DISC Course:
“System and Control Theory of Nonlinear Systems”

TAKE HOME EXAM 3

Hand out: April 21, 2011 Hand in: May 16, 2011

1. A model of an unsaturated induction motor is given by

$$\frac{d}{dt}\begin{pmatrix}\omega\\ \Psi_a\\ \Psi_b\\ i_a\\ i_b\end{pmatrix}=\begin{pmatrix}\mu(\Psi_a i_b-\Psi_b i_a)-T_L/J\\ -\alpha\Psi_a-n_p\omega\Psi_b+\alpha M i_a\\ -\alpha\Psi_b+n_p\omega\Psi_a+\alpha M i_b\\ \alpha\beta\Psi_a+n_p\beta\omega\Psi_b-\gamma i_a\\ \alpha\beta\Psi_b-n_p\beta\omega\Psi_a-\gamma i_b\end{pmatrix}+\begin{pmatrix}0\\0\\0\\1/\sigma\\0\end{pmatrix}u_a+\begin{pmatrix}0\\0\\0\\0\\1/\sigma\end{pmatrix}u_b$$

$$\begin{aligned} y_1 &= \omega\\ y_2 &= \Psi_a^2+\Psi_b^2\end{aligned}$$

Here, ω is the rotor speed, (Ψ_a,Ψ_b) the rotor fluxes, and (i_a,i_b) the stator currents. (u_a,u_b) are the voltages which are the control variables. $\sigma = L_s - M^2/L_r$, $\alpha = R_r/L_r$, $\beta = M/(\sigma L_r)$, $\gamma = M^2R_r/(\sigma L_r^2)+R_s/\sigma$, $\mu = n_pM/(JL_r)$, where T_L the load torque, R_r the rotor resistance, J the rotor inertia, (L_s, L_r) the stator and rotor inductances, M the mutual inductance, R_s the stator resistance, and n_p the number of pole pairs.

(a) Prove that the static input-output decoupling problem is solvable if $\Psi_a^2+\Psi_b^2>0$.

(b) Compute the ”normal form” for the system. Keep in mind that the ”normal form” is local, at least it is required that $y_2=\Psi_a^2+\Psi_b^2>0$.

(c) Let $\omega_r(t)$ and $\|\Psi\|_r^2(t)$ be desired reference signals.

   1. What are the corresponding tracking dynamics, i.e. the unobservable dynamics in case of exact tracking.
   2. Design a suitable feedback that achieves the tracking of the desired reference signals. What can you say about the stability of the tracking dynamics ?
   3. Provide some illustrative simulations for the input-output decoupling, and tracking dynamics.

2. A model for the two dimensional fluid flow using normal modes reads as

$$\begin{aligned}\dot\Psi_0 &= C_1\Psi_1\Psi_2+C_2\Psi_0+C_3F\\ \dot\Psi_1 &= C_4\Psi_2\Psi_0+C_5\Psi_1\\ \dot\Psi_2 &= C_6\Psi_0\Psi_1+C_7\Psi_2.\end{aligned}\tag{1}$$

Take as output the dissipation

$$\xi = A_0\Psi_0^2+A_1\Psi_1^2+A_2\Psi_2^2.\tag{2}$$

In (1) and (2), $C_1,\ldots,C_7,A_0,A_1,A_2$, are arbitrary constants. F denotes the input forcing.

a) Under what conditions on the parameters can you achieve input-output decoupling? Design a decoupling control law.

b) Transform the system to normal form.

c) Compute for $\xi = \xi_0$ =constant, a tracking controller. What are the corresponding (internal) tracking dynamics?

d) Give for the following parameter values some illustrative simulations of the (controlled) system.

$$\begin{array}{llll} C_1 = \frac{k(k^2+3)}{2(k^2+1)} & C_2 = -\frac{k^2+1}{R} & C_3 = -C_2 & C_4 = -\frac{3k}{4} \\ C_5 = -\frac{1}{R} & C_6 = -\frac{k^3}{2(k^2+4)} & C_7 = -\frac{k^2+4}{R} & A_0 = \frac{1}{4}\left(1 + 2k^2 + k^4\right) \\ A_1 = \frac{1}{2} & A_2 = 4 + 2k^2 + \frac{k^4}{4} & \xi_0 = 1.15 & \end{array}$$

Consider $k = 1$ and $R = 10$.

3. Consider a fully actuated mass m with Cartesian coordinates q_1, q_2 moving on a horizontal plane:

$$\begin{aligned} m\ddot{q}_1 &= \bar{u}_1 \\ m\ddot{q}_2 &= \bar{u}_2 \end{aligned}$$

and apply the feedback $\bar{u}_2 = -k(q_2 - q_2^*) + u_2, \bar{u}_1 = u_1$.

(a) Show that the closed-loop system is given by the port-Hamiltonian equations

$$\begin{array}{rclrcl} \dot{q}_1 &=& \frac{\partial H}{\partial p_1} & \dot{q}_2 &=& \frac{\partial H}{\partial p_2} \\ \dot{p}_1 &=& -\frac{\partial H}{\partial q_1} + u_1 & \dot{p}_2 &=& \frac{\partial H}{\partial q_2} + u_2 \\ y_1 &=& \frac{\partial H}{\partial p_1} & y_2 &=& \frac{\partial H}{\partial p_2} \end{array}$$

with $H(q_1, q_2, p_1, p_2) = \frac{1}{2m}p_1^2 + \frac{1}{2m}p_2^2 + \frac{1}{2}k(q_2 - q_2^*)^2$.

(b) Apply now the power-conserving feedback

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & y_1 y_2 \\ -y_1 y_2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Show that the closed-loop system satisfies $\frac{d}{dt}H = 0$, while

$$\frac{d}{dt}H_1 \geq 0, \quad \frac{d}{dt}H_2 \leq 0,$$

where H_1 is the energy corresponding to horizontal motion, and H_2 is the energy corresponding to vertical motion:

$$H(q_1, p_1) = \frac{1}{2m}p_1^2, \quad H(q_2, p_2) = \frac{1}{2m}p_2^2 + \frac{1}{2}k(q_2 - q_2^*)^2$$

The above feedback is therefore called an *energy-router*.
Prove that the above implies that

$$p_2(t) \to 0, \quad q_2(t) \to q_2^*,$$

for $t \to \infty$. What does this imply for the trajectory of the mass ? Argue that we have obtained a form of *path-following* control, and discuss the difference with tracking control.

4. (a) It can be shown that asymptotic stability of the equilibrium 0 of the differential equations $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ implies that for every non-empty neighborhood V of $0 \in \mathbb{R}^n$ the map $f : V \in \mathbb{R}^n \to \mathbb{R}^n$ is onto a neighborhood of 0. With the aid of this prove *Brockett's necessary condition* for local feedback stabilization as formulated in Exercise 10.6.

The equations of motion of a vertical wheel rolling without slipping on a horizontal plane are given as

$$\begin{aligned} \ddot{x} &= \lambda_1 \\ \ddot{y} &= \lambda_2 \\ \ddot{\theta} &= -\lambda_1 \cos\varphi - \lambda_2 \sin\varphi + u_1 \\ \ddot{\varphi} &= u_2 \end{aligned}$$

Here x, y denote the Cartesian coordinates of the point of contact of the wheel with the plane, φ denotes the heading angle of the wheel (with respect to the x-axis), and θ denotes the rotation angle of the wheel. Furthermore, u_1 and u_2 are control torques. Finally, λ_1 and λ_2 are the constraint forces corresponding to the non-slipping constraints

$$\begin{aligned} \dot{x} &= \dot{\theta}\cos\varphi \\ \dot{y} &= \dot{\theta}\sin\varphi \end{aligned}$$

Define the variables

$$z_1 = \theta, z_2 = \varphi, z_3 = x, z_4 = y, z_5 = \dot{\theta}, z_6 = \dot{\varphi}$$

(b) Show that after elimination of the constraints the system reduces to the system

$$\begin{aligned} \dot{z}_1 &= z_5 \\ \dot{z}_2 &= z_6 \\ \dot{z}_3 &= z_5 \cos z_2 \\ \dot{z}_4 &= z_5 \sin z_2 \\ \dot{z}_5 &= \tfrac{1}{2} u_1 \\ \dot{z}_6 &= u_2 \end{aligned}$$

(c) Prove that the system is locally strongly accessible at any point $(z_1, z_2, z_3, z_4, 0, 0)$.

(d) Show that the system does not satisfy Brockett's necessary condition (and thus is not stabilizable using continuous feedback).

(e) Any fully actuated mechanical system with kinematic constraints can be written into port-Hamiltonian form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_n & S(q) \\ -S^T(q) & * \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ I_k \end{bmatrix} u\,,$$

where $q \in \mathbb{R}^n$ is the vector of configuration coordinates, $p \in \mathbb{R}^k, k < n$, is the vector of constrained momenta (with $n - k$ the number of kinematic constraints), $S(q)$ is an $n \times k$ matrix of rank k, and $*$ denotes an unspecified $k \times k$ matrix. Show that such a system never satisfies Brockett's necessary condition.

(**Hint**: Without loss of generality it may be assumed that $S(q)$ is of the form $\begin{bmatrix} * \\ I_k \end{bmatrix}$.)