Catastrophe Theory in Dulac Unfoldings

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Abstract

In this paper, we study a class of families of planar vector fields. Each member of the family consists of an autonomous vector field and a non-autonomous perturbation which is periodic in time. The autonomous part is the sum of a Hamiltonian vector field which does not depend on the parameters and a parameter dependent dissipative part. This setting, modulo suitable rescalings, covers for example the codimension \( k \) Hopf bifurcation and several cases of (subordinate) homoclinic bifurcation. In this setting we consider the non-autonomous part as a perturbation and mainly focus on the ‘unperturbed’ autonomous family of planar vector fields. Our interest is with the geometry of the bifurcation set of limit cycles, in particular in the neighbourhood of certain homoclinic bifurcations. In our approach the corresponding Hamiltonian vector field has a homoclinic loop to a hyperbolic saddle point. It is to be noted that the full system has the solid torus \( \mathbb{R}^2 \times \mathbb{S}^1 \) as its phase space where the ‘unperturbed’ limit cycles correspond to invariant 2-tori with parallel dynamics. It is to be expected that various kinds of quasi-periodic and chaotic dynamics persist for the ‘perturbed’ system. However, it is known that all of this is confined to a narrow neighbourhood of the bifurcation set of limit cycles. Returning to the ‘unperturbed’ family of planar vector fields, the limit cycles are in one to one correspondence with the zeros of a displacement function. The asymptotics of this latter does not take the form of a Taylor series but is a deformation of a Dulac expansion. An exponential scaling is constructed in the parameter space. As a result we generically find an exponentially narrow fine structure in the parameter space in which the bifurcation set of limit cycles can be completely described by Standard Catastrophe Theory.

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Contents

1 Introduction 2
   1.1 Background and setting of the problem 4
       1.1.1 Classical cases of polynomial geometry 5
       1.1.2 The homoclinic loop case 6
       1.1.3 Example: the codimension two saddle connection 7
       1.1.4 Outline 9
   1.2 Results 10
       1.2.1 The Dulac case 11
       1.2.2 The compensator case 14
   1.3 Application: the codimension three saddle connection 18
   1.4 Further applications 20

2 Proofs 22
   2.1 Proof of Theorem 1 26
   2.2 Proof of Theorem 2 31
   2.3 Proof of Theorems 3 and 4 37

1 Introduction

This paper contributes to the study of generic families of planar diffeomorphisms near a degenerate fixed point [1, 17, 47, 48, 49]. Note that such diffeomorphisms can be obtained from time periodic planar vector fields, by taking a Poincaré (return) map. Usually, by normal form or averaging techniques [1, 12], the family becomes autonomous to a large order. In many cases, after suitable rescaling the family is the sum of a Hamiltonian vector field which does not depend on the parameters, a dissipative part and a non autonomous part. The latter part consists of higher order terms in the rescaling parameter [5, 23, 25, 26, 48]. This approach covers many generic cases like the codimension $k$-Hopf bifurcations including degenerate Hopf (or Chenciner) and Hopf-Takens bifurcations [17, 20, 21, 22, 47, 48, 49] but also the Bogdanov-Takens bifurcation for maps [5, 10]. Our interest is with the geometry of the bifurcation set of limit cycles of the autonomous system which correspond to invariant tori in the full system. The latter were studied systematically by Broer, Braaksma, Huijtema and Takens [9]. Limit cycles of the autonomous system correspond to fixed points of the Poincaré map (or zeros of the displacement function) on one-dimensional sections. For values of the parameter that are relatively far from the bifurcation set of limit cycles associated to the autonomous system, the dynamics of the non autonomous system is of Morse-Smale type [43]. The genericity conditions on the original family often lead to certain explicit genericity conditions on the corresponding family of scaled displacement
functions [17, 20, 25, 26], which in our set-up will be translated to suitable hypotheses. By the Division Theorem, the latter lead to structural stability of a scaled displacement function under contact equivalence [31, 33, 40].

Starting point of the present paper will be the family of non-autonomous systems after rescaling, although our results are concerned with bifurcations of limit cycles in the autonomous truncation. In Example 1 below, we show the classical way in which this format is obtained, later on more classes of examples will be discussed. Before that, we note that for the full non-autonomous family, the autonomous bifurcation diagram provides a skeleton indicating the complexity of the actual dynamics [21, 51]. However, as Broer and Roussarie [17] have shown, this dynamical complexity near the bifurcation set of limit cycles, in the real analytic setting, is confined to an exponentially narrow horn. For earlier results in this direction, see [9].

**Example 1 (Bogdanov-Takens bifurcation [5, 12, 15, 24])** Consider the following unfolding of the codimension two bifurcation

\[ X_{\mu, \nu} = y \frac{\partial}{\partial x} + (x^2 + \mu + y \nu \pm y x + R(x, y, t, \mu, \nu)) \frac{\partial}{\partial y}, \]

where \( R \) is a 2\( \pi \)-periodic in the time \( t \) and represents the non autonomous part. We assume that \( |R(x, y, t, \mu, \nu)| = \mathcal{O}(x^2 + y^2)^{\frac{3}{2}} \). Under the classical rescaling [5, 15]

\[ x = \varepsilon^2 \bar{x}, \quad y = \varepsilon^3 \bar{y}, \quad \mu = -\varepsilon^4, \quad \nu = \varepsilon^2 \bar{\nu}, \]

the system above becomes

\[ \varepsilon \bar{Y}_{\varepsilon, \bar{\nu}} = \varepsilon \bar{y} \frac{\partial}{\partial \bar{x}} + \varepsilon \left( (\bar{x}^2 - 1) + \varepsilon \bar{y} \bar{\nu} \pm \varepsilon \bar{y} \bar{x} + \mathcal{O}(\varepsilon^3) \right) \frac{\partial}{\partial \bar{y}}. \]

We see that \( \bar{Y}_{\varepsilon, \bar{\nu}} = X_H + \varepsilon Z_{\varepsilon, \bar{\nu}} + \mathcal{O}(\varepsilon^2) \), where \( X_H = \bar{y} \frac{\partial}{\partial \bar{x}} + (\bar{x}^2 - 1) \frac{\partial}{\partial \bar{y}} \) is Hamiltonian which does not depend on the parameters \((\bar{\nu}, \varepsilon)\) and possesses a homoclinic loop to a hyperbolic saddle. The loop encloses a disc containing a center. Notice that the non-autonomous terms are included in the \( \mathcal{O}(\varepsilon^3) \)-part. See [15, 17] for details.

This example motivates the general perturbation setting of our paper. More examples will be recalled below [12, 15, 18, 25, 26] and certain new examples will be treated in detail. Our focus will be with bifurcations of limit cycles near a homoclinic trajectory of the Hamiltonian vector field. By an explicit exponential scaling in the parameter space, an exponentially narrow horn is created, in which the bifurcation set of limit cycles can be completely described. Indeed the structural stability of a scaled displacement function under contact equivalence, leads to the full complexity of Catastrophe Theory. For the original autonomous family of vector fields, inside the horn, this yields structural stability under weak orbital equivalence [6, 27, 28], where the reparametrizations are \( C^\infty \). In summary, our results reveal
an exponentially narrow fine structure of the parameter space, that regards the dynamics of the autonomous family of planar vector fields. We expect that our results will be of help when studying the full system on the solid torus \( \mathbb{R}^2 \times S^1 \), where the ‘unperturbed’ limit cycles correspond to invariant 2-tori with parallel dynamics. When turning to the non-autonomous ‘perturbed’ system, persistence of these tori becomes a matter of KAM Theory, in particular of quasi-periodic bifurcation theory [6, 11]. Indeed, the 2-tori with Diophantine frequencies form a Cantor foliation of hypersurfaces in the parameter space and we conjecture that most of the Diophantine tori persist, including their bifurcation pattern. The KAM persistence results involve a Whitney smooth reparametrization which is near the identity-map in terms of \( |\varepsilon| \ll 1 \). This means that the perturbed Diophantine 2-tori correspond to a perturbed Cantor foliation in the parameter space. In the complement of this perturbed Cantor foliation of hypersurfaces we expect all the dynamical complexity regarding Cantori, strange attractors, etc., as described in [19, 20, 21, 22, 38, 42].

1.1 Background and setting of the problem

The general perturbation format of the present paper is given by the family of \( C^\infty \) planar, time-periodic vector fields

\[
\begin{align*}
X_{\lambda, \varepsilon} &= \varepsilon' R \\
X_{\lambda, \varepsilon} &= \varepsilon^p X_H + \varepsilon^q Y_{\lambda, \varepsilon},
\end{align*}
\]

where \( X_{\lambda, \varepsilon} \) consists of the autonomous part. The following properties hold. First \( 0 \leq p < q < r \) are integers. Secondly, \( \lambda \in \mathbb{R}^\ell \) is a (multi)-parameter varying near 0, while \( \varepsilon > 0 \) is a perturbation parameter. Thirdly, the vector field \( R \) is 2\( \pi \)-periodic in the time \( t \), and may depend on all the other variables and parameters as well. Finally, the vector field \( X_H \) is Hamiltonian and does not depend on the parameters. Notice that in Example 1, we have \( p = 1, q = 2 \) and \( r = 3 \). We begin discussing the Hamiltonian function \( H \). Our genericity setting implies that \( H \) should be a stable Morse function, which leaves us with three possible cases. \( X_H \) is defined in an annulus, near a center or near a homoclinic loop, see figure 1.

![Figure 1: The Hamiltonian \( X_H \) in a annulus (a), near a center (b), near a homoclinic loop (c).](image)

In each case, we can define a Poincaré return map

\[
\mathcal{P}_{\lambda, \varepsilon} : \Sigma_0 \to \Sigma, \quad \text{with} \quad \mathcal{P}_{\lambda, \varepsilon}(x) = x + \varepsilon^{q-p} B_{\lambda, \varepsilon}(x).
\]
where $\Sigma_0 \subset \Sigma$ is a transversal section and where $B_{\lambda, \varepsilon}(x)$ is the scaled displacement function. Limit cycles are in a one to one correspondence with the zeros of $B_{\lambda, \varepsilon}$. The bifurcation set $\mathcal{B}$ of limit cycles is defined as

$$\mathcal{B} = \{(\lambda, \varepsilon) \in \mathbb{R}^d \times \mathbb{R}^+ | B_{\lambda, \varepsilon}'(x) = 0 \text{ for some } x \in \Sigma_0\}.$$ 

Denote by $\mathcal{B}_{\varepsilon_0} = \mathcal{B} \cap \{\varepsilon = \varepsilon_0\}$. The family $Y_{\lambda, \varepsilon}$ is generic in a sense to be made precise below and which by the Division Theorem [28, 33, 37, 40] implies $C^\infty$ structural stability under contact equivalence of the scaled displacement function $B_{\lambda, \varepsilon}$ in the following sense. In the following definition the families of functions are local, i.e., we consider germs of families.

**Definition 1** (*Contact Equivalence* [6, 27, 37, 44, 50])

Let $f_\lambda : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and $g_\mu : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be two families of functions, with $\lambda, \mu \in (\mathbb{R}^d, 0)$. These families are $C^\infty$-contact equivalent if there exist a $C^\infty$ diffeomorphism $\varphi : (\mathbb{R}^d, 0) \to (\mathbb{R}^d, 0)$, a $C^\infty$ family of diffeomorphisms $h_\lambda : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and a $C^\infty$ function $U : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ with $U(0) \neq 0$ such that

$$f_\lambda \circ h_\lambda(u) = U(u) \cdot g_{\varphi(\lambda)}(u).$$

The definition implies that $h_\lambda$ sends the zero-set of $f_\lambda$ to the zero-set of $g_{\varphi(\lambda)}$ preserving multiplicity. As a consequence for small values of $\varepsilon$, the zero set $\mathcal{B}_{\varepsilon}$ is diffeomorphically equivalent to $\mathcal{B}_0$. Although our main interest is with the homoclinic loop case, we first briefly revisit the simpler cases of an annulus and a center.

### 1.1.1 Classical cases of polynomial geometry

In the case of an annulus (see figure 1-(a)), by our assumption of structural stability under contact equivalence and by the Division Theorem [28, 33, 37, 40], $B_{\lambda, \varepsilon}$ takes the form

$$B_{\lambda, \varepsilon}(x) = U(x, \lambda, \varepsilon)Q_k^{\pm}(x, \beta(\lambda, \varepsilon)).$$

(3)

In (3), $U \neq 0$ is smooth and

$$Q_k^{\pm}(x, \beta(\lambda, \varepsilon)) = \beta_0(\lambda, \varepsilon) + \beta_1(\lambda, \varepsilon)x + \cdots + \beta_{k-2}(\lambda, \varepsilon)x^{k-2} \pm x^k,$$

(4)

which is the *Standard Catastrophe Cuspoid* normal form. Here, the map

$$\lambda \mapsto (\beta_0(\lambda, 0), \ldots, \beta_{k-2}(\lambda, 0))$$

is a submersion. The codimension of the singularity is $k - 1$. For $\varepsilon$ sufficiently small the bifurcation set $\mathcal{B}_{\varepsilon}$ consists of all $\lambda$-values such that

$$Q_k^{\pm}(x, \beta(\lambda, \varepsilon)) = \frac{dQ_k^{\pm}(x, \beta(\lambda, \varepsilon))}{dx}(x, \beta(\lambda, \varepsilon)) = 0,$$

$\$
for some $x$ near 0. Since $Q_k^\pm$ takes the form (4), we say that for sufficiently small $\varepsilon$, the geometry of the bifurcation set of limit cycles $B_\varepsilon$ is polynomial. By studying the zero set $B_0$ using Standard Catastrophe Theory [6, 30, 44, 54], we recover the generic bifurcation theory of limit cycles as it now has become standard [6, 4, 27, 30]. For instance in the case $k = 2$, the bifurcation set is called a fold corresponding to a saddle node of limit cycles, in the case $k = 3$ one obtains a cusp of limit cycles.

In the case of a center (see figure 1-(b)), by similar arguments, the reduced displacement function takes the form

$$B_{\lambda, \varepsilon}(u) = U(u, \lambda, \varepsilon)Q_k^{\pm}(u, \alpha(\lambda)),$$

where $u = r^2$, $r$ being the distance to the origin, with the Pointed Catastrophe Cuspoid normal form

$$Q_k^{\pm}(u, \alpha) = \alpha_0(\lambda, \varepsilon) + \alpha_1(\lambda, \varepsilon)u + \cdots + \alpha_{k-1}(\lambda, \varepsilon)u^{k-1} \pm u^k, \ u \geq 0. \quad (5)$$

Remark that $Q_k^{\pm}$ possesses a term of order $k - 1$ while $Q_k^{\pm}$ does not. For more details, see [17]. Again, the map

$$\lambda \mapsto (\alpha_0(\lambda, 0), \ldots, \alpha_{k-1}(\lambda, 0))$$

is a submersion and the codimension of the singularity is $k$. For each $\varepsilon$ sufficiently small, the set $B_\varepsilon$ consists of all $\lambda$-values such that

$$Q_k^{\pm}(u, \alpha(\lambda, \varepsilon)) = \frac{dQ_k^{\pm}}{du}(u, \alpha(\lambda, \varepsilon)) = 0,$$

for some $u > 0$. As in the annulus case, for each sufficiently small $\varepsilon$, since $Q_k^{\pm}$ takes the form (5), the geometry of the bifurcation set of limit cycles $B_\varepsilon$ is polynomial. $B_\varepsilon$ is now described by the Pointed Catastrophe Theory. This set-up recovers all codimension $k$ Hopf bifurcations [17, 20, 49].

### 1.1.2 The homoclinic loop case

Our main interest in the present paper is with the case where the Hamiltonian $H$ possesses a homoclinic loop $\Gamma$ (see figure 1-(c)). Here the Poincaré return map is defined on a half section $\Sigma_0 \subset \Sigma$ parameterized by $u \geq 0$. The difference between the present and the above cases is that the displacement function is singular at $u = 0$ and then cannot be written as a Taylor expansion at this value. When $\varepsilon = 0$, the displacement function takes the form of a Dulac expansion [3, 34, 35, 45] and when $\varepsilon \neq 0$, the form of a compensator expansion [34, 35, 45] which deforms the Dulac expansion. We shall be more precise in the next section.

The aim of this paper is to present a Division Theorem which is adapted to Dulac and compensator unfoldings. We shall construct a singular scaling in the parameter space so that the corresponding family of displacement functions becomes
structurally stable under contact equivalence, and thanks to the classical Division Theorem, takes the form (3). This implies that the bifurcation set $B_c$ has polynomial geometry. The function $U$ in (3), although smooth in the interior of the horn, contains all nonregularity of the displacement function, i.e., has Dulac asymptotics at the tip of the horn. The region in the phase space to be considered, though arbitrarily close to $\Gamma$, does not contain $\Gamma$. However, the region does contain all the (bifurcating) limit cycles.

1.1.3 Example: the codimension two saddle connection

To fix thoughts, before stating our results, we announce an application in the case of a codimension 2 degenerate homoclinic orbit [17, 25, 26], as it is subordinate to a degenerate Bogdanov-Takens bifurcation.

**Example 2** (DEGENERATE BOGDANOV-TAKENS BIFURCATION [17, 25, 26])

Consider the following unfolding of the codimension three Bogdanov-Takens bifurcation

$$X_{\mu,\nu_0,\nu_1} = y \frac{\partial}{\partial x} + (x^2 + \mu + y\nu_0 + \nu_1 yx \pm yx^3) \frac{\partial}{\partial y}.$$  

This family is studied in [25, 26]. The bifurcation set is a topological cone with vertex at $0 \in \mathbb{R}^3$, which is transverse to the 2 sphere $S^2$. The trace of the bifurcation set on $S^2$ possesses, among various bifurcation points of codimension two, a point of degenerate saddle connection. After suitable rescaling [17], the family here takes the form

$$X_{\nu,\tau_0,\tau_1} = \epsilon \nu \frac{\partial}{\partial \tau} + \epsilon (\tau^2 - 1 + \epsilon^5 (\tau_0^2 + \tau_1 \tau \pm \tau_1 \tau^3)) \frac{\partial}{\partial \eta}.$$ 

In term of our general set-up, the family takes the form (1) with $p = 1, q = 6$ and

$$H(\tau, \eta) = \frac{1}{2} \tau^2 - \frac{1}{3} \tau^3 + \tau.$$ 

The phase portrait of $X_H$ is described in figure 1-(c). The scaled displacement function takes the form

$$B_{\nu,\epsilon}(u) = \alpha_0(\nu_0, \nu_1, \epsilon) + \beta_1(\nu_0, \nu_1, \epsilon)\omega_\epsilon(u) + cu + o(u), \quad (6)$$

where $\omega_\epsilon(u)$ is a compensator function [12, 17, 34, 35, 36] and where $\omega_0(u) = \log u$, where $u$ is the distance to the saddle connection. In this example, it is generic to assume that $\epsilon \neq 0$ and that the map

$$(\nu_0, \nu_1) \mapsto (\alpha_0(\nu_0, \nu_1, 0), \beta_1(\nu_0, \nu_1, 0))$$

is a local diffeomorphism near 0. For simplicity, we restrict to the case $\epsilon = 0$, since the case $\epsilon \neq 0$ is just more complicated without being qualitatively different. Theorem 2 gives the following scaling

$$\Psi : (\mathbb{R}^2, \mathbb{R}^2, \mathbb{R}^2, \mathbb{R}^2, \mathbb{R}^2) \mapsto (0, \tau_0) \mapsto (\gamma_0, \tau) \mapsto (\alpha_0, \beta_1),$$

7
where $A > 0$, $\tau_0 > 0$ and where
\[
\begin{align*}
\beta_1 &= -\frac{c + \tau \log \tau Q_1(\tau)}{1 + \log \tau}, \\
\alpha_0 &= \frac{\gamma_0 \tau}{1 + \log \tau} - \frac{-c\tau + \tau^2 \log \tau R_0(\tau)}{1 + \log \tau},
\end{align*}
\]
where $Q_1$ and $R_0$ are $C^\infty$, see Figure 2. We claim that as an application of our results,
\[
B_{\psi(\gamma_0,\tau),0} \left( \tau(1 + x) \right) = \frac{\tau}{1 + \log \tau} \left( \gamma_0 - \left( \frac{1}{2} + \mathcal{O}(\tau \log \tau) \right)x^2 + \mathcal{O}(x^3) \right),
\]
and by the Division Theorem, we have
\[
B_{\psi(\gamma_0,\tau),0} \left( \tau(1 + x) \right) = \frac{\tau}{1 + \log \tau} U(x,\gamma_0,\tau) \left( \gamma_0 - \frac{x^2}{2} \right),
\]
where $\tilde{U}$ is $C^\infty$ and has Dulac asymptotics at $\tau = 0$, see below. Therefore
\[
B_{\psi(\gamma_0,\tau),0} \left( \tau(1 + x) \right) = U(x,\gamma_0,\tau) \left( \gamma_0 - x^2 \right),
\]
where
\[
U(x,\gamma_0,\tau) = \frac{\tau}{1 + \log \tau} \tilde{U}(x,\gamma_0,\tau),
\]
see Definition 1.

![Figure 2: Image of a rectangle $(-A,A) \times (0,\tau_0)$ under the scaling $\Psi$. On the right hand side the corresponding region is an exponentially flat horn. The resulting bifurcation set $B_0$, located in the dashed region on the right hand side, is the product of a fold bifurcation set and an interval.](image)

The idea of the scaling in (7) is the following. Observe that the form (6) contains a logarithmic term and therefore is singular at $u = 0$. However, the form is no longer singular at $u = \tau$ for any $\tau > 0$ arbitrarily small. By putting $u = \tau(1 + x)$, we localize the study of (7) near $u = \tau$, i.e., near $x = 0$. The map
\[
\Phi_{\psi(\gamma_0,\tau)}(x) = B_{\psi(\gamma_0,\tau),0} \left( \tau(1 + x) \right)
\]
is regular and can be expanded in a Taylor series. All nonregularity of the unfolding is now contained in the parameter dependence (7).

From (7), we have
\[ a_0 \sim -(\gamma_0 - c)\beta_1 \frac{e^{-\frac{1}{\beta_1}}}{e}. \]

Since \( \gamma_0 \) belongs to an interval of length \( 2A \), for fixed \( \beta_1 \), we have the following exponential estimate
\[ a_0 \sim \frac{2A\beta_1}{e} e^{-\frac{1}{\beta_1}}, \beta_1 > 0. \tag{9} \]

The bifurcation set of limit cycles is given by \( B_0 = \Psi(\mathcal{F}) \) where
\[ \mathcal{F} = \{ \gamma_0 = 0 \}, \]

which is the product of a fold and an interval and corresponds to a saddle node bifurcation of limit cycles.

As a consequence of the estimate (9), the image of the rescaling is an exponentially flat region, see Figure 2. Observe that the set of homoclinic bifurcation which is given by \( H = \{ a_0 = 0 \} \) is contained in the image of the rescaling introduced in (7). However, since the study of the displacement function is localized near \( u = \tau \ll \exp\left(-\frac{1}{|\beta_1|}\right) \), the region of the phase space under consideration is an exponentially narrow annulus, which does not contain the homoclinic orbit. Therefore, the subordinate homoclinic bifurcation cannot be studied with help of the form (8). Up to division by \( \beta_1 \), the displacement function is polynomial; indeed it is the normal form of a fold [44, 54].

As mentioned before, the stability of the displacement function under contact equivalence implies weak orbital stability of the vector field unfolding with \( C^\infty \) reparametrization.

Remark: In Example 2, we find a saddle-node of limit cycles inside the horn, corresponding to a fold catastrophe of the scaled displacement function. We mention that in [10, 17] similarly stable limit cycles are discovered that correspond to scaled displacement functions of Morse type.

1.1.4 Outline

This paper is organized as follows. All results will be formulated in section 1.2. We first study the rescaled displacement function \( B_{\lambda,\varepsilon} \) when \( \varepsilon = 0 \), in which case \( B_{\lambda,0} \) has a Dulac expansion. In the Theorems 1 and 2 we present normal forms for \( B_{\lambda,0} \), showing that the geometry of its zero-set again is polynomial. Also we give an explicit algorithm for the normalizing rescaling. We mention that Theorem 2 generalizes the example of section 1.1.3, extending it to all cuspoid bifurcations of limit cycles of even degree. Second we treat the case \( \varepsilon > 0 \), in which the scaled displacement function will have a so-called compensator expansion, which deforms
the above Dulac case. The Theorems 3 and 4 are the analogues of Theorems 1 and 2 for this compensator case.

Section 1.3 contains an example concerning a cusp bifurcation of limit cycles, as it is subordinate to a codimension 4 Bogdanov Takens bifurcation [32, 34, 35]. This example is an application of Theorem 1. In section 1.4 we discuss further applications of our results for the full ‘perturbed’ system, where our statements are mainly conjectural and pointing to future research. All proofs are postponed to section 2.

1.2 Results

We briefly recall the general setting. The family of planar autonomous vector fields to be considered has the form

\[ X_{\lambda, \varepsilon} = \varepsilon^p X_H + \varepsilon^q Y_{\lambda, \varepsilon}, \]

where \( X_H \) possesses a homoclinic orbit \( \Gamma \subset \{ H = 0 \} \). We focus our study near this homoclinic loop. For all \( u < u_0 \) where \( u_0 \) is close to 0, orbits of \( X_H \) contained in \( \Gamma_u \subset \{ H = u \} \) are supposed to be periodic, \( \Gamma_u \) tending to the homoclinic loop \( \Gamma \) as \( u \) tends to 0 in the Hausdorff topology. The subsection \( \Sigma_0 \subset \Sigma \) (see Figure 1), on which the Poincaré return map is well defined, transversally intersects each of these periodic orbits. The Poincaré return map takes the form (2).

The asymptotics of \( B_{\lambda, \varepsilon} \) is given by Roussarie [45]. For convenience, we introduce the 1-form \( \Omega \), which is dual to the vector field \( X_{\lambda, \varepsilon} \) by the standard area form \( \omega \) on \( \mathbb{R}^2 \), i.e.,

\[ \epsilon_{\lambda, \varepsilon} \Omega = \varepsilon^p dH + \varepsilon^q \omega_{\lambda, 0} + o(\varepsilon). \]

We have

\[ B_{\lambda, 0}(u) = \int_{\Gamma_u} \omega_{\lambda, 0}. \] (10)

This Abelian integral expands on a logarithmic scale, to be described below. Such an expansion is called a Dulac expansion. In Theorem 1 and 2 a scaling in the parameter space

\[ \Psi : \mathbb{R}^{k-1} \times \mathbb{R}^+ \rightarrow \mathbb{R}^k, (\gamma, \tau) \mapsto \Psi(\gamma, \tau) \]

is constructed in such a way that

\[ B_{\Psi(\gamma, \tau), 0}(\tau(1 + x)) = \tilde{f}(\tau)(\gamma_0 + \gamma_1 x + \cdots + \gamma_{k-2} x^{k-2} + k(\gamma, \tau) x^k + O(x^{k+1})), \] (11)

where \( \tilde{f}(\tau) = \tau^n \log \tau \), in the case \( k = 2n - 1 \) and \( \tilde{f}(\tau) = \frac{\tau^n}{(1 + \log \tau)} \), in the case \( k = 2n \) and where \( k(0) \neq 0 \). The idea behind the construction of the scaling \( \Psi \) is the following. By putting \( u = \tau(1 + x) \), the map

\[ \Phi_{\alpha, \beta} : \mathbb{R} \rightarrow \mathbb{R}_+ x \mapsto B_{\alpha, \beta, 0}(\tau(1 + x)) \]
becomes $C^\infty$ at the origin, and we expand $\Phi_{\alpha,\beta}$ in a Taylor series. In the generic case, we may identify the coefficients of the Dulac expansion with $\lambda$. First, in the $(\alpha, \beta)$-space, a parametrization by $\tau$ is given of the curve $C$ defined by

$$\mathcal{C} = \{(\alpha, \beta) \in \mathbb{R}^d \mid \Phi_{\alpha,\beta}(0) = \Phi'_{\alpha,\beta}(0) = \cdots = \Phi_{\alpha,\beta}^{(k-1)}(0) = 0, \ \Phi_{\alpha,\beta}^{(k)}(0) \neq 0\}.$$ 

This latter curve emanating from the origin is of 'highest degeneracy', i.e., along this curve

$$\Phi_{\psi(0,\tau)}(x) = \tilde{f}(\tau)(k(0, \tau)x^k + \mathcal{O}(x^{k+1})).$$

Secondly, the scaling $\Psi$ is defined. The image of this latter is a narrow neighbourhood of $\mathcal{C}$ and for each parameter value in this neighbourhood, the map $\Phi_{\psi(\gamma, \tau)}$ takes the form (11). By the Division Theorem [28, 33, 37, 40] applied to (11), we obtain

$$\Phi_{\psi(\gamma, \tau)}(x) = \tilde{U}(x, \gamma, \tau)\tilde{f}(\tau)(\gamma_0 + \gamma_1 x + \cdots + \gamma_{k-2} x^{k-2} \pm x^k),$$

$$= U(x, \tau, \gamma)Q_k^{(n)}(x, \gamma),$$

where $U(x, \tau, \gamma) = \tilde{f}(\tau)\tilde{U}(x, \gamma, \tau)$. All nonregularity of the unfolding is hidden in the function $U$ and after division by this latter, the displacement function is polynomial. In a final step, we consider the complete compensator expansion given in [45]. In Theorem 3, respectively 4, we show that there exists an analytic deformation of the scaling proposed in Theorem 1, respectively 2, such that $B_{\psi(\gamma, \tau), \epsilon}$ takes again the form (11).

### 1.2.1 The Dulac case

We now give a precise description of the Dulac case. The logarithmic scale of functions is given by

$$\mathcal{L} = \{1, u \log u, u, \ldots, u^i, u^{i+1} \log u, \cdots\}.$$ 

The Abelian integral (10) expands on this scale [45, 34, 35, 36]. This means that there exist sequences of $C^\infty$ functions $\alpha_i(\lambda)$ and $\beta_j(\lambda)$ for integers $i \geq 0$ and $j \geq 1$ such that

$$B_{\lambda,0}(u) = \sum_{i=0}^{N} \alpha_i u^i + \sum_{j=1}^{N} \beta_j u^j \log u + o(u^N),$$

for any $N \in \mathbb{N}$.

We say that (12) is a nondegenerate Dulac unfolding of order $2n$ if $\alpha_n \neq 0$ but for each integer $i = 0, \ldots, n - 1$ one has $\beta_{i+1}(0) = 0 = \alpha_i(0)$ and if moreover, the map

$$\lambda \mapsto (\alpha_0(\lambda), \beta_1(\lambda), \ldots, \alpha_{n-1}(\lambda), \beta_n(\lambda))$$
is a submersion. In this case, we rewrite (12) to

\[ B_{\lambda,0}(u) = \sum_{i=0}^{n-1} \alpha_i(\lambda)u^i + \sum_{i=1}^{n} \beta_i(\lambda)u^i \log u + cu^n + o(u^n), \]

where \( c = \alpha_n(\lambda) \).

Similarly we say that (12) is a nondegenerate Dulac unfolding of codimension \( 2n - 1 \) if \( \beta_n(0) \neq 0 \) but \( \alpha_0(0) = 0 \) for each integer \( i = 1, \ldots, n - 1 \) one has \( \beta_i(0) = 0 = \alpha_i(0) \) and if moreover, the map

\[ \lambda \mapsto (\alpha_0(\lambda), \beta_1(\lambda), \ldots, \alpha_{n-2}(\lambda), \beta_{n-1}(\lambda), \alpha_{n-1}(\lambda)) \]

is a submersion. In this case, we rewrite (12) to

\[ B_{\lambda,0}(u) = \sum_{i=0}^{n-1} \alpha_i(\lambda)u^i + \sum_{i=1}^{n-1} \beta_i(\lambda)u^i \log u + cu^n \log^2 + o(u^n), \]

where \( c = \beta_n(\lambda) \). We introduce the following notation. Let \( F : (\mathbb{R},0), \rightarrow \mathbb{R} \) be a Dulac expansion of the form (12). We then write

\[ F(u) = cu^n \log^\ell(u) + \cdots = cu^n \log^\ell(u) + \sum_{m, \ell} c_{\ell, j}u^k \log^j(u), \quad (13) \]

where the order \( << \) on the set

\[ \mathbb{N}^a = \{(a, b) \in \mathbb{N} \times \mathbb{Z}, \; b \leq a\}, \]

is defined as follows

\[ (\ell, j) << (\ell + 1, k) \forall j \leq \ell, \; k \leq \ell + 1, \; \text{and} \; (\ell, j-1) << (\ell, j). \quad (14) \]

In what follows \( \mathbb{R}_+ \) denotes the set of strictly positive real numbers. We now state the first results.

**Theorem 1 (DULAC CASE OF ODD CODIMENSION)** Let

\[ B : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}, \]

\[ (u, \alpha, \beta) \mapsto B_{\alpha,\beta}(u) = \sum_{i=0}^{n-1} \alpha_iu^i + \sum_{i=1}^{n-1} \beta_iu^i \log u + cu^n \log u + \cdots, \]

be a nondegenerate Dulac unfolding of codimension \( 2n - 1 \). Then

\[ B_{\Psi}(\gamma, \tau)(\tau(1+x)) = k(\gamma, \tau)\tau^n \log \tau \left( Q^{\pm}_{2n-1}(x, \gamma) + o(x^{2n}) \right), \quad (15) \]

where \( k(\gamma, \tau) \neq 0 \) depends \( C^\infty \) on \( (\gamma, \tau) \), and where

\[ \Psi : \mathbb{R}^{2n-2} \times \mathbb{R}_+ \rightarrow \mathbb{R}^{2n-1}, \quad (\gamma, \tau) \mapsto \left( \alpha(\gamma, \tau), \beta(\gamma, \tau) \right). \]
The map $\Psi$ is $C^\infty$ and of the form

$$\begin{cases}
\alpha_i = \tau^{n-i}\log^2\tau \tilde{a}_i(\gamma, \tau), & 0 \leq i \leq n - 1, \\
\beta_i = \tau^{n-i}\log \tau \tilde{b}_i(\gamma, \tau), & 1 \leq i \leq n - 1.
\end{cases} \quad (16)$$

We further specify

$$\tilde{b}_i(\gamma, \tau) = -ca_i + V_i(\gamma) - \log^{-1}\tau \tilde{R}_{n-1+i}(\tau),$$

$$\tilde{a}_{n-1}(\gamma, \tau) = -\sum_{j=1}^{n-1}(L_{n-1,j}\log^{-1}\tau + H_{n-1,j})\tilde{b}_j(\gamma, \tau) + d_{n-1}\log^{-1}\tau + \gamma_{n-1}\log^{-1}\tau - (cL_{n-1,n} + \tilde{P}_{n-1}(\tau))\log^{-2}\tau,$$

and for each integer $i = 0, \ldots, n - 2$,

$$\tilde{a}_i(\gamma, \tau) = -\sum_{j=i+1}^{n-1}J_{i,j}\tilde{a}_j(\gamma, \tau) - \sum_{j=1}^{n-1}(L_{i,j}\log^{-1}\tau + H_{i,j})\tilde{b}_j(\gamma, \tau) + d_i\log^{-1}\tau + \gamma_i\log^{-1}\tau - (cL_{i,n} + \tilde{P}_i(\tau))\log^{-2}\tau.$$

For each pair of integers $(i, j)$ under consideration, the coefficients $L_{i,j}, H_{i,j}, J_{i,j}, a_i, d_i$ are real numbers. Moreover $V_i$ is a linear map and

$$\tilde{P}_i(\tau) = \tilde{P}_{i,0} + \tau \log \tau \tilde{P}_{i,1} + \cdots,$$

$$\tilde{R}_{n-1+i}(\tau) = \tilde{R}_{n-1+i,0} + \tau \log \tau \tilde{R}_{n-1+i,1} + \cdots.$$

**Theorem 2** (DULAC CASE OF EVEN CODIMENSION) Let

$$B : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

$$(u, \alpha, \beta) \mapsto B_{\alpha, \beta}(u) = \sum_{i=0}^{n-1} \alpha_i u^i + \sum_{i=1}^{n} \beta_i u^i \log u + [u^n + \cdots],$$

be a nondegenerate Dulac unfolding of codimension $2n$. Then

$$B_{\Psi(\gamma, \tau)} \left( \tau(1 + x) \right) = \frac{k(\gamma, \tau)\tau^n}{1 + a_n \log \tau} \left( Q_{2n}^{\pm}(x, \gamma) + O(x^{2n+1}) \right), \quad (17)$$

where $k(\gamma, \tau) \neq 0$ depends $C^\infty$ on $(\gamma, \tau)$ and

$$\Psi : \mathbb{R}^{2n-1} \times \mathbb{R}_+ \to \mathbb{R}^{2n}, \quad (\gamma, \tau) \mapsto \left( \alpha(\gamma, \tau), \beta(\gamma, \tau) \right).$$

The map $\Psi$ is $C^\infty$ and of the form

$$\begin{cases}
\alpha_i = \tau^{n-i}\tilde{a}_i(\gamma, \tau), & 0 \leq i \leq n - 1, \\
\beta_i = \frac{\tau^{n-i}\tilde{b}_i(\gamma, \tau)}{1 + a_n \log \tau}, & 1 \leq i \leq n.
\end{cases} \quad (18)$$
We further specify
\[
\tilde{\beta}_n(\gamma, \tau) = -\tilde{\alpha}_n + (1 + \tilde{\alpha}_{n,1} \log \tau)^{-1} \psi_n(\gamma)
- \tau \log \tau Q_{2n-1}(\tau),
\]
for each integer \(i = 1, \ldots, n - 1,\)
\[
\tilde{\beta}_i(\gamma, \tau) = -\tilde{\alpha}_i \log \tau (1 + \tilde{\alpha}_{n,1} \log \tau)^{-1} \psi_n(\gamma) + \psi_i(\gamma) - \tilde{\alpha}_i
- \tau \log^2 \tau \tilde{\alpha}_n Q_{n+i-1}(\tau) + \tilde{\alpha}_i \log^2 \tau Q_{2n-1}(\tau) - \tau \log \tau Q_{n+i-1}(\tau),
\]
\[
\tilde{\alpha}_{n-1}(\gamma, \tau) = -\frac{\log \tau}{1 + \tilde{\alpha}_n \log \tau} \sum_{j=1}^{n} (L_{n-1,j} \log^{-1} \tau + H_{n-1,j}) \tilde{\beta}_j(\gamma, \tau)
- c L_{n,0} + \gamma_{n-1}(1 + \tilde{\alpha}_n \log \tau)^{-1} + \tau \log \tau R_{n-1}(\tau),
\]
and for each integer \(i = 0, \ldots, n - 2,\)
\[
\tilde{\alpha}_i(\gamma, \tau) = -\sum_{j=i+1}^{n-1} J_{i,j} \tilde{\alpha}_j(\gamma, \tau)
- \frac{\log \tau}{1 + \tilde{\alpha}_n \log \tau} \sum_{j=1}^{n} (L_{i,j} \log^{-1} \tau + H_{i,j}) \tilde{\beta}_j(\gamma, \tau)
- c L_{i,0} + \gamma_i(1 + \tilde{\alpha}_n \log \tau)^{-1} - \tau \log \tau R_i(\tau).
\]
For each pair of integers \((i, j)\) under consideration, \(L_{i,j}, H_{i,j}, J_{i,j}, \tilde{\alpha}_i,\) are real numbers. Moreover, \(\psi_i\) is a linear map and
\[
Q_i(\tau) = Q_{i,0} + \tau \log \tau Q_{i,1} + \cdots,
R_i(\tau) = R_{i,0} + \tau \log \tau R_{i,1} + \cdots.
\]

### 1.2.2 The compensator case

In the case \(\varepsilon \neq 0\) an asymptotic expansion for \(B_{\lambda, \varepsilon}\) in [45] was found to be
\[
B_{\lambda, \varepsilon}(u) = \alpha_0 + \beta_1 [u \omega + \varepsilon \eta_1(u, \omega)] + \alpha_1 [u + \varepsilon \mu_1(u, \omega)] + \cdots
+ \alpha_n [u^n + \varepsilon \mu_n(u, \omega)] + \beta_{n+1} [u^{n+1} \omega + \varepsilon \eta_{n+1}(u, \omega)]
+ \Psi_n(u, \lambda, \varepsilon),
\]
where \(\alpha_i\) and \(\beta_{j+1}\) for \(0 \leq j \leq n,\) depend \(C^\infty\) on \((\lambda, \varepsilon).\) Either
\[
\alpha_i(0, 0) = \beta_{i+1}(0, 0) = 0, \ 0 \leq i \leq n - 1, \text{ and } \alpha_n(0, 0) \neq 0,
\]
or
\[
\alpha_i(0, 0) = \beta_{i+1}(0, 0) = \alpha_n(0, 0) = 0, \ 0 \leq i \leq n - 2, \text{ and } \beta_n(0, 0) \neq 0.
\]
Furthermore, $\omega = \omega(u)$ is a compensator function of the form

$$\omega(u) = \frac{u^\epsilon \beta - 1}{\epsilon \beta}, \quad \text{if } \epsilon \beta = 0,$$

$$\omega(u) = \log u \text{ if } \epsilon \beta = 0,$$

where $\epsilon \beta = 1 - R(P)$ and where $R(P)$ is the absolute value of the ratio of the negative eigenvalue of the linear part of $\mathcal{X}_{\epsilon, \lambda}$ at the saddle point by the positive one. Moreover, the functions $\eta_i$, $\mu_i$ are polynomial in $u$ and $u \omega(u)$, taking the form

$$\eta_i(u, u \omega(u)) = \eta_{i,0} u^i + \cdots, \quad \mu_i(u, u \omega(u)) = \mu_{i,0} u^{i+1} \omega^{i+1}(u) + \cdots,$$

where $\eta_{i,0}$ and $\mu_{i,0}$ are real numbers. Note that the compensator unfolding deforms the dulac unfolding and is obtained from the latter by replacing $\log(u)$ by $\omega(u)$. The remainder $\Psi_n$ is of class $C^n$ and flat of order $n$ at $u = 0$. An expansion of the form (19) is said to be a nondegenerate compensator unfolding of codimension $k$ if for $\epsilon = 0$, it is a nondegenerate Dulac unfolding of codimension $k$. Theorem 1 and 2 can be extended for compensator unfoldings as follows.

**Theorem 3 (Compensator case of odd codimension)** Let

$$B : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{R},$$

$$(u, \alpha, \beta) \mapsto B_{\alpha, \beta}(u) = \sum_{i=0}^{n-1} \alpha_i \gamma^i u \omega(u) + \cdots + \sum_{i=1}^{n-1} \beta_i \gamma^i u \omega(u) + \cdots$$

be a nondegenerate compensator unfolding of codimension $2n - 1$. Then

$$B_{\Psi(\gamma, \tau)}(\tau (1 + x)) = k(\gamma, \tau, \epsilon) \tau^n \omega(\tau) \left( Q^{s, \pm}_{2n-1}(x, \gamma) + \mathcal{O}(x^{2n}) \right),$$

where $k(\gamma, \tau, \epsilon) \neq 0$ depends $C^\infty$ on $(\gamma, \tau, \epsilon)$ and where

$$\Psi : \mathbb{R}^{2n-2} \times \mathbb{R}_+ \to \mathbb{R}^{2n-1}, \quad (\gamma, \tau) \mapsto \left( \alpha(\gamma, \tau), \beta(\gamma, \tau) \right).$$

The map $\Psi$ is $C^\infty$ and of the form

$$\alpha_i = \tau^{n-i} \omega(\tau) \hat{\alpha}(\gamma, \epsilon), \quad 0 \leq i \leq n - 1,$$

$$\beta_i = \tau^{n-i} \omega(\tau) \hat{\beta}(\gamma, \epsilon), \quad 1 \leq i \leq n - 1.$$
where for each integer $i = n, \ldots, 2n - 2$, $\hat{V}_i$ is linear in $\gamma$ with coefficients depending analytically on $\epsilon$ and $\tau \omega^n(\tau)$. Furthermore

$$\bar{a}_{n-1}(\tau, \gamma, \epsilon) = -\sum_{j=1}^{n-1} (\hat{L}_{n-1,j}(\tau, \epsilon) \omega^{-1}(\tau) + \hat{H}_{n-1,j}(\tau, \epsilon) \tilde{\beta}_j(\tau, \gamma, \epsilon))$$

$$+ \hat{a}_{n-1}(\tau, \epsilon) \omega^{-1}(\tau) + \gamma_{n-1} \omega^{-1}(\tau)$$

$$- (c \hat{L}_{n-1,n}(\tau, \epsilon) + \hat{P}_{n-1}(\tau, \epsilon)) \omega^{-2}(\tau),$$

$$\bar{a}_i(\tau, \gamma, \epsilon) = -\sum_{j=i+1}^{n-1} \hat{J}_{i,j}(\tau, \epsilon) \bar{a}_j(\tau, \gamma, \epsilon)$$

$$- \sum_{j=1}^{n-1} (\hat{L}_{i,j}(\tau, \epsilon) \omega^{-1}(\tau) + \hat{H}_{i,j}(\tau, \epsilon)) \tilde{\beta}_j(\tau, \gamma, \epsilon)$$

$$+ \hat{d}_i(\tau, \epsilon) \omega^{-1}(\tau) + \gamma_i \omega^{-1}(\tau) - (c \hat{L}_{i,n}(\tau, \epsilon) + \hat{P}_i(\tau)) \omega^{-2}(\tau),$$

$$i = 0, \ldots, n - 2.$$

For each pair of integers $(i, j)$ under consideration, $\hat{L}_{i,j}$, $\hat{H}_{i,j}$, $\hat{J}_{i,j}$, $\hat{a}_i$, $\hat{d}_i$, $\hat{P}_i$ and $\tilde{\beta}_i$, are analytic functions in $\epsilon$ and in $\tau \omega^n(\tau)$.

**Theorem 4 (Compensator Case of Even Codimension)** Let

$$B : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R},$$

$$(u, \alpha, \beta) \mapsto B_{\alpha, \beta}(u) = \sum_{i=0}^{n-1} \alpha_i [u^i + \cdots] + \sum_{i=1}^{n} \beta_i [u^i \omega(u) + \cdots]$$

$$+ c[u^n + \cdots]$$

be a nondegenerate compensator unfolding of codimension $2n$. Then

$$B_{\Psi(\gamma, \tau)}(\tau(1 + x)) = \frac{k(\gamma, \tau, \epsilon) \tau^n}{1 + \bar{a}_n(\tau, \epsilon) \omega(\tau)} \left( Q_{2n}^{s,+}(x, \gamma) + O(x^{2n+1}) \right),$$

where $k(\gamma, \tau, \epsilon) \neq 0$ depends $C^\infty$ on $(\gamma, \tau, \epsilon)$, and where

$$\Psi : \mathbb{R}^{2n-1} \times \mathbb{R}_+, (\gamma, \tau) \mapsto \left( \alpha(\gamma, \tau), \beta(\gamma, \tau) \right).$$

The map $\Psi$ is $C^\infty$ and of the form

$$\alpha_i = \tau^{n-i} \bar{a}_i(\tau, \gamma, \epsilon), \quad 0 \leq i \leq n - 1,$$

$$\beta_i = \frac{\tau^{n-i}}{1 + \bar{a}_n \omega(\tau)} \tilde{\beta}_i(\tau, \gamma, \epsilon), \quad 1 \leq i \leq n.$$

We further specify

$$\bar{\beta}_n(\tau, \gamma, \epsilon) = -c \bar{a}_n(\tau, \epsilon) + (1 + \bar{a}_n(\tau, \epsilon) \omega(\tau))^{-1} \Psi_n(\tau, \gamma, \epsilon)$$

$$- \tau \omega(\tau) Q_{2n-1}(\tau, \epsilon).$$
for each integer \( i = 1, \ldots, n - 1 \)
\[
\tilde{\beta}_i(\tau, \gamma, \varepsilon) = -\tilde{a}_i(\tau, \varepsilon)\omega(\tau)(1 + \tilde{a}_n(\tau, \varepsilon)\omega(\tau))^{-1}\tilde{\Psi}_i(\tau, \gamma, \varepsilon) + \tilde{\Psi}_i(\tau, \gamma, \varepsilon) - c\tilde{a}_i(\tau, \varepsilon) - \tau\omega^2(\tau)\tilde{a}_nQ_{n+i-1}(\tau, \varepsilon) + \tilde{a}_i(\tau, \varepsilon)\tau\omega^2(\tau)Q_{2n-1}(\tau, \varepsilon) - \tau\omega(\tau)Q_{n+i-1}(\tau, \varepsilon),
\]
where for each integer \( i = 1, \ldots, n - 1 \), \( \tilde{\Psi}_i \) is linear in \( \gamma \) with coefficients depending analytically on \( \varepsilon \) and \( \tau\omega^n(\tau) \). Furthermore
\[
\tilde{a}_{n-1}(\tau, \gamma, \varepsilon) = -\frac{\omega(\tau)}{1 + \tilde{a}_n(\tau, \varepsilon)\omega(\tau)} \sum_{j=1}^{n} \tilde{L}_{n-1,j}(\tau, \varepsilon)\omega^{-1}(\tau)\tilde{\beta}_j(\tau, \gamma, \varepsilon)
\]
\[
- \frac{\omega(\tau)}{1 + \tilde{a}_n(\tau, \varepsilon)\omega(\tau)} \sum_{j=1}^{n} \tilde{H}_{n-1,j}(\tau, \varepsilon)\tilde{\beta}_j(\tau, \gamma, \varepsilon)
\]
\[
- c\tilde{L}_{n,0}(\tau, \varepsilon) + \gamma_{n-1}(1 + \tilde{a}_n(\tau, \varepsilon)\omega(\tau))^{-1}
\]
\[
+ \tau\omega(\tau)R_{n-1}(\tau, \varepsilon),
\]
and for each \( i = 1, \ldots, n - 2 \),
\[
\tilde{\alpha}_i(\tau, \gamma, \varepsilon) = -\sum_{j=i+1}^{n-1} \tilde{J}_{i,j}(\tau, \varepsilon)\tilde{\alpha}_j(\tau, \gamma, \varepsilon)
\]
\[
- \frac{\omega(\tau)}{1 + \tilde{a}_n(\tau, \varepsilon)\omega(\tau)} \sum_{j=1}^{n} (\tilde{L}_{i,j}(\tau, \varepsilon)\omega^{-1}(\tau) + \tilde{H}_{i,j}(\tau, \varepsilon))\tilde{\beta}_j(\tau, \gamma, \varepsilon)
\]
\[
- c\tilde{L}_{i,0}(\tau, \varepsilon) + \gamma_i(1 + \tilde{a}_n(\tau, \varepsilon)\omega(\tau))^{-1} - \tau\omega(\tau)R_i(\tau, \varepsilon).
\]
For each pair of integers \((i, j)\) under consideration, \( \tilde{L}_{i,j}, \tilde{H}_{i,j}, \tilde{J}_{i,j}, \tilde{\alpha}_i, Q_i \) and \( R_i \) are analytic functions in \( \varepsilon \) and \( \tau\omega^n(\tau) \).

**Remarks:**

- Concerning the topology of the bifurcation set of limit cycles, Mardésic [34, 35] shows that the following properties hold. In the Dulac expansion case \((\varepsilon = 0)\), the bifurcation diagram is homeomorphic to the bifurcation diagram of the model \( Q_k^{\pm}(u, \alpha) \) in (5) and in the compensator case \((\varepsilon > 0)\), the bifurcation diagram is homeomorphic to the product of the interval \((0, \varepsilon_0)\) and the bifurcation diagram of the polynomial model, \( Q_k^{\pm}(u, \beta) \). If we restrict the parameter space to the image of the map \( \Psi \) given in Theorem 1-4, these latter properties also directly follow from our approach. Mardésic’s approach is rather topological and his results are not suitable for more analytical studies. We emphasize here the explicit character of our results that are preparations for further applications. Indeed in [16], we apply these results to study quasi periodic bifurcation of invariant tori for the full system \( X_{\lambda, \varepsilon} + \varepsilon^* R \). For more detailed see sub-section 1.4.
An interesting observation concerns the image of the scaling and its order of contact with the hyperplane \( \{ \alpha_0 = 0 \} \) of homoclinic bifurcation. In the case of odd codimension, if a point \((\alpha, \beta)\) belongs to the boundary of the image of the scaling. From (16) it follows that for all \( r > 0 \) there exists a constant \( C > 1 \) such that for each integer \( i = 1, \ldots, n - 1, \)
\[
C|\beta_{n-1}|^{n-i} \leq \max\{|\alpha_i|, |\beta_i|\} \leq C|\beta_{n-1}|^{n-i-r},
\]
and
\[
C|\beta_{n-1}|^n \leq |\alpha_0| \leq C|\beta_{n-1}|^{n-r}.
\]
This implies that the contact is \( n \)-flat, compare with Example 3. However, in the case of even codimension, from (18) it follows that for all \( r > 0 \) there exists a constant \( \bar{C} > 1 \) such that
\[
|\beta_i| \leq \bar{C} \exp\left(-\frac{n-i}{|\beta_{n-1}|}\right), \ i = 1, \ldots, n - 2,
\]
\[
|\alpha_i| \leq \bar{C} \exp\left(-\frac{n}{|\beta_{n-1}|}\right), \ i = 0, \ldots, n - 1.
\]
This implies that the contact is infinitely flat and that the bifurcation set is located in an exponentially narrow horn, compare with Example 2. Furthermore, the region in the phase space under consideration is exponentially thin annulus. This observation reveals a significant difference between the odd and the even codimension case.

Concerning the geometry of the bifurcation set of limit cycles, from formulae (15) and (17) we clearly obtain the hierarchy of the Standard Catastrophe Theory in terms of the parameter \( \gamma \) and this hierarchy is independent of \( \tau \). In this sense we can say that along the curve \( C \), the singularity has been removed. From this, using formulas (16) and (18), we can deduce a hierarchy for the bifurcation sets in the parameters \((\alpha, \beta)\) which now depends on \( \tau \) in a non regular manner when \( \tau \) approaches 0. It would be of interest, at least in the odd codimension case, to further study the geometry of the bifurcation set of limit cycles, for instance the order of contact between the different subordinate \( d \)-fold bifurcations in a family of cuspoids of degree \( k > d \). However, when working in the range of the scaling, it is appropriate to work in terms of the parameter \( \gamma \) and the variable \( x \).

1.3 Application: the codimension three saddle connection

Consider the following unfolding of the codimension four Bogdanov-Takens bifurcation
\[
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= x^2 + \mu + y(\nu_0 + \nu_1 x + \nu_2 x^2 \pm x^4),
\end{aligned}
\]
This family was partially studied in [25, 32], for more background see [4, 24, 26, 30, 51]. After the rescaling
\[ \mu = -\varepsilon^4, \nu_0 = \varepsilon^8 \lambda_0, \nu_1 = \varepsilon^6 \lambda_1, \nu_2 = \varepsilon^4 \lambda_2, \quad x = \varepsilon^2 \bar{x}, \quad y = \varepsilon^3 \bar{y}, \]
and then omitting all bars, we end up with the following 3-parameter family of vector fields
\[
\begin{cases}
    \dot{x} = \varepsilon y, \\
    \dot{y} = \varepsilon \left( (x^2 - 1) + \varepsilon^5 y(\lambda_0 + \lambda_1 x + \lambda_2 x^3 \pm x^4) + O(\varepsilon^7) \right). 
\end{cases}
\]
In terms of our general system (1) we have \( p = 1 \) and \( q = 8, r > 8 \), i.e.,
\[ X_{\lambda, \varepsilon} = \varepsilon X_H + \varepsilon^8 Y_{\lambda, \varepsilon}, \]
where
\[ H(\bar{x}, \bar{y}) = \frac{1}{2} \bar{y}^2 - \frac{1}{3} \bar{x}^3 + \bar{x}. \]
The scaled displacement function takes the form
\[ B_{\lambda, \varepsilon}(u) = \alpha_0 + \beta_1[u \omega(u) + \cdots] + \alpha_1[u + \cdots] + c[u^2 \omega(u) + \cdots] + O(u^3). \]
For simplicity, we again restrict to the case \( \varepsilon = 0 \). The scaling
\[ \Psi(\gamma, \tau) = \left( \alpha(\gamma, \tau), \beta(\gamma, \tau) \right) \]
of Theorem 1 takes the explicit form
\[
\begin{align*}
    \beta_1 &= -3c\tau - 2c\tau \log \tau + \tau^2 \tilde{R}_2(\tau), \\
    \alpha_0 &= c\tau^2 \log \tau + \gamma_0 \tau^2 \log \tau + \gamma_1 \tau^2 \log \tau + \tau^2 \tilde{R}_1(\tau), \\
    \alpha_1 &= 3c\tau \log \tau + 2c\tau \log^2 \tau + \gamma_1 \tau \log \tau + \tau \tilde{P}_1(\tau). 
\end{align*}
\]
Therefore
\[
B_{\Psi(\gamma, \tau), 0} \left( \tau(1 + x) \right) = \tau^2 \log \tau U(x, \gamma, \tau)(\gamma_0 + \gamma_1 x \pm x^3)
= \tilde{U}(x, \gamma, \tau) Q_3^{\pm}(x, \gamma),
\]
where \( U \) is given by the Division Theorem. The latter function, although smooth if \( \tau > 0 \), has Dulac asymptotics at \( \tau = 0 \). Up to a division by \( \tilde{U} \), the scaled displacement function is the normal form of a cusp. The bifurcation set of limit cycles \( \mathcal{B} \) is given by \( \mathcal{B} = \Psi(V) \) where
\[ V = \{ (\tau, -3x^2, 2x^3), \tau > 0, x \in \mathbb{R} \}, \]
which is the product of a cusp and an interval, see Figure 4. From (22) we observe that if \((\alpha_0, \beta_1, \alpha_1)\) belongs to the bifurcation set of limit cycles, the following estimates hold

\[
|\alpha_1| \leq |\beta_1| \leq |\alpha_1|^{1-r}, \quad |\alpha_1|^2 \leq |\alpha_0| \leq \frac{1}{\epsilon}|\alpha_1|^{2-r},
\]

for any \(r > 0\). This implies that contrary to the even codimension case, see Example 2, the order of contact between the bifurcation set of limit cycles and the hyperplane \(\{\alpha_0 = 0\}\) of homoclinic bifurcation is finite.

![Figure 3: Image of the cube \([0,0.01] \times [-0.1,0.1] \times [-0.1,0.1]\) under the scaling (22). On the right hand side, the resulting set is bounded by surfaces which meet at the origin with finite order.](image)

![Figure 4: The bifurcation set of limit cycles before the scaling (\(V\) on the left hand side) and after the scaling (\(B\) on the right hand side). This latter is the topological product of a cusp and an interval squeezed in a narrow horn.](image)

### 1.4 Further applications

We summarize as follows. Given a family of vector fields of the form (1)

\[
X_{\lambda, \varepsilon}(x, y) + \varepsilon^p R(x, y, t, \lambda, \varepsilon),
\]

with \(x, y, t \in \mathbb{R}, \lambda \in \mathbb{R}^d\) and where \(\varepsilon \geq 0\) is a real perturbation parameter. The autonomous part \(X_{\lambda, \varepsilon} = \varepsilon^p X_H + \varepsilon^q Y_{\lambda, \varepsilon}\) is such that \(X_H\) is Hamiltonian with
Hamilton function $H$, which is of Morse type and $Y_{\lambda,e}$ a dissipative family. Moreover, $X_H$ possesses a homoclinic loop $\Gamma$. The dissipative part is generic in an appropriate sense. In this setting we focus our study near the homoclinic loop $\Gamma$ and our concern is with the autonomous family $X_{\lambda,e}$. By an explicit scaling, a narrow finite structure for the bifurcation set of limit cycles is found in the parameter space $\mathbb{R}^\ell = \{\lambda\}$. We show that the corresponding geometry is polynomial where a scaled displacement function can display the full complexity of the cuspoid family

$$Q_k^{0,\pm}(x, \gamma) = \gamma_0 + \gamma_1 x + \cdots + \gamma_{k-2} x^{k-2} \pm x^k,$$

where $\gamma = \gamma(\lambda)$ is appropriate, compare with (4).

We now discuss certain consequences the above results have for the full non-autonomous family $X_{\lambda,e} + \varepsilon R$, see (1), where $R$ is assumed time-periodic of period $2\pi$ and therefore has the solid torus $\mathbb{R}^2 \times \mathbb{S}^1$ as its phase space. Here the limit cycles of the ‘unperturbed’ part $X_{\lambda,e}$ correspond to invariant 2-tori with parallel dynamics [9, 11]. By normal hyperbolicity these tori persist as long as the multi-parameter $\lambda$ remains outside an exponentially narrow neighbourhood of the bifurcation set of limit cycles found before, compare with [17] for details.

Near the bifurcation set of limit cycles we can apply quasi-periodic bifurcation theory as developed by [2, 7, 8, 9, 11, 13, 20, 39, 52, 53]. To this end we consider ‘unperturbed’ tori which are Diophantine. This means that the two frequencies $\omega_1$ and $\omega_2$ are such that

$$\left|\frac{\omega_1}{\omega_2} - \frac{p}{q}\right| \geq \kappa q^{-\nu},$$

for certain $\kappa > 0$ and $\nu > 2$. For fixed $\kappa$ and $\nu$ formula (24) inside $\mathbb{R}^\ell = \{\lambda\}$ defines a Cantor foliation of hypersurfaces. At the intersection of this foliation with the bifurcation set, the ‘unperturbed’ family $X_{\lambda,e}$ displays a cuspoid bifurcation pattern of 2-tori as can be directly inferred from the above. In the ‘perturbed’ case $X_{\lambda,e} + \varepsilon R$, we conjecture persistence of many of these Diophantine tori, including the bifurcation pattern, provided that $|\varepsilon| \ll 1$. Here $\kappa = \kappa(\varepsilon) = o(\varepsilon)$ has to be chosen appropriately. For instance whenever the Cantor foliation intersects a fold-hyperplane transversally, we expect a quasi-periodic saddle-node bifurcation of 2-tori to occur [9, 11]. These KAM persistence results involve a Whitney smooth reparametrization which is near the identity-map in terms of $|\varepsilon| \ll 1$. This means that the perturbed Diophantine 2-tori correspond to a perturbed Cantor foliation in the parameter space $\mathbb{R}^\ell = \{\lambda\}$. In the complement of this perturbed Cantor foliation of hypersurfaces we expect all the dynamical complexity regarding Cantori, strange attractors, etc., as described in [19, 20, 21, 22, 38, 42]. This program will be the subject of [16] where we aim to apply [9, 11, 52, 53] to establish the occurrence of quasi-periodic cuspoid bifurcations in the three cases (a), (b) and (c) of Figure 1.

A second topic is the generalization of the approach of the present paper to cases where the Hamiltonian $H$ in (1) no longer is a (stable) Morse function, but undergoes simple bifurcations.
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2 Proofs

We first give a proof of Theorem 1 and 2. Theorem 3 and 4 generalize Theorem 1 and 2 to compensator unfoldings and are obtained by a perturbation method. Before we distinguish between the odd and even codimension case we first describe the main idea. Consider a nondegenerate Dulac unfolding of the form

\[ B_{\alpha,\beta}(u) = \sum_{j=0}^{n} \alpha_j u^j + \sum_{j=1}^{n} \beta_j u^j \log u + \tilde{H}(u), \]  

where \( \tilde{H}(u) = cu^{n+1} \log u + \cdots \). Put

\[ u = \tau(1 + x) \text{ and } B_{\alpha,\beta}(\tau(1 + x)) = \Phi_{\alpha,\beta}(x). \]

Using the Taylor formula we get

\[ \Phi_{\alpha,\beta}(x) = \sum_{i=0}^{2n} \Lambda_i x^i + O(x^{2n+1}). \]  

Since the Dulac unfolding is nondegenerate we can see the coefficients of the unfolding as independent. The goal of this section is to show that independent coefficients of the Dulac unfolding yield independent coefficients for the monomials \( x^i \) in (26). It will also be proven that the non zero leading term in (25) \( (\alpha_n \text{ for even and } \beta_n \text{ for odd codimension}) \) leads to a non zero leading term in (26). To illustrate this, we consider the case of codimension \( k = 2n - 1 \), i.e., where \( \beta_{n} \neq 0 \). This case is covered by Theorem 1 and in (11) we have \( \tilde{f}(\tau) = \tau^n \log \tau \). It follows that indeed the leading term in (25) is of degree \( 2n - 1 \); compare with Example 3 for a special case. In the case of codimension \( k = 2n \) covered by Theorem 2, we similarly have \( \tilde{f}(\tau) = \tau^n/(1 + \log \tau) \) in (11); compare with Example 2 for a special case.

To be precise, for each integer \( i = 0, \ldots, 2n \), in (25) we have

\[ \Lambda_i = \frac{\Phi^{(i)}(0)}{i!} = \frac{\tau^i}{i!} B^{(i)}_{\alpha,\beta}(\tau) \]  

and together with (25) we get

\[ \Lambda_i = \frac{\tau^i}{i!} \sum_{j=0}^{n} \alpha_j (u^j)_{u=\tau} + \frac{\tau^i}{i!} \sum_{j=1}^{n} \beta_j (u^j \log u)_{u=\tau} + \frac{\tau^i}{i!} \tilde{f}^{(i)}(\tau). \]
Observe that the following equalities hold

\[
\frac{\tau^i}{i!}\alpha_j(u^j)(u)_{u=\tau}^{(i)} = \alpha_j \tau^j \left( \frac{j!}{i!(j-i)!} \right), \quad \text{if } j \geq i,
\]

\[
\frac{\tau^i}{i!}\alpha_j(u^j)(u)_{u=\tau}^{(i)} = 0, \quad \text{if } j < i,
\]

\[
\frac{\tau^i}{i!}\beta_j(u^j \log u)_{u=\tau}^{(i)} = (-1)^{i-j-1} \frac{j!(i-j-1)!}{i!} \beta_j \tau^j, \quad \text{if } j < i
\]

and

\[
\frac{\tau^i}{i!}H^{(i)}(\tau) = \tau^{n+1} \log \tau R_i(\tau),
\]

where

\[R_i(\tau) = R_{i,0} + R_{i,1}\tau \log \tau + \cdots.\]

We use the Leibniz rule to compute \((u^j \log u)_{u=\tau}^{(i)}\), when \(i \leq j\) and we get

\[
\frac{\tau^i}{i!}\beta_j(u^j \log u)_{u=\tau}^{(i)} = \beta_j \frac{\tau^j}{i!} \sum_{s=0}^{i} \binom{i}{s} (u^j)_{u=\tau}^{(i-s)} (\log u)_{u=\tau}^{(s)},
\]

\[
= \beta_j \frac{\tau^j}{i!} \log \tau \frac{j!}{(j-i)!},
\]

\[
+ \beta_j \frac{\tau^j}{i!} \sum_{s=1}^{i} \binom{i}{s} (u^j)_{u=\tau}^{(i-s)} (\log u)_{u=\tau}^{(s)},
\]

\[
= \beta_j \frac{\tau^j}{i!} \log \tau \frac{j!}{(j-i)!},
\]

\[
+ \beta_j \frac{\tau^j}{i!} \sum_{s=1}^{i} \binom{i}{s} \frac{(-1)^{s-1}(s-1)!j!}{(j-i+s)!}.
\]

Observe that

\[
\frac{\tau^i}{i!}(u^j \log u)_{u=\tau}^{(i)} = \frac{\tau^j}{i!} \log \tau \frac{j!}{(j-i)!} + \frac{\tau^j}{i!}(u^j \log u)_{u=1}^{(i)},
\]

where

\[
\frac{1}{i!}(u^j \log u)_{u=1}^{(i)} = \sum_{s=1}^{i} \binom{i}{s} \frac{(-1)^{s-1}(s-1)!j!}{i!(j-i+s)!},
\]

\[
= \sum_{s=1}^{i} \frac{(-1)^{s-1}j!}{(j+s-i)!(i-s)!s}.
\]
We then introduce the following notation. Let

\[
\mathbf{J} = (J_{i,j})_{i=0,...,n,\ j=0,...,n}, \quad \mathbf{L} = (L_{i,j})_{i=0,...,n,\ j=1,...,n},
\]

\[
\mathbf{H} = (H_{i,j})_{i=0,...,n,\ j=1,...,n} \quad \text{and} \quad \mathbf{T} = (T_{i,j})_{i=n+1,...,2n,\ j=1,...,n},
\]

be respectively the \(n+1 \times n+1\), \(n+1 \times n\), and \(n \times n\) matrices defined as follows

\[
J_{i,j} = H_{i,j} = \binom{j}{i} \quad \text{if} \quad i \leq j, \quad J_{i,j} = H_{i,j} = 0 \quad \text{if} \quad j < i,
\]

\[
L_{i,j} = \frac{(-1)^{i-j-1}}{i!} j!(i-j-1)!, \quad \text{if} \quad j < i,
\]

\[
L_{i,j} = \frac{1}{i!} \left( u^j \log u \right)^{(i)} = \sum_{s=1}^{i} \frac{(-1)^{i-s-1} j!}{(j+s-i)! (i-s)! s!}, \quad \text{if} \quad i \leq j,
\]

\[
T_{i,j} = \frac{(-1)^{i-j+1}}{i!} j!(i-j-1)!.
\]

Finally, \(\mathbf{N} = (\tilde{N}_{i,j})_{i=n,...,2n-1,\ j=1,...,n}\) is defined as follows

\[
\tilde{N}_{n,n} = 1, \quad \tilde{N}_{i,j} = 0, \quad \text{if} \quad (i,j) \neq (n,n).
\]

We also introduce the following matrices

\[
\mathbf{J}^{-} = (J_{i,j}^{-})_{i=0,...,n-1,\ j=0,...,n-1}, \quad \text{where} \quad J_{i,j}^{-} = J_{i,j},
\]

\[
\mathbf{H}^{-} = (H_{i,j}^{-})_{i=0,...,n-1,\ j=1,...,n}, \quad \text{where} \quad H_{i,j}^{-} = H_{i,j}
\]

and

\[
\mathbf{L}^{-} = (L_{i,j}^{-})_{i=0,...,n-1,\ j=1,...,n}, \quad \text{where} \quad L_{i,j}^{-} = L_{i,j}.
\]

With the above notations we get

\[
\Lambda_i = \sum_{j=0}^{n} J_{i,j} \alpha_j \tau^j + \sum_{j=1}^{n} \left( H_{i,j} \log \tau + L_{i,j} \right) \beta_j \tau^j + \tau^{n+1} \log \tau R_i(\tau), \quad (28)
\]

if \(i \leq n\) and

\[
\Lambda_i = \sum_{j=1}^{n} \beta_j \tau^j \frac{(-1)^{i-j-1}}{i!} j!(i-j-1)! + \tau^{n+1} \log \tau R_i(\tau), \quad (29)
\]

\[
= \sum_{j=1}^{n} T_{i,j} \beta_j \tau^j + \tau^{n+1} \log \tau R_i(\tau),
\]

if \(i > n\). We state the following lemma.
Lemma 1 \textbf{Let} \\
\[ \tilde{A} = (\tilde{A}_{i,j})_{i=1,\ldots,2n-1,\ j=1,\ldots,n}, \ \tilde{A}_{i,j} = T_{i,j} \text{ if } i > n, \ \tilde{A}_{n,j} = L_{n,j}, \]
\[ A = (A_{i,j})_{i=n,\ldots,2n-2,\ j=1,\ldots,n-1}, \ A_{i,j} = \tilde{A}_{i,j}, \]
\[ B = (B_{i,j})_{i=n+1,\ldots,2n-1,\ j=1,\ldots,n-1}, \ B_{i,j} = A_{i,j}. \]

\textbf{Then} \( A, B, \tilde{A} \text{ and } T \) \textbf{are invertible}. 

\textbf{Proof:} We show that \( \tilde{A} \) is invertible. The non-degeneracy of the three other matrices follows exactly from the same argument. First of all, we claim that for each integer \( 1 \leq i \leq j \leq n \), \( L_{i,j} > 0 \). To show this, recall that 
\[ L_{i,j} = \frac{1}{i!} \left( u^j \log u \right)_{u=1}^{(i)}. \]
It then follows that 
\[ L_{i+1,j+1} = \frac{1}{(i+1)!} \left( (j+1)u^j \log u + u^j \right)_{u=1}^{(i)} = \frac{j+1}{i+1} L_{i,j} + \frac{j!}{(i+1)!(j-i)!}. \] (30)

In particular, for each integer \( 1 \leq i \leq n \), 
\[ L_{i,i} = \sum_{k=1}^{i} \frac{1}{k}. \]

Since for each integer \( 1 \leq s \leq n \) 
\[ L_{1,s} = \left( u^s \log u \right)_{u=1}^{(1)} = 1, \]
it turns out that for each integer \( s = 1, \ldots, n \), \( L_{1,s} \) is always positive. With (30) it then follows that for each integer \( 1 \leq i \leq j \leq n \), \( L_{i,j} \) is always positive. 

By definition 
\[ (\tilde{A}_{i,j})_{i=n,\ldots,2n-1,\ j=1,\ldots,n} = \frac{1}{i!} \left( u^j \log u \right)_{u=1}^{(i)} \] 
\[ = \prod_{i=n}^{2n-1} \frac{1}{i!} \left( u^j \log u \right)_{u=1}^{(i)}. \]
Thus 
\[ \det(\tilde{A}_{i,j}) = \frac{1}{\prod_{i=n}^{2n-1} i!} \det \left( u^j \log u \right)_{u=1}^{(i)}. \]

For each integer \( j = 1, \ldots, n \), we put \( f_j(u) = (u^j \log u)^{(n)} \). Then 
\[ \det(\tilde{A}_{i,j}) = \frac{1}{\prod_{i=n}^{2n-2} i!} W(f_1, \ldots, f_n)(1), \]

25
where

\[
W(f_1, \ldots, f_n)(u) = \begin{vmatrix}
  f_1(u) & \cdots & f_n(u) \\
  f'_1(u) & \cdots & f'_n(u) \\
  \vdots & \ddots & \vdots \\
  f'_{n-1}(u) & \cdots & f'_{n-1}(u)
\end{vmatrix}.
\]

Let \( g : \mathbb{R} \to \mathbb{R} \), \( u \mapsto g(u) \) be a \( C^\infty \) function. We have that

\[
W(gf_1, \ldots, gf_n)(u) = g^n W(f_1, \ldots, f_n)(u).
\]

This latter property can be shown by induction on \( n \) using the fact that the determinant is multilinear. Take

\[
g(u) = (-1^n)u^{n-1}/(n-1)!
\]

We then get

\[
W(gf_1, \ldots, gf_n)(u) = \\
\begin{vmatrix}
  1 & k_1 u & k_2 u^2 & \cdots & k_{n-1}u^{n-1} + (-1)^n n \log u \\
  0 & k_1 & 2k_2 u & \cdots & (n-1)k_{n-2}u^{n-2} \\
  0 & 0 & 2k_2 & \cdots & (n-1)(n-2)k_{n-3}u^{n-3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & k_{n-1}(n-1)!
\end{vmatrix},
\]

where for each integer \( j = 1, \ldots, n-1 \),

\[
k_j = \frac{(-1)^n}{(n-1)!} (u^j \log u)^n_{u=1} = \frac{(-1)^n j!}{(n-1)!} L_{n,j} \neq 0.
\]

This implies that

\[
W(gf_1, \ldots, gf_n)(1) = \prod_{j=1}^{n-1} j!k_j \neq 0,
\]

this ends the proof of Lemma 1.

From now on we need to distinguish between the odd-codimension case (developed in the next sub-section) and the even-codimension case (developed in the last sub-section).

### 2.1 Proof of Theorem 1

In the odd case \( k = 2n - 1 \), the parameters are \( \alpha_0, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1} \). Moreover we have \( \beta_n = c \neq 0 \) is a constant. This implies that

\[
B_{\alpha,\beta}(u) = \sum_{i=0}^{n-1} \alpha_i u^i + \sum_{i=1}^{n-1} \beta_i u^i \log u + cu^n \log u + u^n \mathcal{F}(u),
\]

where

\[
\mathcal{F}(u) = \mathcal{F}_0 + \mathcal{F}_1 u \log u + \cdots.
\]
In this situation, the term $\alpha_n u^n$ in (25) is included in the term $u^n F(u)$. By putting $x = \tau (1 + x)$ and $B_{\alpha,\beta}(\tau (1 + x)) = \Phi_{\alpha,\beta}(x)$, we get

$$\Phi_{\alpha,\beta}(x) = \Lambda_0 + \Lambda_1 x + \ldots + \Lambda_{2n-2} x^{2n-2} + \Lambda_{2n-1} x^{2n-1} + O(x^{2n}).$$

With (27), (28) and (29) the following equalities hold

$$\begin{pmatrix} \Lambda_0 \\ \vdots \\ \Lambda_{n-1} \end{pmatrix} = J^{-1} \begin{pmatrix} \alpha_0 \\ \alpha_1 \tau \\ \vdots \\ \alpha_{n-1} \tau^{n-1} \end{pmatrix} + \log \tau H^{-1} \begin{pmatrix} \beta_1 \tau \\ \vdots \\ \beta_{n-1} \tau^{n-1} \end{pmatrix} \quad (32)$$

$$+ L^{-1} \begin{pmatrix} \beta_1 \tau \\ \vdots \\ \beta_{n-1} \tau^{n-1} \\ e^\tau \end{pmatrix} + \begin{pmatrix} P_0(\tau) \\ \vdots \\ P_{n-2}(\tau) \\ P_{n-1}(\tau) \end{pmatrix},$$

$$\begin{pmatrix} \Lambda_n \\ \vdots \\ \Lambda_{2n-2} \\ \Lambda_{2n-1} \end{pmatrix} = (\tilde{A} + \log \tau \tilde{N}) \begin{pmatrix} \beta_1 \tau \\ \vdots \\ \beta_{n-1} \tau^{n-1} \\ e^\tau \end{pmatrix} + \begin{pmatrix} P_n(\tau) \\ \vdots \\ P_{2n-2}(\tau) \\ P_{2n-1}(\tau) \end{pmatrix},$$

where for each integer $i \in \{n, n+1, \ldots, 2n-1\}$, $P_i(\tau)$ takes the form

$$P_i(\tau) = \tau^n \tilde{P}_i(\tau) = \tau^n (\tilde{P}_{i,0} + \tau \log \tau \tilde{P}_{i,0} + \cdots).$$

We have the following strategy. We aim to find a scaled reparametrization of the form

$$\Psi(\gamma, \tau) = (\alpha, \beta)$$

linking $\gamma = (\gamma_0, \ldots, \gamma_{2n-3})$ and $\tau$ with $(\alpha, \beta)$. For each integer $i = 1, \ldots, n-1$, we require $\Psi$ to have the components

$$\alpha_i = \tau^{n-i} \log^2 \tau \bar{a}_i(\gamma, \tau), \quad \beta_i = \tau^{n-i} \log \tau \bar{b}_i(\gamma, \tau),$$

leading to

$$\Phi_{\Psi(\gamma, \tau)}(x) = \tilde{f}(\tau) (\gamma_0 + \gamma_1 x + \cdots + \gamma_{2n-3} x^{2n-3} + k(\gamma, \tau) x^{2n-1} + O(x^{2n})).$$

We first determine $\tilde{f}(\tau)$ and show that the leading term $k(\gamma, \tau)$ is such that $k(0, 0) \neq 0$. Note that the set $\Psi(0, \tau)$ parametrizes the curve $C$ of 'highest degeneracy' of $\Phi_{\Psi(\gamma, \tau)}$. We then obtain the desired expression for $\Phi_{\Psi(\gamma, \tau)}$ by solving

$$\Lambda_i = \tilde{f}(\tau) \gamma_i, \quad i = 1, \ldots, 2n-2.$$
To be more precise, we start solving the equation

\[ \Lambda_i = 0, \quad i = n, \ldots, 2n - 2. \]

With (33) and the definition of the matrix \( A \), we get

\[
0 = \begin{pmatrix} \Lambda_n \\ \vdots \\ \Lambda_{2n-2} \end{pmatrix} = A \begin{pmatrix} \beta_1 \tau \\ \vdots \\ \beta_{n-1} \tau^{n-1} \end{pmatrix} + \epsilon_1^n \log \tau \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \epsilon_2^n \begin{pmatrix} \tilde{A}_{n,n} \\ \vdots \\ \tilde{A}_{2n-2,n} \end{pmatrix} + \begin{pmatrix} P_n(\tau) \\ \vdots \\ P_{2n-2}(\tau) \end{pmatrix}. \tag{35}
\]

By Lemma 1, the matrix \( A \) is invertible. Let \( A^{-1} = (a_{ij})_{i,j=1,\ldots,n-1,j=1,\ldots,n-1} \) be its inverse. After multiplication by \( A^{-1} \) on the left hand side and the right hand side of (35) we get

\[
\begin{pmatrix} \beta_1 \tau \\ \vdots \\ \beta_{n-1} \tau^{n-1} \end{pmatrix} = -\epsilon_1^n \log \tau \begin{pmatrix} a_{1,1} \\ \vdots \\ a_{n-1,n} \end{pmatrix} + \tau^n \begin{pmatrix} \tilde{R}_n(\tau) \\ \vdots \\ \tilde{R}_{2n-2}(\tau) \end{pmatrix}, \tag{36}
\]

where

\[
\begin{pmatrix} \tilde{R}_n(\tau) \\ \vdots \\ \tilde{R}_{2n-2}(\tau) \end{pmatrix} = -\epsilon A^{-1} \begin{pmatrix} \tilde{A}_{n,n} \\ \vdots \\ \tilde{A}_{2n-2,n} \end{pmatrix} - A^{-1} \begin{pmatrix} \tilde{P}_n(\tau) \\ \vdots \\ \tilde{P}_{2n-2}(\tau) \end{pmatrix}. \tag{37}
\]

Observe that for each integer \( i = 1, \ldots, n - 2, \)

\[ \tilde{R}_i(\tau) = \tilde{R}_{i,0} + \tau \log \tau \tilde{R}_{i,0} + \cdots . \]

We now set \( \gamma = 0 \) in (34). With (35) and the definition of the matrix \( B \) we have

\[
\begin{pmatrix} \Lambda_{n+1}(0, \tau) \\ \vdots \\ \Lambda_{2n-1}(0, \tau) \end{pmatrix} = B \begin{pmatrix} \beta_1 \tau \\ \vdots \\ \beta_{n-1} \tau^{n-1} \end{pmatrix} + \epsilon_1^n \begin{pmatrix} \tilde{A}_{n+1,n} \\ \vdots \\ \tilde{A}_{2n-1,n} \end{pmatrix} + \begin{pmatrix} P_{n+1}(\tau) \\ \vdots \\ P_{2n-1}(\tau) \end{pmatrix}, \tag{38}
\]

28
Since $A^{-1}$ is invertible we have $(a_{1,1}, \ldots, a_{n-1,1}) \neq 0$. Therefore after division by $\tau^n \log \tau$ of both sides of (36), we get

$$(\hat{b}_1(0, \tau), \ldots, \hat{b}_{n-1}(0, \tau)) \neq (0, \ldots, 0).$$

Moreover, since $B$ is non degenerate, with (38) we must have

$$\sup_{n+1 \leq i \leq 2n-1} |\Lambda_i(0, \tau)| = -\tau^n |\tilde{k}(\tau)| \log \tau,$$

where $\tilde{k}(0) \neq 0$. But

$$(\Lambda_{n+1}(0, \tau), \ldots, \Lambda_{2n-2}(0, \tau)) = (0, \ldots, 0),$$

which implies that

$$\Lambda_{2n-1}(0, \tau) = \tilde{k}(\tau) \tau^n \log \tau,$$

with

$$\tilde{k}(\tau) = \sum_{j=1}^{n-1} B_{n-1,j} \hat{b}_j(0, \tau) + c \log^{-1} \tau (\hat{A}_{2n-1,n} + P_{n-1}(\tau)).$$

As we announced before, we set $\hat{f}(\tau) = \tau^n \log \tau$. We now develop the scaling. For each integer $i = 0, \ldots, 2n - 2$, we put

$$\Lambda_i(\gamma, \tau) = \gamma_i \tau^n \log \tau, \text{ for } i \leq 2n - 3,$$

$$\Lambda_{2n-2}(\gamma, \tau) = 0.$$

From (35) and (37) we then get

$$A \left( \begin{array}{c} \beta_1 \tau \\ \vdots \\ \beta_{n-1} \tau^{n-1} \end{array} \right) = -c \tau^n \log \tau \left( \begin{array}{c} 1 \\ \vdots \\ 0 \end{array} \right) + \tau^n A \left( \begin{array}{c} \tilde{R}_n(\tau) \\ \vdots \\ \tilde{R}_{2n-2}(\tau) \end{array} \right),$$

$$+ \tau^n \log \tau \left( \begin{array}{c} \gamma_n \\ \vdots \\ \gamma_{2n-3} \\ 0 \end{array} \right).$$

29
After multiplication of both sides of (39) by $\mathbf{A}^{-1}$ we get
\[
\begin{pmatrix}
\beta_1 \tau \\
\vdots \\
\beta_{n-1} \tau^{n-1}
\end{pmatrix} = -e^{\tau} \log \tau \begin{pmatrix}
a_{1,1} \\
\vdots \\
a_{n,1}
\end{pmatrix} + \tau^n \begin{pmatrix}
\tilde{R}_n(\tau) \\
\vdots \\
\tilde{R}_{2n-2}(\tau)
\end{pmatrix} + \tau^n \log \tau \mathbf{A}^{-1} \begin{pmatrix}
\gamma_n \\
\vdots \\
\gamma_{2n-3} \\
0
\end{pmatrix},
\] (40)

Next, with (32) one gets
\[
\alpha_{n-1} \tau^{n-1} = -\sum_{j=1}^{n-1} (L_{n-1,j} + \log \tau H_{n-1,j}) \beta_j \tau^j - e^{\tau} \log \tau H_{n-1,n} \tau^n L_{n-1,n} - \tau^n \tilde{P}_{n-1}(\tau),
\]
and for each $i = 1, \ldots, n - 2$,
\[
\alpha_i \tau^i = -\sum_{j=i+1}^{n-1} J_{i,j} \alpha_j \tau^j - \sum_{j=1}^{n-1} (L_{i,j} + \log \tau H_{i,j}) \beta_j \tau^j - e^{\tau} \log \tau H_{i,n} + \gamma_i \tau^n \log \tau - e^{\tau} L_{i,n} - \tau^n \tilde{P}_i(\tau).
\]
Recall that
\[
\begin{pmatrix}
\alpha_i \\
\beta_i
\end{pmatrix} = \begin{pmatrix}
\tau^{n-i} \log^2 \tau \tilde{\alpha}(\gamma, \tau), & 0 \leq i \leq n - 1, \\
\tau^{n-i} \log \tau \tilde{\beta}_i(\gamma, \tau), & 1 \leq i \leq n - 1.
\end{pmatrix}
\] (41)

Putting
\[
\begin{pmatrix}
V_1(\gamma) \\
\vdots \\
V_{n-2}(\gamma) \\
V_{n-1}(\gamma)
\end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix}
\gamma_n \\
\vdots \\
\gamma_{2n-3} \\
0
\end{pmatrix}, \quad a_{i,1} = a_i, -eH_{i,n} = d_i
\]
by (40), this implies that
\[
\tilde{\beta}_i(\gamma, \tau) = -\alpha_i - \log^{-1} \tau \tilde{P}_{n-1+i}(\tau) + V_i(\gamma), 1 \leq i \leq n - 1,
\] (42)
and
\[
\tilde{\alpha}_{n-1}(\gamma, \tau) = -\sum_{j=1}^{n-1} (L_{n-1,j} \log^{-1} \tau + H_{n-1,j}) \tilde{\beta}_j(\gamma, \tau) - (eL_{n-1,n} + \tilde{P}_{n-1}(\tau)) \log^{-2} \tau + d_{n-1} \log^{-1} \tau + \gamma_{n-1} \log^{-1} \tau,
\]
\[
\tilde{\alpha}_i(\gamma, \tau) = -\sum_{j=i+1}^{n-1} J_{i,j} \tilde{\alpha}_j(\gamma, \tau) - \sum_{j=1}^{n-1} (L_{i,j} \log^{-1} \tau + H_{i,j}) \tilde{\beta}_j(\gamma, \tau) - (eL_{i,n} + \tilde{P}_i(\tau)) \log^{-2} \tau + d_i \log^{-1} \tau + \gamma_i \log^{-1} \tau, 0 \leq i \leq n - 2.
\] (43)
Observe that (42), (43) together with (41) define a scaling in the parameter space. In the new parameters $(\gamma, \tau)$, the displacement function takes the form

$$
\Phi_{\psi(\gamma, \tau)}(x) = \tau^n \log \tau \left( \gamma_0 + \gamma_1 x + \cdots + \gamma_{2n-3} x^{2n-3} + k(\gamma, \tau) x^{2n-1} + \mathcal{O}(x^{2n}) \right),
$$

where $k(0, 0) = \tilde{k}(0) \neq 0$. By a rescaling of the form $\tilde{\gamma}_i = -\gamma_i / |k(\gamma, \tau)|$ and after removing the tildes, we get

$$
\Phi_{\psi(\gamma, \tau)}(x) = -|k(\gamma, \tau)| \tau^n \log \tau \left( Q_{2n-1}^\pm(x) + \mathcal{O}(x^{2n}) \right).
$$

This ends the proof of Theorem 1. \(\square\)

In the last subsection we prove Theorem 2. Though the idea is similar, there are differences in the computations that deserve to be mentioned.

### 2.2 Proof of Theorem 2

We now turn to the even case $k = 2n$. The strategy is the same as in the previous section. For $k = 2n$, the expression in (25) holds with $\alpha_n = c \neq 0$ is a constant and we have

$$
\Phi_{\alpha, \beta}(x) = \Lambda_0 + \Lambda_1 x + \cdots + \Lambda_{2n-1} x^{2n-1} + \Lambda_{2n} x^{2n} + \mathcal{O}(x^{2n+1}), \quad (44)
$$

where

$$
\begin{pmatrix}
\Lambda_0 \\
\vdots \\
\Lambda_{n-1}
\end{pmatrix} = J^{-1} \begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\vdots \\
\alpha_{n-1} \tau^{n-1}
\end{pmatrix} + (L^- + \log \tau H^-) \begin{pmatrix}
\beta_1 \\
\vdots \\
\beta_n 
\end{pmatrix} + \tau^{n+1} \log \tau \begin{pmatrix}
R_0(\tau) \\
R_1(\tau) \\
\vdots \\
R_{n-1}(\tau)
\end{pmatrix}, \quad (45)
$$

$$
\begin{pmatrix}
\Lambda_n \\
\vdots \\
\Lambda_{2n-2} \\
\Lambda_{2n-1}
\end{pmatrix} = (A + \log \tau \tilde{N}) \begin{pmatrix}
\beta_1 \\
\vdots \\
\beta_{n-1} \tau^{n-1} \\
\beta_n \tau^n
\end{pmatrix} + c \tau^n \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix} + \tau^{n+1} \log \tau \begin{pmatrix}
R_n(\tau) \\
\vdots \\
R_{2n-2}(\tau) \\
R_{2n-1}(\tau)
\end{pmatrix}, \quad (46)
$$
and
\[
\begin{pmatrix}
\Lambda_{n+1} \\
\vdots \\
\Lambda_{2n-1} \\
\Lambda_{2n}
\end{pmatrix} = T \begin{pmatrix}
\beta_1 \tau \\
\vdots \\
\beta_{n-1} \tau^{n-1} \\
\beta_n \tau^n
\end{pmatrix} + \tau^{n+1} \log \tau \begin{pmatrix}
R_{n+1}(\tau) \\
\vdots \\
R_{2n-1}(\tau) \\
R_{2n}(\tau)
\end{pmatrix}. \quad (47)
\]
From (46), we get
\[
\Lambda_n = \epsilon \tau^n + \log \tau \beta_n \tau^n + \sum_{j=1}^n \tilde{A}_{i,j} \beta_j \tau^j + \tau^{n+1} \log \tau R_n(\tau),
\]
and for each integer \( i = n + 1, \ldots, 2n - 1, \)
\[
\Lambda_i = \sum_{j=1}^n \tilde{A}_{i,j} \beta_j \tau^j + \tau^{n+1} \log \tau R_i(\tau).
\]
By Lemma 1, the matrix \( \hat{A} \) is invertible. Putting
\[
\hat{A}^{-1} = (\tilde{a}_{i,j})_{i=1, \ldots, n, j=1, \ldots, n},
\]
and after multiplication by \( \hat{A}^{-1} \) of both sides of (46), we get
\[
\hat{A}^{-1} \begin{pmatrix}
\Lambda_n \\
\vdots \\
\Lambda_{2n-2} \\
\Lambda_{2n-1}
\end{pmatrix} = \left( \mathbf{id} + \log \tau \hat{A}^{-1} \mathbf{N} \right) \begin{pmatrix}
\beta_1 \tau \\
\vdots \\
\beta_{n-1} \tau^{n-1} \\
\beta_n \tau^n
\end{pmatrix} + \epsilon \tau^n \begin{pmatrix}
\tilde{a}_{1,1} \\
\vdots \\
\tilde{a}_{n-1,1} \\
\tilde{a}_{n,1}
\end{pmatrix} + \tau^{n+1} \log \tau \hat{A}^{-1} \begin{pmatrix}
R_n(\tau) \\
\vdots \\
R_{2n-2}(\tau) \\
R_{2n-1}(\tau)
\end{pmatrix}. \quad (48)
\]
Observe that
\[
\hat{A}^{-1} \mathbf{N} = \begin{pmatrix}
0 & \ldots & 0 & \tilde{a}_{1,1} \\
0 & \ldots & 0 & \tilde{a}_{2,1} \\
\vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & \tilde{a}_{n,1}
\end{pmatrix},
\]
Now, we aim to find a scaled reparametrization of the form
\[
\Psi(\gamma, \tau) = (\alpha, \beta)
\]
linking \( \gamma = (\gamma_0, \ldots, \gamma_{2n-2}) \) and \( \tau \) with \( (\alpha, \beta) \). We require \( \Psi \) to have the components
\[
\alpha_i = \tau^{n-i} \tilde{a}_i(\gamma, \tau), \quad 0 \leq i \leq n - 1,
\]
\[
\beta_i = \frac{\tau^{n-i}}{1 + a_n \log \tau} \tilde{b}_i(\gamma, \tau), \quad 1 \leq i \leq n,
\]
32
where $\bar{a}_n$ is a constant, leading to
\[
\Phi_{\psi(\gamma, \tau)}(x) = \tilde{f}(\tau)(\gamma_0 + \gamma_1 x + \cdots + \gamma_{2n-2} x^{2n-2} + k(\gamma, \tau) x^{2n} + \mathcal{O}(x^{2n+1})).
\]

We first determine $\tilde{f}(\tau)$ and show that the leading term $k(\gamma, \tau)$ is such that $k(0, 0) \neq 0$. We then obtain the desired expression for $\Phi_{\psi(\gamma, \tau)}$ by solving
\[
\Lambda_i = \tilde{f}(\tau)\gamma_i, \ i = 1, \ldots, 2n - 1.
\]

To be more precise, we start solving the equation
\[
\Lambda_i = 0, \ \forall i = n, \ldots, 2n - 1, \ \text{and} \ \Lambda_{2n} \neq 0.
\]

With (48) and for each integer $i = 1, \ldots, n$, we get
\[
\beta_i \tau^i + \log \tau \bar{a}_i \gamma_i \tau^n + c\bar{a}_i \gamma_i \tau^n + \tau^{n+1} \log \tau Q_{n+i-1}(\tau) = 0, \quad (49)
\]
where
\[
\begin{pmatrix}
Q_n(\tau) \\
\vdots \\
 Q_{2n-1}(\tau)
\end{pmatrix} = \tilde{A}^{-1}
\begin{pmatrix}
R_n(\tau) \\
\vdots \\
 R_{2n-1}(\tau)
\end{pmatrix}.
\]

Again for each integer $i = n, \ldots, 2n - 1$,
\[
Q_i(\tau) = Q_{i,0} + Q_{i,1} \tau \log \tau + \cdots.
\]

By putting $i = n$ in (49), we get
\[
\beta_n = -\frac{c\bar{a}_{n,1} + \tau \log \tau Q_{2n-1}(\tau)}{1 + \bar{a}_{n,1} \log \tau}
\]
and for each $i = 1, \ldots, n - 1$,
\[
\beta_i \tau^i = \frac{-c\bar{a}_{i,1} \gamma_i \tau^n \log \tau + \bar{a}_{i,1} \gamma_i \tau^{n+1} \log \tau Q_{2n-1}(\tau)}{1 + \bar{a}_{n,1} \log \tau} = -c\bar{a}_{i,1} \gamma_i \tau^n - \tau^{n+1} \log \tau Q_{n+i-1}(\tau),
\]
which implies that
\[
\beta_i \tau^i = \frac{c\bar{a}_{i,1} \gamma_i \tau^n \log \tau + \bar{a}_{i,1} \gamma_i \tau^{n+1} \log \tau Q_{2n-1}(\tau)}{1 + \bar{a}_{n,1} \log \tau}
\]
\[
+ \frac{-c\bar{a}_{i,1} \gamma_i \tau^n - c\bar{a}_{i,1} \gamma_i \tau^n \log \tau}{1 + \bar{a}_{n,1} \log \tau} + \tau^{n+1} \log \tau Q_{n+i-1}(\tau),
\]
\[
\beta_i \tau^i = -\frac{c\bar{a}_{i,1} \gamma_i \tau^n}{1 + \bar{a}_{n,1} \log \tau} + \frac{\bar{a}_{i,1} \gamma_i \tau^{n+1} \log^2 \tau Q_{2n-1}(\tau)}{1 + \bar{a}_{n,1} \log \tau}
\]
\[
-\tau^{n+1} \log \tau Q_{n+i-1}(\tau)
\]
\[
+ \frac{c\bar{a}_{i,1} \gamma_i \tau^n}{1 + \bar{a}_{n,1} \log \tau} + \tau^{n+1} \log \tau \frac{\tilde{Q}_i(\tau)}{1 + \bar{a}_{n,1} \log \tau}.
\]

33
where $\bar{Q}_t$ is of the form
\[\bar{Q}_t(\tau) = \bar{Q}_{t,-1} \log^{-1} \tau + \bar{Q}_{t,0} + \bar{Q}_{t,1} \log \tau + \cdots.\]

This also implies that
\[
\begin{pmatrix}
\beta_1 \tau \\
\vdots \\
\beta_n \tau^n
\end{pmatrix}
= -\frac{c \tau^n}{1 + a_{n,1} \log \tau}
\begin{pmatrix}
\bar{a}_{1,1} \\
\vdots \\
\bar{a}_{n,1}
\end{pmatrix}
+ \frac{\tau^{n+1} \log^2 \tau}{1 + a_{n,1} \log \tau}
\begin{pmatrix}
\bar{Q}_1(\tau) \\
\vdots \\
\bar{Q}_n(\tau)
\end{pmatrix}.
\]

With (47) we get
\[
\begin{pmatrix}
\Lambda_{n+1}(0, \tau) \\
\vdots \\
\Lambda_{2n}(0, \tau)
\end{pmatrix}
= -\frac{c \tau^n}{1 + a_{n,1} \log \tau}
\begin{pmatrix}
\bar{a}_{1,1} \\
\vdots \\
\bar{a}_{n,1}
\end{pmatrix}
+ \frac{\tau^{n+1} \log^2 \tau}{1 + a_{n,1} \log \tau}
\begin{pmatrix}
\bar{Q}_1(\tau) \\
\vdots \\
\bar{Q}_n(\tau)
\end{pmatrix}.
\]

By Lemma 1, both $\bar{A}^{-1}$ and $T$ are invertible. It follows that
\[\bar{A}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1,1} \\ \vdots \\ a_{n,1} \end{pmatrix} \neq 0, \text{ and } T \begin{pmatrix} \bar{a}_{1,1} \\ \vdots \\ \bar{a}_{n,1} \end{pmatrix} \neq 0,
\]
thus $(\Lambda_{n+1}(0, \tau), \ldots, \Lambda_{2n}(0, \tau))$ does not vanish. Since for each $i = 0, \ldots, 2n-1$, $\Lambda_i = 0$, this implies that
\[\Lambda_{2n}(0, \tau) = \bar{K}(\tau) \frac{\tau^n}{1 + a_{n,1} \log \tau}, \text{ with } \bar{K}(0) \neq 0.\]

We now develop the scaling. Putting
\[\Lambda_i(\gamma, \tau) = \gamma_i \frac{\tau^n}{1 + a_{n,1} \log \tau}, \quad 0 \leq i \leq 2n - 2, \quad \Lambda_{2n-1}(\gamma, \tau) \equiv 0,
\]
with (48) it follows that for each integer $i = 1, \ldots, n$,
\[
\begin{align*}
\beta_i \tau^i + \log \tau \bar{a}_{i,1} \beta_n \tau^n \\
+ c \bar{a}_{i,1} \tau^n + \tau^{n+1} \log \tau Q_{n+i-1}(\tau) &= \sum_{j=n}^{2n-2} \bar{a}_{i,j} \frac{\gamma_j \tau^n}{1 + a_{n,1} \log \tau}.
\end{align*}
\]
In particular we have
\[
\begin{align*}
\beta_n \tau^n + \log \tau \bar{a}_{n,1} \beta_n \tau^n &= -c \bar{a}_{n,1} \tau^n + \sum_{j=n}^{2n-2} \bar{a}_{n,j} \frac{\gamma_j \tau^n}{1 + a_{n,1} \log \tau} \\
&- \tau^{n+1} \log \tau Q_{2n-1}(\tau).
\end{align*}
\]

34
Thus
\[
\beta_n = -\frac{c\bar{a}_{n,1}}{1 + \bar{a}_{n,1} \log \tau} + \frac{1}{(1 + \bar{a}_{n,1} \log \tau)} \sum_{j=1}^{2n-2} \bar{a}_{i,j} \frac{\gamma_j^\tau^n}{1 + \bar{a}_{n,1} \log \tau} \tag{51}
\]
\[- \tau \log \tau Q_{2n-1}(\tau) \frac{1}{1 + \bar{a}_{n,1} \log \tau}.
\]

We now put
\[
\alpha_i = \tau^{n-i} \bar{a}_i(\gamma, \tau), \quad \beta_i = \frac{\tau^{n-i}}{1 + \bar{a}_n \log \tau} \bar{b}_i(\gamma, \tau).
\tag{52}
\]

Write
\[
\begin{pmatrix}
W_1(\gamma) \\
\vdots \\
W_{n-1}(\gamma) \\
W_n(\gamma)
\end{pmatrix} = \bar{A}^{-1} \begin{pmatrix}
\gamma_n \\
\vdots \\
\gamma_{2n-2} \\
0
\end{pmatrix}.
\]

With (50), (51) and (52), it follows that
\[
\bar{\beta}_i(\gamma, \tau) + \bar{a}_{i,1} \bar{b}_n(\gamma, \tau) \log \tau + c\bar{a}_{i,1}(1 + \bar{a}_{n,1} \log \tau)
\]
\[
+ \tau (1 + \bar{a}_{n,1} \log \tau) \log \tau Q_{n+i-1}(\tau) = \sum_{j=n}^{2n-2} \bar{a}_{i,j} \gamma_j,
\]
and we get
\[
\bar{\beta}_n(\gamma, \tau)(1 + \bar{a}_{n,1} \log \tau) = -c\bar{a}_{n,1}(1 + \bar{a}_{n,1} \log \tau)
\]
\[
+ \sum_{j=n}^{2n-2} \bar{a}_{i,j} \gamma_j - \tau (1 + \bar{a}_{n,1} \log \tau) \log \tau Q_{2n-1}(\tau),
\]
\[
\bar{\beta}_n(\gamma, \tau) = -c\bar{a}_n + (1 + \bar{a}_{n,1} \log \tau)^{-1} \sum_{j=1}^{2n-2} \bar{a}_{n,j} \gamma_j
\]
\[
- \tau \log \tau Q_{2n-1}(\tau),
\]
\[
= -c\bar{a}_{n,1} + (1 + \bar{a}_{n,1} \log \tau)^{-1} W_n(\gamma)
\]
\[- \tau \log \tau Q_{2n-1}(\tau).
\]

Moreover, with (53) and (54), for each \(i = 1, \ldots, n-1\), we get
\[
\bar{\beta}_i(\gamma, \tau) + \bar{a}_{i,1} \log \tau \left( -c\bar{a}_{n,1} + (1 + \bar{a}_{n,1} \log \tau)^{-1} W_n(\gamma) - \tau \log \tau Q_{2n-1}(\tau) \right)
\]
\[
= -c\bar{a}_{i,1} + \sum_{j=n}^{2n-2} \bar{a}_{i,j} \gamma_j - \tau \log \tau Q_{n+i-1}(\tau)
\]
\[- c\bar{a}_{i,1} \bar{a}_{n,1} \log \tau - \tau \log^2 \tau \bar{a}_{n,1} Q_{n+i-1}(\tau),
\]
\[
\tilde{b}_i(\gamma, \tau) = -c\bar{a}_i - \bar{a}_i \log \tau (1 + \bar{a}_n \log \tau)^{-1} \mathcal{W}_n(\gamma) + \mathcal{W}_i(\gamma) \\
- \tau \log \tau Q_{n+i-1}(\tau) \\
- \tau \log^2 \tau \bar{a}_n Q_{n+i-1}(\tau) + \bar{a}_i \tau \log^2 \tau Q_{2n-1}(\tau),
\] (54)

where \(\bar{a}_i = \bar{a}_i, 1\). Now with (45), we get

\[
\alpha_{n-1} \tau^{n-1} = \gamma_{n-1} \frac{\tau^n}{1 + a_{n,1} \log \tau} - cL_{n-1,0} \tau^n \\
- \sum_{j=1}^{n} (L_{n-1,j} + \log \tau H_{n-1,j}) \beta_j \tau^j - \tau^n \log \tau R_{n-1}(\tau),
\]

thus

\[
\bar{\alpha}_{n-1}(\gamma, \tau) = -c\ell_n + \gamma_{n-1} (1 + \bar{a}_n \log \tau)^{-1} \\
- \frac{\log \tau}{1 + \bar{a}_n \log \tau} \sum_{j=1}^{n} (L_{n-1,j} \log^{-1} \tau + H_{n-1,j}) \beta_j (\gamma, \tau) \\
- \tau \log \tau R_{n-1}(\tau),
\] (55)

where \(\ell_i = L_{i,0}\) and for each \(i = 1, \ldots, n-2,\)

\[
\alpha_i \tau^i = -c\ell_i \tau^n - \sum_{j=i+1}^{n-1} J_{i,j} \alpha_j \tau^j + \gamma_i \tau^n (1 + \bar{a}_{n,1} \log \tau)^{-1} \\
- \sum_{j=1}^{n} (L_{i,j} + \log \tau H_{i,j}) \beta_j \tau^j - \tau^{n+1} \log \tau R_i(\tau),
\]

thus

\[
\bar{\alpha}_i(\gamma, \tau) = -c\ell_i - \sum_{j=i+1}^{n-1} J_{i,j} \bar{\alpha}_j (\gamma, \tau) + \gamma_i (1 + \bar{a}_{n,1} \log \tau)^{-1} \\
- \frac{\log \tau}{1 + \bar{a}_n \log \tau} \sum_{j=1}^{n} (L_{i,j} \log^{-1} \tau + H_{i,j}) \bar{\beta}_j (\gamma, \tau) - \tau \log \tau R_i(\tau).
\] (56)

Observe that the expressions given in (54), (55) and (56) together with (52) define a scaling. In the new parameters \((\gamma, \tau),\) the displacement function takes the form

\[
\Phi_{\psi(\gamma, \tau)}(x) = \frac{\tau^n}{1 + a_n \log \tau} \left( \gamma_0 + \gamma_1 x + \cdots + \gamma_{2n-2} x^{2n-2} + k(\gamma, \tau) x^{2n} + O(x^{2n+1}) \right),
\]

where \(k(0, 0) = \overline{K}(0) \neq 0.\) By a rescaling of the form \(\hat{\gamma}_i = -\gamma_i / |k(\gamma, \tau)|\) and after removing the tildes, we get

\[
\Phi_{\psi(\gamma, \tau)}(x) = \frac{|k(\gamma, \tau)| \tau^n}{1 + a_n \log \tau} \left( Q_{2n}^{\pm}(x) + O(x^{2n+1}) \right).
\]

This ends the proof of Theorem 2. \(\square\)
2.3 Proof of Theorems 3 and 4

To construct the scaling, we first recall that

\[
B_{\lambda, \varepsilon}(u) = \alpha_0 + \beta_1[u\omega + \varepsilon \eta_1(u, u\omega)] + \alpha_1[u + \varepsilon \mu_1(u, u\omega)] + \cdots + \alpha_{n-1}[u^{n-1} + \varepsilon \mu_{n-1}(u, u\omega)] + \beta_n[u^n\omega + \varepsilon \eta_n(u, u\omega)] + \Psi_n(u, \lambda, \varepsilon).
\]

For simplicity we assume that, \(\beta_n = \varepsilon \neq 0\) and we are treating an unfolding of odd codimension. In the even codimension case, the proof is completely similar.

Let \(\tau > 0\), and put \(u = \tau(1 + x)\) where \(x\) close to 0. Recall that

\[
\omega(u) = \frac{u^{\varepsilon \beta} - 1}{\varepsilon \beta}.
\]

It follows that

\[
\omega[\tau(1 + x)] = \frac{1}{\varepsilon \beta} \left( \tau^{\varepsilon \beta}(1 + x)^{\varepsilon \beta} - (1 + x)^{\varepsilon \beta} + (1 + x)^{\varepsilon \beta} - 1 \right),
\]

\[
= \omega(\tau)(1 + x)^{\varepsilon \beta} + \omega(1 + x),
\]

\[
= \omega(\tau)(1 + \varepsilon A_1(x, \varepsilon)) + \log(1 + x) + \varepsilon A_2(x, \varepsilon), \tag{57}
\]

where both \(A_1\) and \(A_2\) are analytic functions. Consider the maps

\[
u \mapsto G_j(u) = \beta_j(u^j\omega(u) + \varepsilon \eta_j(u, u\omega(u))), 1 \leq j \leq n - 1
\]

and

\[
u \mapsto F_i(u) = \alpha_i(u^i + \varepsilon \mu_i(u, u\omega)), 0 \leq i \leq n - 1.
\]

For each integers \(i, j\) under consideration, we have

\[
G_j(\tau(1 + x)) = \beta_j \left[ \tau^j(1 + x)^j\omega(\tau(1 + x)) \right. + \varepsilon \eta_j \left[ \tau(1 + x), \left( \tau(1 + x)\omega(\tau(1 + x)) \right) \right],
\]

\[
F_i(\tau(1 + x)) = \alpha_i \left[ \tau^i(1 + x)^i + \varepsilon \mu_i \left[ \tau(1 + x), \left( \tau(1 + x)\omega(\tau(1 + x)) \right) \right] \right].
\]

With (57) and knowing that

\[
\eta_j(u, u\omega(u)) = \eta_{j,0} u^j + \cdots, 1 \leq j \leq n - 1,
\]

\[
\mu_i(u, u\omega(u)) = \mu_{i,0} u^{i+1} \omega(u) + \cdots, 0 \leq i \leq n - 1,
\]
it follows that
\[
G_j(\tau(1 + x)) = \beta_j \tau^j \left((1 + x)^j \left(\omega(\tau)(1 + \epsilon A_1(x, \epsilon) + \log(1 + x) + \epsilon A_2(x, \epsilon)\right) + \epsilon C_j(x, \epsilon, \tau, \omega(\tau))\right), \quad j = 1, \ldots, n - 1,
\]
where \(C_j\) and \(D_i\) are smooth in \(x\) and \(\epsilon\), polynomial in \(\tau\) and \(\omega^n(\tau)\) and satisfy
\[
C_j = C_{j0} + \cdots, \quad D_i = D_{i0} \tau^{\omega_i + 1}(\tau) + \cdots.
\]

Write \(B_{\lambda, \epsilon}(\tau(1 + x)) = \Phi_{\alpha, \epsilon}(x)\). We get
\[
\Phi_{\alpha, \epsilon}(x) = \Lambda_0 + \Lambda_1 x + \cdots + \Lambda_{2n-1} x^{2n-2} + \Lambda_{2n-1} x^{2n-1} + \mathcal{O}(x^{2n}).
\]
From (58) and (59), there exist matrices
\[
\bar{J}^{-} = (\bar{J}_{i,j}^{-})_{i=0, \ldots, n, \ j=0, \ldots, n}, \quad \bar{L}^{-} = (\bar{L}_{i,j}^{-})_{i=0, \ldots, n, \ j=1, \ldots, n},
\]
\[
\bar{H}^{-}(\epsilon) = (\bar{H}_{i,j}^{-}(\epsilon))_{i=0, \ldots, n, \ j=1, \ldots, n}, \quad \bar{T} = (T_{i,j})_{i=n+1, \ldots, 2n, \ j=1, \ldots, n},
\]
and \(\mathcal{N}\) with entries which are polynomial in \(\tau\) and \(\omega^n(\tau)\) and analytic in \(\epsilon\) such that
\[
\bar{J}_{i,j}^{-} = J_{i,j}^{-} + \cdots, \quad \bar{H}_{i,j}^{-} = H_{i,j}^{-} + \cdots, \quad \bar{L}_{i,j}^{-} = L_{i,j}^{-} + \cdots,
\]
\[
\bar{T}_{i,j} = T_{i,j} + \cdots, \quad \bar{N}_{i,j} = N_{i,j} + \cdots,
\]
and such that
\[
\begin{pmatrix}
\Lambda_0 & \\
\vdots & \\
\Lambda_{n-1}
\end{pmatrix} = \bar{J}^{-} \begin{pmatrix}
\alpha_0 \\
\alpha_1 \tau \\
\vdots \\
\alpha_{n-1} \tau^{n-1}
\end{pmatrix} + \omega(\tau) \bar{H}^{-} \begin{pmatrix}
\beta_1 \tau \\
\vdots \\
\beta_{n-1} \tau^{n-1}
\end{pmatrix} + \bar{L}^{-} \begin{pmatrix}
\beta_1 \tau \\
\vdots \\
\beta_{n-1} \tau^{n-1}
\end{pmatrix} + \begin{pmatrix}
P_0(\tau, \epsilon) \\
\vdots \\
P_{n-2}(\tau, \epsilon) \\
P_{n-1}(\tau, \epsilon)
\end{pmatrix},
\]
where for each integer \(i = 0, \ldots, n - 1\),
\[
P_i(\tau, \epsilon) = \tau^n \bar{P}_i(\tau, \epsilon) = \tau^n [\bar{P}_i(0, \epsilon) + \cdots.
\]

38
Define also the following matrices

\[
\tilde{A} = (\tilde{A}_{i,j})_{i=n,...,2n-1, j=1,...,n}, \quad \tilde{A}_{i,j} = T_{i,j} \text{ if } i > n, \quad \tilde{A}_{n,j} = I_{n,j},
\]

\[
\tilde{A} = (A)_{i=n,...,2n-2, j=1,...,n-1}, \quad \tilde{A}_{i,j} = A_{i,j},
\]

\[
\tilde{B} = (B)_{i=n+1,...,2n-1, j=1,...,n-1}, \quad \tilde{B}_{i,j} = A_{i,j}.
\]

From Lemma 1, these 3 matrices remain invertible for \( \varepsilon \) sufficiently small. We get

\[
\begin{pmatrix}
\Lambda_n \\
\vdots \\
\Lambda_{2n-2} \\
\Lambda_{2n-1}
\end{pmatrix} = \left( \tilde{A} + \log \tau \tilde{N} \right) \begin{pmatrix}
\beta_1 T \\
\vdots \\
\beta_{n-1} \tau^{n-1} \\
\beta_n \tau^n
\end{pmatrix} + \begin{pmatrix}
P_n(\tau, \varepsilon) \\
P_{2n-2}(\tau, \varepsilon) \\
F
\end{pmatrix},
\]

where for each integer \( i = n, \ldots, 2n - 1, \)

\[
P_i(\tau, \varepsilon) = \tau^n P_i(\tau, \varepsilon) = \tau^n [\tilde{F}_i(\tau, \varepsilon) + \cdots].
\]

We follow the same strategy as in the previous section. We claim that for each integer \( i = 1, \ldots, n - 1, \) there exist maps \( \tilde{R}_{i-1,i}(\tau, \varepsilon), \tilde{V}_i(\tau, \gamma, \varepsilon) \) and coefficients \( \tilde{a}_i(\tau, \varepsilon), \tilde{d}_i(\tau, \varepsilon) \) which are smooth in \( \varepsilon \) and \( \tau \omega^n(\tau), (\tilde{V}_i \text{ is linear in } \gamma), \) such that

\[
\tilde{a}_i(\tau, \varepsilon) = a_i + \cdots, \quad \tilde{d}_i(\tau, \varepsilon) = d_i + \cdots,
\]

\[
\tilde{R}_{i-1,i}(\tau, \varepsilon) = \tilde{R}_{i-1,i} + \cdots, \quad \tilde{V}_i(\tau, \gamma, \varepsilon) = V_i + \cdots
\]

and such that by putting

\[
\beta_i = \tau^{n-i} \omega(\tau) \tilde{a}_i(\tau, \gamma, \varepsilon), \quad \alpha_i = \tilde{\alpha}_i(\tau, \gamma, \varepsilon) \tau^{n-i} \omega^2(\tau),
\]

\[
\tilde{\alpha}_i(\tau, \gamma, \varepsilon) = -c\tilde{a}_i(\tau, \varepsilon) - \omega^{-1}(\tau) \tilde{R}_{i-1,i}(\tau, \varepsilon) + \tilde{V}_i(\tau, \gamma, \varepsilon)
\]

and

\[
\tilde{\alpha}_{n-1}(\tau, \gamma, \varepsilon) = -\sum_{j=1}^{n-1} (\tilde{\alpha}_{n-1,j}(\tau, \varepsilon) \omega^{-1}(\tau) + \tilde{H}_{n-1,j}(\tau, \varepsilon)) \tilde{\beta}_j(\tau, \gamma, \varepsilon)
\]

\[
- \left( c\tilde{L}_{n-1,n}(\tau, \varepsilon) + \tilde{P}_{n-1}(\tau, \varepsilon) \right) \omega^{-2}(\tau)
\]

\[
+ \tilde{d}_{n-1}(\tau, \varepsilon) \omega^{-1}(\tau) + \gamma_{n-1} \omega^{-1}(\tau),
\]

\[
\tilde{\alpha}_i(\tau, \gamma, \varepsilon) = -\sum_{j=i+1}^{n-1} \tilde{H}_{i,j}(\tau, \varepsilon) \tilde{\alpha}_j(\tau, \gamma, \varepsilon)
\]

\[
- \sum_{j=1}^{n-1} (\tilde{H}_{i,j}(\tau, \varepsilon) \omega^{-1}(\tau) + \tilde{H}_{i,j}(\tau, \varepsilon)) \tilde{\beta}_j(\tau, \gamma, \varepsilon)
\]

\[
- \left( c\tilde{L}_{i,n}(\tau, \varepsilon) + \tilde{P}_i(\tau, \varepsilon) \right) \omega^{-2}(\tau) + \tilde{d}_i(\tau, \varepsilon) \omega^{-1}(\tau)
\]

\[
+ \gamma_i \omega^{-1}(\tau), \quad i = 0, \ldots, n - 2,
\]

39
we get
\[ \Phi_{\psi}(\gamma, \tau)(x) = \tau^n \omega(\tau)(\gamma_0 + \gamma_1 + \cdots + \gamma_{2n-3}x^{2n-3} + \mathcal{K}(\tau, \gamma, \varepsilon)x^{2n-1} + O(x^{2n}), \]
where \(\mathcal{K}(0, 0, 0) \neq 0\).

\[ \square \]

References


