Systems and Control Theory of Nonlinear Systems

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Addendum on CONTROLLED INVARIANCE

Let us consider a nonlinear system. We say that the distribution D on the state space manifold $\mathcal{X}$ is *invariant* for the system if

$$\begin{aligned} &[f, D] \subset D \\ &[g_j, D] \subset D, \quad j = 1, \cdots, m \end{aligned} \tag{1}$$

It follows from Frobenius' theorem that if D is involutive and constant dimensional then there exist coordinates $(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n) = (x^1, x^2)$ such that the system takes the form

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = \begin{bmatrix} f^1(x^1, x^2) \\ f^2(x^2) \end{bmatrix} + \sum_{j=1}^{m} \begin{bmatrix} g_j^1(x^1, x^2) \\ g_j^2(x^2) \end{bmatrix} u_j \tag{2}$$

We derive

Proposition 0.1. Consider the nonlinear system affected by disturbances

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)q \quad , u \in \mathbb{R}^m, x \in \mathcal{X}, q \in \mathbb{R}^s, \\ y &= h(x), \qquad\qquad\qquad\quad , y \in \mathbb{R}^p \end{aligned} \tag{3}$$

on the state space manifold $\mathcal{X}$. The disturbance q does not influence the output y, irrespectively of the initial condition $x(0)$ and input u, if there exists an involutive and constant dimensional distribution D on M such that

$$\begin{array}{ll} (i) & D \text{ is invariant for } \dot{x} = f(x) + g(x)u \\ (ii) & D \subset ker\ dh \\ (iii) & p_j \in D, j = 1, \cdots, s \end{array} \tag{4}$$

Proof By Frobenius' theorem (i) implies that locally $\dot{x} = f(x) + g(x)u$ takes the form (2). By (ii) we have

$$L_{\frac{\partial}{\partial x_i}} h(x) = 0, \quad i = 1, \cdots, k \tag{5}$$

implying that h only depends on x^2. Furthermore by (ii) all the disturbance vectorfields p_j are linear combinations of the vectorfields $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}$ spanning D. It follows that (3) has the form

$$\begin{aligned} \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} &= \begin{bmatrix} f^1(x^1, x^2) \\ f^2(x^2) \end{bmatrix} + \begin{bmatrix} g^1(x^1, x^2) \\ g^2(x^2) \end{bmatrix} u + \begin{bmatrix} p^1(x^1, x^2) \\ 0 \end{bmatrix} q \\ y &= h(x^2) \end{aligned} \tag{6}$$

and thus q only affects the x^1-dynamics, and therefore does not affect y. $\square$

The next thing to do is to find conditions such that a distribution D satisfying conditions (ii) and (iii) of (4) can be *rendered* invariant by static state feedback. In analogy with Definition 2.2.2 we give

Definition 0.2. A distribution D on $\mathcal{X}$ is called *controlled invariant* for the system $\dot{x} = f(x) + g(x)u$ if there exists a regular state feedback

$$u = \alpha(x) + \beta(x)v, \quad \det \beta(x) \neq 0, \ \ \forall x \in \mathcal{X}, \tag{7}$$

such that, denoting the closed-loop system by

$$\begin{aligned} \dot{x} \quad &= \quad \tilde{f}(x) + \sum_{j=1}^{m} \tilde{g}_j(x)v_j \\ \tilde{f}(x) \quad &= \quad f(x) + g(x)\alpha(x), \ \ \tilde{g}_j(x) = [g(x)\beta(x)]_{j-th} \text{ column} \end{aligned} \tag{8}$$

we have

$$[\tilde{f}, D] \subset D, \quad [\tilde{g}_j, D] \subset D, \quad j = 1, \cdots, m \tag{9}$$

The notion of controlled invariance as given above is not easily checked. However, analogously to Proposition 2.2.3, we have the following condition for "local" controlled invariance. Define the distribution $G(x) := \text{span}\{g_1(x), \cdots, g_m(x)\}$.

Theorem 0.3. *Consider the system $\dot{x} = f(x)+g(x)u$ on $\mathcal{X}$, and let D be a constant dimensional and involutive distribution. Suppose that*

$$\begin{aligned} &(i) \quad \dim[D(x) + G(x)] = \ \textit{constant} \\ &(ii) \quad [f, D] \subset D + G \\ &(iii) \quad [g_j, D] \subset D + G, \ \ j = 1, \cdots, m \end{aligned} \tag{10}$$

Then for every point $x^o \in \mathcal{X}$ there exists a neighborhood V of x^o and a feedback (7) defined on it such that (9) holds on V. A distribution D satisfying (10) is called locally controlled invariant.

Proof Take local coordinates $x = (x_1, \cdots, x_k, x_{k+1}, \cdots, x_n) = (x^1, x^2)$ around x^o such that $D = \text{span}\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}\} =: \text{span}\{\frac{\partial}{\partial x^1}\}$. Write correspondingly

$$f = \begin{bmatrix} f^1 \\ f^2 \end{bmatrix}, g = \begin{bmatrix} g^1 \\ g^2 \end{bmatrix} \tag{11}$$

By (i) the $(n-k, m)$-matrix $g^2(x)$ has constant rank, say ℓ. Without loss of generality we may assume that the first ℓ rows of $g^2(x)$ are independent. First we state

Lemma 0.4. There exists, locally around x^o, an invertible matrix $\beta(x)$ such that for some $R(x)$

$$g^2(x)\beta(x) \quad = \quad \begin{array}{cc} \begin{bmatrix} I_\ell & 0_{\ell \times (m-\ell)} \\ R(x) & 0_{(n-k-\ell)\times(m-\ell)} \end{bmatrix} & \begin{array}{c} \uparrow \\ n-k \\ \downarrow \end{array} \\ \longleftarrow m \quad \longrightarrow & \end{array} \tag{12}$$

We postpone the proof of Lemma 0.4 till after the remaining proof of Theorem 0.3.
It follows from (12) that (unspecified elements denoted by $*$)

$$\tilde{g}(x) := g(x)\beta(x) = \left[\begin{array}{ll} * & * \\ I_\ell & 0 \\ R(x) & 0 \end{array}\right] \begin{array}{lc} \} & k \\ \} & \ell \\ \} & n-k-\ell \end{array} \tag{13}$$

Now (10 iii) yields for $i = 1, \cdots, k$

$$-\left[\tilde{g}, \frac{\partial}{\partial x_i}\right](x) = \left[\begin{array}{ll} * & * \\ 0 & 0 \\ \frac{\partial R}{\partial x_i}(x) & 0 \end{array}\right] \subset Im \left[\begin{array}{c} I_k \\ 0 \\ 0 \end{array}\right] + Im \left[\begin{array}{ll} * & * \\ I_\ell & 0 \\ R(x) & 0 \end{array}\right] \tag{14}$$

which necessarily implies that $\frac{\partial R}{\partial x_i}(x) = 0, i = 1, \cdots, k$, and thus that

$$\left[\tilde{g}, \frac{\partial}{\partial x_i}\right](x) \subset D(x), \quad i = 1, \cdots, k$$

or equivalently,

$$[\tilde{g}_j, D] \subset D, \quad j = 1, \cdots, m$$

Hence $\beta(x)$ is as required.
Finally, in order to construct $\alpha(x)$ we write f^2 according to the splitting (12) as

$$f^2 = \left[\begin{array}{c} f^{21} \\ f^{22} \end{array}\right]$$

Now we simply define

$$\alpha(x) = -\beta(x) \left[\begin{array}{c} f^{21}(x) \\ 0 \end{array}\right] \begin{array}{ll} \updownarrow & \ell \\ \updownarrow & m-\ell \end{array} \tag{15}$$

Then

$$\tilde{f} = f + g\alpha = \left[\begin{array}{c} f^1 \\ f^{21} \\ f^{22} \end{array}\right] - \left[\begin{array}{ll} * & * \\ I_\ell & 0 \\ R & 0 \end{array}\right] \left[\begin{array}{c} f^{21} \\ 0 \end{array}\right] = \left[\begin{array}{l} * \\ 0 \\ f^{22} - Rf^{21} \end{array}\right]$$

and (10 ii) implies for $i = 1, \cdots, k$

$$-\left[\tilde{f}, \frac{\partial}{\partial x_i}\right] = \left[\begin{array}{l} * \\ 0 \\ \frac{\partial}{\partial x_i}(f^{22} - Rf^{21}) \end{array}\right] \subset Im \left[\begin{array}{c} I_k \\ 0 \\ 0 \end{array}\right] + Im \left[\begin{array}{ll} * & * \\ I_\ell & 0 \\ R & 0 \end{array}\right] \tag{16}$$

showing, as above, that $\frac{\partial}{\partial x_i}(f^{22} - Rf^{21}) = 0, i = 1, \cdots, k$, or equivalently

$$[\tilde{f}, D] \subset D$$

$\square$

Proof of Lemma 0.4 Denote the matrix consisting of the first ℓ (independent) rows of $g^2(x)$ by $g^{21}(x)$. Then consider the equation

$$g^{21}(x)\beta(x) - [I_\ell \; 0_{\ell\times(m-\ell)}] = 0 \tag{17}$$

Clearly in x^0 this has an invertible solution $\beta(x^o)$ (this is just linear algebra; the crucial fact is that $g^{21}(x^o)$ has full row rank). Then by the implicit function theorem it follows that locally around x^o there exists an invertible solution $\beta(x)$ of (17), so that

$$g^2(x)\beta(x) = \begin{bmatrix} I_\ell & 0_{\ell\times(m-\ell)} \\ R(x) & * \end{bmatrix} \tag{18}$$

However since the rank of $g^2(x)$ is ℓ, also the rank of $g^2(x)\beta(x)$ is ℓ, and thus necessarily the unspecified elements $*$ have to be zero. $\square$

Example 0.5. Consider a linear system $\dot{x} = Ax + Bu$ on $\mathcal{X} = \mathbb{R}^n$, and let D be the distribution corresponding to a linear subspace $V \subset \mathbb{R}^n$, i.e., if $V = \text{span}\{e_1, \cdots, e_k\}$, then $D = \text{span}\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}\}$, where x are the linear coordinates corresponding to the basis choice $e_1, \cdots, e_n$. Then condition (10 i) is always satisfied (since $D(x)+G(x)$ corresponds to the linear subspace $V+ImB$), while (ii) amounts to $AV \subset V + ImB$. Finally (iii) is automatically satisfied since $[b_j, \frac{\partial}{\partial x_i}] = 0$ for the constant columns b_j of B.

Combining Proposition 0.1 and Theorem 0.3 leads to the following geometric condition for the (local) solvability of the disturbance decoupling problem.

Theorem 0.6. *Consider the nonlinear system (3). Suppose there exists an involutive and constant dimensional distribution D on $\mathcal{X}$ such that*

$$\begin{aligned} &(i) && [f, D] \subset D + G, \quad [g_j, D] \subset D + G \\ &(ii) && \dim(D+G) \text{ is constant} \\ &(iii) && D \subset \ker dh \\ &(iv) && p_j \in D, \quad j = 1, \cdots, s \end{aligned} \tag{19}$$

Then around each $x^o \in \mathcal{X}$ there exists a regular state feedback $u = \alpha(x) + \beta(x)v$ such that the closed-loop system is disturbance decoupled.

Example 0.7. Consider the following extension to Example 1.0.1: a cart with a fixed rear axis.

Assuming that the driving velocity and the steering velocity can be directly controlled we obtain the following model:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \varphi \\ \theta \end{bmatrix} = \begin{bmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \\ \sin\theta \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2 \tag{20}$$

Figure 1: Simple model of a cart

with u_1 the driving input, and u_2 the steering input.
Suppose now that there is additionally a disturbance vector field $p = (0\ 0\ 1\ \ -1)^T$ (this corresponds to sideways slipping of the rear axis). It can be checked that the distribution D spanned by p, that is

$$D = span\{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}\} \tag{21}$$

is involutive and constant dimensional, and satisfies (19). We may thus compute a decoupling feedback law using the approach of the proof of Theorem 3.3.23. Define $x_3 := \varphi + \theta, x_4 := \theta$. Then in the coordinates x_1, x_2, x_3, x_4 the dynamics including the disturbances are given as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \cos x_3 \\ \sin x_3 \\ \sin x_4 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} d \tag{22}$$

while $D = span\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\}$. Thus modulo permutations we have obtained coordinates as in the proof of Theorem 3.3.23, with

$$g^2(x) = \begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ \sin x_4 & 1 \end{bmatrix} \tag{23}$$

Consider now an arbitrary point $x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$. Suppose $\cos x_3^0 \neq 0$. Then the first and third row of $g^2(x)$ are independent, and we construct $\beta(x)$ such that

$$\begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ \sin x_4 & 1 \end{bmatrix} \beta(x) = \begin{bmatrix} 1 & 0 \\ * & 0 \\ 0 & 1 \end{bmatrix} \tag{24}$$

A possible solution is

$$\beta(x) = \begin{bmatrix} \frac{1}{\cos x_3} & 0 \\ & \\ -\frac{\sin x_4}{\cos x_3} & 1 \end{bmatrix} \tag{25}$$

If $\cos x_3^0 = 0$, then $\sin x_3^0 \neq 0$, in which case the second and third row of $g^2(x)$ are independent, and we construct $\beta(x)$ such that

$$\begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ \sin x_4 & 1 \end{bmatrix} \beta(x) = \begin{bmatrix} * & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{26}$$

yielding as possible solution

$$\beta(x) = \begin{bmatrix} \frac{1}{\sin x_3} & 0 \\ & \\ -\frac{\sin x_4}{\sin x_3} & 1 \end{bmatrix} \tag{27}$$

Note that both expressions (25) and (27) are only *locally* defined. In the present case, however, we may also find a *globally defined* $\beta(x)$ which solves the disturbance decoupling problem; in fact we may take

$$\beta(x) = \begin{bmatrix} 1 & 0 \\ & \\ -\sin x_4 & 1 \end{bmatrix} \tag{28}$$

transforming the input vectorfields g_1 and g_2 to

$$\tilde{g}_1 = \begin{bmatrix} \cos(\varphi + \theta) \\ \\ \sin(\varphi + \theta) \\ \\ \sin \theta \\ \\ -\sin \theta \end{bmatrix}, \tilde{g}_2 = \begin{bmatrix} 0 \\ \\ 0 \\ \\ 0 \\ \\ 1 \end{bmatrix} \tag{29}$$

Note that the new driving input vectorfield $\tilde{g}_1$ has the property that $\dot{\varphi} + \dot{\theta} = 0$, that is, $\varphi + \theta$ remains constant. This has the physical interpretation that along this vector field the front axis moves in the same direction, and that the rear axis tends to a position parallel to the front axis.