# Do Diophantine vectors form a Cantor bouquet?

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We introduce the set of Diophantine vectors in $\mathbb{R}^n$, which is a standard notion in KAM Theory. The problem is whether this set forms a Cantor bouquet.

## 1 Definition and elementary properties

Let $n \geq 2$ and for $\tau > 0$ and $\gamma > 0$ consider the set of $(\tau, \gamma)$-Diophantine frequency vectors by

$$\mathbb{R}^n_{\tau,\gamma} = \{\omega \in \mathbb{R}^n \mid \forall k \in \mathbb{Z}^n \setminus \{0\} : |\langle \omega, k \rangle| \geq \gamma |k|^{-\tau}\}.$$

Here $\langle \omega, k \rangle = \sum_{j=1}^n \omega_j k_j$ is the usual inner product and $|k| = \sum_{j=1}^n |k_j|$ is the length of the integer vector $k$.

We summarize a few properties of $\mathbb{R}^n_{\tau,\gamma}$, for details see [3, 4]. First of all, the set $\mathbb{R}^n_{\tau,\gamma}$ is a *union of closed half lines* in the sense that for $\omega \in \mathbb{R}^n_{\tau,\gamma}$, then $s\,\omega \in \mathbb{R}^n_{\tau,\gamma}$ for all $s \geq 1$. Moreover, if $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, we consider the intersection $\mathbb{R}^n_{\tau,\gamma} \cap \mathbb{S}^{n-1}$. This is a closed (and hence compact) set that by the Cantor-Bendixson Theorem is the union of a perfect and a discrete set. Due to the fact that for any $k \in \mathbb{Z}$ the resonant hyperplane

$$\{\omega \in \mathbb{R}^n \mid \langle \omega, k \rangle = 0\}$$

is in its complement, the perfect set is totally disconnected and hence is a Cantor set.

For $\tau > n - 1$ the intersection $\mathbb{R}^n_{\tau,\gamma} \cap \mathbb{S}^{n-1}$ has positive measure in $\mathbb{S}^{n-1}$ for sufficiently small $\gamma > 0$. Indeed, one even has that the measure of the complement $\mathbb{S}^{n-1} \setminus \mathbb{R}^n_{\tau,\gamma}$ is of order $O(\gamma)$ as $\gamma \downarrow 0$. For a sketch of the planar case $\mathbb{R}^2_{\tau,\gamma} \subset \mathbb{R}^2$, see Figure 1.

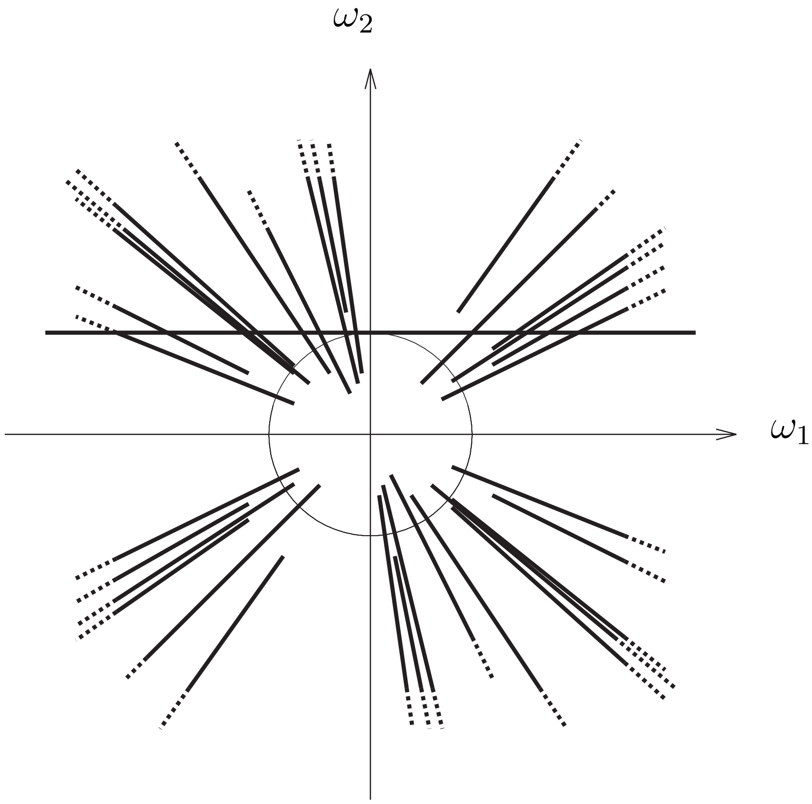


Figure 1: The Diophantine set $\mathbb{R}^n_{\tau,\gamma}$ has closed half line geometry and the intersection $\mathbb{S}^{n-1} \cap \mathbb{R}^n_{\tau,\gamma}$ is a Cantor set of measure $\mathbb{S}^{n-1} \setminus \mathbb{R}^n_{\tau,\gamma} = O(\gamma)$ as $\gamma \downarrow 0$.

For $n = 2$, by taking the ratio $\varrho = \omega_1/\omega_2$, one finds a relationship with the Diophantine numbers $\varrho \in \mathbb{R}$ satisfying

$$\left| \varrho - \frac{p}{q} \right| \geq \gamma q^{-(\tau+1)}$$

for all rationals $p/q \in \mathbb{Q}$, which is a subset of the Bruno numbers [7]. Compare with the horizontal line in Figure 1. Generally, for $n \geq 3$, a similar relationship is established between $\omega \in \mathbb{R}^n$ and $[\omega_1 : \omega_2, \ldots, \omega_n] \in \mathbb{P}^{n-1}(\mathbb{R})$, the $(n-1)$-dimensional projective space [2].

## 2 The problem

The problem is to show that $\mathbb{R}^n_{\tau,\gamma}$ is a Cantor bouquet [1, 5]. For this it is needed to know more about the endpoints of the closed half lines in $\mathbb{R}^n_{\tau,\gamma}$, in particular one has to prove that the set of endpoints accumulates on every point of $\mathbb{R}^n_{\tau,\gamma}$.

For $n = 2$ the answer may be related to a similar result that concerns the graph of the Bruno numbers, which uses continued fraction expansions [6]. For $n \geq 3$ the number theoretic aspects of the sets $\mathbb{R}^n_{\tau,\gamma}$ and $\mathbb{R}^n_{\tau,\gamma} \cap \mathbb{S}^{n-1}$ may be a lot more involved.

## References


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