On Parametrized KAM Theory

Henk Broer, University of Groningen

Johann Bernoulli Institute
for Mathematics and Computer Science
University of Groningen
PO Box 407
NL-9700 AK GRONINGEN

email: `h.w.broer@rug.nl`
URL: `http:\\www.math.rug.nl\~broer`

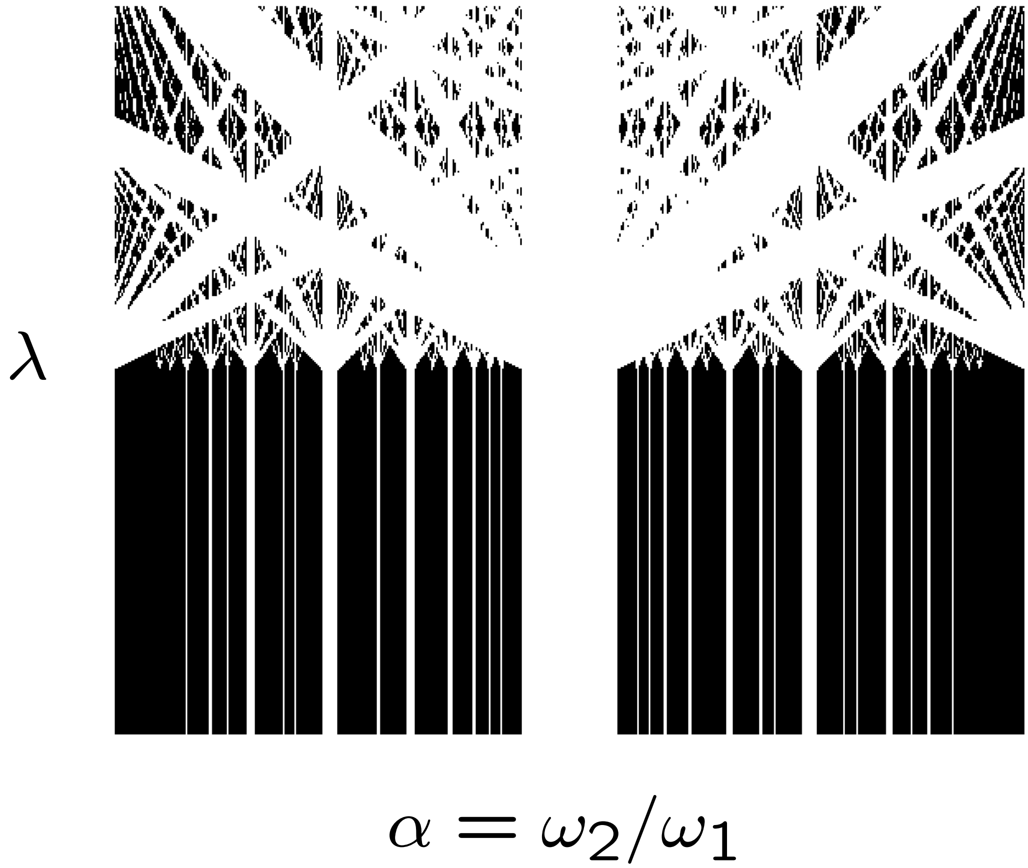
AIMS Dresden, May 2010

Motivation

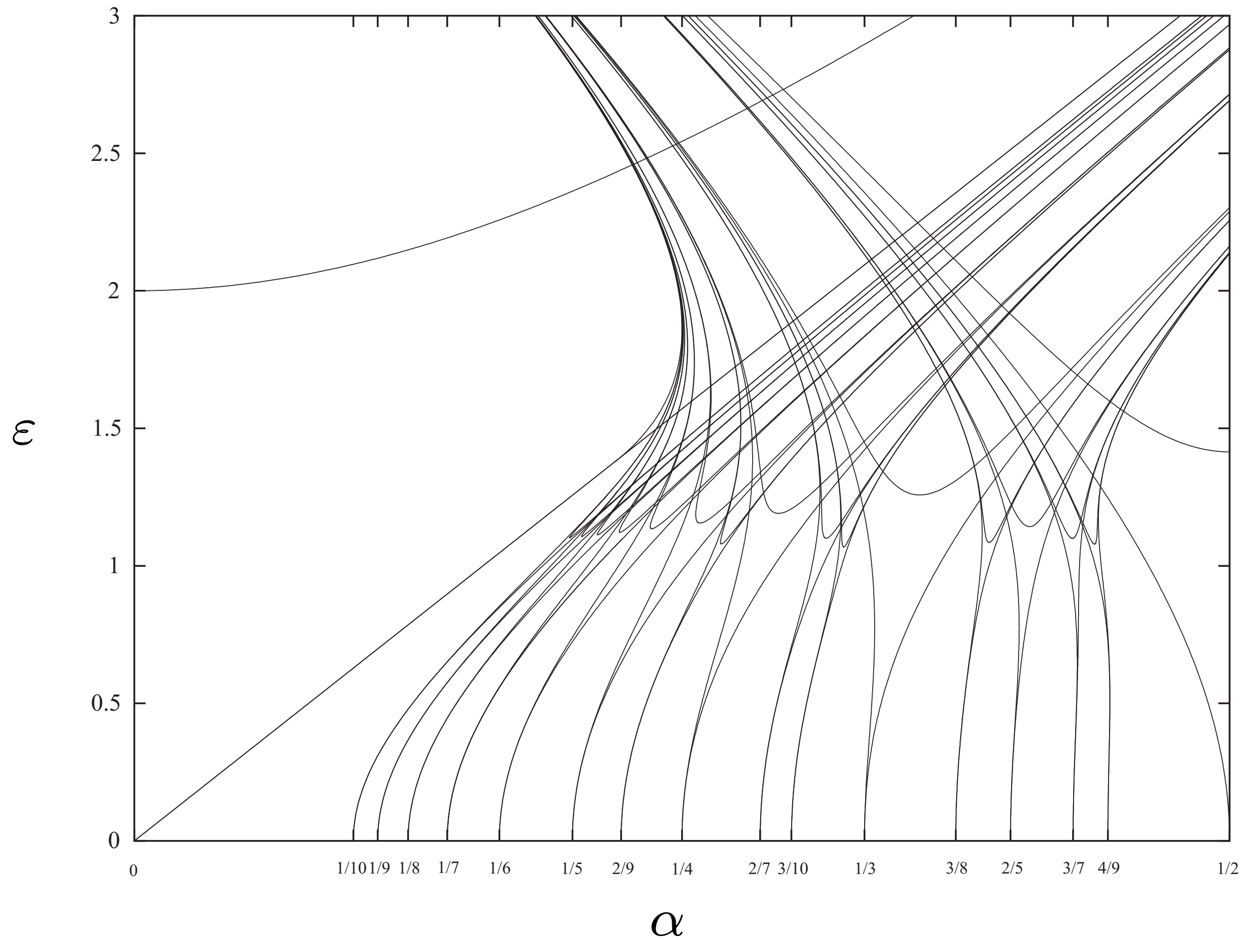
Parametrized KAM Theory and Quasi-Periodic Bifurcation Theory developed in:

Moser (1966, 1967), Chenciner-Iooss (1979), Chenciner (1985), Braaksma-B (1987), Huitema (1988), B-Huitema-Takens-Braaksma (1990). B-Huitema (1995), B-Huitema-Sevryuk (1996), Hanßmann (1998, 2006, 2006), B-Hanssmann-Jorba-Villanueva-Wagener (2003), Hoo (2005), Wagener (2002 and to appear), B-Hoo-Naudot (2007), B-Hanßmann-You (2005, 2006 and in preparation), B-Hanßmann-Hoo (2007), B-Ciocci-Hanßmann (2007), B-Ciocci-Hanßmann-Vanderbauwhede (2009), B-Takens (2009), B-Sevryuk (to appear), B-Hanßmann-Wagener (in preparation)

Marriage of KAM– and Singularity Theory $\rightsquigarrow$ Cantorisation of semi-algebraic stratifications (or Cantor stratifications)



Cantorized fold in quasi-periodic bifo's. E.g. in q-p (Hamiltonian) center-saddle bifo and in 'Ruelle-Takens' bifo of (dissipative) maps from invariant circles to invariant 2-tori. Gaps (white) contain 'interesting' dynamics, like periodicity and chaos, generically coexisting with positive measure of quasi-periodicity.



Resonance tongues in the Arnol'd family of circle maps

$$A_{\alpha,\varepsilon}(x) = x + 2\pi\alpha + \varepsilon \sin x$$

for $\varepsilon < 1$ circle diffeomorphisms associated to the lower half of the previous figure

Parametrized KAM Theory (dissipative case)

Phase space $M = \mathbb{T}^n \times \mathbb{R}^m = \{(x, z)\}$
parameter space $P \subseteq \mathbb{R}^s = \{\mu\}$ (open)

Starting point:
integrable C^∞-family $X = X_\mu(x, z)$

$$\begin{aligned} \dot{x} &= \omega(\mu) + \mathrm{O}(|z|) \\ \dot{z} &= \Omega(\mu) z + \mathrm{O}(|z|^2), \end{aligned}$$

with O-estimates (locally) uniform in μ,
$\omega(\mu) \in \mathbb{R}^n$ and $\Omega(\mu) \in gl(m, \mathbb{R})$

Integrability of $X \Leftrightarrow x$-independence

Interest: persistence properties of the family $T_\mu = \mathbb{T}^n \times \{0\}$ of invariant n-tori with parallel dynamics

BHT nondegeneracy (1990)

Nondegeneracy condition on product map

$$\omega \times \Omega : P \to \mathbb{R}^n \times gl(m, \mathbb{R})$$

at $\mu = \mu_0 \in P$: 'simultaneously'

1. $\mu \in \mathbb{R}^s \mapsto \omega(\mu) \in \mathbb{R}^n$ submersion
   (think of Kolmogorov-nondegeneracy)

2. $\mu \in \mathbb{R}^s \mapsto \Omega(\mu) \in gl(m, \mathbb{R})$ versal
   unfolding of $\Omega(\mu_0)$

   i.e., transversal to orbit of $\Omega(\mu_0)$
   under the adjoint action of $GL(m, \mathbb{R})$

Assumption throughout: $\Omega(\mu_0)$ invertible
(relaxed by Wagener (to appear)
and B-Hanßmann-Wagener (in preparation))

Case: $\Omega(\mu_0)$ simple eigenvalues

Eigenvalues $\Omega(\mu)$ (where $N_1 + 2N_2 = m$):

$$(\delta_1, \ldots, \delta_{N_1}, \, \alpha_1 \pm \mathrm{i}\beta_1, \ldots, \alpha_{N_2} \pm \mathrm{i}\beta_{N_2})$$

with $\beta_j > 0$ for $1 \leq j \leq N_2$
Call $\beta = (\beta_1, \beta_2, \ldots, \beta_{N_2})$ *normal frequencies*

By simpleness of eigenvalues, map

spec : $gl(m, \mathbb{R}) \to \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_2}; \quad \Omega \mapsto (\delta, \alpha, \beta)$,

parametrizes $GL(m, \mathbb{R})$ orbit space near $\Omega(\mu_0)$

BHT nondegeneracy condition means that

$P \ni \mu \mapsto (\omega \times (\text{spec} \circ \Omega))(\mu) \in \mathbb{R}^n \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_2}$

is submersion (at $\mu = \mu_0$) !!

Inverse Function Theorem
$\rightsquigarrow$ ‘reparametrization’

$$\mu \leftrightarrow (\omega, \delta, \alpha, \beta)$$

on $A \subseteq P$ (open)

Diophantine conditions

For $\tau > n-1$ and $\gamma > 0$ define (τ, γ)-Diophantine normal-internal frequency vectors by

$$\mathrm{D}_{\tau,\gamma}(\mathbb{R}^{\mathrm{n}}; \mathbb{R}^{\mathrm{N}_2}) = \\ \{(\omega, \beta) \in \mathbb{R}^n \times \mathbb{R}^{N_2} \mid |\langle \omega, k\rangle + \langle \beta, \ell\rangle| \geq \gamma |k|^{-\tau}, \\ \forall k \in \mathbb{Z}^n \setminus \{0\} \text{ and } \forall \ell \in \mathbb{Z}^{N_2} \text{ with } |\ell| \leq 2\}$$

Nowhere dense, positive measure, closed half line property

Defining $\Gamma = (\omega \times (\mathrm{spec} \circ \Omega))(A)$, w.l.o.g. assume that Γ has 'product form' $\Gamma = \Gamma_\omega \times \Gamma_\delta \times \Gamma_\alpha \times \Gamma_\beta$

Shrunken version

$$\Gamma_\gamma = \{(\omega, \delta, \alpha, \beta) \in \Gamma \mid \mathrm{dist}((\omega, \delta, \alpha, \beta), \partial\Gamma) \geq \gamma\}$$

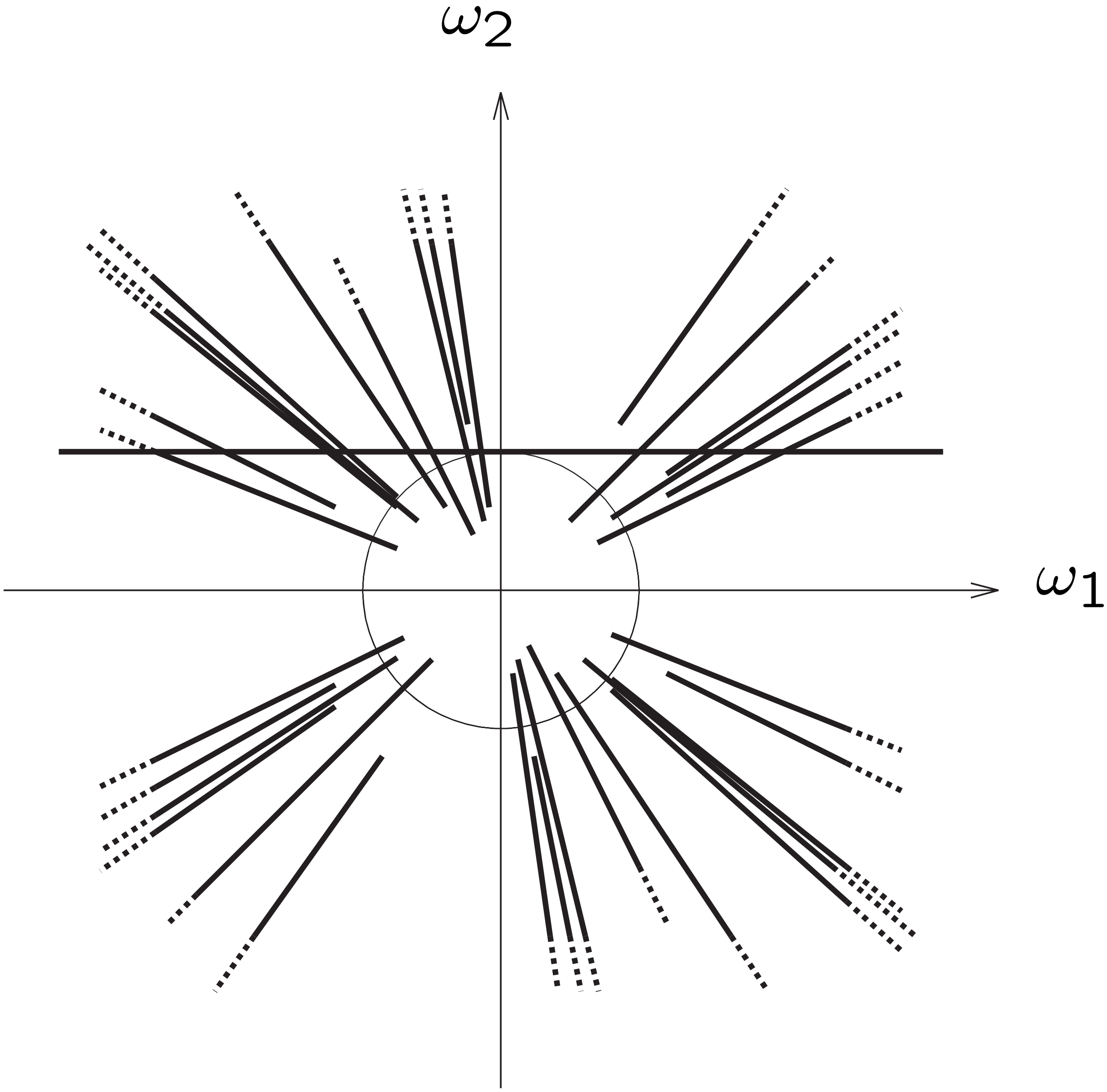
of Γ as well as

$$\mathrm{D}_{\tau,\gamma}(\Gamma_\gamma) = \Gamma_\gamma \bigcap (\mathrm{D}_{\tau,\gamma}(\mathbb{R}^{\mathrm{n}}; \mathbb{R}^{\mathrm{N}_2}) \times \Gamma_\delta \times \Gamma_\alpha)$$

and $\mathrm{D}_{\tau,\gamma}(\mathrm{A}_\gamma) \subset \mathrm{A}$

Closed half lines turn into closed linear half spaces of dimension $1 + N_1 + N_2$

KAM theory: this geometry, up to diffeo's, is inherited by perturbations !!



Diophantine set $\mathsf{D}_{\tau,\gamma}(\mathbb{R}^n)$ has closed half line geometry For $\tau > n-1$ intersection $\mathsf{D}_{\tau,\gamma}(\mathbb{R}^n) \cap \mathbb{S}^{n-1}$ is Cantor set; measure complement $\mathbb{S}^{n-1} \setminus \mathsf{D}_{\tau,\gamma}(\mathbb{R}^n) = O(\gamma)$ as $\gamma \downarrow 0$

Parametrized KAM ('dissipative')

Thm. Let integrable C^∞-family $X = X_\mu(x, z)$ be BHT nondegenerate on $\mathbb{T}^n \times \{0\} \times A$
Assume $\det \Omega(\mu) \neq 0$
Then, for $0 < \gamma \ll 1$, $\exists C^\infty$-nbhd of X, such that, $\forall \tilde{X} \in \mathcal{O}$, $\exists$ (local) C^∞-map $\Phi : \mathbb{T}^n \times \mathbb{R}^m \times A \to \mathbb{T}^n \times \mathbb{R}^m \times A$, with:

1. Φ is a C^∞-near identity diffeo preserving projections to the parameter space P

2. Image $\Phi(\mathbb{T}^n \times \{0\} \times \mathsf{D}_{\tau,\gamma}(\mathsf{A}_\gamma))$ is $\tilde{X}$-invariant, and restriction $\widehat{\Phi} = \Phi|_{\mathbb{T}^n \times \{0\} \times \mathsf{D}_{\tau,\gamma}(\mathsf{A}_\gamma)}$ conjugates X to $\tilde{X}$, that is
$$\widehat{\Phi}_* X = \tilde{X}$$

3. Φ preserves normal linear behavior of the tori T_μ for $\mu \in \mathsf{D}_{\tau,\gamma}(\mathsf{A}_\gamma)$

Discussion I

1. Family X is quasi-periodically stable on torus union $\mathbb{T}^n \times \{0\} \times \mathsf{D}_{\tau,\gamma}(\mathsf{A}_\gamma)$
   (Item 3 $\sim$*normal linear stability*)

2. Preservation of structures allowed:
   Hamiltonian, volume preserving,
   equivariant and reversible,
   with or without external parameters,
   combinations of the above
   Axiomatic: *admissible* structures

3. Simpleness of eigenvalues of $\Omega(\mu)$ can be dropped in full 'admissible' generality

   Arnold's theory of matrices depending on parameters carries over: Normalization to *Linear Centralizer Unfoldings*
   Example in normal $1 : -1$ -resonance

Normally trivial cases

In these cases $m = 0$ ($\Rightarrow \Omega(\mu) \equiv 0$):

1. Normally Hyperbolic Dissipative KAM (families of quasi-periodic attractors)

2. 'Classical' Lagrangean KAM (Hamiltonian, a similar reversible case exists too)
   $X(x, y) = \omega(y)\frac{\partial}{\partial x}$,
   Compare with proof Pöschel (1982)

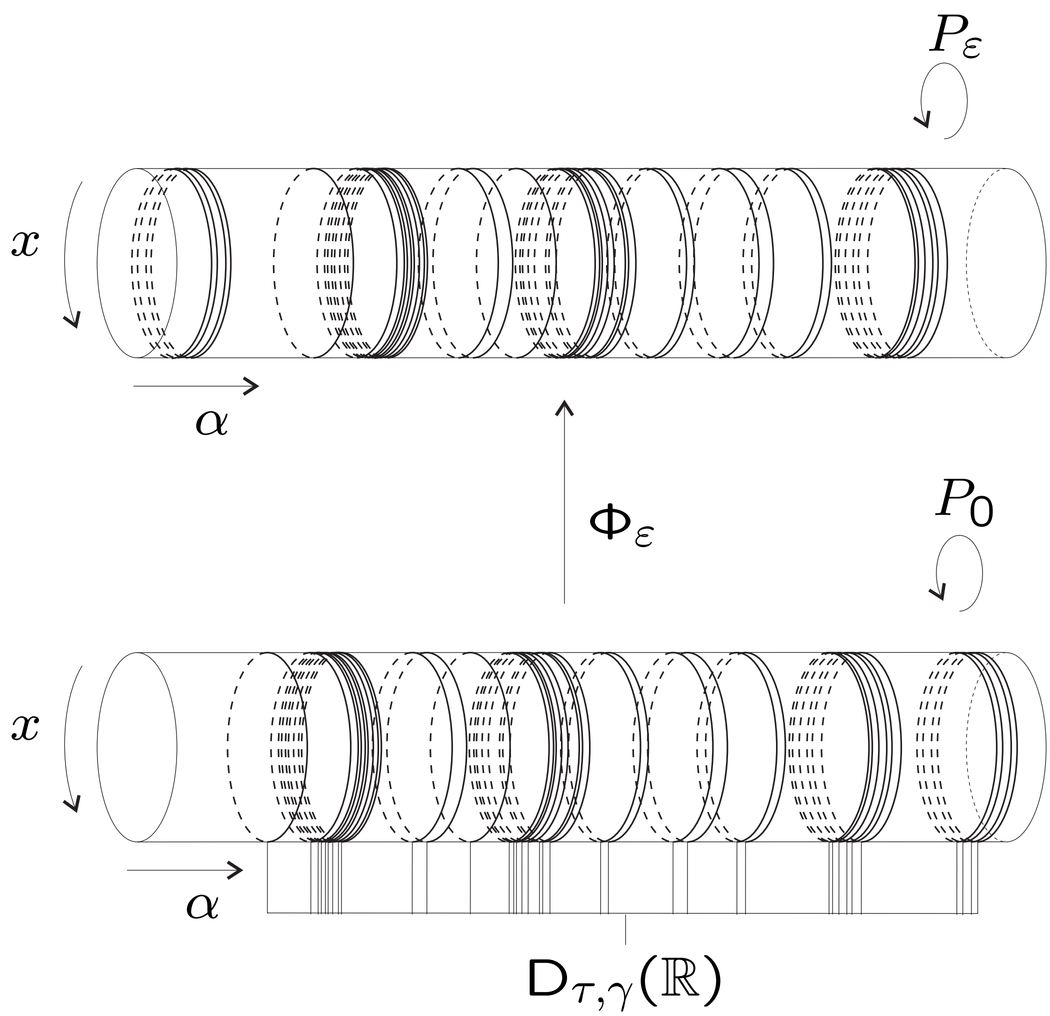
3. KAM for codimension 1 tori in volume preserving setting

4. &c.

Latter two *direct* consequence by 'localisation': invariant torus given by $y = \nu$, then define $y_{loc} = y - \nu$ and pass to

$$X_{loc}(x, y_{loc}, \nu) := X(x, y_{loc} + \nu)$$

NB: ν is *distinguished* parameter



Family of quasi-periodic attractors and quasi-periodic stability: simplest case for circle maps

$$P_{\varepsilon;\alpha}(x) = x + 2\pi\alpha + \varepsilon f(x, \varepsilon; \alpha)$$

Isotropic Hamiltonian tori

Slightly adapted format family X valid for Hamiltonian subtori:

- $x \in \mathbb{T}^n, y \in \mathbb{R}^n, z \in \mathbb{R}^m$ with m even and symplectic form $dx \wedge dy + dz^2$
  $X(x, y, z) = \omega(y)\frac{\partial}{\partial x} + \Omega(y)z\frac{\partial}{\partial z}$

- 'Actions' y distinguished parameters
  Compare with proof Pöschel (1989)

- Analogous for reversible setting

Idea also used by Herman (lecture 1990's) compare with [BHS (1996), BS (to appear)]

On the Axioms

Given: Pair (G, V) with
$G \neq \{\mathrm{I}_m\}$ Lie subgroup of $GL(m, \mathbb{R})$
$V \neq \{0\}$ linear subspace of $gl(m, \mathbb{R})$ such that

$$Q \in G, \& \Omega \in V \Rightarrow Q\Omega Q^{-1} \in V$$

(invariance under G-action)
G-orbit: $O(\Omega, G) = \{Q\Omega Q^{-1} : Q \in G\} \subseteq V$

$O(\Omega, G)$ is smooth manifold, e.g., if
if G is semi-algebraic subset (**Axiom A1**)

Also assume: $\Omega \in V \Rightarrow \Omega^T \in V$ (**Axiom A2**)

Pair (G, V) called *linear structure*
NB: **Axiom A3** later

Applies to $G = GL(m, \mathbb{R})$, $SL(m, \mathbb{R})$, $SP(m, \mathbb{R})$
or $GL_{+R}(m, \mathbb{R})$ with V their Lie algebra,

and to reversible case
$(G, V) = (GL_{+R}(m, \mathbb{R}), gl_{-R}(m, \mathbb{R}))$,
with $R \in Gl(m, \mathbb{R})$ an involution, where

$gl_{-R}(m, \mathbb{R}) := \{A \in gl(m, \mathbb{R}) \mid AR = -RA\}$

Unfoldings within (G, V)

Let $\Pi_p(V)$ be the set of all smooth p-parameter families on V (i.e., germs at $\mu = 0$)

Unfolding $\Omega \in \Pi_p(V)$ of Ω_0 *versal* if $\forall B \in \Pi_q(V)$ with $B(0) = \Omega_0$, $\exists$ local map $(\rho, Q) : (\mathbb{R}^q, 0) \to (\mathbb{R}^p, 0) \times (G, \mathrm{I}_m)$ such that

$$B(\mu) = Q(\mu)\Omega(\rho(\mu))Q^{-1}(\mu)$$

Thm. [BHN] Family Ω is *versal* at $\mu = 0 \iff D_0\Omega(\mathbb{R}^p) + T_{\Omega_0}(O(\Omega_0, G)) = V$ (transversal)

Minimal p called *codimension*
$\leadsto$ *universal* unfolding

NB: Invariance of eigenvalues:
$\leadsto$ always $\mathrm{cod}\,\Omega_0 > 0$

Lemma. $T_{\Omega_0}(O(\Omega_0, G)) = \mathrm{ad}\Omega_0(\mathbf{g})$, with $\mathbf{g}$ *Lie algebra* of G and $\mathrm{ad}\Omega_0(Y) = [\Omega_0, Y]$

Versal Unfolding Stability Thm

Linear unfolding $\Omega \in \Pi_p(V)$ of Ω_0 is called Linear Centralizer Unfolding (LCU) if $p = \mathrm{cod}\Omega_0$ and if

$$D_0\Omega(\mathbb{R}^p) = \ker \mathrm{ad}\, \Omega_0^T \subseteq V$$

NB: $\ker \mathrm{ad}\ ^T \oplus \mathrm{im}\, \mathrm{ad} = V$

Thm. $\Omega \in \Pi_p(V)$ versal within linear structure (G, V). Then $\exists C^1$-neighbourhood $\mathcal{V}$ of Ω in $\Pi_p(V)$, and $\forall B \in \mathcal{V}$, $\exists$ map (ρ, Q), such that

i. $B(\mu) = Q(\mu)\Omega(\rho(\mu))Q^{-1}(\mu)$, for small μ;

ii. Reparametrization ρ is C^1-near identity

Proof: IFT

Linear stability of equilibria (dissipative case)

Given: family $X_\mu(z) = [\Omega(\mu)z + O(|z|^2)]\frac{\partial}{\partial z}$
$\Omega(\mu) \in gl(m,\mathbb{R}), \mu \in P$
Assume $\Omega_0 = \Omega(0)$ invertible and $\mu \mapsto \Omega(\mu)$ versal (nondegeneracy)

Zero set $\mathcal{Z}(X) \subset \mathbb{R}^m \times P$ of family X :
$\mathcal{Z}(X) = \{z = 0, \mu \text{ in full nbhd of } 0 \in P\}$

Lemma. $\forall C^2$-small perturbation $\tilde{X}$ of X, $\exists$ diffeomorphism $\Phi : \mathbb{R}^m \times P \to \mathbb{R}^m \times P$ (of form $\Phi(z,\mu) = (\phi_\mu(z), \rho(\mu))$), with

i. Φ is C^1-near identity;

ii. $\Phi(\mathcal{Z}(X)) = \mathcal{Z}(\tilde{X})$;

iii. Φ preserves the linear behaviour of X

Discussion II

Conclusion of Lemma $\longleftrightarrow$ *linear stability* of X at equilibrium $(z, \mu) = (0, 0)$

By IFT $\mathcal{Z}(\tilde{X})$ locally is smooth graph $(z(\mu), \mu)$ in $\mathbb{R}^m \times P$ over P

Trivial application to Parametrized KAM Theory when perturbation $\tilde{X}$ of integrable family X *also* is integrable !

Full KAM Thm as before. Linear stability solves small divisor problems. Newtonian iteration with *homological equation*

Similar statements for other linear structures: *e.g.*, symplectic, volume preserving, equivariant, reversible, etc.
Proof: Moser-Pöschel-Huitema-Hoo [BHN (2007), BCHV (2009)]

Admissibility: linear structure Axioms A1, A2 PLUS **Axiom A3**: 'homogeneity' (Taylor-Fourier truncations preserve structure)

Normal $1 : -1$ -resonance I

B, Hoo, Ciocci, Hanßmann, Naudot, Vanderbauwhede (2007-09)

$\mathbb{R}^4 = \{z_1, z_2, z_3, z_4\}$ with symplectic form $\sigma = dz_1 \wedge dz_3 + dz_2 \wedge dz_4$

Let Y be Hamiltonian vector field given by

$$Y(z) = \Omega_0 z \frac{\partial}{\partial z} \in \mathcal{X}^\sigma(\mathbb{R}^4),$$

where $\Omega_0 \in sp(4, \mathbb{R})$
In $1 : -1$ *resonance* if symplectically similar to

$$N_\varepsilon = \begin{pmatrix} 0 & -\lambda_0 & 0 & 0 \\ \lambda_0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & -\lambda_0 \\ 0 & \varepsilon & \lambda_0 & 0 \end{pmatrix},$$

where $\lambda_0 \neq 0$ and $\varepsilon = 0, \pm 1$

Generic case: $\varepsilon = \pm 1$, just take $\varepsilon = 1$

Normal $1 : -1$ -resonance II

For $\varepsilon = 1$ (generic case),
an LCU of N_ε is given by family

$$\Omega(\mu) = \begin{pmatrix} 0 & -\lambda_0 - \mu_1 & -\mu_2 & 0 \\ \lambda_0 + \mu_1 & 0 & 0 & -\mu_2 \\ 1 & 0 & 0 & -\lambda_0 - \mu_1 \\ 0 & 1 & \lambda_0 + \mu_1 & 0 \end{pmatrix}$$

where $\mu_1, \mu_2 \in \mathbb{R}$ (codimension = 2)

Eigenvalues $\Omega(\mu)$:

$$\pm(\mathrm{i}(\lambda_0 + \mu_1) \pm \sqrt{\mu_2}),$$

1. Elliptic for $\mu_2 < 0$
2. Parabolic (or Doubly Elliptic) for $\mu_2 = 0$:
   $\rightsquigarrow \mathrm{i}(\lambda_0 - \mu_1), -\mathrm{i}(\lambda_0 - \mu_1)$
3. Hyperbolic for $\mu_2 > 0$

Similar for the semisimple case: cod = 4
Similar for the reversible case:
same codimensions

Normal $1:-1$ -resonance III

Recall: eigenvalues of $\Omega(\mu)$

$$\pm \mathrm{i}(\lambda_0 + \mu_1 \pm \sqrt{\mu_2})$$

Normal frequencies of X given by
$\mu_2 \leq 0 \Rightarrow$
$\omega^N(\mu) = (\lambda_0 + \mu_1, \lambda_0 + \mu_1)$;
$\mu_2 > 0 \Rightarrow$
$\omega^N(\mu) = (\lambda_0 + \mu_1 + \sqrt{\mu_2}, \lambda_0 + \mu_1 - \sqrt{\mu_2})$

(Generalized) frequency map $\mathcal{F}$ given by:

$$\mathcal{F}(\omega,\mu) = \begin{cases} (\omega,\ \lambda_0 + \mu_1,\ \lambda_0 + \mu_1) \text{ for } \mu_2 \leq 0 \\ (\omega,\ \lambda_0 + \mu_1 + \sqrt{\mu_2}\ ,\ \lambda_0 + \mu_1 - \sqrt{\mu_2}) \\ \text{ for } \mu_2 > 0. \end{cases}$$

Corresponding Diophantine conditions

$$\begin{aligned} |\langle \omega, k\rangle + \ell(\lambda_0 + \mu_1)| &\geq \gamma |k|^{-\tau} \\ |\langle \omega, k\rangle + \ell_1(\lambda_0 + \mu_1) + \ell_2\sqrt{|\mu_2|}| &\geq \gamma |k|^{-\tau}, \end{aligned}$$

$\forall k \in \mathbb{Z}^n \setminus \{0\}$ and $\forall |\ell| = 0, 1, 2$ $\forall |\ell_1|, |\ell_2| \leq 2$ with $(\ell_1 \pm \ell_2)$ even

Nonlinear aspects I

Quasi-periodic Hamiltonian Hopf bifurcation

Higher order terms (VdMeer):
$\widehat{H} = (\nu_1(\mu) + \lambda_0)S + N + \nu_2(\mu)M + \frac{1}{2}b(\mu)M^2 + c_1(\mu)SM + c_2(\mu)S^2 + \cdots$

$$M = \frac{1}{2}(z_1^2 + z_2^2), N = \frac{1}{2}(z_1^3 + z_2^4), S = z_1 z_4 - z_2 z_3$$

Rotational symmetry

Integrable (swallowtail) versus
nearly-integrable (Cantorized swallowtail)
$H(x, y, z, \mu) = \langle \omega(\mu), y \rangle + \frac{1}{2}\langle Jz, \Omega(\mu)z \rangle + \cdots$

Nearly integrable KAM theory:
$\rightsquigarrow$ preservation of *most degenerate* m-torus, !!
with hyperbolic and elliptic m-tori along crease and thread

elliptic $(m + 1)$-tori on 2-dimensional sheet
& *Lagrangean* $(m + 2)$-tori above sheet
(both by 'standard' KAM)

Nonlinear aspects II

Quasi-periodically driven Lagrange top
(various devices)

Cushman-VdMeer
B-Hanßmann-Hoo (2007)

NB: All parameters ‘distinguished’:
$\rightsquigarrow$ weak stability
(only control frequency ratios)

Similarly: qp reversible Hopf

Knobloch-Vanderbauwhede
B-Ciocci-Hanßmann (2007)

Fewer parameters

More or less direct consequences of Parametrized KAM Theory:

1. Submersivity of frequency ratio
   $[\omega_1 : \omega_2 : \ldots \omega_n] \in \mathrm{P}^{n-1}(\mathbb{R})$
   $\rightsquigarrow$ *weak* quasi-periodic stability
   (all admissible structures)

2. Related to isoenergetic KAM
   in Hamiltonian systems
   B-Huitema (1991)

3. Rüssmann nondegeneracy too:
   M.R. Herman lectures 1990's
   B-Huitema-Sevryuk (1996), B-Sevryuk (to appear)

Discussion III

GAMES:
'Compensate' frequencies +
unfolding parameters
by 'distinguished' ones or by 'coefficients'

Play effectively with $\gamma = \gamma(\mu)$:
as small 'as perturbation allows'

CONCLUSIONS:
Cantor stratifications and Measure Theory:
many Hausdorff density points in appropriate dimension (*e.g.*, exponential condensation)
see [B-Sevryuk (1996, to appear)]
for references

EXAMPLES (in various admissible structures):

Quasi-periodic BIFO's (Hopf, period doubling)
With $\det \Omega_0 = 0$: qp saddle-node, cuspoids, saddle-center, normally parabolic
(*e.g.*, $H_\mu(x, y, t) = \frac{1}{2}y^2 + V_\mu(x) + \varepsilon f(x, \mu, t)$)
and umbilic cases [B-Hanßmann-You (2005) and (2006)] Destruction of Lagrangean tori by internal resonance [BHY (to appear)]
Global problems B-Cushman Fassò-Takens (2007)