial Z} \end{bmatrix} \right\} \end{aligned} \quad (55)$$

where $|_{\partial Z}$ denotes restriction to the boundary ∂Z , and $r := pq + 1$. Then $D = D^\perp$, that is, D is a Dirac structure.

For the proof of this theorem we refer to [37].

Remark 3.6. The spatial *compositionality* properties of the Stokes-Dirac structure immediately follow from its definition. Indeed, let Z_1, Z_2 be two n -dimensional manifolds with boundaries $\partial Z_1, \partial Z_2$, such that

$$\begin{aligned}\partial Z_1 &= \Gamma \cup \Gamma_1, & \Gamma \cap \Gamma_1 &= \phi \\ \partial Z_2 &= \Gamma \cup \Gamma_2, & \Gamma \cap \Gamma_2 &= \phi\end{aligned}\tag{56}$$

for certain $(n-1)$ -dimensional manifolds $\Gamma, \Gamma_1, \Gamma_2$ (that is, Z_1 and Z_2 have boundary Γ in common). Then the Stokes-Dirac structures D_1, D_2 on Z_1 , respectively Z_2 , compose to the Stokes-Dirac structure on the manifold $Z_1 \cup Z_2$ with boundary $\Gamma_1 \cup \Gamma_2$ if we equate on Γ the boundary variables f_b^1 (corresponding to D_1) with $-f_b^2$ (corresponding to D_2). (Note that a minus sign is inserted in order to ensure that the power flowing *into* Z_1 via Γ is equal to the power flowing *out* of Z_2 via Γ .)

4 Hamiltonian formulation of distributed-parameter systems with boundary energy flow

4.1 Boundary port-Hamiltonian systems

The definition of a distributed-parameter Hamiltonian system with respect to a Stokes-Dirac structure can now be stated as follows. Let Z be an n -dimensional manifold with boundary ∂Z , and let D be a Stokes-Dirac structure as in Subsection 2.2. Consider furthermore a *Hamiltonian density* (energy per volume element)

$$\mathcal{H} : \Omega^p(Z) \times \Omega^q(Z) \times Z \rightarrow \Omega^n(Z)\tag{57}$$

resulting in the total energy

$$H := \int_Z \mathcal{H} \in \mathbb{R}\tag{58}$$

Let $\alpha_p, \partial\alpha_p \in \Omega^p(Z)$, $\alpha_q, \partial\alpha_q \in \Omega^q(Z)$. Then (with $z \in Z$)

$$\begin{aligned}H(\alpha_p + \partial\alpha_p, \alpha_q + \partial\alpha_q) &= \\ &= \int_Z \mathcal{H}(\alpha_p + \partial\alpha_p, \alpha_q + \partial\alpha_q, z) = \\ &= \int_Z \mathcal{H}(\alpha_p, \alpha_q, z) + \int_Z [\delta_p \mathcal{H} \wedge \partial\alpha_p + \delta_q \mathcal{H} \wedge \partial\alpha_q] \\ &\quad + \text{higher order terms in } \partial\alpha_p, \partial\alpha_q\end{aligned}\tag{59}$$

for certain uniquely defined differential forms

$$\begin{aligned}\delta_p \mathcal{H} &\in \Omega^{n-p}(Z) \\ \delta_q \mathcal{H} &\in \Omega^{n-q}(Z)\end{aligned}\tag{60}$$

This means that $(\delta_p H, \delta_q H) \in \Omega^{n-p}(Z) \times \Omega^{n-q}(Z)$ can be regarded as the *variational derivative* of H at $(\alpha_p, \alpha_q) \in \Omega^p(Z) \times \Omega^q(Z)$.

Now consider a time-function

$$(\alpha_p(t), \alpha_q(t)) \in \Omega^p(Z) \times \Omega^q(Z), \quad t \in \mathbb{R}, \quad (61)$$

and the Hamiltonian $H(\alpha_p(t), \alpha_q(t))$ evaluated along this trajectory. It follows that at any time t

$$\frac{dH}{dt} = \int_Z \left[\delta_p H \wedge \frac{\partial \alpha_p}{\partial t} + \delta_q H \wedge \frac{\partial \alpha_q}{\partial t} \right] \quad (62)$$

The differential forms $\frac{\partial \alpha_p}{\partial t}, \frac{\partial \alpha_q}{\partial t}$ represent the generalized velocities of the energy variables α_p, α_q . They are connected to the Stokes-Dirac structure D by setting

$$f_p = -\frac{\partial \alpha_p}{\partial t} \quad (63)$$

$$f_q = -\frac{\partial \alpha_q}{\partial t}$$

(again the minus sign is included to have a consistent energy flow description). Since the right-hand side of (62) is the rate of increase of the stored energy H , we set

$$e_p = \delta_p H \quad (64)$$

$$e_q = \delta_q H$$

Now we come to the general Hamiltonian description of a distributed-parameter system with boundary energy flow. In order to emphasize that the boundary variables are regarded as *interconnection variables*, which can be interconnected to other systems and whose product represents *power*, we call these models *port-Hamiltonian* systems. (This terminology comes from network modelling, see e.g. [23], [35], [34].)

Definition 4.1. The *boundary port-Hamiltonian system* with n -dimensional manifold of spatial variables Z , state space $\Omega^p(Z) \times \Omega^q(Z)$ (with $p + q = n + 1$), Stokes-Dirac structure D given by (55), and Hamiltonian H , is given as (with $r = pq + 1$)

$$\begin{bmatrix} -\frac{\partial \alpha_p}{\partial t} \\ -\frac{\partial \alpha_q}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} \delta_p H \\ \delta_q H \end{bmatrix} \quad (65)$$

$$\begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} \delta_p H|_{\partial Z} \\ \delta_q H|_{\partial Z} \end{bmatrix}$$

By the power-conserving property (47) of any Dirac structure it immediately follows that for any $(f_p, f_q, f_b, e_p, e_q, e_b)$ in the Stokes-Dirac structure D

$$\int_Z [e_p \wedge f_p + e_q \wedge f_q] + \int_{\partial Z} e_b \wedge f_b = 0 \quad (66)$$

Hence by substitution of (63), (64) and using (62) we obtain

Proposition 4.2. Consider the distributed parameter port-Hamiltonian system (65). Then

$$\frac{dH}{dt} = \int_{\partial Z} e_b \wedge f_b, \quad (67)$$

expressing that the increase in energy on the domain Z is equal to the power supplied to the system through the boundary ∂Z .

The system (65) can be called a (nonlinear) *boundary control* system in the sense of e.g. [9]. Indeed, we could interpret f_b as the boundary control inputs to the system, and e_b as the measured outputs (or the other way around). In Section 6 we shall further elaborate on this point of view.

4.2 Boundary port-Hamiltonian systems with distributed ports and dissipation

Energy exchange through the boundary is not the only way a distributed-parameter system may interact with its environment. An example of this is provided by Maxwell's equations (Example 5.1), where interaction may also take place via the current density J , which directly affects the electric charge distribution in the domain Z . In order to cope with this situation we augment the spaces $\mathcal{F}_{p,q}, \mathcal{E}_{p,q}$ as defined in (50), (52) to

$$\begin{aligned} \mathcal{F}_{q,p}^a &:= \mathcal{F}_{p,q} \times \Omega^d(S) \\ \mathcal{E}_{q,p}^a &:= \mathcal{E}_{p,q} \times \Omega^{n-d}(S) \end{aligned} \quad (68)$$

for some m -dimensional manifold S and some $d \in \{0, 1, \dots, m\}$, with $f^d \in \Omega^d(S)$ denoting the externally supplied distributed control flow, and $e^d \in \Omega^{n-d}(S)$ the conjugate distributed quantity, corresponding to an energy exchange

$$\int_S e^d \wedge f^d \quad (69)$$

The Stokes-Dirac structure (55) is now extended to

$$\begin{aligned} \begin{bmatrix} f_p \\ f_q \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^{rd} \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix} + G(f_d) \\ \begin{bmatrix} f_b \\ e_b \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_{p|\partial Z} \\ e_{q|\partial Z} \end{bmatrix} \\ e_d &= -G^* \begin{bmatrix} e_p \\ e_q \end{bmatrix} \end{aligned} \quad (70)$$

with G denoting a linear map

$$G = \begin{pmatrix} G_p \\ G_q \end{pmatrix} : \Omega^d(S) \rightarrow \Omega^p(Z) \times \Omega^q(Z) \quad (71)$$

with dual map

$$G^* = (G_p^*, G_q^*) : \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \rightarrow \Omega^{n-d}(S) \quad (72)$$

satisfying

$$\int_Z [e_p \wedge G_p(f_d) + e_q \wedge G_q(f_d)] = \int_S [G_p^*(e_p) + G_q^*(e_q)] \wedge f_d \quad (73)$$

for all $e_p \in \Omega^{n-p}(Z)$, $e_q \in \Omega^{n-q}(Z)$, $f_d \in \Omega^d(S)$.

The following proposition can be easily checked.

Proposition 4.3. Equations (70) determine a Dirac structure $D^a \subset \mathcal{F}_{p,q}^a \times \mathcal{E}_{p,q}^a$ with respect to the augmented bilinear form on $\mathcal{F}_{p,q}^a \times \mathcal{E}_{p,q}^a$ which is obtained by adding to the bilinear form (53) on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ the term

$$\int_S [e_d^1 \wedge f_d^2 + e_d^2 \wedge f_d^1] \quad (74)$$

By making now the substitutions (63), (64) into D^a given by (70) we obtain a port-Hamiltonian system with external variables (f_b, f_d, e_b, e_d) , with f_b, e_b the *boundary* external variables and f_d, e_d the *distributed* external variables. Furthermore, the energy balance (67) extends to

$$\frac{dH}{dt} = \int_{\partial Z} e_b \wedge f_b + \int_S e_d \wedge f_d, \quad (75)$$

with the first term on the right-hand side denoting the power flow through the boundary, and the second term denoting the distributed power flow.

Finally, *energy dissipation* can be incorporated in the framework of distributed-parameter port-Hamiltonian systems by *terminating* some of the ports (boundary or distributed) with a *resistive relation*. For example, for distributed dissipation, let $R : \Omega^{n-d}(S) \rightarrow \Omega^d(S)$ be a map satisfying

$$\int_S e_d \wedge R(e_d) \geq 0, \quad \forall e_d \in \Omega^{n-d}(S) \quad (76)$$

Then by adding the relation

$$f_d = -R(e_d) \quad (77)$$

to the port-Hamiltonian system defined with respect to the Dirac structure D^a , we obtain a port-Hamiltonian system *with dissipation*, satisfying the energy inequality

$$\frac{dH}{dt} = \int_{\partial Z} e_b \wedge f_b - \int_S e_d \wedge R(e_d) \leq \int_{\partial Z} e_b \wedge f_b \quad (78)$$

5 Examples

In this section we show how the framework of distributed-parameter port-Hamiltonian systems admits the representation of Maxwell's equations, the telegraph equations of an ideal transmission line, the vibrating string, and the Euler equations of an ideal isentropic fluid.

5.1 Maxwell's equations

We closely follow the formulation of Maxwell's equations in terms of differential forms as presented in [12], and show how this directly leads to the formulation as a distributed-parameter port-Hamiltonian system.

Let $Z \subset \mathbb{R}^3$ be a 3-dimensional manifold with boundary ∂Z , defining the spatial domain, and consider the electromagnetic field in Z . The energy variables are the *electric field induction* 2-form $\alpha_p = \mathcal{D} \in \Omega^2(Z)$:

$$\mathcal{D} = \frac{1}{2} D_{ij}(t, z) dz^i \wedge dz^j \quad (79)$$

and the *magnetic field induction* 2-form $\alpha_q = \mathcal{B} \in \Omega^2(Z)$:

$$\mathcal{B} = \frac{1}{2} B_{ij}(t, z) dz^i \wedge dz^j \quad (80)$$

The corresponding Stokes-Dirac structure ($n = 3, p = 2, q = 2$) is given as (cf. (55))

$$\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \quad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{p|\partial Z} \\ e_{q|\partial Z} \end{bmatrix} \quad (81)$$

Usually in this case one does *not* start with the definition of the total energy (Hamiltonian) H , but instead with the co-energy variables $\delta_p H, \delta_q H$, given, respectively, as the electric field intensity $\mathcal{E} \in \Omega^1(Z)$:

$$\mathcal{E} = E_i(t, z) dz^i \quad (82)$$

and the magnetic field intensity $\mathcal{H} \in \Omega^1(Z)$:

$$\mathcal{H} = H_i(t, z) dz^i \quad (83)$$

They are related to the energy variables through the constitutive relations of the medium (or material equations)

$$\begin{aligned} *\mathcal{D} &= \epsilon \mathcal{E} \\ *\mathcal{B} &= \mu \mathcal{H} \end{aligned} \quad (84)$$

with the scalar functions $\epsilon(t, z)$ and $\mu(t, z)$ denoting the electric permittivity, respectively magnetic permeability, and $*$ denoting the Hodge star operator (corresponding to a Riemannian metric on Z), converting 2-forms into 1-forms. Then one *defines* the Hamiltonian H as

$$H = \int_Z \frac{1}{2} (\mathcal{E} \wedge \mathcal{D} + \mathcal{H} \wedge \mathcal{B}), \quad (85)$$

and one immediately verifies that $\delta_p H = \mathcal{E}, \delta_q H = \mathcal{H}$.

Nevertheless there are other cases (corresponding to a *nonlinear* theory of the electromagnetic field, such as the Born-Infeld theory, see e.g. [12]) where one starts with a more general Hamiltonian $H = \int_Z h$, with the energy density $h(\mathcal{D}, \mathcal{B})$ being a more general expression than $\frac{1}{2}(\epsilon^{-1} * \mathcal{D} \wedge \mathcal{D} + \mu^{-1} * \mathcal{B} \wedge \mathcal{B})$.

Assuming that there is no current in the medium Maxwell's equations can now be written as (see [12])

$$\begin{aligned}\frac{\partial \mathcal{D}}{\partial t} &= d\mathcal{H} \\ \frac{\partial \mathcal{B}}{\partial t} &= -d\mathcal{E}\end{aligned}\tag{86}$$

Explicitly taking into account the behavior at the boundary, Maxwell's equations on a domain $Z \subset \mathbb{R}^3$ are then represented as the port-Hamiltonian system with respect to the Stokes-Dirac structure given by (81), as

$$\begin{aligned}\begin{bmatrix} -\frac{\partial \mathcal{D}}{\partial t} \\ -\frac{\partial \mathcal{B}}{\partial t} \end{bmatrix} &= \begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix} \begin{bmatrix} \delta_D H \\ \delta_B H \end{bmatrix} \\ \begin{bmatrix} f_b \\ e_b \end{bmatrix} &= \begin{bmatrix} \delta_D H|_{\partial Z} \\ \delta_B H|_{\partial Z} \end{bmatrix}\end{aligned}\tag{87}$$

Note that the first line of (86) is nothing else than (the differential version of) Ampère's law, while the second line of (86) is Faraday's law. Hence the Stokes-Dirac structure in (86), (87) expresses the basic physical laws connecting \mathcal{D} , \mathcal{B} , \mathcal{H} and \mathcal{E} .

The energy-balance (67) in the case of Maxwell's equations takes the form

$$\frac{dH}{dt} = \int_{\partial Z} \delta_B H \wedge \delta_D H = \int_{\partial Z} \mathcal{H} \wedge \mathcal{E} = - \int_{\partial Z} \mathcal{E} \wedge \mathcal{H}\tag{88}$$

with $\mathcal{E} \wedge \mathcal{H}$ a 2-form corresponding to the *Poynting vector* (see [12]).

In the case of a non-zero *current density* we have to modify the first matrix equation of (87) to

$$\begin{bmatrix} -\frac{\partial \mathcal{D}}{\partial t} \\ -\frac{\partial \mathcal{B}}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix} \begin{bmatrix} \delta_D H \\ \delta_B H \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} J\tag{89}$$

with I denoting the identity operator from $J \in \Omega^2(Z)$ to $\Omega^2(Z)$. (Thus, in the notation of (71), $f_d = J$, $S = Z$, and $\Omega^d(S) = \Omega^2(Z)$.) Furthermore, we add the equation

$$e_d = -[I \ 0] \begin{bmatrix} \delta_D H \\ \delta_B H \end{bmatrix} = -\mathcal{E},\tag{90}$$

yielding the augmented energy balance

$$\frac{dH}{dt} = - \int_{\partial Z} \mathcal{E} \wedge \mathcal{H} - \int_Z \mathcal{E} \wedge J\tag{91}$$

which is known as *Poynting's theorem*.

Finally, in order to incorporate energy dissipation we write $J = J_d + \bar{J}$, and we impose Ohm's law

$$*J_d = \sigma \mathcal{E}\tag{92}$$

with $\sigma(t, z)$ the specific conductivity of the medium.

5.2 Telegraph equations

Consider an ideal lossless transmission line with $Z = [0, 1] \subset \mathbb{R}$. The energy variables are the charge density 1-form $Q = Q(t, z)dz \in \Omega^1([0, 1])$, and the flux density 1-form $\varphi = \varphi(t, z)dz \in \Omega^1([0, 1])$; thus $p = q = n = 1$. The total energy stored at time t in the transmission line is given as

$$H(Q, \varphi) = \int_0^1 \frac{1}{2} \left(\frac{Q^2(t, z)}{C(z)} + \frac{\varphi^2(t, z)}{L(z)} \right) dz \quad (93)$$

with co-energy variables

$$\begin{aligned} \delta_Q H &= \frac{Q(t, z)}{C(z)} = V(t, z) \quad (\text{voltage}) \\ \delta_\varphi H &= \frac{\varphi(t, z)}{L(z)} = I(t, z) \quad (\text{current}) \end{aligned} \quad (94)$$

where $C(z), L(z)$ are respectively the distributed capacitance and distributed inductance of the line.

The resulting port-Hamiltonian system is given by the telegraph equations

$$\begin{aligned} \frac{\partial Q}{\partial t} &= -\frac{\partial I}{\partial z} \\ \frac{\partial \varphi}{\partial t} &= -\frac{\partial V}{\partial z} \end{aligned} \quad (95)$$

together with the boundary variables

$$\begin{aligned} f_b^0(t) &= V(t, 0), & f_b^1(t) &= V(t, 1) \\ e_b^0(t) &= -I(t, 0), & e_b^1(t) &= -I(t, 1) \end{aligned} \quad (96)$$

The resulting energy-balance is

$$\frac{dH}{dt} = \int_{\partial([0, 1])} e_b f_b = -I(t, 1)V(t, 1) + I(t, 0)V(t, 0), \quad (97)$$

in accordance with (67).

5.3 Vibrating string

Consider an elastic string subject to traction forces at its ends. The spatial variable z belongs to the interval $Z = [0, 1] \subset \mathbb{R}$. Let us denote by $u(t, z)$ the displacement of the string. The elastic potential energy is a function of the *strain* given by the 1-form

$$\alpha_q(t) = \epsilon(t, z)dz = \frac{\partial u}{\partial z}(t, z)dz \quad (98)$$

The associated co-energy variable is the *stress* given by the 0-form

$$\sigma = T * \alpha_q \quad (99)$$

with T the elasticity modulus and $*$ the Hodge star operator. Hence the potential energy is the quadratic function

$$U(\alpha_q) = \int_0^1 \sigma \alpha_q = \int_0^1 T * \alpha_q \wedge \alpha_q = \int_0^1 T \left(\frac{\partial u}{\partial z} \right)^2 dz \quad (100)$$

and $\sigma = \delta_q U$.

The kinetic energy K is a function of the kinetic *momentum* defined as the 1-form

$$\alpha_p(t) = p(t, z) dz \quad (101)$$

given by the quadratic function

$$K(\alpha_p) = \int_0^1 \frac{p^2}{\mu} dz \quad (102)$$

The associated co-energy variable is the *velocity* given by the 0-form

$$v = \frac{1}{\mu} * \alpha_p = \delta_p K \quad (103)$$

In this case the Dirac structure is the Stokes-Dirac structure for $n = p = q = 1$, with an *opposite sign* convention leading to the equations (with $H := U + K$)

$$\begin{bmatrix} -\frac{\partial \alpha_p}{\partial t} \\ -\frac{\partial \alpha_q}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & -d \\ -d & 0 \end{bmatrix} \begin{bmatrix} \delta_p H \\ \delta_q H \end{bmatrix} \quad (104)$$

$$\begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_p H|_{\partial Z} \\ \delta_q H|_{\partial Z} \end{bmatrix}$$

or, in more down-to-earth notation

$$\begin{aligned} \frac{\partial p}{\partial t} &= \frac{\partial \sigma}{\partial z} = \frac{\partial}{\partial z} (T \epsilon) \\ \frac{\partial \epsilon}{\partial t} &= \frac{\partial v}{\partial z} = \frac{\partial}{\partial z} \left(\frac{1}{\mu} p \right) \\ f_b &= v|_{\{0,1\}} \\ e_b &= \sigma|_{\{0,1\}} \end{aligned} \quad (105)$$

with boundary variables the velocity and stress at the ends of the string. Of course, by substituting $\epsilon = \frac{\partial u}{\partial z}$ into the 2nd equation of (105) one obtains $\frac{\partial}{\partial z} \left(\frac{\partial u}{\partial t} - \frac{p}{\mu} \right) = 0$, implying that

$$p = \mu \frac{\partial u}{\partial t} + \mu f(t) \quad (106)$$

for some function f , which may be set to zero. Substitution of (106) into the first equation of (105) then yields the wave equation

$$\mu \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z} \left(T \frac{\partial u}{\partial z} \right) \quad (107)$$

6 Extension of port-Hamiltonian systems defined on Stokes-Dirac structures

6.1 Burger's equations

Consider the inviscid Burger's equation as discussed in Section 2.1. Consider Z to be a bounded interval of \mathbb{R} , then *Burger's inviscid equations* are:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = 0$$

which is a scalar convex conservation equation.

It may be formulated as a boundary control system as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\partial}{\partial x} (\delta_u H) \\ w_b &= \delta_u H |_{\partial Z} \end{aligned}$$

where $\delta_u H$ denotes the variational derivative of the Hamiltonian functional $H(u) = \frac{1}{6}u^3$. Defining the power-conjugated variables to be $f = \frac{\partial u}{\partial t}$, $e = \delta_u H$ and on the boundary w_b , one may define an infinite-dimensional Dirac structure which is different from the the Stokes-Dirac structure. With regard to this Dirac structure the inviscid Burger's equation is represented as a distributed port-Hamiltonian system. For details we refer to [15].

6.2 Ideal isentropic fluid

Consider an ideal compressible isentropic fluid in three dimensions, described in Eulerian representation by the standard Euler equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho v) \\ \frac{\partial v}{\partial t} &= -v \cdot \nabla v - \frac{1}{\rho} \nabla p \end{aligned} \tag{108}$$

with $\rho(z, t) \in \mathbb{R}$ the mass density at the spatial position $z \in \mathbb{R}^3$ at time t , $v(z, t) \in \mathbb{R}^3$ the (Eulerian) velocity of the fluid at spatial position z and time t , and $p(z, t)$ the pressure function, derivable from an internal energy function $U(\rho)$ as

$$p(z, t) = \rho^2(z, t) \frac{\partial U}{\partial \rho}(\rho(z, t)) \tag{109}$$

Much innovative work has been done regarding the Hamiltonian formulation of (113) and more general cases; we refer in particular to [24, 17, 18, 25, 14]. However, in these treatments only *closed* fluid dynamical systems are being considered with no energy exchange through the boundary of the spatial domain. As a result, a formulation in terms of Poisson structures can be given, while as argued before, the general inclusion of boundary variables necessitates the use of Dirac structures.

The formulation of (108) as a port-Hamiltonian system is given as follows. Let $\mathcal{D} \subset \mathbb{R}^3$ be a given domain, filled with the fluid. We assume the existence of a

Riemannian metric \langle, \rangle on \mathcal{D} ; usually the standard Euclidean metric on \mathbb{R}^3 . Let $Z \subset \mathcal{D}$ be any 3-dimensional manifold with boundary ∂Z .

We identify the mass-density ρ with a 3-form on Z (see e.g. [17, 18]), that is, with an element of $\Omega^3(Z)$. Furthermore, we identify the Eulerian vector field v with a 1-form on Z , that is, with an element of $\Omega^1(Z)$. (By the existence of the Riemannian metric on Z we can, by “index raising” or “index lowering”, identify vector fields with 1-forms and vice versa.) The precise motivation for this choice of variables will become clear later on. As a result we consider as the carrier spaces for the port-Hamiltonian formulation of (108) the linear spaces $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$ for $n = 3, p = 3, q = 1$; that is

$$\mathcal{F}_{p,q} = \Omega^3(Z) \times \Omega^1(Z) \times \Omega^0(\partial Z) \quad (110)$$

and

$$\mathcal{E}_{p,q} = \Omega^0(Z) \times \Omega^2(Z) \times \Omega^2(\partial Z) \quad (111)$$

Since $p + q = n + 1$ we can define the corresponding Stokes-Dirac structure D given by (55) on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$. However, as will become clear later on, due to 3-dimensional convection we need to *modify* this Stokes-Dirac structure with an additional term into the following modified Stokes-Dirac structure

$$\begin{aligned} D^m &:= \{(f_p, f_v, f_b, e_\rho, e_v, e_b) \in \\ &\Omega^3(Z) \times \Omega^1(Z) \times \Omega^0(\partial Z) \times \Omega^0(Z) \times \Omega^2(Z) \times \Omega^2(\partial Z) \\ &\left[\begin{array}{c} f_\rho \\ f_v \end{array} \right] = \left[\begin{array}{c} de_v \\ de_\rho + \frac{1}{*\rho} * ((*dv) \wedge (*e_v)) \end{array} \right] \\ &\left. \left[\begin{array}{c} f_b \\ e_b \end{array} \right] = \left[\begin{array}{c} e_{\rho|\partial Z} \\ -e_{v|\partial Z} \end{array} \right] \right\} \end{aligned} \quad (112)$$

where $*$ denotes the Hodge star operator (corresponding to the Riemannian metric on Z), converting k -forms on Z to $(3 - k)$ -forms. A fundamental difference of the modified Stokes-Dirac structure D^m with respect to the standard Stokes-Dirac structure D is that D^m explicitly depends on the energy variables ρ and v (via the terms $*\rho$ and dv in the additional term $\frac{1}{*\rho} * ((*dv) \wedge (*e_v))$).

Completely similar to the proof of Theorem 5 it is shown that $(D^m(\rho, v))^\perp = D^m(\rho, v)$ for all ρ, v ; the crucial additional observation is that the expression

$$e_v^2 \wedge *((*dv) \wedge (*e_v^1)) \quad (113)$$

is *skew-symmetric* in $e_v^1, e_v^2 \in \Omega^2(Z)$.

Remark 6.1. In the standard Euclidean metric, identifying via the Hodge star operator 2-forms β_i with 1-forms, and representing 1-forms as vectors, we have in vector calculus notation the equality

$$\beta_2 \wedge *(\alpha \wedge *\beta_1) = \alpha \cdot (\beta_1 \times \beta_2) \quad (114)$$

for all 2-forms β_1, β_2 and 1-forms α . This shows clearly the skew-symmetry of (113).

The Eulerian equations (108) for an ideal isentropic fluid are obtained in the port-Hamiltonian representation by considering the Hamiltonian

$$H(\rho, v) := \int_Z \left[\frac{1}{2} \langle v^\sharp, v^\sharp \rangle \rho + U(*\rho) \rho \right] \quad (115)$$

with v^\sharp the vector field corresponding to the 1-form v (“index lowering”), and $U(*\rho)$ the potential energy. Indeed, by making the substitutions (63), (64) in D^m , and noting that

$$\text{grad } H = (\delta_\rho H, \delta_v H) = \left(\frac{1}{2} \langle v^\sharp, v^\sharp \rangle + \frac{\partial}{\partial \tilde{\rho}} (\tilde{\rho} U(\tilde{\rho})), i_{v^\sharp} \rho \right) \quad (116)$$

with $\tilde{\rho} := *\rho$, the port-Hamiltonian system takes the form

$$\begin{aligned} -\frac{\partial \rho}{\partial t} &= d(i_{v^\sharp} \rho) \\ -\frac{\partial v}{\partial t} &= d\left(\frac{1}{2} \langle v^\sharp, v^\sharp \rangle + w(*\rho)\right) + \frac{1}{*\rho} ((*dv) \wedge (*i_{v^\sharp} \rho)) \\ f_b &= \left[\frac{1}{2} \langle v^\sharp, v^\sharp \rangle + w(*\rho)\right]_{|\partial Z} \\ e_b &= -i_{v^\sharp} \rho|_{\partial Z} \end{aligned} \quad (117)$$

with

$$w(\tilde{\rho}) := \frac{\partial}{\partial \tilde{\rho}} (\tilde{\rho} U(\tilde{\rho})) \quad (118)$$

the *enthalpy*. The expression $\delta_\rho H = \frac{1}{2} \langle v^\sharp, v^\sharp \rangle + w(\tilde{\rho})$ is known as the *Bernoulli function*.

The first two equations of (117) can be seen to represent the Eulerian equations (108). The first equation corresponds to the basic law of *mass-balance*

$$\frac{d}{dt} \int_{\varphi_t(V)} \rho = 0 \quad (119)$$

where V denotes an arbitrary volume in Z , and φ_t is the flow of the fluid (transforming the material volume V at $t = 0$ to the volume $\varphi_t(V)$ at time t). Indeed, (119) for any V is equivalent to

$$\frac{\partial \rho}{\partial t} + L_{v^\sharp} \rho = 0 \quad (120)$$

Since by Cartan’s magical formula $L_{v^\sharp} \rho = d(i_{v^\sharp} \rho) + i_{v^\sharp} d\rho = d(i_{v^\sharp} \rho)$ (since $d\rho = 0$) this yields the first line of (117). It also makes clear the interpretation of ρ as a 3-form on Z .

For the identification of the second equation of (117) with the second equation of (113) we note the following (see [36] for further details). Interpret $\nabla \cdot$ in (108) as the covariant derivative corresponding to the assumed Riemannian metric \langle, \rangle on Z . For a vector field u on Z , let u^\flat denote the corresponding 1-form $u^\flat := i_u \langle, \rangle$

(“index raising”). The covariant derivative ∇ is related to the Lie derivative by the following formula (see for a proof [14], p. 202)

$$L_u u^b = (\nabla_u u)^b + \frac{1}{2} d \langle u, u \rangle \quad (121)$$

Since by Cartan’s magical formula $L_u u^b = i_u du^b + d(i_u u^b) = i_u du^b + d \langle u, u \rangle$, (121) can be also written as

$$(\nabla_u u)^b = i_u du^b + \frac{1}{2} d \langle u, u \rangle \quad (122)$$

(This is the coordinate-free analog of the well-known vector calculus formula $u \cdot \nabla u = \text{curl } u \times u + \frac{1}{2} \nabla |u|^2$.) Furthermore we have the identity

$$i_{v^\sharp} dv = \frac{1}{* \rho} * ((* dv) \wedge (* i_{v^\sharp} \rho)) \quad (123)$$

Finally, we have the following well-known relation between enthalpy and pressure (obtained from (114) and (118))

$$\frac{1}{\tilde{\rho}} dp = d(w(\tilde{\rho})). \quad (124)$$

Hence by (122) (with $u = v^\sharp$), (98) and (124), we may rewrite the 2nd equation of (117) as

$$-\frac{\partial v}{\partial t} = \left(\nabla_{v^\sharp} v^\sharp \right)^b + \frac{1}{* \rho} dp \quad (125)$$

which is the coordinate-free formulation of the 2nd equation of (108).

The boundary variables f_b and e_b given in (117) are respectively the *stagnation pressure* at the boundary divided by ρ , and the (incoming) *mass flow* through the boundary. The energy-balance (67) can be written out as

$$\begin{aligned} \frac{dH}{dt} &= \int_{\partial Z} e_b \wedge f_b = - \int_{\partial Z} i_{v^\sharp} \rho \wedge \left[\frac{1}{2} \langle v^\sharp, v^\sharp \rangle + w(*\rho) \right] \\ &= - \int_{\partial Z} i_{v^\sharp} \left[\frac{1}{2} \langle v^\sharp, v^\sharp \rangle \rho + w(*\rho) \rho \right] \\ &= - \int_{\partial Z} i_{v^\sharp} \left[\frac{1}{2} \langle v^\sharp, v^\sharp \rangle \rho + U(*\rho) \rho \right] - \int_{\partial Z} i_{v^\sharp} (*p) \end{aligned} \quad (126)$$

where for the last equality we have used the relation (following from (109), (118))

$$w(*\rho) \rho = U(*\rho) \rho + *p \quad (127)$$

The first term in the last line of (126) corresponds to the convected energy through the boundary ∂Z , while the second term is (minus) the external work (static pressure times velocity).

Usually, the second line of the Euler equations (108) (or equivalently equation (125)) is obtained from the basic conservation law of momentum-balance together

with the first line of (108). Alternatively, emphasizing the interpretation of v as a 1-form, we may obtain it from *Kelvin's circulation theorem*

$$\frac{d}{dt} \int_{\varphi_t(C)} v = 0 \quad (128)$$

where C denotes any *closed* contour. Indeed, (128) for any closed C is equivalent to the 1-form $\frac{\partial v}{\partial t} + L_{v^\#}v$ being *closed*. By (121) this is equivalent to requiring

$$\frac{\partial v}{\partial t} + \left(\nabla_{v^\#} v^\# \right)^\flat \quad (129)$$

to be closed, that is

$$\frac{\partial v}{\partial t} + \left(\nabla_{v^\#} v^\# \right)^\flat = -dk \quad (130)$$

for some (possibly locally defined) $k : Z \rightarrow \mathbb{R}$. Now additionally requiring that this function k depends on z through ρ , that is

$$k(z) = w(\rho(z)) \quad (131)$$

for some function w , we recover (125) with $\frac{1}{*\rho}dp$ replaced by dw (the differential of the enthalpy).

Remark 6.2. In the case of a one- or two-dimensional fluid flow the extra term in the Dirac structure D^m as compared with the standard Stokes-Dirac structure D vanishes, and so in these cases we are back to the standard definition of a distributed-parameter port-Hamiltonian system (with ρ being a 1-form, respectively, a 2-form).

Furthermore, if in the 3-dimensional case the 2-form $dv(t)$ happens to be zero at a certain time-instant $t = t_0$ (*irrotational flow*), then it *continues* to be zero for all time $t \geq t_0$. Hence also in this case the extra term (113) in the modified Stokes-Dirac structure D^m vanishes, and the port-Hamiltonian system describing the Euler equations reduces to the standard distributed-parameter port-Hamiltonian system given in Definition 4.1.

7 Conserved quantities of port-Hamiltonian systems

Let us consider the distributed-parameter port-Hamiltonian system Σ , as defined in Definition 4.1, on an n -dimensional spatial domain Z having state space $\Omega^p(Z) \times \Omega^q(Z)$ (with $p + q = n + 1$) and Stokes-Dirac structure D given by (55).

Conservation laws for Σ , which are *independent* from the Hamiltonian H , are obtained as follows. Let

$$C : \Omega^p(Z) \times \Omega^q(Z) \times Z \rightarrow \mathbb{R} \quad (132)$$

be a function satisfying

$$d(\delta_p C) = 0, \quad d(\delta_q C) = 0, \quad (133)$$

where $d(\delta_p C), d(\delta_q C)$ are defined similarly to (60). Then the time-derivative of C along the trajectories of Σ is given as (in view of (133), and using similar calculations as in the proof of Theorem 3.5

$$\begin{aligned}
\frac{d}{dt}C &= \int_Z \delta_p C \wedge \dot{\alpha}_p + \int_Z \delta_q C \wedge \dot{\alpha}_q \\
&= - \int_Z \delta_p C \wedge (-1)^r d(\delta_q H) - \int_Z \delta_q C \wedge d(\delta_p H) \\
&= -(-1)^{n-q} \int_Z d(\delta_q H \wedge \delta_p C) - (-1)^{n-q} \int_Z d(\delta_q C \wedge \delta_p H) \\
&= \int_{\partial Z} e_b \wedge f_b^C + \int_{\partial Z} e_b^c \wedge f_b
\end{aligned} \tag{134}$$

where we have denoted, in analogy with (55),

$$f_b^C := \delta_p C|_{\partial Z}, \quad e_b^c := -(-1)^{n-q} \delta_q C|_{\partial Z} \tag{135}$$

In particular, if additionally to (133) the function C satisfies

$$\delta_p C|_{\partial Z} = 0, \quad \delta_q C|_{\partial Z} = 0 \tag{136}$$

then $\frac{dC}{dt} = 0$ along the system trajectories of Σ for any Hamiltonian H . Therefore a function C satisfying (133), (136) is called a *Casimir* function. If C satisfies (133) but not (136) then C is called a *conservation law* for Σ : its time-derivative is determined by the boundary conditions of Σ .

Example 7.1. In the case of the telegraph equations (Example 3.2) the total charge

$$C_Q = \int_0^1 Q(t, z) dz$$

as well as the total magnetic flux

$$C_\varphi = \int_0^1 \varphi(t, z) dz$$

are both conservation laws. Indeed

$$\begin{aligned}
\frac{d}{dt}C_Q &= - \int_0^1 \frac{\partial I}{\partial z} = I(0) - I(1) \\
\frac{d}{dt}C_\varphi &= - \int_0^1 \frac{\partial V}{\partial z} dz = V(0) - V(1)
\end{aligned}$$

Similarly, in the case of the vibrating string (Example 5.3) conservation laws are $= \int_0^1 \epsilon(t, z) dz = u(t, 1) - u(t, 0)$,

$$\frac{d}{dt} \int_0^1 \epsilon(t, z) dz = \frac{d}{dt} (u(t, 1) - u(t, 0)) = v(t, 1) - v(t, 0)$$

□

$$\frac{d}{dt} \int_0^1 p(t, z) dz = \sigma(t, 1) - \sigma(t, 0)$$

Conservation laws C for Σ which are *dependent* on the Hamiltonian H are obtained by replacing (133) by the weaker condition

$$\delta_q H \wedge d(\delta_p C) + (-1)^r \delta_p H \wedge d(\delta_q C) = 0 \quad (137)$$

Indeed, it immediately follows from the computation in (134) that under this condition (134) continues to hold.

In the case of the modified Stokes-Dirac structure D^m defined in (112), for any function $C : \Omega^3(Z) \times \Omega^1(Z) \times Z \rightarrow \mathbb{R}$ satisfying

$$\delta_v H \wedge d(\delta_p C) + \delta_p H \wedge d(\delta_v C) = 0, \quad \rho \in \Omega^3(Z), \quad v \in \Omega^1(Z) \quad (138)$$

equation (134) takes the form

$$\begin{aligned} \frac{d}{dt} C &= \int_Z \delta_p C \wedge d(\delta_v H) + \int_Z \delta_v C \wedge \left[d(\delta_p H) + \frac{1}{*\rho} * ((*dv) \wedge (*\delta_v H)) \right] \\ &= \int_{\partial Z} \delta_p C \wedge \delta_v H + \int_{\partial Z} \delta_v C \wedge \delta_p H \\ &\quad + \int_Z \frac{1}{*\rho} \delta_v C * ((*dv) \wedge (*\delta_v H)) \end{aligned} \quad (139)$$

Hence we conclude that in order to obtain a conservation law we need to impose an extra condition eliminating the last \int_Z integral. A specific example of a conservation law in this case is the *helicity*

$$C = \int_Z v \wedge dv \quad (140)$$

with time-derivative

$$\frac{d}{dt} C = - \int_{\partial Z} f_b \wedge dv \quad (141)$$

A *second* class of conserved quantities corresponding to the Stokes-Dirac structure D (55) is identified by noting that by (65)

$$\begin{aligned} -d \left(\frac{\partial \alpha_p}{\partial t} \right) &= (-1)^r d(d\delta_q H) = 0 \\ -d \left(\frac{\partial \alpha_q}{\partial t} \right) &= d(d\delta_p H) = 0 \end{aligned} \quad (142)$$

and thus the differential forms $d\alpha_p$ and $d\alpha_q$ do not depend on time. Therefore, the component functions of $d\alpha_p$ and $d\alpha_q$ are conserved quantities of any port-Hamiltonian system corresponding to D .

Example 7.2. In the case of Maxwell's equations (Example 3.1) this yields that $d\mathcal{D}$ and $d\mathcal{B}$ are constant 3-forms. The 3-form $d\mathcal{D}$ is the *charge density* (Gauss' electric law), while by Gauss' magnetic law $d\mathcal{B}$ is actually zero.

In the case of an ideal isentropic fluid (Example 3.4) for which the vorticity $dv(t_0, z)$ is *zero* at a certain time t_0 we obtain by the same reasoning (since the additional term in the Stokes-Dirac structure D^m is zero for t_0) that $dv(t, z)$ is *zero* for all $t \geq t_0$ (irrotational flow); cf. Remark 6.2.

8 Conclusions and final remarks

In this paper we have exposed a framework for the compositional modelling of distributed-parameter systems, based on our papers [37, 20, 22]. This allows the Hamiltonian formulation of a large class of distributed-parameter systems with boundary energy-flow, including the examples of the telegraph equations, Maxwell's equations, vibrating strings and ideal isentropic fluids. It has been argued that in order to incorporate boundary variables into this formulation the notion of a Dirac structure provides the appropriate generalization of the more commonly used notion of a Poisson structure for evolution equations. The employed Dirac structure is based on Stokes' theorem, and emphasizes the geometrical content of the variables as being differential k -forms. From a physical point of view the Stokes-Dirac structure captures the balance laws inherent to the system, like Faraday's and Ampère's law (in Maxwell's equations), or mass-balance (in the case of an ideal fluid). This situation is quite similar to the lumped-parameter case where the Dirac structure incorporates the topological interconnection laws (Kirchhoff's laws, Newton's third law) and other interconnection constraints (see e.g. [19] [19] [35]). Hence the starting point for the Hamiltonian description is different from the more common approach of deriving Hamiltonian equations from a variational principle and its resulting Lagrangian equations, or (very much related) a Hamiltonian formulation starting from a state space being a co-tangent bundle endowed with its natural symplectic structure. In the case of Maxwell's equations this results in the use of the basic physical variables D and B (the electric and magnetic field inductions), instead of the use of the variable D (or E) together with the vector potential A (with $dA = B$) in the symplectic formulation of Maxwell's equations. It should be of interest to compare both approaches more closely, also in the context of the natural multi-symplectic structures which have been formulated for the Hamiltonian formulation of Lagrangian field equations; see e.g. [5], [15].

A prominent and favorable property of Dirac structures is that they are closed under power-conserving interconnection. This has been formally proven in the finite-dimensional case, but the result carries through to the infinite-dimensional case as well. It is a property of fundamental importance since it enables to link port-Hamiltonian systems (lumped- or distributed-parameter) to each other to obtain an interconnected port-Hamiltonian system with total energy being the sum of the Hamiltonians of its constituent parts. Clearly, this is equally important in modelling (coupling e.g. solid components with fluid components, or finite-dimensional electric components with transmission lines), as in control. First of all, it enables to formulate directly distributed-parameter systems with constraints as (implicit) Hamiltonian systems, like this has been done in the finite-dimensional case for mechanical systems with kinematic constraints, multi-body systems, and general electrical networks. Secondly, from the control perspective the notion of feedback control can be understood on its most basic level as the coupling of given physical components with additional control components (being themselves physical systems, or software components linked to sensors and actuators). A preliminary study from this point of view of a control scheme involving transmission lines has been provided in [30]. Among others, this opens up the way for the application of passivity-based control

techniques, which have been proven to be very effective for the control of lumped-parameter physical systems modelled as port-Hamiltonian systems.

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