Bernoulli’s lightray solution for the brachistochrone problem from Hamilton’s viewpoint

Henk Broer

Johann Bernoulli Institute for Mathematics and Computer Science
Rijksuniversiteit Groningen
Summary

i. Introduction

ii. Bernoulli’s solution

iii. Translation into Hamilton’s terms

vi. Concluding remarks

A study in anachronism . . .


Introduction

Johann Bernoulli (1667 - 1748) ‘his’ brachistochrone problem

- Groningen period 1695 – 1705
- Acta Eruditorum 1697

*Opera Johannis Bernoullii.* G. Cramer (ed.; 4 delen), Genève 1742
Bernoulli’s lightray solution

Fermat principle of least time for light rays ⇾ optical solution in mechanical setting

- Discrete approximation in horizontal layers
  each layer homogeneous and isotropic
  ⇾ refraction index \( n = 1/v \ (c = 1) \)

- Fermat principle ⇒
  - inside layer:
    ray = straight line with velocity \( v = 1/n \)
  - at boundary: Snell’s law

Fermat and Snell

\[ n_1 \sin \alpha = n_2 \sin \beta \]

**Proof:** Let \( x = x_C \) indicate position of \( C \)

\[ t_{AC} = n_1 |A - C| \quad \text{and} \quad t_{CB} = n_2 |C - B| \]

where \( |-| \) denotes Euclidean metric
Fermat and Snell, ctd

By Pythagoras theorem:

$$|A - C| = \sqrt{x^2 + b^2}, \quad |C - B| = \sqrt{(a - x)^2 + c^2}$$

Differentiating $t_{AC}$ and $t_{CB}$ w.r.t. $x$:

$$\frac{d}{dx} t_{AC}(x) = \frac{n_1 x}{\sqrt{x^2 + b^2}} = n_1 \sin \alpha$$

$$\frac{d}{dx} t_{AC}(x) = -\frac{n_2 (a - x)}{\sqrt{(a - x)^2 + c^2}} = -n_2 \sin \beta$$

$$\leadsto \frac{d}{dx} (t_{AC} + t_{CB})(x) = 0 \Leftrightarrow n_1 \sin \alpha = n_2 \sin \beta \quad \square$$

- locally minimum
- globally caustics (think of focal points ellipse)
Bernoulli and Snell

- Snell’s law: $n_j \sin \alpha_j = n_{j+1} \sin \alpha'_{j}$
- Euclidean geometry: $\alpha'_j = \alpha_{j+1}$

$\rightsquigarrow$ Conservation law: $n_j \sin \alpha_j = n_{j+1} \sin \alpha_{j+1}$

just take limits . . . $\rightsquigarrow$ inclination $\alpha$ with vertical and conserved quantity $n \sin \alpha$
The cycloid as brachistochrone

Two conserved quantities as a function of \( y \)

\[
S = n(y) \sin \alpha(y) \quad (1)
\]
\[
E = \frac{1}{2} m(v(y))^2 + mg y \quad (2)
\]

Better look for profile \( \alpha \mapsto (x(\alpha), y(\alpha)) \)!

\( \mapsto \) Theorem.

\[
x(\alpha) = x_0 - \frac{1}{4S^2g} (2\alpha - \sin(2\alpha))
\]
\[
y(\alpha) = y_0 + \frac{1}{4S^2g} \cos(2\alpha)
\]

Note: \( E = \frac{1}{2} m(v(y_0))^2 + mg y_0 \),
Possible choices: \( v(y_0) = 0 \) and even \( y_0 = 0 \) \( \mapsto \) falling rate \( v(y) = \sqrt{-2gy} \)
A few high school computations

**Lemma.** (abbreviating \( v' = dv/dy \))

\[ v' = -\frac{g}{v(y)} \quad \text{and} \quad \cos \alpha = Sv'(y) \frac{dy}{d\alpha}, \]

**Proof:** Differentiate (2) with respect to \( y \) and (1) with respect to \( \alpha \)

Thus

\[ \frac{dy}{d\alpha} \text{ lemma} = \frac{1}{Sv'(y)} \cos \alpha \text{ lemma} = -\frac{v(y)}{Sg} v \cos \alpha = -\frac{1}{2S^2g} \sin(2\alpha) \quad (3) \]

while:

\[ \frac{dx}{d\alpha} = \tan \alpha \frac{dy}{d\alpha} \quad (3) = -\frac{v(y)}{Sg} \sin \alpha \quad (1) = -\frac{1}{2S^2g}(1 - \cos(2\alpha)) \]

Integration then gives the desired result.

**Cycloid** with radius \( \frac{1}{4S^2g} \) and rolling angle \( 2\alpha \)
Radius and rolling angle

Roll wheel (radius $\rho$) along ceiling $\leadsto$ cycloid:

$$x(\varphi) = \rho(\varphi + \sin \varphi),$$
$$y(\varphi) = \rho(1 - \cos \varphi),$$

parameter $\varphi$ called **rolling angle**
Christiaan Huygens (1629-1695)

Christiaan Huygens (by Gaspar Netscher)
and page from *Horologium Oscillatorium* 1673
- Challenge in Acta Eruditorum (1697)
  Many contemporaries published solutions
  Newton’s was anonymous, however, 
ex uNGue leONem COgnavi

- Isochrony and tautochrony cycloid dynamics
  by harmonic oscillations

William Rowan Hamilton

- Many contributions to mathematics, physics and astronomy, principle of least action

Fermat principle revisited

- Given curve $\tau \mapsto q(\tau)$ consider $\dot{q} = dq/d\tau$ and

$$dt = n(q(\tau)) \| \dot{q}(\tau) \| d\tau \quad \text{magic formula}$$

- To (locally) minimize $\int_{\tau_1}^{\tau_2} n(q(\tau)) \| \dot{q}(\tau) \| d\tau$ or, rather and equivalently,

$$\mathcal{I}(q) := \int_{\tau_1}^{\tau_2} \frac{1}{2} n^2(q(\tau)) \| \dot{q}(\tau) \|^2 d\tau$$

- Lagrangian $L(q, \dot{q}) = \frac{1}{2} n^2(q) \| \dot{q} \|^2$

($= \text{energy} = \text{kinetic energy}$)
Calculus of variations

- If \( q = (x, y) \) then Euler–Lagrange equations

\[
\frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \\
\frac{d}{d\tau} \frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y}
\]

necessary & sufficient for local \textbf{optimality} of \( \mathcal{I} \)

- under small variations of \( q \),
  while keeping endpoints \( q(\tau_1) \) and \( q(\tau_2) \) fixed

- in current optical setting also \textbf{minimal}

---

L. Euler, \textit{Leonhardi Euleri Opera Omnia}. 72 vols., Bern 1911-1975

Hamilton’s canonical formalism

- Legendre transformation:
  \[ \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \times \mathbb{R}^2, \]
  \[ (q, \dot{q}) \mapsto (q, p) \text{ with } p = n^2(q)\dot{q} \]

- Hamiltonian
  \[ H(q, p) = L(q, p/n^2(q)) = \frac{1}{n^2(q)} \|p\|^2 \]

**Theorem.** The light rays are the projections of the solutions of the **canonical equations**

\[ \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \quad (j = 1, 2) \]

- modern lingo in terms of (co-) tangent bundles
Laws, properties

- Energy conservation $\dot{H} \equiv 0$, say $H(q, p) = E$
- $q = (x, y)$ and $p_x = n^2(y) \dot{x}$, $p_y = n^2(y) \dot{y}$

$$H(x, y, p_x, p_y) = \frac{1}{2n^2(y)}(p_x^2 + p_y^2)$$

- $H$ independent of $x$ (cyclic variable): $\dot{p}_x \equiv 0$
  conservation of momentum, say $p_x = I$

- translation symmetry, Noether principle

- $\sim$ conservation of

$$\frac{I}{\sqrt{2E}}(x, y, \dot{x}, \dot{y}) = n(y) \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = n(y) \sin \alpha(\dot{x}, \dot{y})$$

$(= S$ for Snell, this is familiar !)
Reduction of the symmetry

**Theorem.** Given \( p_x = I \) can reduce to planar system

\[
H_I(y, p_y) = \frac{1}{2n^2(y)} p_y^2 + V_I(y) \quad \text{with} \quad V_I(y) = \frac{I^2}{2n^2(y)}
\]

*(effective potential)*

Reduced system:

\[
\begin{align*}
\dot{y} &= \frac{1}{n^2(y)} p_y \\
\dot{p}_y &= -\frac{n'(y)}{n^3(y)} \left( I^2 + p_y^2 \right)
\end{align*}
\]

Take \( I \neq 0 \)
Gravitational refraction index

Back to brachistochrone problem
Gravitational energy \( \frac{1}{2}(v(y))^2 + gy = 0 \mapsto \)

\[
(n(y))^2 = \frac{1}{-2gy}, \text{ for } y < 0
\]

\( \mapsto \) reduced canonical equations

\[
\begin{align*}
\dot{y} &= -2gyp_y \\
\dot{p}_y &= g(p_y^2 + I^2)
\end{align*}
\]

Level curve \( H_I(y, p_y) = E \) has form

\[
y = -\frac{1}{g} \left(\frac{E}{p_y^2 + I^2}\right) \quad (4)
\]
Reduced phase-portrait

Reduced phase-portrait
brachistochrone lightray dynamics
Reconstruction I

Time parametrization reduced system, for given $I \neq 0$ completely determined by $E, N$

- Time parametrization

$$d\tau = \frac{dp_y}{g(p_y^2 + I^2)} \leadsto \tau = \frac{1}{gI} \arctan \left( \frac{p_y}{I} \right)$$

- Gives parametrization (with $p_y(0) = 0$):

$$p_y = I \tan(gI\tau) \quad \leadsto \quad y = -\frac{E}{gI^2} \cos^2(gI\tau) \quad (5)$$

PS: Similar action for pendulum system leadsto elliptic integrals . . .
Reconstruction II

Cycloids recovered

- $I = p_x = n^2(y)x \Rightarrow$

\[
\dot{x} = \frac{1}{n^2(y)}I = -2Igy \quad (5)
\]

\[
\frac{2E}{I} \cos^2(gI\tau) = \frac{E}{I} (1 + \cos(2gI\tau))
\]

- And so

\[
x = x_0 + \frac{E}{2gI^2} (2gI\tau + \sin(2gI\tau))
\]

\[
y = -\frac{E}{2gI^2} (1 + \cos(2gI\tau))
\]

Cycloid radius $\varphi = E/2gI^2$, rolling angle $\varphi = 2gI\tau$

Inclination $\alpha = gI\tau$ (proportionality)
- Reminds of how Newton ‘principia’ yield the Keplerian orbits in central force field
- Nowadays brachistochrone problem exercise in course calculus of variations:

Look for curve $y = y(x)$

arclength

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (y')^2} \, dx$$

energy conservation in gravitational field

$$\sim \frac{ds}{dt} = \sqrt{-2gy}$$

so have to optimize . . .
Scholium I: ctd

- ... time

\[ dt = \frac{\sqrt{1 + (y')^2}}{\sqrt{-2gy}} \sim L(y, y') = \sqrt{-\frac{1 + (y')^2}{2gy}} \]

- Euler–Lagrange equations

\[ \sim (1 + (y')^2) y = \text{constant} \]

substitution of \((x, y) = (x(\theta), y(\theta))\) again gives cycloid result


Scholium II: lightrays geodesics

Georg Friedrich Bernhard Riemann 1826-1866

- Riemannian metric \( G_q(\dot{q}_1, \dot{q}_2) = n^2(q)\langle \dot{q}_1, \dot{q}_2 \rangle \), where latter brackets are Euclidean note

\[
L(q, \dot{q}) = \frac{1}{2} G_q(\dot{q}, \dot{q})
\]

- \( \rightsquigarrow \) lightrays are geodesics of metric \( G \)

Atmospherical optics

- blank strip in the setting Sun
- illusions: fata morgana’s, Nova Zembla phenomena


H.W. Broer, Aardse en hemelse luchtspiegelingen, met Bernoulli, Wegener en Minnaert. To be published Epsilon Uitgaven 2013
Arclength cycloid

arclength \( s = s(\varphi) \) (using Pythagoras):

\[
\begin{align*}
\arclength \ ds & = \sqrt{\arclength^2 \arclength + \arclength^2} \\
& = \sqrt{(\arclength^2 \arclength \arclength/ \arclength \varphi)^2 + (\arclength^2 \arclength \arclength \arclength \arclength / \arclength \varphi)^2} \arclength \varphi \\
& = 2 \rho \sqrt{2} \sqrt{1 + \cos \varphi} \arclength \varphi = 2 \rho \cos \frac{\varphi}{2} \arclength \varphi
\end{align*}
\]

\[\arclength \sim s(\varphi) = 4 \rho \sin \frac{\varphi}{2}\]
Cycloidal wire profile

Vertical height

\[ y(\varphi) = 2\rho \sin^2 \frac{\varphi}{2} = \frac{1}{8\rho} (s(\varphi))^2 \]

potential energy \( "V = mg y" \) : \( V(s) = \frac{mg}{8\rho} s^2 \)

\[ \rightarrow \text{equation of motion bead} \]

\[ s'' = -\frac{g}{4\rho} s : \]

a harmonic oscillator with \( \omega = \sqrt{g/(4\rho)} \)

Conclusion: cycloid isochronous curve (also tautochronous . . .)