

Multiperiodic dynamics – overview and some recent results

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1. Complex Linearization I

Problem Given $F(z) = \lambda z + f(z)$
holomorphic map (germ) $F : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$
 $f(0) = f'(0) = 0$;
to find $\Phi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$
biholomorphic germ such that

$$\Phi \circ F = \lambda \cdot \Phi$$

Such Φ is called linearization of F

Formal solution

Given $f(z) = \sum_{j \geq 2} f_j z^j$
want $\Phi(z) = z + \sum_{j \geq 2} \phi_j z^j$

\exists solution for $\lambda \neq 0$ no root of 1
Proof recursive / inductive:

For $j = 2$: get $\lambda(1 - \lambda)\phi_2 = f_2$

For $j = 3$: get $\lambda(1 - \lambda^2)\phi_3 = f_3 + 2\lambda f_2 \phi_2$

For $j = n$: get $\lambda(1 - \lambda^{n-1})\phi_n = f_n + \text{known}$

1. Complex Linearization II: Convergent Solutions

Hyperbolic case $0 < |\lambda| \neq 1$:

Solved by H. Poincaré (proof by iteration)

Elliptic case $\lambda \in \mathbb{T}^1$ (SMALL DIVISORS):

Solved by C.L. Siegel when

for some $\gamma > 0$ and $\tau > 2$

$$|\lambda - e^{2\pi i \frac{p}{q}}| \geq \gamma |q|^{-\tau} \quad (1)$$

Diophantine (nonresonance) conditions:
set of full measure in \mathbb{T}^1

1. Complex Linearization III: H. Cremer's topological counter example (1928), version by M.R. Herman

$$F(z) = \lambda z + z^2, \lambda \in \mathbb{T}^1 \text{ not a root of unity}$$
$$F^q(z) = \lambda^q z + \dots + z^{2^q}$$

Point z is periodic of period q iff $F^q(z) = z$

$$F^q(z) - z = z(\lambda^q - 1 + \dots + z^{2^q-1})$$

Call $N = 2^q - 1$

$$z_1 \cdot z_2 \cdot \dots \cdot z_N = \lambda^q - 1$$

where z_j nontrivial solutions

$\Rightarrow \exists$ solution within radius

$$|\lambda^q - 1|^{1/N}$$

of $z = 0$. Consider $\lambda \in \mathbb{T}^1$ with

$$\liminf_{q \rightarrow \infty} |\lambda^q - 1|^{1/N} = 0$$

form residual set (dense G_δ , 2nd Baire)
Measure and category, Oxtoby (1970)

For such λ linearization is impossible:

- Linearization $z \mapsto \lambda z$ 'irrational' rotation
- \exists periodic points in any nbhd. of $z = 0$

1. Circle Maps I

Problem Given

$$P_{\alpha, \varepsilon} : \mathbb{T}^1 \rightarrow \mathbb{T}^1 \quad x \mapsto x + 2\pi\alpha + \varepsilon a(x, \alpha, \varepsilon)$$

C^∞ -family of circle maps

$$P_\varepsilon : \mathbb{T}^1 \times [0, 1] \rightarrow \mathbb{T}^1 \times [0, 1],$$

$$(x, \alpha) \mapsto (x + 2\pi\alpha + \varepsilon a(x, \alpha, \varepsilon), \alpha)$$

Question What is fate of the rigid rotations of P_0 under perturbation ‘as a family’ ?

Naively look for conjugation Φ_ε

$$\begin{array}{ccc} \mathbb{T}^1 \times [0, 1] & \xrightarrow{P_0} & \mathbb{T}^1 \times [0, 1] \\ \downarrow \Phi_\varepsilon & & \downarrow \Phi_\varepsilon \\ \mathbb{T}^1 \times [0, 1] & \xrightarrow{P_\varepsilon} & \mathbb{T}^1 \times [0, 1] \end{array}$$

i.e., with

$$P_\varepsilon \circ \Phi_\varepsilon = \Phi_\varepsilon \circ P_0$$

1. Circle Maps II

Small divisors again

(Skew) shape of Φ_ε :

$$\Phi_\varepsilon(x, \alpha) = (x + \varepsilon U(x, \alpha, \varepsilon), \alpha + \varepsilon \sigma(\alpha, \varepsilon))$$

\leadsto *nonlinear* equation:

$$\begin{aligned} U(x + 2\pi\alpha, \alpha, \varepsilon) - U(x, \alpha, \varepsilon) &= \\ &= 2\pi\sigma(\alpha, \varepsilon) + a(x + \varepsilon U(x, \alpha, \varepsilon), \alpha + \varepsilon\sigma(\alpha, \varepsilon), \varepsilon) \end{aligned}$$

Expand in powers of ε

comparing lowest order \leadsto linear equation

$$U_0(x + 2\pi\alpha, \alpha) - U_0(x, \alpha) = 2\pi\sigma_0(\alpha) + a_0(x, \alpha)$$

Write in Fourier-series

$$a_0(x, \alpha) = \sum_{k \in \mathbf{Z}} a_{0k}(\alpha) e^{ikx}, U_0(x, \alpha) = \sum_{k \in \mathbf{Z}} U_{0k}(\alpha) e^{ikx}$$

Solutions $\sigma_0 = -\frac{1}{2\pi} a_{00}$ (parameter shift) and

$$U_{0k}(\alpha) = \frac{a_{0k}(\alpha)}{e^{2\pi ik\alpha} - 1}$$

\exists formal solution $\Leftrightarrow \alpha$ irrational

Still small divisors ...

1. Circle Maps III

Measure and Category

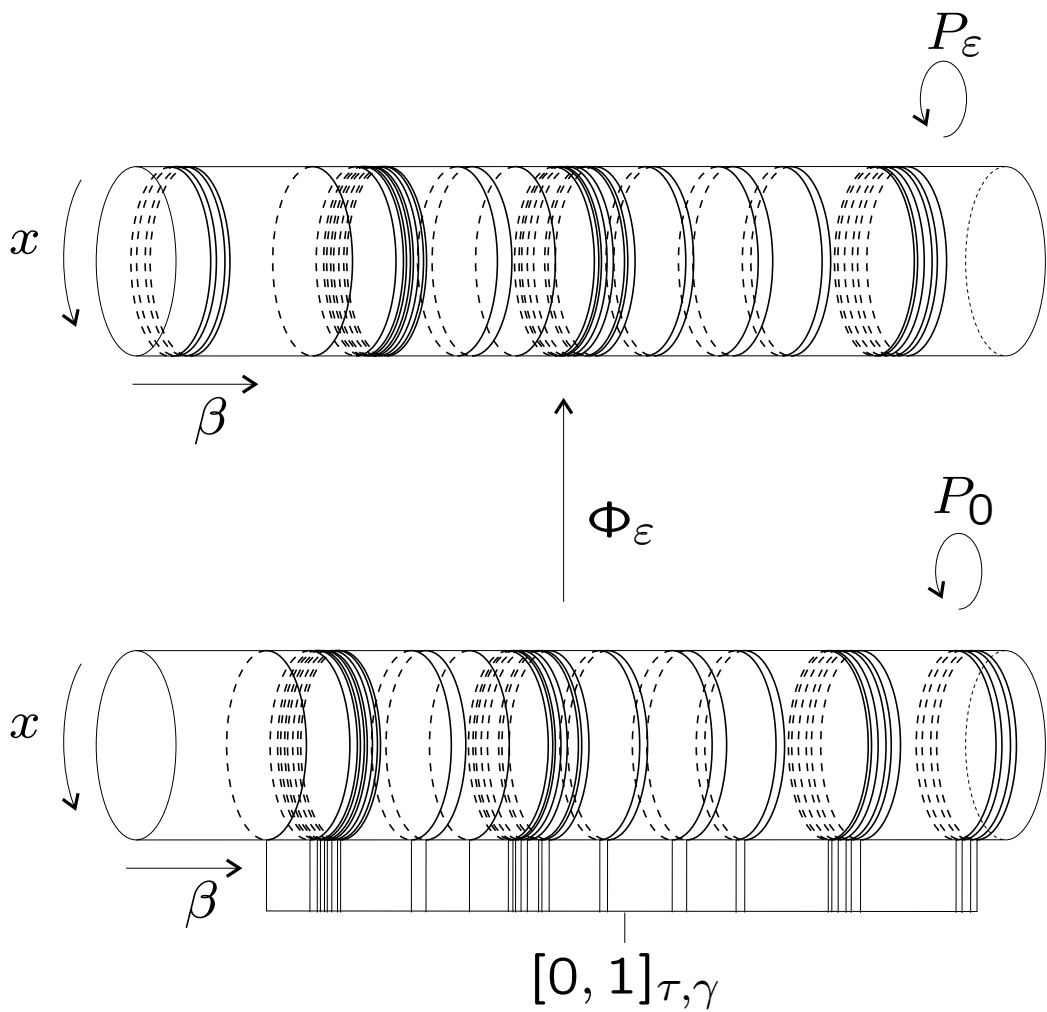
Diophantine nonresonance conditions
for $\tau > 2$ and $\gamma > 0$ consider $\alpha \in [0, 1]$ such
that for all rationals p/q

$$\left| \alpha - \frac{p}{q} \right| \geq \gamma q^{-\tau}, \quad (2)$$

Defines Cantor set $[0, 1]_{\tau, \gamma}$:
nowhere dense, yet large measure,
Oxtoby again ...

Circle map theorem γ small and ε a small
in C^∞ -topology $\Rightarrow \exists C^\infty$ transformation $\Phi_\varepsilon :$
 $\mathbb{T}^1 \times [0, 1] \rightarrow \mathbb{T}^1 \times [0, 1]$, conjugating the re-
striction $P_0|_{[0, 1]_{\tau, \gamma}}$ to a subsystem of P_ε

Formulation of Broer-Huitema-Takens (1990)
in terms of 'quasi-periodic stability'



Smooth map of the cylinder, conjugating
Diophantine invariant circles

1. Circle Maps IV

Discussion

Irrational rotation \leftrightarrow quasi-periodicity

Smoothness $\Phi_\varepsilon \Rightarrow$

Quasi-periodicity typically occurs with *positive* measure in parameter space

Meaning of Φ_ε in gaps ?

Perfectness Cantor set \Rightarrow

non-isolated occurrence of quasi-periodicity

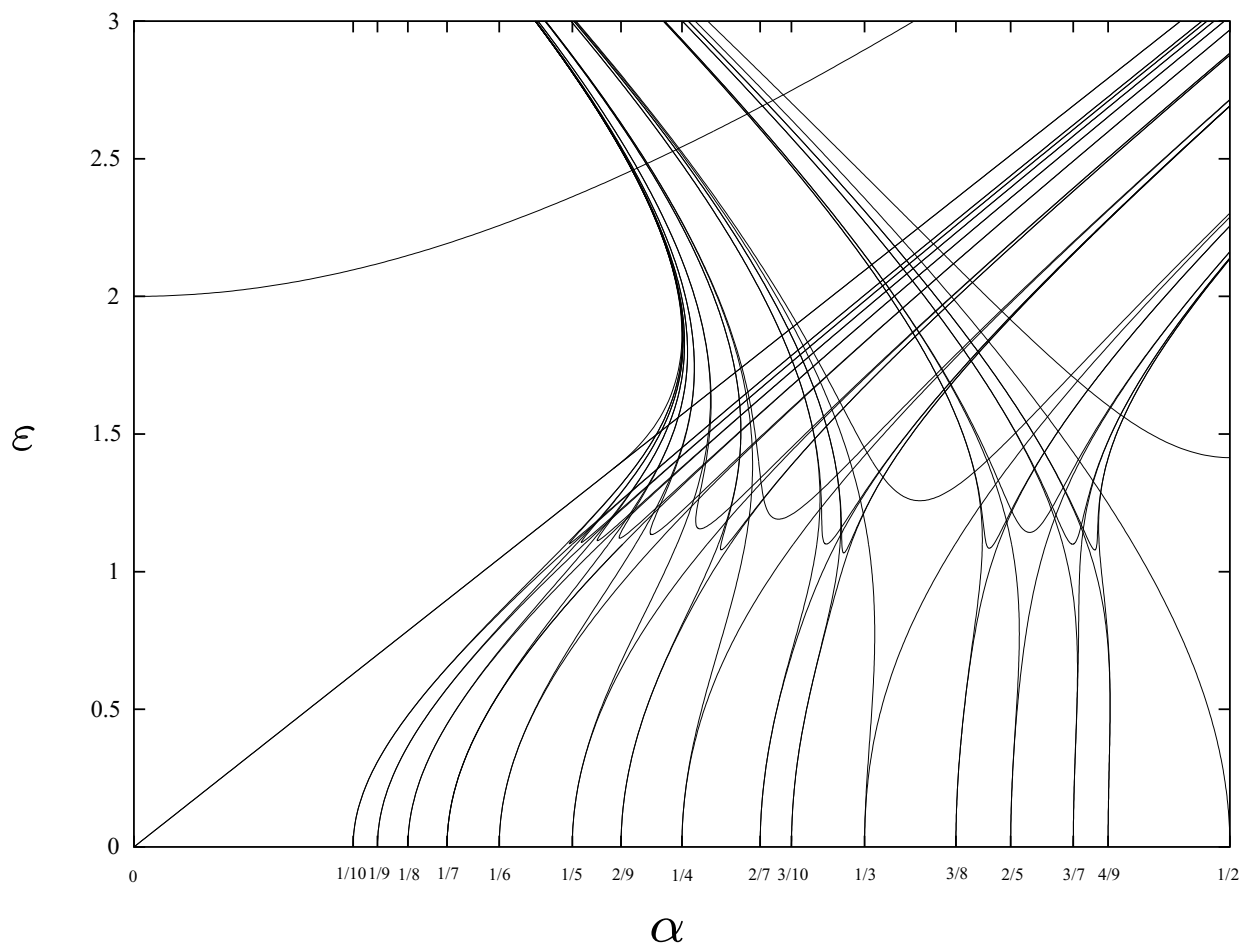
Example: Arnold family of circle maps

$$P_\varepsilon(x) = x + 2\pi\alpha + \varepsilon \sin x$$

periodic tongues (generic)

in between Diophantine quasi-periodic 'hairs'
(positive measure)

Herman-Yoccoz non-perturbative version



Arnold resonance tongues; for $\varepsilon \geq 1$ the maps are endomorphic

1. Circle Maps V

Discussion

Examples Coupled Van der Pol type oscillators

$$\ddot{y}_1 + c_1\dot{y}_1 + a_1y_1 + f_1(y_1, \dot{y}_1) = \varepsilon g_1(y_1, y_2, \dot{y}_1, \dot{y}_2)$$

$$\ddot{y}_2 + c_2\dot{y}_2 + a_2y_2 + f_2(y_2, \dot{y}_2) = \varepsilon g_2(y_1, y_2, \dot{y}_1, \dot{y}_2)$$

\leadsto 4-D vector field in $\mathbb{R}^2 \times \mathbb{R}^2 = \{(y_1, z_1), (y_2, z_2)\}$

with $\dot{y}_j = z_j, j = 1, 2$

\leadsto 2-torus attractor in 4-D

Poincaré map P in 2-torus attractor \Rightarrow

similar results for vector fields / ODE's:

phase-lock inside resonance tongues

quasi-periodicity on Diophantine 'hairs'

Huygens's synchronised clocks \leftrightarrow 1 : 1 tongue

Families of quasi-periodic attractors

2. Area preserving twist maps I J.K. Moser (1962)

Annulus $\Delta \subseteq \mathbb{R}^2$ coordinates $(\varphi, I) \in \mathbb{T}^1 \times \mathbf{K}$,
 \mathbf{K} interval; area form $\sigma = d\varphi \wedge dI$

σ -preserving smooth map $P_\varepsilon : \Delta \rightarrow \Delta$

$$P_\varepsilon(\varphi, I) = (\varphi + 2\pi\alpha(I), I) + O(\varepsilon)$$

Assume:

'frequency' map $I \mapsto \alpha(I)$ diffeomorphism
(twist condition \leftrightarrow Kolmogorov-nondegeneracy)

Question What is fate of the P_0 -dynamics
for $|\varepsilon| \ll 1$?

Diophantine conditions as before:

for $\tau > 2, \gamma > 0$ and for all rationals p/q

$$\left| \alpha - \frac{p}{q} \right| \geq \gamma q^{-\tau},$$

$\leadsto \Delta_{\tau, \gamma} \subseteq \Delta$

Twist theorem γ small and perturbation small
in C^∞ -topology $\Rightarrow \exists$ smooth transformation
 $\Phi_\varepsilon : \Delta \rightarrow \Delta$, conjugating the restriction $P_0|_{\Delta_{\tau, \gamma}}$
to a subsystem of P_ε

2. Area preserving twist maps II: Discussion and applications

Compare Theorems 1 and 2:

parameter $\alpha \longleftrightarrow$ coordinate I

Typically quasi-periodicity occurs
with positive measure in phase space

In gaps periodicity and ‘chaos’
(generically \exists horseshoes \Rightarrow
positive topological entropy)

Examples: coupled pendula as before,
but now *conservative*

Map P (iso-energetic) Poincaré map on
families of 2-tori

2. Conservative KAM theory

Kolmogorov (Amsterdam 1954) -Arnold (1963)
-Moser (1962)

KAM theorem *In Hamiltonian systems with n degrees of freedom typically n quasi-periodic Lagrangean tori occur with positive measure in phase space*

Format:

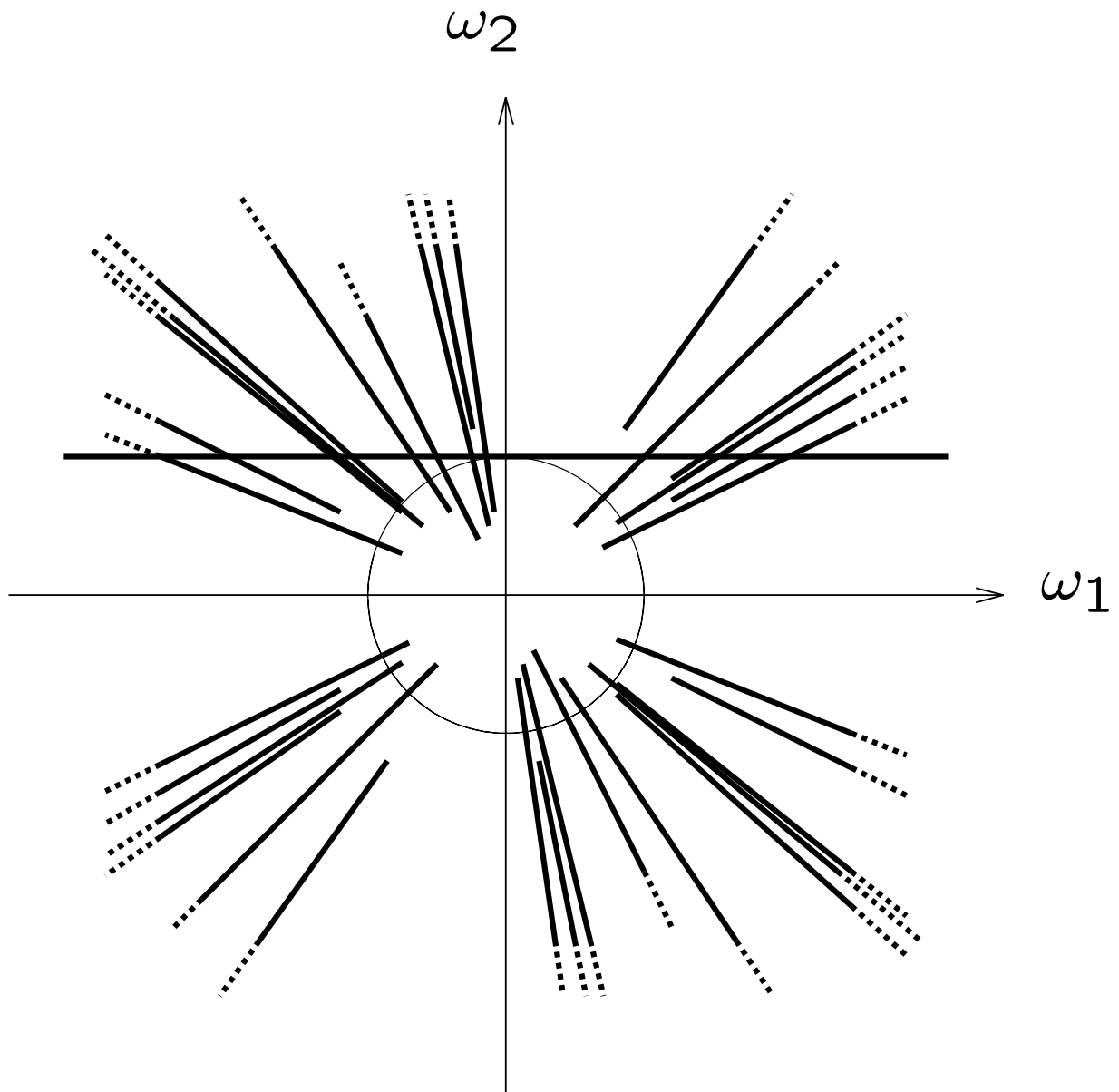
$$\begin{aligned}\dot{\varphi} &= \omega(I) + \varepsilon f(I, \varphi) \\ \dot{I} &= \varepsilon g(I, \varphi),\end{aligned}$$

Kolmogorov nondegeneracy: frequency map $I \mapsto \omega(I)$ local diffeomorphism

Diophantine set of frequencies: $\mathbb{R}_{\tau, \gamma}^n = \{\omega \in \mathbb{R}^n \mid |\langle \omega, k \rangle| \geq \gamma |k|^{-\tau}, k \in \mathbb{Z}^n \setminus \{0\}\}$,
 $\tau > n - 1, \gamma > 0$

(More precise formulation below)

Stability problems Example: Solar system?



Diophantine set $\mathbb{R}_{T,\gamma}^n$ has closed half line geometry

$\mathbb{S}^{n-1} \cap \mathbb{R}_{T,\gamma}^n$ Cantor set

measure $\mathbb{S}^{n-1} \setminus \mathbb{R}_{T,\gamma}^n = O(\gamma)$ as $\gamma \downarrow 0$

3. Quasi-periodic bifurcations I

Bifurcations of vector fields and diffeomorphisms:

Equilibria: saddle-node and Hopf

Periodic solutions / fixed points: saddle-node, period-doubling, Hopf-Neimark-Sacker
Compare with Arnold family:
describe with **frequency parameter**

Tori: \mathbb{T}^n -symmetric (= 'integrable') systems:
take product with above cases
Perturbation by KAM \rightsquigarrow (C^∞ -) typicality
restricted to Diophantine tori

Example: quasi-periodic bifurcation in diffeomorphism from circle to 2-torus
(mind: Hopf-Landau-Lifschitz-Ruelle-Takens)

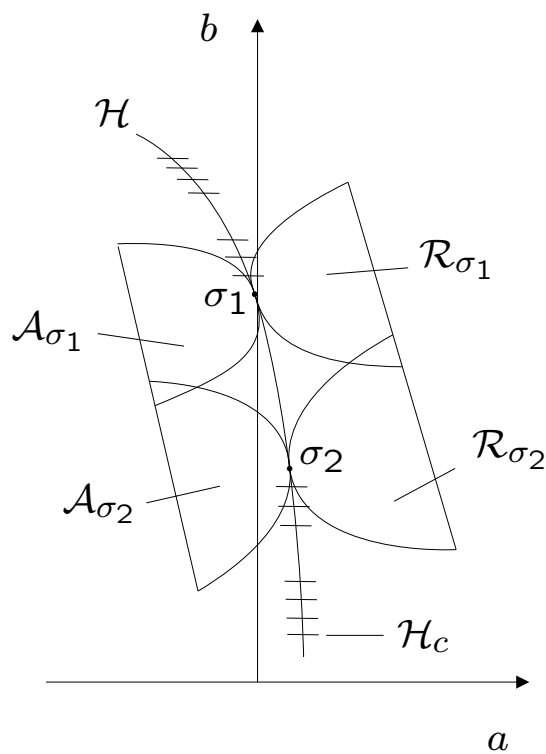
'Cantorise' above bifurcation geometry
(Whitney, Thom, Arnold, *et al.*)



In the gaps ‘interesting’ dynamics may occur, like chaotic attractors. In generic family this coexists with positive measure of quasi-periodicity.

3. Quasi-periodic bifurcations II

Fattening torus domains by hyperbolicity
~> Chenciner-bubbles around resonances



4. Theoretical background I

Definition of Q-P by smooth conjugation

Fix $n \in \mathbb{N}, n \geq 2$

Standard n -torus $\mathbb{T}^n = \mathbb{R}^n / (2\pi\mathbb{Z})^n$

coordinates $\varphi_1, \varphi_2, \dots, \varphi_n \bmod 2\pi$

For $\omega \in \mathbb{R}^n$ consider $\mathbb{X}_\omega = \sum_{j=1}^n \omega_j \frac{\partial}{\partial \varphi_j}$

$\omega_1, \omega_2, \dots, \omega_n$ called frequencies

Given smooth vector field X on manifold M ,
with $T \subseteq M$ an X invariant n -torus

Restriction $X|_T$ called *parallel* iff

$\exists \omega \in \mathbb{R}^n$ and $\Phi : T \rightarrow \mathbb{T}^n$, smooth diffeo, such
that $\Phi_*(X|_T) = \mathbb{X}_\omega$

$X|_T$ is *quasi-periodic* if $\omega_1, \omega_2, \dots, \omega_n$ are in-
dependent over \mathbb{Q}

Remark. Composing Φ by translation on \mathbb{T}^n
does not change ω

However, composition with linear invertible
 $S \in GL(n, \mathbb{Z})$ gives $S_*\mathbb{X}_\omega = \mathbb{X}_{S\omega}$

4. Theoretical background II

Integer affine structure

In Q-P case:

Self conjugations of $\mathbb{X}_\omega \leftrightarrow$ translations of \mathbb{T}^n
 \leftrightarrow affine structure on \mathbb{T}^n

Thus given $\Phi : T \rightarrow \mathbb{T}^n$ with $\Phi_*(X|_T) = \mathbb{X}_\omega$,
self conjugations of $X|_T$ determine natural
affine structure on T

Conjugation Φ unique modulo translations in
 T and \mathbb{T}^n

Remarks

Structure on T *integer* affine:

transition maps: translations and $GL(n, \mathbb{Z})$

Fits with integrable Hamiltonian case where
affine structure on all parallel or Liouville tori

5. Global KAM theory I

Setting: Example: Spherical pendulum
consider Hamiltonian perturbations

Question: What is fate of nontrivial 2-torus bundle ?

Background: Liouville-Arnold and KAM
Gives KAM 2-tori in charts

Tools: Unicity of KAM tori (restricted Diophantine)
Partition of Unity: convex combinations of KAM conjugations in charts

Result (Rink, Broer-Cushman-Fassò-Takens):
Extension of KAM theorem to bundles of Lagrangean tori (all near-identity Whitney interpolations isomorphic)

BCFT-approach preserves closed half line geometry

5. Global KAM theory II

Example: spherical pendulum

Configuration space $\mathbb{S}^2 = \{q \in \mathbb{R}^3 \mid \langle q, q \rangle = 1\}$

Phase space

$$T^*\mathbb{S}^2 \cong \{(q, p) \in \mathbb{R}^6 \mid \langle q, q \rangle = 1 \ \& \ \langle q, p \rangle = 0\}$$

Energy–momentum map $\mathcal{EM} : T^*\mathbb{S}^2 \rightarrow \mathbb{R}^2$

$$(q, p) \mapsto (I, E) = \left(q_1 p_2 - q_2 p_1, \frac{1}{2} \langle p, p \rangle + q_3 \right)$$

Lagrangian fibration, fibers \mathbb{T}^2

Equilibria (de and ude)

$$(q, p) = ((0, 0, \pm 1), (0, 0, 0)) \mapsto (I, E) = (0, \pm 1)$$

singularities

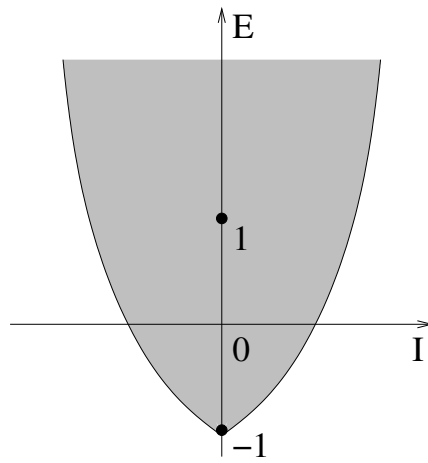
Duistermaat (1980): Circle around $(I, E) = (0, 1)$ collects nontrivial *monodromy*

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{R})$$

(in suitable bases of period lattice)

\Rightarrow *nontrivial* \mathbb{T}^2 -bundle (Liouville-Arnold)

& nonexistence of *global* action angles



Range of the energy-momentum map of the spherical pendulum

5. Global KAM theory III

Liouville-Arnold theorem

(M, σ) symplectic manifold, $\dim M = 2n$,
 B n -dimensional manifold

GIVEN

integral $\mathcal{F} : M \rightarrow B$,

$\mathcal{F} = (f_1, f_2, \dots, f_n)$ (locally)

with $\{f_i, f_j\} = 0$, $d\mathcal{F}$ maximal rank

and compact fibres $\mathcal{F}^{-1}(b)$

THEN

$\forall b \in B \exists$ open $b \in V \subseteq B$ such that

$\mathcal{F}^{-1}(V) \subseteq M$ open and invariant under X_{f_i}

and

\exists symplectic diffeo

$(I, \varphi) : \mathcal{F}^{-1}(V) \rightarrow U \times (\mathbb{R}/\mathbb{Z})^n$ with $U \subseteq \mathbb{R}^n$
open, i.e., such that:

$$(I, \varphi)^* \left(\sum_{j=1}^n d\varphi_j \wedge dI_j \right) = \sigma|_{\mathcal{F}^{-1}(V)}$$

Global KAM virtual

Singularity ude *saddle* (double eigenvalues ± 1)

Singular fiber: *pinched* 2-torus ($= W^{u/s}(\text{ude})$)

Theorem (Matveev 1996, Nguyen 1997):

Given 4D symplectic manifold M fibered
by level sets of map \mathcal{EM}

ASSUME

- \mathcal{EM} has only one critical value
- fibers of \mathcal{EM} compact & connected
- singular fiber has k singular points,
all real or complex saddle points

THEN

singular fiber is pinched torus

& Liouville-Arnold \mathbb{T}^2 -bundle is non-trivial
with monodromy similar to

$$\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \in Sl(2, \mathbb{R})$$

Remark Quantum monodromy *exists*

in semi-classical quantizations of classical in-
tegrable system with nontrivial \mathbb{T}^n -bundles
(San Vu Ngọc 1999)

Global KAM theory V

Back to Conservative KAM theorem:

Phase space $\mathbb{T}^n \times A$

$A \subset \mathbb{R}^n$ is bounded, open & connected
action angle coordinates (I, φ)

symplectic form $\mu = \sum_{j=1}^n dI_j \wedge d\varphi_j$

$h : \mathbb{T}^n \times A \rightarrow \mathbb{R}$ C^∞ Hamilton function,
integrable (not depending on φ) \rightarrow
Hamiltonian vector field X_h

$$X_h(\varphi, I) = \sum_{j=1}^n \omega_j(I) \frac{\partial}{\partial \varphi_j}, \text{ with } \omega(I) := \frac{\partial h}{\partial I}$$

Kolmogorov nondegeneracy frequency map

$\omega : A \rightarrow \mathbb{R}^n$ is diffeomorphism

Diophantine frequencies

Given $\tau > n - 1$ and $\gamma > 0$, recall definition

$$\mathbb{R}_{\tau, \gamma}^n := \{\omega \in \mathbb{R}^n \mid |\langle \omega, k \rangle| \geq \gamma |k|^{-\tau}, \forall k \in \mathbb{Z}^n \setminus \{0\}\}$$

Global KAM theory VI

For $\Gamma := \omega(A)$ let

$\Gamma' := \{\omega \in \Gamma \mid \text{dist}(\omega, \partial\Gamma) > \gamma\}$ and $\Gamma'_\gamma := \Gamma' \cap \mathbb{R}_\gamma^n$

also define $A'_\gamma \subset A$ by $A'_\gamma := \omega^{-1}(\Gamma'_\gamma)$

NB measure $(A \setminus A'_\gamma) = O(\gamma)$ as $\gamma \downarrow 0$

Perturb to $h + f : \mathbb{T}^n \times A \rightarrow \mathbb{R}$, all of class C^∞
with C^∞ -topology

Local KAM Theorem (Pöschel (82),
Broer-Huitema-Takens (90))

*If $\gamma > 0$ is sufficiently small and $h + f$ is
sufficiently near h in C^∞ -topology*

*then \exists map $\Phi : \mathbb{T}^n \times A \rightarrow \mathbb{T}^n \times A$
with following properties:*

- Φ is C^∞ diffeomorphism
- for $\widehat{\Phi} := \Phi|_{\mathbb{T}^n \times D_\gamma(A'_\gamma)}$ we have

$$\widehat{\Phi}_* X_h = X_{h+f}$$

Global KAM theory VII

Discussion

Map Φ

- is generally *not* symplectic
- is near-identity in the C^∞ -topology

In global KAM theorem

- convex combinations on *unique* tori
keeps closed half line geometry
- near-identity torus automorphisms are all translations
- Whitney extensions near Id
 \Rightarrow all extended \mathbb{T}^n -bundles isomorphic
- Quantum monodromy in semi-classical versions of integrable systems with monodromy
- How about the nearly-integrable case ?