Normal Forms in Perturbation Theory

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Article Outline

Glossary

Normal form procedure This is the stepwise 'simplification' by changes of coordinates, of the Taylor series at an equilibrium point, or of similar series at periodic or quasi-periodic solutions.

Preservation of structure The normal form procedure is set up in such a way that all coordinate changes preserve a certain appropriate structure. This applies to the class of **Hamiltonian** or **volume preserving** systems, as well as to systems that are **equivariant** or **reversible** with respect to a symmetry group. In all cases the systems may also depend on **parameters**.

Symmetry reduction The truncated normal form often exhibits a toroidal symmetry that can be factored out, thereby leading to a lower dimensional reduction.

Perturbation theory The attempt to extend properties of the (possibly reduced) normal form truncation, to the full system.

Definition of the Subject

Nonlinear dynamical systems are notoriously hard to tackle by analytic means. One of the few approaches that has been effective for the last couple of centuries, is Perturbation Theory. Here systems are studied, which in an appropriate sense, can be seen as perturbations of a given system with 'well-known' dynamical properties. Such 'well-known' systems usually are systems with a great amount of symmetry (like integrable Hamiltonian systems [1]) or very low-dimensional systems. The methods of Perturbation Theory then try to extend the 'well-known' dynamical properties to the perturbed system. Methods to do this are often based on the Implicit Function Theorem, on normal hyperbolicity [85,87] or on Kolmogorov–Arnold–Moser theory [1,10,15,18,38].

To obtain a perturbation theory set-up, normal form theory is a vital tool. In its most elementary form it amounts to 'simplifying' the Taylor series of a dynamical system at an equilibrium point by successive changes of coordinates. The lower order truncation of the series often belongs to the class of 'well-known' systems and by the Taylor formula the original system then locally can be viewed as a perturbation of this 'well-known' truncation, that thus serves as the 'unperturbed' system. More involved versions of normal form theory exist at periodic or quasi-periodic evolutions of the dynamical system.

Introduction

We review the formal theory of normal forms of dynamical systems at equilibrium points. Systems with continuous time, i. e., of vector fields, or autonomous systems of ordinary differential equations, are considered extensively. The approach is universal in the sense that it applies to many cases where a structure is preserved. In that case also the normalizing transformations preserve this structure. In particular this includes the Hamiltonian and the volume preserving case as well as cases that are equivariant or reversible with respect to a symmetry group. In all situations the systems may depend on parameters. Related topics are being dealt with concerning a vector field at a periodic solution or a quasi-periodic invariant torus as well as the case of a diffeomorphism at a fixed point. The paper is concluded by discussing a few non-formal aspects and some applications.

Motivation

The term 'normal form' is widely used in mathematics and its meaning is very sensitive for the context. In the case of linear maps from a given vector space to itself, for example, one may consider all possible choices of a basis. Each choice gives a matrix-representation of a given linear map. A suitable choice of basis now gives the well-known Jordan canonical form. This normal form, in a simple way displays certain important properties of the linear map, concerning its spectrum, its eigenspaces, and so on.

Presently the concern is with dynamical systems, such as vector fields (i. e., systems of autonomous ordinary differential equations), or diffeomorphisms. The aim is to simplify these systems near certain equilibria and (quasi-periodic) solutions by a proper choice of coordinates. The purpose of this is to better display the local dynamical properties. This normalization is effected by a stepwise simplification of formal power series (such as Taylor series).

Reduction of Toroidal Symmetry

In a large class of cases, the simplification of the Taylor series induces a toroidal symmetry up to a certain order, and truncation of the series at that order gives a local approximation of the system at hand. From this truncation we can reduce this toroidal symmetry, thereby also reducing the dimension of the phase space. This kind of procedure is reminiscent of the classical reductions in classical mechanics related to the Noether Theorem, compare with [1], ► Dynamics of Hamiltonian Systems.

This leads us to a first perturbation problem to be considered. Indeed, by truncating and factoring out the torus symmetry we get a polynomial system on a reduced phase, and one problem is how persistent this system is with respect to the addition of higher order terms. In the case where the system depends on parameters, the persistence of a corresponding bifurcation set is of interest.

In many examples the reduced phase space is 2-dimensional where the dynamics is qualitatively determined by a polynomial and this perturbation problem can be handled by Singularity Theory. In quite a number of cases the truncated and reduced system turns out to be structurally stable [67,86].

A Global Perturbation Theory

When returning to the original phase space we consider the original system as a perturbation of the de-reduced truncation obtained so far. In the ensuing perturbation problem, several types of resonance can play a role.

A classical example [2,38] occurs when in the reduced model a Hopf bifurcation takes place and we have reduced by a 1-torus. Then in the original space generically a Hopf–Neĭmark–Sacker bifurcation occurs, where in the parameter space the dynamics is organized by resonance *tongues*. In cases where we have reduced by a torus of dimension larger than 1, we are dealing with quasi-periodic Hopf bifurcation, in which the bifurcation set gets 'Cantorized' by Diophantine conditions, compare with [6,15,21,38]. Also the theory of homo- and heteroclinic bifurcations then is of importance [68].

We use the two perturbation problems as a motivation for the Normal Form Theory to be reviewed, for details, however, we just refer to the literature. For example, compare with ► Perturbation Theory and references therein.

The Normal Form Procedure

The subject of our interest is the simplification of the Taylor series of a vector field at a certain equilibrium point. Before we explore this, however, let us first give a convenient normal form of a vector field near a non-equilibrium point.

Theorem 1 (Flow box [81]) *Let the C^∞ vector field X on $\mathbb{R}^n$ be given by $\dot{x} = f(x)$ and assume that $f(p) \neq 0$. Then there exists a neighborhood of p with local C^∞ coordinates $y = (y_1, y_2, \ldots, y_n)$, such that in these coordinates X has the form*

$$\dot{y}_1 = 1$$
$$\dot{y}_j = 0\,,$$

for $2 \leq j \leq n$.

Such a local chart usually is called a *flowbox* and the above theorem the Flowbox Theorem. A proof simply can be given using a transversal local C^∞ section that cuts the flow of the vector field transversally. For the coordinate y_1 then use the time-parametrization of the flow, while the coordinates y_j, for $2 \leq j \leq n$ come from the section, compare with, e. g., [81].

Background, Linearization

The idea of simplification near an equilibrium goes back at least to Poincaré [69], also compare with Arnold [2]. To be definite, we now let X be a C^∞ vector field on $\mathbb{R}^n$, with the origin as an equilibrium point. Suppose that X has the form $\dot{x} = Ax + f(x), x \in \mathbb{R}^n$, where A is linear and where $f(0) = 0$, $D_0 f = 0$. The first idea is to apply successive C^∞ changes of coordinates of the form $\mathrm{Id} + P$, with P a homogeneous polynomial of degree $m = 2, 3, \ldots$, 'simplifying' the Taylor series step by step.

The most 'simple' form that can be obtained in this way, is where *all* higher order terms vanish. In that case the normal form is formally linear. Such a case was treated by Poincaré, and we shall investigate this now.

We may even assume to work on $\mathbb{C}^n$, for simplicity assuming that the eigenvalues of A are distinct. A collection $\lambda = (\lambda_1, \ldots, \lambda_n)$ of points in $\mathbb{C}$ is said to be *resonant* if there exists a relation of the form

$$\lambda_s = \langle r, \lambda \rangle ,$$

for $r = (r_1, \ldots, r_n) \in Z^n$, with $r_k \geq 0$ for all k and with $\sum r_k \geq 2$. The *order* of the resonance then is the number $|r| = \sum r_k$. The Poincaré Theorem now reads

Theorem 2 (Formal linearization [2,69]) *If the (distinct) eigenvalues $\lambda_1, \ldots, \lambda_n$ of A have no resonances, there exists a formal change of variables $x = y + O(|y|^2)$, transforming the above vector field X, given by*

$$\dot{x} = Ax + f(x)$$

to

$$\dot{y} = Ay .$$

We include a proof [2], since this will provide the basis for almost all further considerations.

Proof The formal power series $x = y + O(|y|^2)$ is obtained in an inductive manner. Indeed, for $m = 2, 3, \ldots$ a polynomial transformation $x = y + P(y)$ is constructed, with P homogeneous of degree m, which removes the terms of degree m from the vector field. At the end we have to take the composition of all these polynomial transformations.

1. The basic tool for the mth step is the following. Let v be homogeneous of degree m, then if the vector fields $\dot{x} = Ax + v(x) + O(|x|^{m+1})$ and $\dot{y} = Ay$ are related by the transformation $x = y + P(y)$ with P also homogeneous of degree m, then

$$D_x PAx - AP(x) = v(x) .$$

This relation usually is called the *homological equation*, the idea being to determine P in terms of v: by this choice of P the term v can be transformed away.

The proof of this relation is straightforward. In fact,

$$\begin{aligned} \dot{x} &= (\mathrm{Id} + D_y P)Ay \\ &= (\mathrm{Id} + D_y P)A(x - P(x) + O(|x|^{m+1})) \\ &= Ax + \{D_x PAx - AP(x)\} + O(|x|^{m+1}) , \end{aligned}$$

where we used that for the inverse transformation we know $y = x - P(x) + O(|x|^{m+1})$.

2. For notational convenience we introduce the linear operator $\mathrm{ad}A$, the so-called adjoint operator, by

$$\mathrm{ad}A(P)(x) := D_x PAx - AP(x) ,$$

then the homological equation reads $\mathrm{ad}A(P) = v$. So the question is reduced to whether v is in the image of the operator $\mathrm{ad}A$. It turns out that the eigenvalues of $\mathrm{ad}A$ can be expressed in those of A. If $x_1, x_2, \ldots, x_n$ are the coordinates corresponding to the basis $e_1, e_2, \ldots, e_n$, again it is a straightforward computation to show that for $P(x) = x^r e_s$, one has

$$\mathrm{ad}A(P) = (\langle r, \lambda \rangle - \lambda_s)P .$$

Here we use the multi-index notation $x^r = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$. Indeed, for this choice of P one has $AP(x) = \lambda_s P(x)$, while

$$\frac{\partial x^r}{\partial x} Ax = \sum_j \frac{r_j}{x_j} x^r \lambda_j x_j = \langle r, \lambda \rangle x^r .$$

We conclude that the monomials $x^r e_s$ are eigenvectors corresponding to the eigenvalues $\langle r, \lambda \rangle - \lambda_s$. We conclude that the operator $\mathrm{ad}A$ is *semisimple*, since it has a basis of eigenvectors. Therefore, if $\ker \mathrm{ad}A = 0$, the operator is surjective. This is exactly what the non-resonance condition on the eigenvalues of A amounts to. More precisely, the homological equation $\mathrm{ad}A(P) = v$ can be solved for P for each homogeneous part v of degree m, provided that there are no resonances up to order m.

3. The induction process now runs as follows. For $m \geq 2$, given a form

$$\dot{x} = Ax + v_m(x) + O(|x|^{m+1}) ,$$

we solve the homological equation

$$\mathrm{ad}A(P_m) = v_m ,$$

then carrying out the transformation $x = y + P_m(y)$. This takes the above form to

$$\dot{y} = Ay + w_{m+1}(y) + O(|y|^{m+2}) .$$

The composition of all the polynomial transformations then gives the desired formal transformation. □

Remark

- It is well-known that the formal series usually diverge. Here we do not go into this problem, for a brief discussion see below.
- If resonances are excluded up to a finite order N, we can linearize up to that order, so obtaining a normal form.

  $$\dot{y} = Ay + O(|y|^{N+1}) .$$

  In this case the transformation can be taken as a polynomial.
- If the original problem is real, but with the matrix A having non-real eigenvalues, we still can keep all transformations real by also considering complex conjugate eigenvectors.

We conclude this introduction by discussing two further linearization theorems, one due to Sternberg and the other to Hartman–Grobman. We recall the following for a vector field $X(x) = Ax + f(x)$, $x \in \mathbb{R}^n$, with 0 as an equilibrium point, i. e., with $f(0) = 0$, $D_0 f = 0$. The equilibrium 0 is *hyperbolic* if the matrix A has no purely imaginary eigenvalues. Sternberg's Theorem reads

Theorem 3 (Smooth linearization [82]) *Let X and Y be C^∞ vector fields on $\mathbb{R}^n$, with 0 as a hyperbolic equilibrium point. Also suppose that there exists a formal transformation $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ taking the Taylor series of X at 0 to that of Y. Then there exists a local C^∞-diffeomorphism $\Phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, such that $\Phi_* X = Y$.*

We recall that $\Phi_* X(\Phi(x)) = D_x \Phi X(x)$. This means that X and Y are locally *conjugated* by Φ, the evolution curves of X are mapped to those of Y in a time-preserving manner. In particular Sternberg's Theorem applies when the conclusion of Poincaré's Theorem holds: for Y just take the linear part $Y(x) = Ax(\partial/\partial x)$.

Combining these two theorems we find that in the hyperbolic case, under the exclusion of all resonances, the vector field X is linearizable by a C^∞-transformation. The Hartman–Grobman Theorem moreover says that the nonresonance condition can be omitted, provided we only want a C^0-linearization.

Theorem 4 (Continuous linearization e. g., [67]) *Let X be a C^∞ vector field on $\mathbb{R}^n$, with 0 as a hyperbolic equilibrium point. Then there exists a local homeomorphism $\Phi\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, locally conjugating X to its linear part.*

Preliminaries from Differential Geometry

Before we develop a more general Normal Form Theory we recall some elements from differential geometry. One central notion used here is that of the *Lie derivative*. For simplicity all our objects will be of class C^∞. Given a vector field X we can take any tensor τ and define its Lie-derivative $\mathcal{L}_X \tau$ with respect to X as the infinitesimal transformation of τ along the flow of X. In this way $\mathcal{L}_X \tau$ becomes a tensor of the same type as τ. To be more precise, for τ a real *function* f one so defines

$$\mathcal{L}_X f(x) = X(f)(x) = \mathrm{d}f(X)(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(X_t(x)) ,$$

i. e., the directional derivative of f with respect to X. Here X_t denotes the flow of X over time t. For τ a *vector field* Y one similarly defines

$$\begin{aligned}\mathcal{L}_X Y(x) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (X_{-t})_* Y(x) \\ &= \lim_{t\to 0} \frac{1}{t}\{(X_{-t})_* Y(x) - Y(x)\} \\ &= \lim_{h\to 0} \frac{1}{h}\{Y(x) - (X_h)_* Y(x)\} ,\end{aligned}$$

and similarly for *differential forms*, etc. Another central notion is the *Lie-brackets* $[X, Y]$ defined for any two vector fields X and Y on $\mathbb{R}^n$ by

$$[X, Y](f) = X(Y(f)) - Y(X(f)) .$$

Here f is any real function on $\mathbb{R}^n$ while, as before, $X(f)$ denotes the directional derivative of f with respect to X.

We recall the expression of Lie-brackets in coordinates. If X is given by the system of differential equations $\dot{x}_j = X_j(x)$, with $1 \le j \le n$, then the directional derivative $X(f)$ is given by $X(f) = \sum_{j=1}^n X_j \partial f / \partial x_j$. Then, if $[X, Y]$ is given by the system $\dot{x}_j = Z_j(x)$, for $1 \le j \le n$, of differential equations, one directly shows that

$$Z_j = \sum_{k=1}^{n} \left(X_k \frac{\partial Y_j}{\partial x_k} - Y_k \frac{\partial X_j}{\partial x_k} \right) .$$

Here Y_j relates to Y as X_j does to X. We list some useful properties.

Proposition 5 (Properties of the Lie-derivative [81])

1. $\mathcal{L}_X(Y_1 + Y_2) = \mathcal{L}_X Y_1 + \mathcal{L}_X Y_2$ *(linearity over $\mathbb{R}$)*
2. $\mathcal{L}_X(fY) = X(f) \times Y + f \times \mathcal{L}_X Y$ *(Leibniz rule)*
3. $[Y, X] = -[X, Y]$ *(skew symmetry)*
4. $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$ *(Jacobi identity)*
5. $\mathcal{L}_X Y = [X, Y]$
6. $[X, Y] = 0 \Leftrightarrow X_t \circ Y_s = Y_s \circ X_t$

Proof The first four items are left to the reader.

5. The equality $\mathcal{L}_X Y = [X, Y]$ can be proven by observing the following. Both members of the equality are defined intrinsically, so it is enough to check it in any choice of (local) coordinates. Moreover, we can restrict our attention to the set $\{x | X(x) \neq 0\}$. By the Flowbox Theorem 1 we then may assume for the component functions of X that $X_1(x) = 1, X_j(x) = 0$ for $2 \leq j \leq n$. It is easy to see that both members of the equality now are equal to the vector field Z with components

$$Z_j = \frac{\partial Y_j}{\partial x_1} .$$

6. Remains the equivalence of the commuting relationships. The commuting of the flows, by a very general argument, implies that the bracket vanishes. In fact, fixing t we see that X_t conjugates the flow of Y to itself, which is equivalent to $(X_t)_* Y = Y$. By definition this implies that $\mathcal{L}_X Y = 0$.

Conversely, let $c(t) = ((X_t)_* Y)(p)$. From the fact that $\mathcal{L}_X Y = 0$ it then follows that $c(t) \equiv c(0)$. Observe that the latter assertion is sufficient for our purposes, since it implies that $(X_t)_* Y = Y$ and therefore $X_t \circ Y_s = Y_s \circ X_t$. Finally, that $c(t)$ is constant can be shown as follows.

$$\begin{aligned}
c'(t) &= \lim_{h\to 0} \frac{1}{h}\{c(t+h) - c(t)\} \\
&= \lim_{h\to 0} \frac{1}{h}\{((X_{t+h})_* Y)(p) - ((X_t)_* Y)(p)\} \\
&= (X_t)_* \lim_{h\to 0} \frac{1}{h}\{((X_h)_* Y)(X_{-t}(p)) - Y(X_{-t}(p))\} \\
&= (X_t)_* \mathcal{L}_X Y(X_{-t}(p)) \\
&= (X_t)_*(0) \\
&= 0 .
\end{aligned}$$

□

'Simple' in Terms of an Adjoint Action

We now return to the setting of Normal Form Theory at equilibrium points. So given is the vector field $\dot{x} = X(x)$, with $X(x) = Ax + f(x), x \in \mathbb{R}^n$, where A is linear and where $f(0) = 0$, $D_0 f = 0$. We recall that it is our general aim to 'simplify' the Taylor series of X at 0.

However, we have not yet said what the word 'simple' means in the present setting. In order to understand what is going on, we reintroduce the adjoint action associated to the linear part A, defined on the class of all C^∞ vector fields on $\mathbb{R}^n$. To be precise, this adjoint action ad$A$ is defined by the Lie-bracket

$$\mathrm{ad}A \colon Y \mapsto [A, Y] ,$$

where A is identified with the linear vector field $\dot{x} = Ax$. It is easily seen that this fits with the notation introduced in Theorem 2.

Let $H^m(\mathbb{R}^n)$ denote the space of polynomial vector fields, homogeneous of degree m. Then the Taylor series of X can be viewed as an element of the product $\prod_{m=1}^{\infty} H^m(\mathbb{R}^n)$. Also, it directly follows that ad$A$ induces a linear map $H^m(\mathbb{R}^n) \to H^m(\mathbb{R}^n)$, to be denoted by $\mathrm{ad}_m A$. Let

$$B^m := \mathrm{im}\, \mathrm{ad}_m A ,$$

the image of the map $\mathrm{ad}_m A$ in $H^m(\mathbb{R}^n)$. Then for *any* complement G^m of B^m in $H^m(\mathbb{R}^n)$, in the sense that

$$B^m \oplus G^m = H^m(\mathbb{R}^n) ,$$

we define the *corresponding* notion of 'simpleness' by requiring the homogeneous part of degree m to be in G^m. In the case of the Poincaré Theorem 2, since $B^m = H^m(\mathbb{R}^n)$, we have $G^m = \{0\}$.

We now quote a theorem from Sect. 7.6.1. in Dumortier et al. [16]. Although its proof is very similar to the one of Theorem 2, we include it here, also because of its format.

Theorem 6 ('Simple' in terms of G^m [74,82]) *Let X be a C^∞ vector field, defined in the neighborhood of $0 \in \mathbb{R}^n$, with $X(0) = 0$ and $D_0 X = A$. Also let $N \in \mathbb{N}$ be given and, for $m \in \mathbb{N}$, let B^m and G^m be such that $B^m \oplus G^m = H^m(\mathbb{R}^n)$. Then there exists, near $0 \in \mathbb{R}^n$, an analytic change of coordinates $\Phi \colon \mathbb{R}^n \to \mathbb{R}^n$, with $\Phi(0) = 0$, such that*

$$\Phi_\star X(y) = Ay + g_2(y) + \cdots + g_N(y) + O(|y|^{N+1}) ,$$

with $g_m \in G^m$, for all $m = 2, 3, \ldots, N$.

Proof We use induction on N. Let us assume that

$$X(x) = Ax + g_2(x) + \cdots + g_{N-1}(x) + f_N(x) \\ + O(|x|^{N+1}) ,$$

with $g_m \in G^m$, for all $m = 2, 3, \ldots, N-1$ and with f_N homogeneous of degree N.

We consider a coordinate change $x = y + P(y)$, where P is polynomial of degree N, see above. For any such P, by substitution we get

$$(\mathrm{Id} + D_y P)\dot{y} = A(y + P(y)) + g_2(y) + \cdots \\ + g_{N-1}(y) + f_N(y) + O(|y|^{N+1}) ,$$

or

$$\begin{aligned}\dot{y} &= (\mathrm{Id} + D_y P)^{-1}(A(y + P(y)) + g_2(y) \\ &\quad + \cdots + g_{N-1}(y) + f_N(y) + O(|y|^{N+1})) \\ &= Ay + g_2(y) + \cdots + g_{N-1}(y) + f_N(y) + AP(y) \\ &\quad - D_y PAy + O(|y|^{N+1}) ,\end{aligned}$$

using that $(\mathrm{Id} + D_y P)^{-1} = \mathrm{Id} - D_y P + O(|y|^N)$. We conclude that the terms up to order $N-1$ are unchanged by this transformation, while the Nth order term becomes

$$f_N(y) - \mathrm{ad}_N A(P)(y) .$$

Clearly, a suitable choice of P will put this term in G^N. This is the present version of the *homological equation* as introduced before. □

Remark For simplicity the formulations all are in the C^∞-context, but obvious changes can be made for the case of finite differentiability. The latter case is of importance for applications of Normal Form Theory after reduction to a center manifold [85,87].

Torus Symmetry

As a special case let the linear part A be semisimple, in the sense that it is diagonizable over the complex numbers. It then directly follows that also $\mathrm{ad}_m A$ is semisimple, which implies that

$$\mathrm{im}\, \mathrm{ad}_m A \oplus \ker \mathrm{ad}_m A = H^m(\mathbb{R}^n) .$$

The reader is invited to provide the eigenvalues of $ad_m A$ in terms of those of A, compare with the proof of Theorem 2. In the present case the obvious choice for the complementary space defining 'simpleness' is $G^m = \ker \mathrm{ad}_m A$. Moreover, the fact that the normalized, viz. simplified, terms g_m are in G^m by definition means that

$$[A, g_m] = 0 .$$

This, in turn, implies that N-jet of $\Phi_\star X$, i. e., the *normalized part* of $\Phi_\star X$, by Proposition 5 is invariant under all linear transformations

$$\exp tA, \quad t \in \mathbb{R}$$

generated by A. For further reading also compare with Sternberg [82], Takens [84], Broer [7,8] or [12].

More generally, let $A = A_s + A_n$ be the Jordan canonical splitting in the semisimple and nilpotent part. Then one directly shows that $\mathrm{ad}_m A = \mathrm{ad}_m A_s + \mathrm{ad}_m A_n$ is the Jordan canonical splitting, whence, by a general argument [89] it follows that

$$\mathrm{im}\, \mathrm{ad}_m A + \ker \mathrm{ad}_m A_s = H^m(\mathbb{R}^n) ,$$

so where the sum splitting in general no longer is direct. Now we can choose the complementary spaces G^m such that

$$G^m \subset \ker \mathrm{ad}_m A_s ,$$

ensuring equivariance of the normalized part of $\Phi_\star X$, with respect to all linear transformations

$$\exp tA_s, \quad t \in \mathbb{R} .$$

The choice of G^m can be further restricted, e. g., such that

$$G^m \subseteq \ker \mathrm{ad}_m A_s \setminus \mathrm{im}\, \mathrm{ad}_m A_n \subseteq H^m(\mathbb{R}^n) ,$$

compare with Van der Meer [88]. For further discussion on the choice of G^m, see below.

Example (Rotational symmetry [84]) Consider the case $n = 2$ where

$$A = A_s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

So, the eigenvalues of A are $\pm i$ and the transformations $\exp tA_s$, $t \in \mathbb{R}$, form the rotation group $\mathrm{SO}(2, \mathbb{R})$. From this, we can arrive at once at the general format of the normal form. In fact, the normalized, Nth order part of $\Phi_\star X$ is rotationally symmetric. This implies that, if we pass to polar coordinates (r, φ), by

$$y_1 = r\cos\varphi, \quad y_2 = r\sin\varphi ,$$

the normalized truncation of $\Phi_\star X$ obtains the form

$$\begin{aligned}\dot{\varphi} &= f(r^2) \\ \dot{r} &= rg(r^2) ,\end{aligned}$$

for certain polynomials f and g, with $f(0) = 1$ and $g(0) = 0$.

Remark

- A more direct, 'computational' proof of this result can be given as follows, compare with the proof of Theorem 2 and with [16,84,87]. Indeed, in vector field notation we can write

$$\begin{aligned}A &= -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} \\ &= i\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) ,\end{aligned}$$

where we complexified putting $z = x_1 + ix_2$ and use the well-known Wirtinger derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right).$$

Now a basis of eigenvectors for $\mathrm{ad}_m A$ can be found directly, just computing a few Lie-brackets, compare the previous subsection. In fact, it is now given by all monomials

$$z^k \bar{z}^l \frac{\partial}{\partial z}, \quad \text{and} \quad z^k \bar{z}^l \frac{\partial}{\partial \bar{z}},$$

with $k + \ell = m$. The corresponding eigenvalues are $i(k - \ell - 1)$ viz. $i(k - \ell + 1)$. So again we see, now by a direct inspection, that $\mathrm{ad}_m A$ is semisimple and that we can take $G^m = \ker \mathrm{ad}_m A$. This space is spanned by

$$(z\bar{z})^r \left(z\frac{\partial}{\partial z} \pm \bar{z}\frac{\partial}{\partial \bar{z}}\right),$$

with $2r + 1 = m - 1$, which indeed proves that the normal form is rotationally symmetric.

- A completely similar case occurs for $n = 3$, where

$$A = A_s = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case the normalized part in cylindrical coordinates (r, φ, z), given by

$$y_1 = r\cos\varphi, \quad y_2 = r\sin\varphi, \quad y_3 = z,$$

in general gets the axially symmetric form

$$\begin{aligned} \dot{\varphi} &= f(r^2, z) \\ \dot{r} &= rg(r^2, z) \\ \dot{z} &= h(r^2, z), \end{aligned}$$

for suitable polynomials f, g and h.

As a generalization of this example we state the following proposition, where the normalized part exhibits an m-torus symmetry.

Proposition 7 (Toroidal symmetry [84]) *Let X be a C^∞ vector field, defined in the neighborhood of $0 \in \mathbb{R}^n$, with $X(0) = 0$ and where $A = D_0 X$ is semisimple with the eigenvalues $\pm i\omega_1, \pm i\omega_2, \dots, \pm i\omega_m$ and 0. Here $2m \le n$. Suppose that for given $N \in \mathbb{N}$ and all integer vectors $(k_1, k_2, \dots, k_m)$,*

$$1 \le \sum_{j=1}^{m} |k_j| \le N + 1 \Rightarrow \sum_{j=1}^{m} k_j \omega_j \neq 0, \tag{1}$$

(i. e., there are no resonances up to order $N + 1$). Then there exists, near $0 \in \mathbb{R}^n$, an analytic change of coordinates $\Phi: \mathbb{R}^n \to \mathbb{R}^n$, with $\Phi(0) = 0$, such that $\Phi_\star X$, up to terms of order N has the following form. In suitable generalized cylindrical coordinates $(\varphi_1, \dots, \varphi_m, r_1^2, \dots, r_m^2, z_{n-2m+1}, \dots, z_n)$ it is given by

$$\begin{aligned} \dot{\varphi}_j &= f_j(r_1^2, \dots, r_m^2, z_{n-2m+1}, \cdots, z_n) \\ \dot{r}_j &= r_j g_j(r_1^2, \dots, r_m^2, z_{n-2m+1}, \cdots, z_n) \\ \dot{z}_\ell &= h_\ell(r_1^2, \dots, r_m^2, z_{n-2m+1}, \cdots, z_n), \end{aligned}$$

where $f_j(0) = \omega_j$ and $h_\ell(0) = 0$ for $1 \le j \le m$, $n - 2m + 1 \le \ell \le n$.

Proofs can be found, e. g., in [16,84,85,87]. In fact, if one introduces a suitable complexification, it runs along the same lines as the above remark. For the fact that finitely many non-resonance conditions are needed in order to normalize up to finite order, also compare a remark following Theorem 2.

Since the truncated system of $\dot{r}_j$- and $\dot{z}_\ell$-equations is independent of the angles φ_j, this can be studied separately. A similar remark holds for the earlier examples of this section. As indicated in the introduction, this kind of 'reduction by symmetry' to lower dimension can be of great importance when studying the dynamics of X : as it enables us to consider X, viz. $\Phi_\star X$, as an N-flat perturbation of the normalized part, which is largely determined by this lower dimensional reduction. For more details see below.

On the Choices of the Complementary Space and of the Normalizing Transformation

In the previous subsection we only provided a general (symmetric) format of the normalized part. In concrete examples one has to do more. Indeed, given the original Taylor series, one has to *compute* the coefficients in the normalized expansion. This means that many choices have to be made explicit.

To begin with there is the choice of the spaces G^m, which define the notion of 'simple'. We have seen already that this choice is not unique.

Moreover observe that, even if the choice of G^m has been fixed, still P usually is not uniquely determined. In the semisimple case, for example, P is only determined modulo the kernel $G^m = \ker \mathrm{ad}_m A$.

Remark To fix thoughts, consider the former of the above examples, on $\mathbb{R}^2$, where the normalized truncation has the rotationally symmetric form

$$\dot{\varphi} = f(r^2)$$
$$\dot{r} = rg(r^2)\,.$$

Here $g(r^2) = cr^2 + O(|r|^4)$. The coefficient c dynamically is important, just think of the case where the system is part of a family that goes through Hopf-bifurcation. The computation of (the sign of) c in a concrete model can be quite involved, as it appears from, e. g., Marsden and McCracken [56]. Machine-assisted methods largely have taken over this kind of work.

One general way to choose G^m is the following, compare with e. g., Sect. 7.6 in [16] and Sect. 2.3 in [87], :

$$G^m := \ker(\mathrm{ad}_m A^{\mathrm{T}})\,.$$

Here A^{T} is the *transpose* of A, defined by the relation $\langle A^{\mathrm{T}}x, y\rangle = \langle x, Ay\rangle$, where $\langle .,.\rangle$ is an inner product on $\mathbb{R}^n$. A suitable choice for an inner product on $H^m(\mathbb{R}^n)$ then directly gives that

$$G^m \oplus \mathrm{im}(\mathrm{ad}_m A) = H^m(\mathbb{R}^n)\,,$$

as required. Also here the normal form can be interpreted in terms of symmetry, namely with respect to the group generated by A^{T}. In the semisimple case, this choice leads to exactly the same symmetry considerations as before.

The above algorithms do not provide methods for computing the normal form yet, i. e., for actually solving the homological equation. In practice, this is an additional computation. Regarding the corresponding algorithms we give a few more references for further reading, also referring to their bibliographies. General work in this direction is Bruno [35,36], Sect. 7.6 in Dumortier et al. [16], Takens [84] or Vanderbauwhede [87], Part 2. In the latter reference also a brief description is given of the sl(2, $\mathbb{R}$)-theory of Sanders and Cushman [39], which is a powerful tool in the case where the matrix A is nilpotent [84]. For a thorough discussion on some of these methods we refer to Murdock [60]. Later on we shall come back to these aspects.

Preservation of Structure

It goes without saying that Normal Form Theory is of great interest in special cases where a given structure has to preserved. Here one may think of a symplectic or a volume form that has to be respected. Also a given symmetry group can have this rôle, e. g., think of an involution related to reversibility. Another, similar, problem is the dependence of external parameters in the system.

A natural language for preservation of such structures is that of Lie-subalgebra's of general Lie-algebra of vector fields, and the corresponding Lie-subgroup of the general Lie-group of diffeomorphisms.

The Lie-Algebra Proof

Fortunately the setting of Theorem 6 is almost completely in terms of Lie-brackets. Let us briefly reconsider its proof.

Given is a C^∞ vector field $X(x) = Ax + f(x)$, $x \in \mathbb{R}^n$, where A is linear and where $f(0) = 0$, $D_0 f = 0$. We recall that in the inductive procedure a transformation

$$h = \mathrm{Id} + P\,,$$

with P a homogeneous polynomial of degree $m = 2, 3, \ldots$, is found, putting the homogeneous mth degree part of the Taylor series of X into the 'good' space G^m.

Now, nothing changes in this proof if instead of $h = \mathrm{Id} + P$, we take $h = P_1$, the flow over time 1 of the vector field P : indeed, the effect of this change is not felt until the order $2m - 1$. Here we use the following formula for $X^t := (P_t)_\star X$:

$$[X^t, P] = \mathrm{ad}_m A(P) + O(|y|^{m+1})$$
$$\text{and} \quad \frac{\partial X^t}{\partial t} = [X^t, P],$$

compare with Sect. "Preliminaries from Differential Geometry". In fact, exactly this choice $h = P_1$ was taken by Roussarie [74], also see the proof of Takens [84]. Notice that $\Phi = h_N \circ h_{N-1} \circ \cdots \circ h_3 \circ h_2$. Moreover notice that if the vector field P is in a given Lie-algebra of vector fields, its time 1 map P_1 is in the corresponding Lie-group. In particular, if P is Hamiltonian, P_t is canonical or symplectic, and so on.

For the validity of this set-up for a more general Lie-subalgebra of the Lie-algebra of all C^∞ vector fields, one has to study how far the *grading*

$$\prod_{m=1}^{\infty} H^m(\mathbb{R}^n)$$

of the formal power series, as well as the *splittings*

$$B^m \oplus G^m = H^m(\mathbb{R}^n)\,,$$

are compatible with the Lie-algebra at hand. This issue was addressed by Broer [7,8] axiomatically, in terms of graded and filtered Lie-algebra's. Moreover, the methods concerning the choice of G^m briefly mentioned at the end of the previous section, all carry over to a more general Lie-algebra set-up. As a consequence there exists a version of Theorem 6 in this setting. Instead of pursuing this further, we discuss its implications in a few relevant settings.

The Volume Preserving and Symplectic Case On $\mathbb{R}^n$, resp. $\mathbb{R}^{2n}$, we consider a volume form or a symplectic form, both denoted by σ. We assume, that

$$\sigma = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n, \quad \text{resp.} \quad \sigma = \sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}x_{j+n} .$$

In both cases, let $\mathcal{X}_\sigma$ denote the Lie-algebra of σ-preserving vector fields, i. e., vector fields X such that $\mathcal{L}_X \sigma = 0$. Here $\mathcal{L}$ again denotes the Lie-derivative, see Sect. "The Normal Form Procedure".

Indeed, one defines

$$\begin{aligned} \mathcal{L}_X \sigma(x) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (X_t)^* \sigma(x) \\ &= \lim_{t \to 0} \frac{1}{t} \{(X_t)^* \sigma(x) - \sigma(x)\} . \end{aligned}$$

Properties, similar to Proposition 5, hold here. Since in both cases σ is a closed form, one shows by 'the magic formula' [81] that

$$\begin{aligned} \mathcal{L}_X \sigma(x) &= \mathrm{d}(\iota_X \sigma) + \iota_X \, \mathrm{d}\sigma \\ &= \mathrm{d}(\iota_X \sigma) . \end{aligned}$$

Here $\iota_X \sigma$ denotes the flux-operator defined by $\iota_X \sigma(Y) = \sigma(X, Y)$. In the volume-preserving case the latter expression denotes $\mathrm{div}(X)\sigma$ and we see that preservation of σ exactly means that $\mathrm{div}(X) = 0$: the divergence of X vanishes. In the Hamiltonian case we conclude that the 1-form $\iota_X \sigma$ is closed and hence (locally) of the form $\mathrm{d}H$, for a Hamilton function H. In both cases, the fact that for a transformation h the fact that $h^* \sigma = \sigma$ implies that with X also $h_* X$ is σ-preserving. Moreover, for a σ-preserving vector field P and $h = P_1$ one can show that indeed $h^* \sigma = \sigma$.

One other observation is, that by the *homogeneity* of the above expressions for σ, the homogeneous parts of the Taylor series of σ-preserving vector fields are again σ-preserving. This exactly means that

$$H^m(\mathbb{R}^{(2)n}) \cap \mathcal{X}_\sigma ,$$

$m = 1, 2, \ldots$, *grades* the formal power series corresponding to $\mathcal{X}_\sigma$. Here, notice that

$$H^1(\mathbb{R}^{(2)n}) \cap \mathcal{X}_\sigma = \mathrm{sl}(n, \mathbb{R}), \quad \text{resp.} \quad \mathrm{sp}(2n, \mathbb{R}) ,$$

the *special*- resp. the *symplectic* linear algebra.

In summary we conclude that both the symplectic and the volume preserving setting are covered by the axiomatic approach of [7,8] and that an appropriate version of Theorem 6 holds here. Below we shall illustrate this with a few examples.

External Parameters A C^∞ family $X = X^\lambda(x)$ of vector fields on $\mathbb{R}^n$, with a multi-parameter $\lambda \in \mathbb{R}^p$, can be regarded as one C^∞ vector field on the product space $\mathbb{R}^n \times \mathbb{R}^p$. Such a vector field is *vertical*, in the sense that it has no components in the λ-direction. In other words, if $\pi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ is the natural projection on the second component, X is tangent to the fibers of π. It is easily seen that this property defines a Lie-subalgebra of the Lie-algebra of all C^∞ vector fields on $\mathbb{R}^n \times \mathbb{R}^p$. Again, by the linearity of this projection, the gradings and splittings are compatible. The normal form transformations Φ preserve the parameter λ, i. e., $\Phi \circ \pi = \Phi$.

When studying a bifurcation problem, we often consider systems $X = X^\lambda(x)$ locally defined near $(x, \lambda) = (0, 0)$, considering series expansions both in x and in λ. Then, in the Nth order normalization $\Phi_\star X$, the normalized part consists of a polynomial in y and λ, while the remainder term is of the form $O(|y|^{N+1} + |\lambda|^{N+1})$.

As in the previous case, we shall not formulate the present analogue of Theorem 6 for this case, but illustrate its meaning in examples.

The Reversible Case In the reversible case a linear involution R is given, while for the vector fields we require $R_\star X = -X$. Let $\mathcal{X}_R$ denote the class of all such reversible vector fields. Also, let C denote the class of all X such that $R_\star X = X$. Then, both $\mathcal{X}_R$ and C are linear spaces of vector fields. Moreover, C is a Lie-subalgebra. Associated to C is the group of diffeomorphisms that commute with R, i. e., the R-*equivariant* transformations. Also it is easy to see that for each of these diffeomorphisms Φ one has $\Phi_\star(\mathcal{X}_R) \subset \mathcal{X}_R$. The above approach applies to this situation in a straightforward manner. The gradings and splittings fit, while we have to choose the infinitesimal generator P from the set C. For details compare with [29].

Remark In the case with parameters, it sometimes is possible to obtain an alternative normal form where the normalized part is polynomial in y alone, with coefficients that depend smoothly on λ. A necessary condition for this is that the origin $y = 0$ is an equilibrium for all values of λ in some neighborhood of 0. To be precise, at the Nth order we van achieve smooth dependence of the coefficients on λ for $\lambda \in \Lambda_N$, where Λ_N is a neighborhood of $\lambda = 0$, that may shrink to $\{0\}$ as $N \to \infty$. So, for $N \to \infty$ only the formal aspect remains, as is the case in the above approach. This alternative normal form can be obtained by a proper use of the Implicit Function Theorem in the spaces $H^m(\mathbb{R}^n)$; for details e. g., see Section 2.2 in Vanderbauwhede [87]. For another discussion on this topic, cf. Section 7.6.2 in Dumortier et al. [16].

In Sect. "The Normal Form Procedure" the role of symmetry was considered regarding the semisimple part of the matrix A. A question is how this discussion generalizes to Lie-subalgebra's of vector fields.

Example (Volume preserving, parameter dependent axial symmetry [7,8]*)* On $\mathbb{R}^3$ consider a 1-parameter family X^λ of vector fields, preserving the standard volume $\sigma = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$. Assume that $X^0(0) = 0$ while the spectrum of $D_0 X^0$ consists of the eigenvalues $\pm i$ and 0. For the moment regarding λ as an extra state space coordinate, we obtain a vertical vector field on $\mathbb{R}^4$ and we apply a combination of the above considerations. The 'generic' Jordan normal form then is

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

with an obvious splitting in semisimple and nilpotent part. The considerations of Sect. "The Normal Form Procedure" then directly apply to this situation. For any N this yields a transformation $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$, with $\Phi(0) = 0$, preserving both the projection to the 1-dimensional parameter space and the volume of the 3-dimensional phase space, such that the normalized, Nth order part of $\Phi_\star X(y, \lambda)$, in cylindrical coordinates $y_1 = r\cos\varphi$, $y_2 = r\sin\varphi$, $y_3 = z$, has the rotationally symmetric form

$$\begin{aligned} \dot\varphi &= f(r^2, z, \lambda) \\ \dot r &= r g(r^2, z, \lambda) \\ \dot z &= h(r^2, z, \lambda)\,, \end{aligned}$$

again, for suitable polynomials f, g and h. Note, that in cylindrical coordinates the volume has the form $\sigma = r\mathrm{d}r \wedge \mathrm{d}\varphi \wedge \mathrm{d}z$. Again the functions f, g and h have to fit with the linear part. In particular we find that $h(r^2, z, \lambda) = \lambda + az + \cdots$, observing that for $\lambda \neq 0$ the origin is no equilibrium point.

Remark If $A = A_s + A_n$ is the canonical splitting of A in $H^1(\mathbb{R}^n) = \mathrm{gl}(n, \mathbb{R})$, then automatically both A_s and A_n are in the subalgebra under consideration. In the volume preserving setting this can be seen directly. In general the same holds true as soon as the corresponding linear Lie-group is algebraic, see [7,8] and the references given there.

The Hamiltonian Case

The Normal Form Theory in the Hamiltonian case goes back at least to Poincaré [69] and Birkhoff [5]. Other references are, for instance, Gustavson [47], Arnold [1,2,3], Sanders, Verhulst and Murdock [75], Van der Meer [88], Broer, Chow and Kim [17]. In Sect. "Preservation of Structure" we already saw that the axiomatic Lie algebra approach of [7,8] applies here, especially since the symplectic group $\mathrm{SP}(2n, \mathbb{R})$ is algebraic. The canonical form here usually goes with the name Williamson, compare Galin [44], Koçak [54] and Hoveijn [49]. We discuss how the Lie algebra approach compares to the literature.

The Lie algebra of Hamiltonian vector fields can be associated to the Poisson-algebra of Hamilton functions as follows, even in an arbitrary symplectic setting. As before, the symplectic form is denoted by σ. We recall, that for any Hamiltonian H the corresponding Hamiltonian vector field X_H is given by $\mathrm{d}H = \sigma(X_H, .)$, Now, let H and K be Hamilton functions with corresponding vector fields X_H resp. X_K. Then

$$X_{\{H,K\}} = [X_H, X_K]\,,$$

implying that the map $H \mapsto X_H$ is a morphism of Lie algebra's. By definition this map is surjective, while its kernel consists of the (locally) constant functions.

This implies, that the normal form procedure can be completely rephrased in terms of the Poisson-bracket. We shall now demonstrate this by an example, similar to the previous one.

Example (Symplectic parameter dependent rotational symmetry [17]*)* Consider $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) and the standard symplectic form $\sigma = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2$, considering a C^∞ family of Hamiltonian functions H^λ, where $\lambda \in \mathbb{R}$ is a parameter. The fact that $\mathrm{d}H = \sigma(X_H, .)$ in coordinates means

$$\dot x_j = \frac{\partial}{H}\partial y_j\,, \quad \dot y_j = -\frac{\partial}{H}\partial x_j\,,$$

for $j = 1, 2$. We assume that for $\lambda = 0$ the origin of $\mathbb{R}^4$ is a singularity. Then we expand as a Taylor series in (x, y, λ)

$$H^\lambda(x, y) = H_2(x, y, \lambda) + H_3(x, y, \lambda) + \ldots\,,$$

where the H_m is homogeneous of degree m in (x, y, λ). It follows for the corresponding Hamiltonian vector fields that

$$X_{H_m} \in H^{m-1}(\mathbb{R}^4)\,,$$

in particular $X_{H_2} \in \mathrm{sp}(4, \mathbb{R}) \subset H^1(\mathbb{R}^4)$. Let us assume that this linear part X_{H_2} at $(x, y, \lambda) = (0, 0, 0)$ has eigenvalues $\pm i$ and a double eigenvalue 0. One 'generic' Willamson's normal form then is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

compare with (2), corresponding to the quadratic Hamilton function

$$H_2(x, y, \lambda) = I + \frac{1}{2}{y_2}^2 ,$$

where $I := \frac{1}{2}({x_1}^2 + {y_1}^2)$. It is straightforward to give the generic matrix for the linear part in the extended state space $\mathbb{R}^5$, see above, so we will leave this to the reader.

The semisimple part $A_s = X_I$ now can be used to obtain a rotationally symmetric normal form, as before. In fact, for any $N \in \mathbb{N}$ there exists a canonical transformation $\Phi(x, y, \lambda)$, which keeps the parameter fixed, and a polynomial $F(I, x_2, y_2, \lambda)$, such that

$$(H \circ \Phi^{-1})(x, y, \lambda) = F(I, x_2, y_2, \lambda) + O(I + x_2^2 + y_2^2 + \lambda^2)^{(N+1)/2} .$$

Instead of using the adjoint action ad_A on the spaces $H^{m-1}(\mathbb{R}^5)$, we also may use the adjoint action

$$\mathrm{ad} H_2 : f \mapsto \{H_2, f\} ,$$

where f is a polynomial function of degree m in $(x_1, y_1, x_2, y_2, \lambda)$. In the vector field language, we choose the 'good' space $G^{m-1} \subset \ker \mathrm{ad}_{m-1} A_s$, which, in the function language, translates to a 'good' subset of $\ker \mathrm{ad} I$.

Whatsoever, the normalized part of the Hamilton function $H \circ \Phi^{-1}$, viz. the vector field $\Phi_\star X_H = X_{H\circ\Phi^{-1}}$, is rotationally symmetric. The fact that the Hamilton function F Poisson-commutes with I exactly amounts to invariance under the action generated by the vector field X_I, in turn implying that I is an integral of X_F. Indeed, if we define a 2π-periodic variable φ as follows:

$$x_1 = \sqrt{2I} \sin \varphi , \quad y_1 = \sqrt{2I} \cos \varphi ,$$

then $\sigma = \mathrm{d}I \wedge \mathrm{d}\varphi + \mathrm{d}x_2 \wedge \mathrm{d}y_2$, implying that the normalized vector field X_F has the canonical form

$$\dot{I} = 0, \quad \dot{\varphi} = -\frac{\partial F}{\partial I} ,$$
$$\dot{x}_2 = \frac{\partial F}{\partial y_2}, \quad \dot{y}_2 = -\frac{\partial F}{\partial x_2} .$$

Notice that φ is a cyclic variable, making the fact that I is an integral clearly visible. Also observe that $\dot{\varphi} = -1 + \cdots$. As before, and as in, e. g., the central force problem, this enables a reduction to lower dimension. Here, the latter two equations constitute the *reduction to 1 degree of freedom*: it is a family of planar Hamiltonian vector fields, parametrized by I and λ.

Remark

- This example has many variations. First of all it can be simplified by omitting parameters and even the zero eigenvalues. The conclusion then is that a planar Hamilton function with a nondegenerate minimum or maximum has a formal rotational symmetry, up to canonical coordinate changes.
- It also can be easily made more complicated, compare with Proposition 7 in Sect. "The Normal Form Procedure". Given an equilibrium with purely imaginary eigenvalues $\pm i\omega_1, \pm i\omega_2, \dots, \pm i\omega_m$ and with $2(n-m)$ zero eigenvalues. Provided that there are no resonances up to order $N + 1$, see (1), also here we conclude that Hamiltonian truncation at the order N, by canonical transformations can be given the form

$$F(I_1, \dots, I_m, x_{m+1}, y_{m+1} \dots, x_n, y_n) ,$$

where $I_j := \frac{1}{2}({x_j}^2 + {y_j}^2)$. As in Proposition 7 this normal form has a toroidal symmetry. Writing $x_j = \sqrt{2I_j} \sin \varphi_j$, $y_j = \sqrt{2I_j} \cos \varphi_j$, we obtain the canonical system of equations

$$\dot{I}_j = 0, \quad \dot{\varphi}_j = -\frac{\partial F}{\partial I_j} ,$$
$$\dot{x}_\ell = \frac{\partial F}{\partial y_l}, \quad \dot{y}_\ell = -\frac{\partial F}{\partial x_l} ,$$

$1 \le j \le m, \quad m + 1 \le l \le n$. Note that $\dot{\varphi}_j = -\omega_j + \cdots$. In the case $m = n$ we deal with an elliptic equilibrium. The corresponding result usually is named the Birkhoff normal form [1,5,47], ► Dynamics of Hamiltonian Systems. Then, the variables (I_j, φ_j), $1 \le j \le m$, are a set of *action-angle variables* for the truncated part of order N, [1].
- Returning to algorithmic issues, in addition to Subsect. "On the Choices of the Complementary Space and of the Normalizing Transformation", we give a few further references in cases where a structure is being preserved: Broer [7], Deprit [40,41], Meyer [57] and Ferrer, Hanßmann, Palacián and Yanguas [43]. It is to be noted that this kind of Hamiltonian result also can be obtained, where the coordinate changes are constructed using *generating functions*, compare with, e. g., [1,3,77], see also the discussion in Broer, Hoveijn, Lunter and Vegter [20] and in particular Section 4.4.2 in [23]. For another, computationally very effective normal form algorithm, see Giorgilli and Galgani [45]. Also compare with references therein.

Semi-local Normalization

This section roughly consists of two parts. To begin with, a number of subsections are devoted to related formal normal form results, near fixed points of diffeomorphisms and near periodic solutions and invariant tori of vector fields.

A Diffeomorphism Near a Fixed Point

We start by formulating a result by Takens:

Theorem 8 (Takens normal form [83]) *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ diffeomorphism with $T(0) = 0$ and with a canonical decomposition of the derivative $D_0 T = S + N$ in semisimple resp. nilpotent part. Also, let $N \in \mathbb{N}$ be given, then there exists diffeomorphism Φ and a vector field X, both of class C^∞, such that $S_\star X = X$ and*

$$\Phi^{-1} \circ T \circ \Phi = S \circ X_1 + O(|y|^{N+1}) .$$

Here, as before, X_1 denotes the flow over time 1 of the vector field X. Observe that the vector field X necessarily has the origin as an equilibrium point. Moreover, since $S_\star X = X$, the vector field X is invariant with respect to the group generated by S.

The proof is a bit more involved than Theorem 6 in Sect. "The Normal Form Procedure", but it has the same spirit, also compare with [37]. In fact, the Taylor series of T is modified step-by-step, using coordinate changes generated by homogeneous vector fields of the proper degree.

After a reduction to center manifolds the spectrum of S is on the complex unit circle and Theorem 8 especially is of interest in cases where this spectrum consists of roots of unity, i. e., in the case of resonance. Compare with [32,37,83].

Again, the result is completely phrased in terms of Lie algebra's and groups and therefore bears generalization to many contexts with a preserved structure, compare Sect. "Preservation of Structure". The normalizing transformations of the induction process then are generated from the corresponding Lie algebra. For a symplectic analogue see Moser [59]. Also, both in Broer, Chow and Kim [17] and in Broer and Vegter [14], symplectic cases with parameters are discussed, where $S = \mathrm{Id}$, the Identity Map, resp. $S = -\mathrm{Id}$, the latter involving a period-doubling bifurcation.

Remark Let us consider a symplectic map T of the plane, which means that T preserves both area and orientation. Assume that T is fixing the origin, while the eigenvalues of $S = D_0 T$ are on the unit circle, without being roots of unity. Then S generates the rotation group $\mathrm{SO}(2, \mathbb{R})$, so the vector field X, which has divergence zero, in this case is rotationally symmetric. Again this result often goes with the name of Birkhoff.

Near a Periodic Solution

The Normal Form Theory at a periodic solution or closed orbit has a lot of resemblance to the local theory we met before.

To fix thoughts, let us consider a C^∞ vector field of the form

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) , \end{aligned} \tag{4}$$

with $(x, y) \in \mathbb{T}^1 \times \mathbb{R}^n$. Here $\mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Assuming $y = 0$ to be a closed orbit, we consider the formal Taylor series with respect to y, with x-periodic coefficients. By Floquet Theory, we can assume that the coordinates (x, y) are such that

$$f(x, y) = \omega + O(|y|), \quad g(x, y) = \Omega y + O(|y|^2) ,$$

where $\omega \in \mathbb{R}$ is the frequency of the closed orbit and $\Omega \in \mathrm{gl}(n, \mathbb{R})$ its Floquet matrix. Again, the idea is to 'simplify' this series further. To this purpose we introduce a grading as before, letting $H^m = H^m(\mathbb{T}^1 \times \mathbb{R}^n)$ be the space of vector fields

$$Y(x, y) = L(x, y)\frac{\partial}{\partial x} + \sum_{j=1}^{n} M_j(x, y)\frac{\partial}{\partial y_j} ,$$

with $L(x, y)$ and $M(x, y)$ homogeneous in y of degree $m - 1$ resp. m. Notice, that this space H^m is infinite-dimensional. However, this is not at all problematic for the things we are doing here. By this definition, we have that

$$A := \omega\frac{\partial}{\partial x} + \Omega y\frac{\partial}{\partial y}$$

is a member of H^1 and with this normally linear part we can define an adjoint representation $\mathrm{ad}A$ as before, together with linear maps

$$\mathrm{ad}_m A : H^m \to H^m .$$

Again we assume to have a decomposition

$$G^m \oplus \mathrm{Im}(\mathrm{ad}_m A) = H^m ,$$

where the aim is to transform the terms of the series successively into the G^m, for $m = 2, 3, 4, \ldots$.

The story now runs as before. In fact, the proof of Theorem 6 in Sect. "The Normal Form Procedure", as well as its Lie algebra versions indicated in Sect. "Preservation of Structure", can be repeated almost verbatim for this case. Moreover, if $\Omega = \Omega_s + \Omega_n$ is the canonical splitting in semisimple and nilpotent part, then

$$\omega \frac{\partial}{\partial x} + \Omega_s y \frac{\partial}{\partial} y$$

gives the semisimple part of $\mathrm{ad}_m A$, as can be checked by a direct computation. From this computation one also deduces the non-resonance conditions needed for the present torus-symmetric analogue of Proposition 7 in Sect. "The Normal Form Procedure".

There are different cases *with* resonance either between the imaginary parts of the eigenvalues of Ω (normal resonance) or between the latter and the frequency ω (normal-internal resonance). All of this extends to the various settings with preservation of structure as discussed before. In all cases direct analogues of the Theorems 6 and 8 are valid. General references in this direction are Arnold [2,3], Bruno [35,36], Chow, Li and Wang [37], Iooss [50], Murdock [60] or Sanders, Verhulst and Murdock [75].

Remark

- This approach also is important for non-autonomous systems with periodic time dependence. Here the normalization procedure includes averaging. As a special case of the above form, we obtain a system

$$\begin{aligned} \dot{x} &= \omega \\ \dot{y} &= g(x, y) , \end{aligned}$$

so where $x \in \mathbb{T}^1$ is proportional to the time. Apart from the general references given above, we also refer to, e. g., Broer and Vegter [14] and to Broer, Roussarie and Simó [9,19]. The latter two applications also contain parameters and deal with bifurcations. Also compare with Verhulst ► Perturbation Analysis of Parametric Resonance.
- A geometric tool that can be successfully applied in various resonant cases is a *covering space*, obtained by a Van der Pol transformation (or by passing to co-rotating coordinates). This involves equivariance with respect to the corresponding deck group. This setting (with or without preservation of structure) is completely covered by the general Lie algebra approach as described above.
For the Poincaré map this deck group symmetry directly yields the normal form symmetry of Theorem 8. For applications in various settings, with or without preservation of structure, see [14,24,32]. This normalization technique is effective for studying bifurcation of subharmonic solutions. In the case of period doubling the covering space is just a double cover. In many cases Singularity Theory turns out to be useful.

Near a Quasi-periodic Torus

The approach of the previous subsection also applies at an invariant torus, provided that certain requirements are met. Here we refer to Braaksma, Broer and Huitema [15], Broer and Takens [12] and Bruno [35,36].

Let us consider a C^∞-system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \tag{5}$$

as before, with $(x, y) \in \mathbb{T}^m \times \mathbb{R}^n$. Here $\mathbb{T}^m = \mathbb{R}^m/(2\pi\mathbb{Z})^m$. We assume that $f(x, y) = \omega + O(|y|)$, which implies that $y = 0$ is an invariant m-torus, with on it a constant vector field with frequency-vector ω. We also assume that $g(x, y) = \Omega y + O(|y|^2)$, which is the present analogue of the Floquet form as known in the periodic case, with $\Omega \in \mathrm{gl}(n, \mathbb{R})$, independent of x. Contrarious to the situations for $m = 1$ and $n = 1$, in general reducibility to Floquet form is not possible. Compare with [10,15,18,38] and references therein. For a similar approach of a system that is not reducible to Floquet form, compare with [22].

Presently we assume this reducibility, expanding in formal series with respect to the variables y, where the coefficients are functions on $\mathbb{T}^m$. These coefficients, in turn, can be expanded in Fourier series. The aim then is, to 'simplify' this combined series by successive coordinate changes, following the above procedure. As a second requirement it then is needed that certain *Diophantine* conditions are satisfied on the pair (ω, Ω). Below we give more details on this. Instead of giving general results we again refer to [12,15,26,29,35,36]. Moreover, to fix thoughts, we present a simple example with a parameter. Here again a direct link with averaging holds.

Example (Toroidal symmetry with small divisors [38]*)* Given is a family of vector fields $\dot{x} = X(x, \lambda)$ with $(x, \lambda) \in \mathbb{T}^m \times \mathbb{R}$. We assume that $X = X(x, \lambda)$ has the form

$$X(x, \lambda) = \omega + f(x, \lambda) ,$$

with $f(x, 0) \equiv 0$. It is assumed that the frequency vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ has components that satisfy the Diophantine non-resonance condition

$$|\langle \omega, k \rangle| \geq \gamma |k|^{-\tau} , \tag{6}$$

for all $k \in \mathbb{Z} \setminus \{0\}$. Here $\gamma > 0$ and $\tau > n-1$ are prescribed constants. We note that $\tau > n-1$ implies that this condition in $\mathbb{R}^n = \{\omega\}$ excludes a set of Lebesgue measure $O(\gamma)$ as $\gamma \downarrow 0$, compare with [38]. It follows, that by successive transformations of the form

$$h\colon (x, \lambda) \mapsto (x + P(x, \lambda), \lambda)$$

the x-dependence of X can be pushed away to ever higher order in λ, leading to a formal normal form

$$\dot{\xi} = \omega + g(\lambda)\,,$$

with $g(0) = 0$. Observe that in this case 'simple' means x- (or ξ-) independent. Therefore, in a proper formalism, x-independent systems constitute the spaces G^m. Indeed, in the induction process we get

$$X(x, \lambda) = \omega + g_2(\lambda) + \cdots + g_{N-1}(\lambda) + f_N(x, \lambda) + O(|\lambda|^{N+1})\,,$$

compare the proof of Theorem 6 in Sect. "The Normal Form Procedure". Writing $\xi = x + P(x, \lambda)$, with $P(x, \lambda) = \tilde{P}(x, \lambda)\lambda^N$, we substitute

$$\begin{aligned}\dot{\xi} &= (\mathrm{Id} + D_\xi P)\dot{x}\\ &= \omega + g_2(\lambda) + \cdots + g_{N-1}(\lambda)\\ &\quad + f_N(x, \lambda) + D_x P(x, \lambda)\,\omega + O(|\lambda|^{N+1})\,,\end{aligned}$$

Where we express the right-hand side in x. So we have to satisfy an equation

$$D_x P(x, \lambda)\omega + f_N(x, \lambda) \equiv c\lambda^N \mathrm{mod} O(|\lambda|^{N+1})\,,$$

for a suitable constant c. Writing $f_N(x, \lambda) = \tilde{f}_N(x, \lambda)\lambda^N$, this amounts to

$$D_x \tilde{P}(x, \lambda)\omega = -\tilde{f}_N(x, \lambda) + c\,,$$

which is the present form of the homological equation. If

$$\tilde{f}_N(x, \lambda) = \sum_{k \in \mathbb{Z}^n} a_k(\lambda) \mathrm{e}^{i\langle x, k\rangle}\,,$$

then $c = a_0$, i. e., the m-torus average

$$a_0(\lambda) = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} \tilde{f}_N(x.\lambda)\mathrm{d}x\,.$$

Moreover,

$$\tilde{P}(x, \lambda) = \sum_{k \neq 0} \frac{a_k(\lambda)}{i\langle \omega, k\rangle} \mathrm{e}^{i\langle x, k\rangle}\,.$$

This procedure formally only makes sense if the frequencies $(\omega_1, \omega_2, \ldots, \omega_m)$ have no resonances, which means that for $k \neq 0$ also $\langle \omega, k\rangle \neq 0$. In other words this means that the components of the frequency vector ω are rationally independent. Even then, the denominator $i\langle \omega, k\rangle$ can become arbitrarily small, so casting doubt on the convergence. This problem of *small divisors* is resolved by the Diophantine conditions (6). For further reference, e. g., see [2,10,15,18,38]: for real analytic X, by the Paley–Wiener estimate on the exponential decay of the Fourier coefficients, the solution P again is real analytic. Also in the C^∞-case the situation is rather simple, since then the coefficients in both cases decay faster than any polynomial.

Remark

- The discussion at the end of Subsect. "Near a Periodic Solution" concerning normalization at a periodic solution, largely extends to the present case of a quasi-periodic torus. Assuming reducibility to Floquet form, we now have a normally linear part

  $$\omega \frac{\partial}{\partial x} + \Omega y \frac{\partial}{\partial y}\,,$$

  with a frequency vector $\omega \in \mathbb{R}^n$. As before $\Omega \in \mathrm{gl}(m, \mathbb{R})$. For the corresponding KAM Perturbation Theory the Diophantine conditions (6) on the frequencies are extended by including the normal frequencies, i. e., the imaginary parts $\omega_1^N, \ldots, \omega_s^N$ of the eigenvalues of Ω. To be precise, for $\gamma > 0$ and $\tau > n-1$, above the conditions (6), these extra Melnikov conditions are given by

  $$\begin{aligned}|\langle \omega, k\rangle + \omega_j^N| &\geq \gamma |k|^{-\tau}\\ |\langle \omega, k\rangle + 2\omega_j^N| &\geq \gamma |k|^{-\tau}\\ |\langle \omega, k\rangle + \omega_j^N + \omega_\ell^N| &\geq \gamma |k|^{-\tau}\,,\end{aligned} \qquad (7)$$

  for all $k \in \mathbb{Z}^n \setminus \{0\}$ and for $j, \ell = 1, 2, \ldots, s$ with $\ell \neq j$. See below for the description of an application to KAM Theory (and more references) in the Hamiltonian case.
- In this setting normal resonances can occur between the ω_j^N and normal-internal resonances between the ω_j^N and ω. Certain strong normal-internal resonances occur when one of the left-hand sides of (7) vanishes. For an example see below. Apart from these now also internal resonances between the components of ω come into play. The latter generally lead to destruction of the invariant torus.
- As in the periodic case also here covering spaces turn out to be useful for studying various resonant

bifurcation scenarios often involving applications of both Singularity Theory and KAM Theory. Compare with [15,25,31,33].

Non-formal Aspects

Up to this moment (almost) all considerations have been formal, i. e., in terms of formal power series. In general, the Taylor series of a C^∞-function, say, $\mathbb{R}^n \to \mathbb{R}$ will be divergent. On the other hand, any formal power series in n variables occurs as the Taylor series of some C^∞-function. This is the content of a theorem by É. Borel, cf. Narasimhan [61].

We briefly discuss a few aspects regarding convergence or divergence of the normalizing transformation or of the normalized series. We recall that the growth rate of the formal series, including the convergent case, is described by the Gevrey index, compare with, e. g., [4,55,72,73].

Normal Form Symmetry and Genericity

For the moment we assume that all systems are of class C^∞. As we have seen, if the normalization procedure is carried out to some finite order N, the transformation Φ is a real analytic map. If we take the limit for $N \to \infty$, we only get formal power series $\hat{\Phi}$, but, by the Borel Theorem, a 'real' C^∞-map Φ exists with $\hat{\Phi}$ as its Taylor series.

Let us discuss the consequences of this, say, in the case of Proposition 7 in Sect. "The Normal Form Procedure". Assuming that there are no resonances at all between the ω_j, as a corollary, we find a C^∞-map $y = \Phi(x)$ and a C^∞ vector field $p = p(y)$, such that:

- The vector field $\Phi_\star X - p$, in corresponding generalized cylindrical coordinates has the symmetric C^∞-form
$$\begin{aligned} \dot{\varphi}_j &= f_j(r_1^2, \dots, r_m^2, z_{n-2m+1}, \dots, z_n) \\ \dot{r}_j &= r_j g_j(r_1^2, \dots, r_m^2, z_{n-2m+1}, \dots, z_n) \\ \dot{z}_\ell &= h_\ell(r_1^2, \dots, r_m^2, z_{n-2m+1}, \dots, z_n)\,, \end{aligned}$$
where $f_j(0) = \omega_j$ and $h_\ell(0) = 0$ for $1 \le j \le m$, $n - 2m + 1 \le \ell \le n$.
- The Taylor series of p identically vanishes at $y = 0$.

Note, that an ∞-ly flat term p can have component functions like e^{-1/y_1^2}. We see, that the m-torus symmetry only holds up to such flat terms. Therefore, this symmetry, if present at all, also can be destroyed again by a *generic* flat 'perturbation'. We refer to Broer and Takens [12], and references therein, for further consequences of this idea. The main point is, that by a Kupka–Smale argument, which generically forbids so much symmetry, compare with [67].

Remark The Borel Theorem also can be used in the reversible, the Hamiltonian and the volume preserving setting. In the latter two cases we exploit the fact that a structure preserving vector field is generated by a function, resp. an $(n-2)$-form. Similarly the structure preserving transformations have such a generator. On these generators we then apply the Borel Theorem. Many Lie algebra's of vector fields have this 'Borel Property', saying that a formal power series of a transformation can be represented by a C^∞ map in the same structure preserving setting.

On Convergence

The above topological ideas also can be pursued in many real analytic cases, where they imply a generic *divergence* of the normalizing transformation. For an example in the case of the Hamiltonian Birkhof normal form [5,47,77] compare with Broer and Tangerman [13] and its references. As an example we now deal with the linearization of a holomorphic germ in the spirit of Sect. "The Normal Form Procedure".

Example (Holomorphic linearization [34,58,92]*)* A holomorphic case concerns the linearization of a local holomorphic map $F\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the form $F(z) = \lambda z + f(z)$ with $f(0) = f'(0) = 0$. The question is whether there exists a local biholomorphic transformation $\Phi\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that

$$\Phi \circ F = \lambda \cdot \Phi\,.$$

We say that Φ linearizes F near its fixed point 0. A formal solution as in Sect. "The Normal Form Procedure" generally exists for all $\lambda \in \mathbb{C} \setminus \{0\}$, not equal to a root of unity. The elliptic case concerns λ on the complex unit circle, so of the form $\lambda = e^{2\pi i\alpha}$, where $\alpha \notin \mathbb{Q}$, and where the approximability of α by rationals is of importance, cf. Siegel [76] and Bruno [34]. Siegel introduced a sufficient Diophantine condition related to (6), which by Bruno was replaced by a sufficient condition on the continued fraction approximation of α. Later Yoccoz [92] proved that the latter condition is both necessary and sufficient. For a description and further comments also compare with [2,10,42,58].

Remark

- In certain real analytic cases Neĭshtadt [63], by truncating the (divergent) normal form series appropriately, obtains a remainder that is exponentially small in the perturbation parameter. Also compare with,

e. g., [2,3,4,78]. For an application in the context of the Bogdanov–Takens bifurcations for diffeomorphisms, see [9,19]. It follows that chaotic dynamics is confined to exponentially narrow horns in the parameter space.

- The growth of the Taylor coefficients of the usually divergent series of the normal form and of the normalizing transformation can be described using Gevrey symptotics [4,55,70,71,72,73]. Apart from its theoretical interest, this kind of approach is extremely useful for computational issues also compare with [4,46,78, 79,80].

Applications

We present two main areas of application of the Normal Form Theory in Perturbation Theory. The former of these deals with more globally qualitative aspects of the dynamics given by normal form approximations. The latter class of applications concerns the Averaging Theorem, where the issue is that solutions remain close to approximating solutions given by a normal form truncation.

'Cantorized' Singularity Theory

We return to the discussion of the motivation in Sect. "Motivation", where there is a toroidal normal form symmetry up to a finite or infinite order. To begin with let us consider the quasi-periodic Hopf bifurcation [6,15], which is the analogue of the Hopf bifurcation for equilibria and the Hopf–Neĭmark–Sacker bifurcation for periodic solutions, in the case of quasi-periodic tori. For a description comparing the differences between all these cases we refer to [38]. For a description of the resonant dynamics in the resonant gaps see [30] and references therein. Apart from this, a lot of related work has been done in the Hamiltonian and reversible context as well, compare with [25,26,29,48].

To be more definite, we consider families of systems defined on $\mathbb{T}^m \times \mathbb{R}^m \times \mathbb{R}^{2p} = \{x, y, z\}$, endowed with the symplectic form $\sigma = \sum_{j=1}^m \mathrm{d}x_j \wedge \mathrm{d}y_j + \mathrm{d}z^2$. We start with 'integrable', i. e., x-independent, families of systems of the form

$$\begin{aligned} \dot{x} &= f(y, z; \lambda) \\ \dot{y} &= g(y, z; \lambda) \\ \dot{z} &= h(y, z; \lambda)\,, \end{aligned} \tag{8}$$

compare with (5), to be considered near an invariant m-torus $\mathbb{T}^m \times \{y_0\} \times \{z_0\}$, meaning that we assume $g(y_0, z_0) = 0 = h(y_0, z_0)$. The general interest is with the persistence of such a torus under nearly integrable perturbations of (8), where we include λ as a multiparameter. This problem belongs to the Parametrized KAM Theory [10,15,18] of which we sketch some background now. For y near y_0 and λ near 0 consider $\Omega(\lambda, y) = D_z h(y, z_0; \lambda)$, noting that $\Omega(\lambda, y) \in \mathrm{sp}(2p, \mathbb{R})$. Also consider the corresponding normal linear part

$$\omega(\lambda, y)\frac{\partial}{\partial x} + \Omega(\lambda, y) z \frac{\partial}{\partial z}\,.$$

As a first case assume that the matrix $\Omega(0, y_0)$ has only simple non-zero eigenvalues. Then a full neighborhood of $\Omega(0, y_0)$ in $\mathrm{sp}(2p, \mathbb{R})$ is completely parametrized by the eigenvalues of the matrices, where – in this symplectic case – we have to refrain from 'counting double'. We roughly quote a KAM Theorem as this is known in the present circumstances [15]. As a nondegeneracy condition assume that the product map

$$(\lambda, y) \mapsto (\omega(\lambda, y), \Omega(\lambda, y))$$

is a submersion near $(\lambda, y) = (0, y_0)$. Also assume all Diophantine conditions (6), (7) to hold. Then the parametrized system (8) is *quasi-periodically stable*, which implies persistence of the corresponding Diophantine tori near $\mathbb{T}^m \times \{y_0\} \times \{z_0\}$ under nearly integrable perturbations.

Remark

- A key concept in the KAM Theory is that of *Whitney smoothness* of foliations of tori over nowhere dense 'Cantor' sets. In real analytic cases even Gevrey regularity holds, and similarly when the original setting is Gevrey; compare with [71,90]. For general reference see [10,15,18] and references therein.
- The gaps in the 'Cantor sets' are centered around the various resonances. Their union forms an open and dense set of small measure, where perturbation series diverge due to small divisors. In each gap the considerations mentioned at the end of Subsects. "Near a Periodic Solution" and "Near a Quasi-Periodic Torus" apply. In [25,31,33,38] the differences between period and the quasi-periodic cases are highlighted.

Example (Quasi-periodic Hamiltonian Hopf bifurcation [26]) As a second case we take $p = 2$, considering the case of normal $1:-1$ resonance where the eigenvalues of $\Omega(0, y_0)$ are of the form $\pm i\mu_0$, for a positive μ_0. For simplicity we only consider the non-semisimple Williamson normal form

$$\Omega(0, y_0) \sim \begin{pmatrix} 0 & -\mu_0 & 1 & 0 \\ \mu_0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\mu_0 \\ 0 & 0 & \mu_0 & 0 \end{pmatrix},$$

where $\sim$ denotes symplectic similarity. The present format of the nondegeneracy condition regarding the product map $(\lambda, y) \mapsto (\omega(\lambda, y), \Omega(\lambda, y)$ is as before, but now it is required that the second component $(\lambda, y) \mapsto \Omega(\lambda, y)$ is a *versal unfolding* of the matrix $\Omega(0, y_0)$ in $\mathrm{sp}(4, \mathbb{R})$ with respect to the adjoint $\mathrm{SP}(4, \mathbb{R})$-action, compare with [2,26,27,29,44,49,54]. It turns out that a standard normalization along the lines of Sect. "Preservation of Structure" can be carried out in the z-direction, generically leading to a Hamiltonian Hopf bifurcation [88], characterized by its swallowtail geometry [86], which 'governs' families of invariant m-, $(m+1)$- and $(m+2)$-tori (the latter being Lagrangean). Here for the complementary spaces, see Sect. "The Normal Form Procedure", the $\mathrm{sl}(2, \mathbb{R})$-Theory is being used [39,60].

As before the question is what happens to this scenario when perturbing the system in a non-integrable way? In that case we need quasi-periodic Normal Form Theory, in the spirit of Sect. "Semi-Local Normalization". Observe that by the $1:-1$ resonance, difficulties occur with the third of the three Melnikov conditions (7). In a good set of Floquet coordinates the resonance can be written in the form $\omega_j^N + \omega_\ell^N = 0$. Nevertheless, another application of Parametrized KAM Theory [10,15,18,27] yields that the swallowtail geometry is largely preserved, when leaving out a dense union of resonance gaps of small measure. Here perturbation series diverge due to small divisors. What remains is a 'Cantorized' version of the swallowtail [26,27]. For a reversible analogue see [29,31].

Remark

- The example concerns a strong resonance and it fits in some of the larger gaps of the 'Cantor' set described in the former application of Parametrized KAM Theory. Apart from this, the previous remarks largely apply again. It turns out that in these and many other cases there is an infinite regression regarding the resonant bifurcation diagram.
- The combination of KAM Theory with Normal Form Theory generally has been very fruitful. In the example of Sect. "Applications" it implies that each KAM torus corresponds to a parameter value λ_0 that is the Lebesgue density point of such quasi-periodic tori. In the real analytic case by application of [63], it can be shown that the relative measure with non-KAM tori is exponentially small in $\lambda - \lambda_0$.
  Similar results in the real analytic Hamiltonian KAM Theory (often going by the name of exponential condensation) have been obtained by [51,52,53], also compare with [35,36], ► Diagrammatic Methods in Classical Perturbation Theory and with [3,10] and references therein.

On the Averaging Theorem

Another class of applications of normalizing-averaging is in the direction of the Averaging Theorem. There is a wealth of literature in this direction, that is not in the scope of the present paper, for further reading compare with [1,2,3,75] and references therein.

Example (A simple averaging theorem [1]) Given is a vector field

$$\dot{x} = \omega(y) + \varepsilon f(x, y)$$
$$\dot{y} = \varepsilon g(x, y)$$

with $(x, y) \in \mathbb{T}^1 \times \mathbb{R}^n$, compare with (4). Roughly following the recipe of the normalization process, a suitable near-identity transformation

$$(x, y) \mapsto (x, \eta)$$

of $\mathbb{T}^1 \times \mathbb{R}^n$ yields the following reduction, after truncating at order $O(\varepsilon^2)$:

$$\dot{\eta} = \varepsilon \bar{g}(\eta), \quad \text{where} \quad \bar{g}(\eta) = \frac{1}{2\pi}\int_0^{2\pi} g(x, \eta)\mathrm{d}x\,.$$

We now compare $y = y(t)$ and $\eta = \eta(t)$ with coinciding initial values $y(0) = \eta(0)$ as t increases. The Averaging Theorem asserts that if $\omega(\eta) > 0$ is bounded away from 0, it follows that, for a constant $c > 0$, one has

$$|y(t) - \eta(t)| < c\varepsilon, \quad \text{for all } t \quad \text{with } 0 \le t \le \frac{1}{\varepsilon}\,.$$

This theory extends to many classes of systems, for instance to Hamiltonian systems or, in various types of systems, in the immediate vicinity of a quasi-periodic torus. Further normalizing can produce sharper estimates that are polynomial in ε, while in the analytic case this even extends over exponentially long time intervals, usually known under the name of Nekhorochev estimates [64,65,66], for a description and further references also see [10], ► Nekhoroshev Theory. Another direction of generalization concerns passages through resonance, which in the example implies that the condition on $\omega(\eta)$ is no longer valid. We here mention [62], for further references and descriptions referring to [2,3] and to [75].

Future Directions

The area of research in Normal Form Theory develops in several directions, some of which are concerned with computational aspects, including the nilpotent case. Although for smaller scale projects much can be done with computer packages, for the large scale computations, e. g., needed in Celestial Mechanics, single purpose formula manipulators have to be built. For an overview of such algorithms, compare with [60,78,79,80] and references therein. Here also Gevrey asymptotics is of importance.

Another direction of development is concerned with applications in Bifurcation Theory, often in particular Singularity Theory. In many of these applications, certain coefficients in the truncated normal form are of interest and their computation is of vital importance. For an example of this in the Hamiltonian setting see [11], where the Giorgili–Galgani [45] algorithm was used to obtain certain coefficients at all arbitrary order in an efficient way. For other examples in this direction see [28,32].

Related to this is the problem how to combine the normal form algorithms as related to the present paper, with the polynomial normal forms of Singularity Theory. The latter have a universal (i. e., context independent) geometry in the product of state space and parameter space. A problem of relevance for applications is to pull-back the Singularity Theory normal form back to the original system. For early attempts in this direction see [20,23], which, among other things, involve Gröbner basis techniques.

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