QUASI-PERIODIC HÉNON-LIKE ATTRACTORS
IN THE LORENZ-84 CLIMATE MODEL
WITH SEASONAL FORCING

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A class of strange attractors is described, occurring in a low-dimensional model of general atmospheric circulation. The differential equations of the system are subject to periodic forcing, where the period is one year – as suggested by Lorenz in 1984. The dynamics of the system is described in terms of a Poincaré map, computed by numerical means. It is conjectured that certain strange attractors observed in the Poincaré map are of quasi-periodic Hénon-like type, i.e., they coincide with the closure of the unstable manifold of a quasi-periodic invariant circle of saddle type. A route leading to the formation of such strange attractors is presented. It involves a finite number of quasi-periodic period doubling bifurcations, followed by the destruction of an invariant circle due to homoclinic tangency.

1. Introduction

In this note we examine a class of strange attractors occurring in the model

\[
\begin{align*}
\dot{x} &= -ax - y^2 - z^2 + aF(1 + \varepsilon \cos(\omega t)), \\
\dot{y} &= -y + xy - bxz + G(1 + \varepsilon \cos(\omega t)), \\
\dot{z} &= -z + bxy + xz.
\end{align*}
\]

This is a variation on an autonomous model proposed by Lorenz in 1984\(^1\) for the long term atmospheric circulation at mid latidude of the northern hemisphere. The autonomous Lorenz-84 model, given by Eq. (1) with \(\varepsilon = 0\),
is used in climatological research, e.g. by coupling it with a low-dimensional model for ocean dynamics.\(^2\) See Ref. 3 for the bifurcation diagram of the autonomous system and Ref. 4 for its derivation from the Navier-Stokes equations.

The variable \(x\) in (1) stands for the strength of a symmetric, globally averaged westerly wind current. The variables \(y\) and \(z\) are the strengths of cosine and sine phases of a chain of superposed waves transporting heat poleward. The terms in \(F\) and \(G\) are thermal forcings: \(F\) is the symmetric cross-latitude heating contrast and \(G\) accounts for the asymmetric heating contrast between oceans and continents. The periodic forcing of frequency \(\omega = 2\pi/T\), where the period \(T\) is fixed at 73, simulates a seasonal variation of \(F\) and \(G\). Indeed, \(T = 73\) corresponds to one year in the time-scale unit of Eq. (1), estimated to be five days. As in Refs. 1–4, the coefficients \(a\) and \(b\) are fixed at \(a = 1/4\) and \(b = 4\).

In this note we only consider one of the dynamical phenomena observed by numerical simulations in system (1), namely the occurrence of attractors which we conjecture to be of \textit{quasi-periodic Hénon-like} type. Moreover, only \(G\) is used here as control parameter, while \(\varepsilon\) and \(F\) are kept fixed at 0.5 and 11 respectively. See Refs. 5–6 for a more detailed study of the bifurcation diagram of Eq. (1) in the three-dimensional parameter space \(\{F, G, \varepsilon\}\).

The dynamics of the forced system (1) is described in terms of the one-parameter family of diffeomorphisms given by the Poincaré map \(P_G : \mathbb{R}^3 \to \mathbb{R}^3\), also called stroboscopic, first return or period map. The map \(P_G\) is computed by numerical integration of Eq. (1) over a period \(T\), see Refs. 5–6 for the methods used.

2. The dynamics on quasi-periodic Hénon-like attractors

Let \(H : \mathbb{R}^m \to \mathbb{R}^m\) be a map and \(\mathcal{A} \subset \mathbb{R}^m\). Then \(\mathcal{A}\) is called an attractor if \(\mathcal{A}\) is compact and \(H\)-invariant, if the stable set (basin of attraction) \(W^s(\mathcal{A})\) has nonempty interior and if there exists a point \(p \in \mathcal{A}\) such that the orbit \(\text{Orb}(p) = \{H^j(p)\}_{j \geq 0}\) is dense in \(\mathcal{A}\). The attractor \(\mathcal{A}\) is called \textit{Hénon-like}\(^7,8,9\) if there exist a saddle periodic orbit \(\text{Orb}(s) = \{s, H(s), \ldots, H^n(s)\}\), a point \(p\) in the unstable manifold \(W^u(\text{Orb}(s))\), and a tangent vector \(v \in T_p \mathbb{R}^m\) such that the orbit of \(p\) is dense in \(\mathcal{A}\) and

\[
\mathcal{A} = \overline{W^s(\text{Orb}(s))},
\]

\[
\|DH^n(p)v\| \geq \kappa \lambda^n \quad \text{for} \ n \geq 0,
\]

where overline denotes topological closure. Condition (3) means that the dense orbit \(\text{Orb}(p)\) has a positive Lyapunov exponent. We say that the
attractor $\mathcal{A}$ is quasi-periodic Hénon-like if there exist a quasi-periodic invariant circle $\mathcal{C}$ of saddle type, a point $p \in W^s(\mathcal{C})$, and a vector $v \in T_p \mathbb{R}^m$ such that condition (3) holds while

$$\mathcal{A} = W^u(\mathcal{C}).$$

The conjectural occurrence of this type of attractors in the family $P_G$ is now illustrated by numerical results. An attractor $\mathcal{D}$ of the map $P_G$ is plotted in Fig. 1 (A), where $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ is the union of two disjoint circles $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $P_G(\mathcal{D}_1) = \mathcal{D}_2$ and $P_G^2(\mathcal{D}_j) = \mathcal{D}_j$ for $j = 1, 2$. Upon a slight variation of the parameters, this period two circle becomes resonant (i.e., phase-locked to a periodic attractor) and it gets eventually destroyed by homoclinic tangency$^{10,11}$ of a periodic saddle point. For nearby parameter values the attractor $\mathcal{A}$ in Fig. 1 (B) is detected. The attractor $\mathcal{A}$

![Figure 1](image-url)

Figure 1. (A) Projection on $(x, z)$ of the attracting period two circle $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ occurring at $G = 0.4969$. (B) Same as (A) for the strange attractor $\mathcal{A}$, at $G = 0.4972$. (a) Power spectrum of the attractor in (A). The square modulus of the Fourier coefficients (vertical axis) is plotted against the Fourier frequencies $f_k = k/N$ for $k = 1, \ldots, N/2$ (horizontal axis). Here $N = 2^{16}$ is the sample length of the time series given by the $y$-coordinate along an orbit on the attractor. The first six harmonics $g_k = k g_1$ of the internal frequency $g_1$ are labelled, and up to order $k = 35$ the harmonics are marked by asterisks. (b) Power spectrum of the attractor in (B).
Table 1. Numerical values of the Lyapunov dimension $D_L$ and Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of the attractors in Fig. 1.

<table>
<thead>
<tr>
<th>Fig. 1</th>
<th>attractor</th>
<th>$D_L$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>$\mathcal{D}$</td>
<td>1</td>
<td>0</td>
<td>-0.18</td>
<td>-14.5</td>
</tr>
<tr>
<td>(B)</td>
<td>$\mathcal{A}$</td>
<td>2.016</td>
<td>0.24</td>
<td>0</td>
<td>-14.9</td>
</tr>
</tbody>
</table>

is contained inside a Möbius strip which is slightly fattened in the normal direction. Indeed, the Lyapunov dimension $D_L$ of $\mathcal{A}$ is quite close to 2 (Table 1). This is due to the large absolute value of the negative Lyapunov exponent $\lambda_3$, corresponding to strong normal contraction, and to the fact that $\lambda_1$ is positive, the dynamics on $\mathcal{A}$ is sensitive to initial conditions. However, the property $\lambda_2 \simeq 0$ suggests that the dynamics on $\mathcal{A}$ still contains a quasi-periodic direction.

This idea is also supported by examination of power spectra, displayed for $\mathcal{D}$ and $\mathcal{A}$ in Fig. 1 (a) and (b) respectively. The period two circle $\mathcal{D}$ has two internal frequencies, $g_1 = 0.328$ and $h_1 = \frac{1}{2}$. The second harmonic $g_2 = 2g_1$ is the internal frequency of $P^2_G$ on $\mathcal{D}_1$ and $\mathcal{D}_2$. Only a few harmonics of $g_1$ persist in the spectrum of $\mathcal{A}$ (Fig. 1 (b)), all others having turned into broad band. Power spectra like in Fig. 1 (b) are of mixed type: they contain marked peaks (atoms of the spectral density) but also have a broad band component (locally continuous density).

The process leading to the formation of attractors like $\mathcal{A}$ (Fig. 1 (B)) passes through a finite number of quasi-periodic period doubling bifurcations. A whole quasi-periodic period doubling cascade does not take place, since the attracting periodic circles are eventually destroyed due to homoclinic tangencies of a saddle periodic point. An attracting invariant circle $\mathcal{C}$ of $P_G$ occurs at $G = 0.4872$ (Fig. 2 (A)). As $G$ increases, $\mathcal{C}$ loses stability through a quasi-periodic period doubling, and a circle attractor $\mathcal{C}^2$ is created (Fig. 2 (B)). The circle $\mathcal{C}$ still exists, is of saddle type and its two-dimensional unstable manifold is a Möbius strip with $\mathcal{C}^2$ as its boundary. Through another quasi-periodic doubling, the attracting period two circle $\mathcal{D}$ is born (Fig. 1 (A)), and $\mathcal{C}^2$ also becomes of saddle type. We stress that the latter bifurcation is different from the previous "length-doubling", since two disjoint circles $\mathcal{D}_1$ and $\mathcal{D}_2$ are now created.

A strange attractor $\mathcal{B}$, consisting of a single fattened Möbius strip, is plotted in Fig. 3 (A). This attractor is formed as the two "belts" of $\mathcal{A}$ (Fig. 1 (B)) melt together. Sections of $\mathcal{B}$ and $\mathcal{A}$ have a planar Hénon-like structure, compare the slice $\Sigma$ in Fig. 3 (B) and condition (2). To
illustrate the dynamics inside $\mathcal{B}$, we computed the image under $P_G$ of all points in the slice $\Sigma$. The image $P_G(\Sigma)$ is stretched and folded (Fig. 3 (A)), and again has a planar Hénon-like structure.

The main point of this note is the conjecture that the strange attractors $\mathcal{A}$ Fig. 1 (B) and $\mathcal{B}$ Fig. 3 (A) are indeed quasi-periodic Hénon-like. To be more precise, there exists a positive measure subset of the parameter space for which $\mathcal{A}$ (resp. $\mathcal{B}$) occurs. For such parameter values:

1. the circle $C^2$ coexists with $\mathcal{A}$ (resp. $C$ coexists with $\mathcal{B}$).
2. $C^2$ is quasi-periodic and of saddle type (resp. $C$ is);
3. $\mathcal{A} = \overline{W^u(C^2)}$ (resp. $\mathcal{B} = \overline{W^u(C)}$).

3. Concluding remarks

Quasi-periodic Hénon-like attractors are also numerically observed in a model map for the Hopf-saddle-node bifurcation of fixed point of diffeomorphisms.\textsuperscript{14} This bifurcation is one of the organizing centers of the Poincaré map $P_{F,G,\varepsilon}$ for $\varepsilon$ not too large.\textsuperscript{5,6}

However, for the above models a rigorous proof for the existence of quasi-periodic Hénon-like attractors is out of reach, though a computer-assisted proof may be possible. So in Ref. 15 we turn to the setting of product maps on $\mathbb{R}^2 \times S^1$, which is easier to deal with. In particular, a new result on Hénon-like attractors is obtained for maps given by the skew-product of a planar Hénon map\textsuperscript{7,9} with the Arnol’d map on $S^1$. We also consider perturbations of the product of certain dissipative planar maps with a rigid rotation on $S^1$. In this case, it is proved that there exists a quasi-periodic saddle-like invariant circle $\mathcal{C}$ such that the closure $\overline{W^u(\mathcal{C})}$ attracts an open set of points. However, the characterization of quasi-periodic Hénon-like

Figure 2. (A) Projection on $(x, z)$ of the circle attractor $\mathcal{C}$ of $P_G$, occurring at $G = 0.4872$. (B) Same as (A), for the circle $C^2$ at $G = 0.4874$. 


attractors largely remains open even in this simplified setting.

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**References**