Normal Linear Stability of Quasi-Periodic Tori

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Abstract. We consider families of dynamical systems having invariant tori that carry quasi-periodic motions. Our interest is the persistence of such tori under small, nearly-integrable perturbations. This persistence problem is studied in the dissipative, the Hamiltonian and the reversible setting, as part of a more general KAM theory for classes of structure preserving dynamical systems. This concerns the parametrized KAM theory as initiated by Moser [88, 89] and further developed in [67, 31, 28]. The corresponding nondegeneracy condition involves certain (trans-) versality conditions on the normal linear, leading, part at the invariant tori. We show that as a consequence, a Cantor family of Diophantine tori with positive Hausdorff measure is persistent under nearly-integrable perturbations. This result extends the above references since presently the case of multiple Floquet exponents is included. Our leading example is the normal $1 : -1$ resonance, which occurs a lot in applications, both Hamiltonian and reversible. As an illustration of this we briefly describe the Lagrange top coupled to an oscillator.

Keywords. KAM theory, nearly-integrable system, perturbation theory, quasi-periodic stability, (uni-) versal unfolding, linear centralizer unfolding, normal linear part, quasi-periodic bifurcations, Hamiltonian and reversible Hopf bifurcation, the Lagrange top.

1 Introduction

We consider dynamical systems, in particular, vector fields with invariant tori that carry quasi-periodic dynamics. These systems may depend on parameters and also may respect a given structure such as a symplectic 2-form, a volume form, or a symmetry, including reversibility, or combinations of these. Starting with an integrable family, the persistence of of such tori is investigated, when the perturbation is not necessarily integrable. This is a part of KAM theory, where it turns out that persistence is closely related to the notion of (trans-) versality with respect to the...
normal linear (leading) part at the tori. In [88, 89, 82, 83, 98, 99, 67, 31, 104] this persistence problem was addressed for the case where all Floquet exponents are simple. According to [67, 31, 28], the ‘majority’ of invariant tori of an integrable family of vector fields is persistent under small perturbation, whenever this family satisfies the following conditions:

- The Floquet exponents (or normal eigenvalues) are simple;
- The Floquet matrix is non-singular;
- The family depends on parameters in a generic way, only affecting the normal linear (leading) part;
- The internal and normal frequencies satisfy Diophantine conditions.

These various concepts will be explained below. The main purpose of the present paper is to generalize the result of [67, 31, 28], by extending it to the case of multiple Floquet exponents. We begin by briefly reviewing its setting, which is based on [88, 89, 98, 99] and on [64, 39]. Also see [82, 83, 104, 18] and [30, 32].

Our phase space will be a smooth, finite dimensional manifold $M$ and the parameter space $P \subset \mathbb{R}^p$ as an open and connected domain. Let $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ denote the standard $n$-torus group. We assume that $\mathbb{T}^n$ acts on $M$ by a smooth free action, that is, there is a smooth map $\Phi : \mathbb{T}^n \times M \to M$, such that for all $x \in M$,

- $\Phi(0, x) = x$;
- $\Phi(\theta_1, \Phi(\theta_2, x)) = \Phi(\theta_1 + \theta_2, x)$, for all $\theta_1, \theta_2 \in \mathbb{T}^n$;
- The induced map $\Phi(\cdot, x) : \mathbb{T}^n \to M$ is injective, for all $x \in M$.

The latter condition means that the action $\Phi$ is free. A vector field on $M$ is called integrable or $\mathbb{T}^n$-symmetric if it is equivariant with respect to the $\mathbb{T}^n$-action. Consider a smooth family $X = X_\mu(x)$ of $\mathbb{T}^n$-symmetric vector fields on $M$, where $\mu \in P$, with an invariant submanifold $T \subseteq M \times P$ which is assumed to have the form

$$T = \bigcup_{\mu \in P} T_\mu \times \{\mu\} \subseteq M \times P,$$

where $T_\mu \subseteq M$ is an $X_\mu$-invariant $\mathbb{T}^n$-orbit for each $\mu$. Since the action is free, each $T_\mu$ is diffeomorphic to $\mathbb{T}^n$, carrying conditionally periodic (or parallel) dynamics. It is known that in the Hamiltonian and reversible setting, the invariant tori generically occur as continua. However, in §§1.2 and 1.3 below, we shall see that one can always reduce to the above situation, where each parameter value
\( \mu \) corresponds to exactly one \( T^n \)-orbit \( T_\mu \). This can be achieved by introducing extra distinguished parameters that serve to single out individual invariant tori and introduce local coordinates around these, compare with [67, 31] §5a and [28] §2.1.

In the sequel we study the persistence of the invariant tori \( T_\mu \) when the integrable family \( X \) is perturbed into a not necessarily integrable one, usually referred to as nearly-integrable. The present setting is axiomatic, thereby including both the Lie algebra approach of [89, 31, 67] and the reversible case of [28], also compare with [64, 39]. See [32] for a survey, also compare with [30, 113]. We briefly discuss a few consequences of this approach. The ensuing parametrized KAM theory has many applications, in various symplectic, volume preserving and equivariant (including reversing) settings, or in combinations of these.

- The nondegeneracy condition as used in [31, 28], which is based on generic unfolding of the normal linear part, often is referred to as Broer-Huiitema-Takens (or BHT) nondegeneracy, which contains the classical Kolmogorov nondegeneracy condition as a special case.

- Although BHT nondegeneracy may need quite a lot of parameters, it leads to transparent proofs of KAM persistence results for individual systems, for instance in the Hamiltonian isotropic case under Rüssmann nondegeneracy conditions using Herman’s method. Here the geometric and algebraic properties of the Diophantine conditions are exploited, for a description see [30, 113, 32]. Here it should be mentioned that in the cases where the integrable invariant tori occur in continua, many of the parameters are distinguished, which means that these can be ‘compensated’ by phase space variables.

- On the other hand the parametrized KAM theory allows to develop typical models for quasi-periodic bifurcation in all the various settings, e.g., compare with [14, 30, 56]. For an extensive survey with many applications, we refer to [22]. Typicality here means that the persistence covers an open subset in the \( C^k \)-topology, for \( k \) large, including the cases \( C^\infty \) and real-analytic. For details see below.

- As said earlier, the main contribution of the present paper is generalization of the parametrized KAM theory to the case with multiple Floquet exponents. The present results include the cases of [31, 28] and the Lie algebra proof of [67, 31] is largely followed, where the in the linearization used for Newtonian iteration, the linear versal unfolding theory of matrices [4, 52, 50] is being used. The case of zero Floquet exponents is not covered here, but is dealt with in [121]; for case studies in this respect, also see, e.g., [14, 55, 23, 24, 120, 57].
In the presentation we shall treat three of the settings in detail, namely the dissipative, the Hamiltonian and the reversible setting, taking as a leading example the normal $1 : -1$ resonance, which in the reversible case coincides with the normal $1 : 1$ resonance. The corresponding nonlinear theory leads to quasi-periodic versions of the Hamiltonian and reversible Hopf bifurcations, compare with [20, 17, 64, 39].

For simplicity we stick to the real analytic setting, meaning that the manifold $M$, all families of vector fields $X = X_\mu(x), x \in M, \mu \in P \subset \mathbb{R}^\mu$ are assumed real analytic. The corresponding topology is the compact-open topology on complex-analytic extensions, for details see below. We note that the theory has extensions to the case of $C^k$, for $k \in \mathbb{N}$ sufficiently large [98], also compare with [31, 30].

**Notation**

We here collect a few notations needed later on. For a real-analytic manifold $N$, by $\mathcal{X}(N)$ and $\mathcal{X}_P(N)$ we denote the sets of all real-analytic vector fields on $N$ and the set of all real-analytic $p$-parameter families of vector fields on $N$, parametrized over $P$, respectively. For $m \in \mathbb{N}$, the set of all real $m \times m$ matrices is denoted by $\mathfrak{gl}(m, \mathbb{R})$ and the set of invertible matrices in $\mathfrak{gl}(m, \mathbb{R})$ by $GL(m, \mathbb{R})$. We write $I_m$ for the identity matrix in $GL(m, \mathbb{R})$. For vector fields $X; Y \in \mathcal{X}(N)$, the Lie bracket $[X; Y]$ is defined by $[X; Y](\phi) = Y(X(\phi)) - X(Y(\phi))$, for all smooth functions $\phi : N \to \mathbb{R}$, which is in accordance with [1, 117]. Throughout we use $|\cdot|$ for maximum norm of vectors $u \in \mathbb{R}^m$ or $\mathbb{C}^n$ (i.e., $|u| = \max |u_j|$) and similarly for the $\ell_1$-norm of integer vectors $u \in \mathbb{Z}^n$ (i.e., $|u| = \sum |u_j|$).

### 1.1 Normal linear stability: nearly-integrable dissipative case

Since the $\mathbb{T}^n$-action on $M$ was assumed free, for all $\mu \in U \subset P$, where $U$ is an open domain, a full neighbourhood of the $\mathbb{T}^n$-orbit $T_\mu$ is diffeomorphic to $\mathbb{T}^n \times \mathbb{R}^m$, for $m \in \mathbb{Z}_{\geq 0}$. From now on we take $m > 0$. Moreover, such a neighbourhood can be given coordinates $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{T}^n \times \mathbb{R}^m$, so with the angular variables $x_j (j = 1, \ldots, n)$ counted modulo $2\pi \mathbb{Z}$, such that

$$T_\mu = \{(x, y, \mu) \in \mathbb{T}^n \times \mathbb{R}^m \times U : y = 0\},$$

while the vector fields $\partial/\partial x_j (j = 1, \ldots, n)$ are the infinitesimal generators of the $\mathbb{T}^n$-action, see [67, 31]. Integrability now implies that, for each $\mu$, the vector field $X_\mu$ is independent of the angles $x$. Hence, the family $X = X_\mu(x, y)$ can be locally written as

$$\begin{cases} \dot{x} &= \omega(\mu) + O(|y|) \\ \dot{y} &= \Omega(\mu)y + O(|y|^2), \end{cases}$$
where $\omega(\mu) = (\omega_1(\mu), \ldots, \omega_n(\mu)) \in \mathbb{R}^n$ are the internal frequencies of the invariant torus $T_\mu$ and where $\Omega(\mu) \in \mathfrak{gl}(n, \mathbb{R})$ is the Floquet matrix. The eigenvalues of $\Omega(\mu)$ are called the Floquet exponents of the torus $T_\mu$. In vectorial shorthand notation, we write

$$X_\mu(x, y) = [\omega(\mu) + O(|y|)] \frac{\partial}{\partial x} + [\Omega(\mu)y + O(|y|^2)] \frac{\partial}{\partial y}. \tag{1}$$

**Definition 1.1** The family $X = X_\mu(x, y)$ as in (1) is nondegenerate at the invariant torus $T_0$, if the following is satisfied:

(a) The Floquet matrix $\Omega_0 = \Omega(0)$ is invertible;

(b) The local product map $\omega \times \Omega : (\mathbb{R}^p, 0) \rightarrow (\mathbb{R}^n \times \mathfrak{gl}(m, \mathbb{R}), (\omega(0), \Omega(0)))$ has the following property. The first component $\omega$ is submersive, while the second component $\Omega$ is a versal unfolding of $\Omega_0$ in $\mathfrak{gl}(m, \mathbb{R})$, in the sense of [4, 5, 6, 52].

**Remarks 1.2**

1. By the Inverse Function Theorem, the invertibility of $\Omega_0$ implies that, for each $\mu$ in a neighbourhood of 0, the vector field $X_\mu$ locally has exactly one single invariant torus $T_\mu$. Again by the Inverse Function Theorem, up to a $\mu$-dependent translation, we have $T_\mu \cong \mathbb{T}^n \times \{0\}$. From now on we shall assume this simplification to have taken place.

2. The nondegeneracy condition means that the normal linear, leading part

$$NX_\mu(x, y) = \omega(\mu) \frac{\partial}{\partial x} + \Omega(\mu)y \frac{\partial}{\partial y}$$

of $X$ is transversal to the conjugacy class of $NX_0$ in the space of normally affine vector fields. Such a generic condition plays a role in persistence results in the following sense. At the level of affine conjugacies and affine vector fields, the transversality condition provides the persistence of the tori $\{y = 0\}$ by the unfolding theorem [4, 5, 6, 52]. Without the invertibility of $\Omega_0$, this transversality property in general will be lost. As a result, the persistence of $X$-invariant tori is not guaranteed. For instance, consider $m = 1$ and

$$X_\mu(x, y) = \omega(\mu) \frac{\partial}{\partial x} + [\varepsilon \mu y + y^2] \frac{\partial}{\partial y},$$

where $\mu$ is in a small neighbourhood of $0 \in \mathbb{R}$ and $\varepsilon > 0$. Observe that the family $X$ has $V = \{y = 0\} \cup \{y = -\varepsilon \mu\}$ as invariant tori. The Floquet matrices $\Omega(\mu) = \varepsilon \mu$ form a versal unfolding of the zero matrix $\Omega_0 = 0$. But
\( V \) is not persistent under perturbations of \( X \). Indeed, for any small value \( \varepsilon > 0 \), a perturbed family \( \tilde{X} \) of the form

\[
\tilde{X}_\mu(x, y) = \omega(\mu) \frac{\partial}{\partial x} + \left[ \varepsilon \mu y + y^2 + \varepsilon \left( \frac{\varepsilon \mu^2}{4} + 1 \right) \right] \frac{\partial}{\partial y},
\]

has no (relative) equilibria. Hence, the versality of the matrix unfolding in this case does not give persistence of \( V \). We plan to come back to this problem in the future. For partial solutions see [14, 120, 121].

For the persistence of invariant tori, we restrict to a special class of quasi-periodic dynamics: we impose Diophantine conditions on the frequencies of the unperturbed integrable systems as follows.

**Diophantine conditions.** We denote by \( \omega^N(\mu) \) the array of the positive imaginary parts \( \omega^N_1(\mu), \ldots, \omega^N_r(\mu) \) of eigenvalues of \( \Omega(\mu) \). These positive parts are referred to as the normal frequencies of the torus \( T_\mu \). In contrast to the case of simple eigenvalues, see [67, 31], here the number \( r \) of normal frequencies in general may depend on \( \mu \). Let \( s \leq m/2 \) be the maximal number of these frequencies for \( \mu \in \mathcal{U} \). We introduce the generalized frequency map \( F : P \to \mathbb{R}^n \times \mathbb{R}^s \) defined by

\[
F(\mu) = (\omega(\mu), \omega^N(\mu), 0),
\]

where 0 denotes the zero vector of \( \mathbb{R}^s \). We abbreviate \( \tilde{\omega}^N(\mu) = (\omega^N(\mu), 0) \in \mathbb{R}^s \), and write \( F(\mu) = (\omega(\mu), \tilde{\omega}^N(\mu)) \) when \( s = r \). We say that the frequency vectors \( (\omega(\mu), \tilde{\omega}^N(\mu)) \) are \((\tau, \gamma)\)-Diophantine if the following holds. For a constant \( \tau > n - 1 \) and a ‘parameter’ \( \gamma > 0 \), we require that

\[
|\langle \omega(\mu), k \rangle + \langle \tilde{\omega}^N(\mu), \ell \rangle| \geq \gamma |k|^{-\tau}, \tag{2}
\]

for all \( k \in \mathbb{Z}^n \setminus \{0\} \) and for all \( \ell \in \mathbb{Z}^s \), with \(|\ell| \leq 2\).

**Remark 1.3** By \((\mathbb{R}^n \times \mathbb{R}^s)_{\tau, \gamma}\) we denote the set of all \( (\omega, \tilde{\omega}^N) \in \mathbb{R}^n \times \mathbb{R}^s \) subject to the Diophantine conditions (2). This set is a nowhere dense, uncountable union of closed half lines. It intersects the unit sphere \( \mathbb{S}^{n+s-1} \subset \mathbb{R}^n \times \mathbb{R}^s \) in a closed set, which by the Cantor-Bendixon theorem [60] is the union of a perfect set and a countable set. The complement of \((\mathbb{R}^n \times \mathbb{R}^s)_{\tau, \gamma} \cap \mathbb{S}^{n+s-1} \) in \( \mathbb{S}^{n+s-1} \) contains the dense resonant web which causes the perfect set to be totally disconnected and hence a Cantor set. In \( \mathbb{S}^{n+s-1} \) the measure of this Cantor set tends to full Lebesgue measure as \( \gamma \downarrow 0 \). Colloquially we call the set of frequencies \( (\omega(\mu), \tilde{\omega}^N(\mu)) \) satisfying (2) (and also the corresponding set of parameters \( \mu \in P \)) a ‘Cantor set’, i.e., a foliation of manifolds (with boundary) over a Cantor set.
For any subset \( U \subseteq P \), we introduce the ‘Cantor set’ \( \Gamma_{\tau,\gamma}(U) \) in \( U \) defined by
\[
\Gamma_{\tau,\gamma}(U) = \mathcal{F}^{-1}((\mathbb{R}^n \times \mathbb{R}^r)_{\tau,\gamma}) \cap U,
\]
This set \( \Gamma_{\tau,\gamma}(U) \) has positive Lebesgue measure \([89, 98, 99, 67, 31, 47]\). Also we need the subset \( \Gamma_{\tau,\gamma}(U') \subset \Gamma_{\tau,\gamma}(U) \), where the set \( U' \) is defined by
\[
U' = \{ \mu \in U : \text{dist.}(\mathcal{F}(\mu), \partial \mathcal{F}(U)) \geq \gamma \}.
\]
Note that if \( U \) is an open neighbourhood of \( 0 \in P \), then the subset \( U' \subset U \) is still an open neighbourhood of \( 0 \) for \( \gamma > 0 \) sufficiently small.

Now we are ready to announce our main KAM theorem in the dissipative setting, which reads

**Theorem 1.4** (NORMAL LINEAR STABILITY: NEARLY-INTEGRABLE DISSIPATIVE CASE) Let \( X \in \mathcal{X}_P(M) \) be a \( p \)-parameter real-analytic family of integrable vector fields given by (1). Assume that \( X \) is nondegenerate at the invariant central torus \( T_0 = \{(x, y, \mu) : y = 0, \mu = 0\} \). Then, for \( \gamma > 0 \) sufficiently small and for any real-analytic family \( \hat{X} \in \mathcal{X}_P(M) \) sufficiently close to \( X \) in the compact-open topology on complex analytic extensions, there exists a domain \( U \) around \( 0 \in P \) and a map
\[
\Phi : M \times U \to M \times P,
\]
defined near the invariant torus \( T_0 \), such that,

i. \( \Phi \) is a \( C^\infty \)-near-identity diffeomorphism onto its image;

ii. The image of the Diophantine tori \( V = \bigcup_{\mu \in \Gamma_{\tau,\gamma}(U')} (T_\mu \times \{\mu\}) \) under \( \Phi \) is \( \tilde{X} \)-invariant, and for the restriction \( \tilde{\Phi} = \Phi|_V \) we have
\[
\tilde{\Phi}_*(X) = \hat{X},
\]
that is, \( \tilde{\Phi} \) conjugates \( X \) to \( \hat{X} \);

iii. The restriction \( \Phi|_V \) preserves the normal linear part of \( X \).

The conclusion of Theorem 1.4 is referred to as normal linear stability of the family \( X \) at \((y, \mu) = (0, 0)\) in the space \( \mathcal{X}_P(M) \). We call the Diophantine torus union \( V \) (as well as its diffeomorphic image \( \Phi(V) \)) a Cantor family of invariant \( n \)-tori, since it is parametrized over a ‘Cantor set’ in a Whitney smooth sense \([123, 126, 127, 98]\), also compare with \([31, 30]\).

**Remarks 1.5**
1. Suppose that the family $X$ has a holomorphic extension to a complex domain $\mathcal{N}$. A compact-open neighbourhood of the family $X$ consists of families in $\mathcal{X}_p(M)$, which are close to $X$ in the supremum norm on all compacta in $\mathcal{N}$. We will specify such a neighbourhood in the proof of Theorem 1.4, see Appendix B.1 (b) below.

2. It turns out that in the allowed perturbation size of $\tilde{X} - X$ the positive Diophantine ‘parameter’ $\gamma$ shows up in a linear way, compare with formula (67) in Appendix B.1 (b). Measure theoretically the amount of $\tilde{X}$-invariant $n$-tori is larger for smaller $\gamma$, so the ‘game’ in applications of Theorem 1.4 (as well as its companion Theorems 1.6 and 1.8 below), is to take $\gamma > 0$ as small as the perturbation size allows. In fact, $\gamma$ can be chosen as a function of the distinguished and external parameters of $X$ in an appropriate way, tending to 0 at a boundary component of the domain.\footnote{In the Lagrangian KAM Theory near an elliptic Hamiltonian equilibrium, $\gamma$ is taken as a positive power of the distance to the equilibrium, which here acts as a distinguished parameter, leading to a ‘small twist’ version, compare with, e.g., [98].}

As a result for the perturbation $\tilde{X}$ sets of density points of quasi-periodicity can be obtained, in the sense of measure theory. Compare with, e.g., [31, 30]. In certain cases one also speaks of exponential condensation, for a discussion of these subjects and many references, see [32]. At the end of §5 we come back to this.

3. In the proof of Theorem 1.4 and all its counterparts, we use a Hölder condition on the spectra $\text{Spec} \Omega(\mu)$ of the Floquet matrices $\Omega(\mu)$. In fact, since the family $X = X_\mu(x, y)$ is real analytic in all variables, the map $\Omega : U \to \mathfrak{gl}(m, \mathbb{R})$ has a holomorphic extension to a complex domain

$$U + r_0 = \{ \tilde{\mu} \in \mathbb{C}^p : \exists \mu \in U \text{ such that } |\mu - \tilde{\mu}| \leq r_0 \} \subset \mathbb{C}^p \quad (4)$$

for a certain value $r_0 > 0$. Then for any $\tilde{\mu} \in U + r_0$ and $\mu \in U$ and for any $\hat{\lambda} \in \text{Spec} \Omega(\tilde{\mu})$, there exists an eigenvalue $\lambda \in \text{Spec} \Omega(\mu)$ such that

$$|\text{Im} \lambda - \text{Im} \hat{\lambda}| \leq L |\mu - \tilde{\mu}|^{1/m}. \quad (5)$$

See Appendix A, Theorem A.1. A full proof is included as well, which is closely related to [53] and based on an application of the Rouché lemma to the characteristic polynomial of $\Omega(\mu)$.

4. The present KAM theory allows generalizations to the world of $C^k$-systems endowed with the $C^k$-topology [62], for $k \in \mathbb{N}$ sufficiently large. Compare with [98] and for discussions on this subject also see, e.g., [31, 30, 67, 90, 

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To give an idea, for \( k > 4\tau + 2 \), the conjugation is at least of class \( C^{k-2\tau} \). Therefore in the \( C^\infty \)-case, no losses of differentiability occur and the conjugations also are of class \( C^\infty \). For an elaborate discussion and more references, compare with [32].

1.2 Normal linear stability: nearly-integrable symplectic case

In the symplectic setting, the even-dimensional manifold \( M \) (i.e., where \( n + m \) is even) is endowed with a symplectic 2-form \( \sigma \) [1, 8]. The infinitesimal generators of the free \( \mathbb{T}^n \)-action are assumed to be Hamiltonian with respect to \( \sigma \) in a full neighbourhood of the central \( n \)-torus \( T \), which is a \( \mathbb{T}^n \)-orbit.\(^2\) Then, the \( \mathbb{T}^n \)-orbit \( T \) is isotropic in the sense that the restriction \( \sigma|_T = 0 \) [122, 1]. By the generalized Darboux theorem [92, 1], near the submanifold \( T \), there exist local coordinates \((x, y, z) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q}\) such that \( T \) is given by

\[
T = \{(x, y, z) : (y, z) = (0, 0)\},
\]

where \( \sigma \) takes the local form

\[
\sigma = \sum_{i=1}^n dx_i \wedge dy_i + \sum_{j=1}^q dz_j \wedge dz_{q+j} = dx \wedge dy + dz^2. \tag{6}
\]

Moreover, the \( \mathbb{T}^n \)-action is infinitesimally generated by the vector fields \( \partial/\partial x_j \) \((j = 1, \ldots, n)\). Let us first consider a single integrable Hamiltonian vector field \( X = X_H \) on \((M, \sigma)\) with \( H : M \to \mathbb{R} \) as the corresponding Hamiltonian function and with \( T \) as invariant submanifold. Then, the local form of \( X \) is given by

\[
X(x, y, z) = \sum_{i=1}^n \frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i} + \sum_{i=1}^q \frac{\partial H}{\partial z_i} \frac{\partial}{\partial z_{q+i}} - \sum_{j=1}^q \frac{\partial H}{\partial z_{q+j}} \frac{\partial}{\partial z_j},
\]

in shorthand notation, where \( z_+ = (z_1, \ldots, z_q) \) and \( z_- = (z_{q+1}, \ldots, z_{2q}) \).

Since that the Hamiltonian \( H \) is constant along the \( \mathbb{T}^n \)-orbit \( T \), it follows that \( \partial H/\partial x \equiv 0 \). Hence, in the local coordinates \((x, y, z)\), the integrable Hamiltonian vector field \( X \) has the form

\[
X(x, y, z) = f(y, z) \frac{\partial}{\partial x} + h(y, z) \frac{\partial}{\partial z},
\]

\(^2\)If the corresponding \( \mathbb{T}^n \)-bundle has nontrivial monodromy, the actions in general are not globally Hamiltonian.
where \( h(0, 0) = 0 \) and where \( h(y, z) \frac{\partial}{\partial z} \) is Hamiltonian with respect to the 2-form \( dz^2 = \sum_{j=1}^{q} dz_j \wedge dz_{q+j} \). Under the assumption that \( \det(\partial h / \partial z)(0, 0) \neq 0 \), by the Implicit Function Theorem it follows that the equation \( h(y, z) = 0 \) has a continuum \( z = z(y) \), with \( z(0) = 0 \), of solutions, locally parametrized by \( y \). This zero-set corresponds to a continuum \( X \)-invariant tori all carrying conditionally periodic (or parallel) dynamics with frequency vector \( f(y, z(y)) \). After a suitable translation, this continuum, locally coincides with \( \{ z = 0 \} \). From now on we assume to be in this normalized situation. It follows that \( \partial h / \partial z(y, 0) \in \text{sp}(2q, \mathbb{R}) \) for all small values of \( y \). Recall that for \( A \in \text{gl}(2q, \mathbb{R}) \) one has that \( A \in \text{sp}(2q, \mathbb{R}) \) if and only if \( J_{2q}A \) is symmetric, where \( J_{2q} \) denotes the standard symplectic \( 2q \times 2q \)-matrix, that is,

\[
J_{2q} = \begin{pmatrix} 0 & I_q \\
-I_q & 0 \end{pmatrix}.
\]

compare with, e.g., [37, 1, 8].

Next, we introduce the multi-parameter \( \mu \in \mathbb{R}^p \) by considering a (real analytic) family \( X = X_\mu(x, y, z) \) of the form

\[
X_\mu(x, y, z) = f(y, z, \mu) \frac{\partial}{\partial x} + h(y, z, \mu) \frac{\partial}{\partial z},
\]

where now \( h(0, 0, \mu) \equiv 0 \). Proceeding as above for the case of individual vector fields and assuming that \( \det(\partial h / \partial z)(0, 0, 0) \neq 0 \), we again obtain a continuum of \( X \)-invariant \( n \)-tori normalized to \( \{ z = 0 \} \), locally parametrized by both \( y \) and \( \mu \). This means that \( h(y, 0, \mu) \equiv 0 \), where \( \partial h / \partial z(y, 0, \mu) \in \text{sp}(2q, \mathbb{R}) \) for each \( (y, \mu) \).

However, as in the dissipative case, see §1.1, we prefer to work with one \( n \)-torus per parameter value. To achieve this we define a distinguished parameter \( \nu \), that varies over a neighbourhood of \( 0 \in \mathbb{R}^n \), introducing a localized coordinate \( y_{\text{loc}} = y - \nu \). For this localization procedure compare with [67, 31]. This leads to the \( n \)-torus family

\[
T_{\text{loc}} = \{(x, y, z, \mu, \nu) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times \mathbb{R}^p \times \mathbb{R}^n \mid y = \nu, z = 0 \}
\]

\[
= \{(x, y_{\text{loc}}, z, \mu, \nu) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times \mathbb{R}^p \times \mathbb{R}^n \mid y_{\text{loc}} = 0, z = 0 \},
\]

where \( (\mu, \nu) \) varies over a neighbourhood of \( (0, 0) \in \mathbb{R}^p \times \mathbb{R}^n \), which is invariant for the extended integrable family \( X_{\mu, \nu}(x, y_{\text{loc}}, z) = X_\mu(x, y_{\text{loc}} + \nu, z) \). We so have

\[
X_{\mu, \nu}(x, y_{\text{loc}}, z) = f(y_{\text{loc}} + \nu, z, \mu) \frac{\partial}{\partial x} + h(y_{\text{loc}} + \nu, z, \mu) \frac{\partial}{\partial z},
\]
observing that $\sigma = dx \wedge dy_{\text{loc}} + \sum_{j=1}^{q} dz_j \wedge dz_{q+j}$. As before we abbreviate $\omega(\mu, \nu) = f(\nu, 0, \mu)$ and $\Omega(\mu, \nu) = (\partial h/\partial z)(\nu, 0, \mu)$.

Now, an analogue of Theorem 1.4 can be formulated for the extended integrable family $X = X_{\mu,\nu}(x, u, z)$ as in (10), regarding the invariant tori $T_{\text{loc}}$ as in (9). To this end, we need to adapt the nondegeneracy condition for the present setting: we say that the (extended) family $X = X_{\mu,\nu}$ is nondegenerate at the torus $T_{\text{loc}}^{0,0} = T_{\text{loc}} \cap \{(\mu, \nu) = (0, 0)\}$, if the following holds:

(a) The Floquet matrix $\Omega_0 = \Omega(0, 0)$ is invertible;

(b) The local product map $\omega \times \Omega : (\mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathfrak{sp}(2q, \mathbb{R}), (\omega(0, 0), \Omega(0, 0))$ has the following property. The first component $\omega$ is submersive, while the second component $\Omega$ is a versal unfolding of $\Omega(0, 0)$ in $\mathfrak{sp}(2q, \mathbb{R})$, in the sense of [4, 6, 50, 52].

Let $\mathcal{X}_{P_{\text{loc}}}^p(M)$ be the set of real-analytic families of Hamiltonian vector fields of $(M, \sigma)$, locally parametrized by $(\mu, \nu) \in P_{\text{loc}} = \mathbb{R}^p \times \mathbb{R}^n$. Then, we have the following Hamiltonian counterpart of Theorem 1.4.

**Theorem 1.6** (NORMAL LINEAR STABILITY: NEARLY-INTEGRABLE SYMPLECTIC CASE) Let $X \in \mathcal{X}_{P_{\text{loc}}}^p(M)$ be the real-analytic $(p + n)$-parameter family of Hamiltonian vector fields given by (10). Assume that $X$ is nondegenerate at the invariant torus $T_{\text{loc}}^{0,0} = \{(x, y_{\text{loc}}, z, \mu, \nu) : y_{\text{loc}} = 0, (\mu, \nu) = (0, 0)\}$. Then, for $\gamma > 0$ sufficiently small and for any $\tilde{X} \in \mathcal{X}_{P_{\text{loc}}}^p(M)$ sufficiently close to $X$ in the compact-open topology on complex analytic extensions, there exists a domain $U$ around $(0, 0) \in P_{\text{loc}}$ and a map

$$\Phi : M \times U \to M \times P_{\text{loc}},$$

defined near the torus $T_{\text{loc}}^{0,0}$, such that

i. $\Phi$ is a $C^\infty$-near-identity diffeomorphism onto its image;

ii. The image of the Diophantine tori $V = \bigcup_{(\mu, \nu) \in \Gamma_{\gamma}(U')} \left( T_{\text{loc}}^{\mu,\nu} \times \{(\mu, \nu)\} \right)$ under $\Phi$ is $\tilde{X}$-invariant, and for the restriction $\tilde{\Phi} = \Phi|_V$ we have

$$\tilde{\Phi}_*(X) = \tilde{X},$$

that is, $\tilde{\Phi}$ conjugates $X$ to $\tilde{X}$;

iii. The restriction $\tilde{\Phi}|_V$ is symplectic and preserves the (symplectic) normal linear part of $X$. 

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Remarks 1.7

1. Theorem 1.6 for the extended family \( X = X_{\mu, \nu} \) as in (10) induces a similar stability result for the non-extended case (i.e., for the family \( X = X_\mu \) given by (8)) via the projection \( M \times P_{loc} \to M \times P \) given by

\[
(x, y_{loc}, z, \mu, \nu) \mapsto (x, y_{loc} + \nu, z, \mu).
\]

However, in the non-extended case, the induced conjugacy between the integrable family \( X = X_\mu(x, y, z) \) and its perturbation \( \tilde{X} = \tilde{X}_\mu(x, y, z) \) in general is not symplectic, see [67, 31].

2. The symplectic normal linear part \( N^\nu X \) of the extended family \( X = X_{\mu, \nu} \) at \( V \) given by (10) is of the form

\[
N^\nu X(x, y, z) = \omega(\nu, \mu) \frac{\partial}{\partial x} + \Omega(\nu, \mu)z \frac{\partial}{\partial z}.
\]  \hspace{1cm} (11)

The situation is more involved here than in the dissipative case. For a discussion compare with [31] §6b; also see §3.3 and Appendix C below.

3. For the conclusions (ii) and (iii) of Theorem 1.6, it is sufficient to show that \( \Phi^* \tilde{X} = N X + O(|y|, |z|) \frac{\partial}{\partial y} + O(|y|, |z|) \frac{\partial}{\partial z} + O(|y|, |z|^2) \frac{\partial}{\partial z}. \) Indeed, first we observe that this relation implies that the Cantor family \( V \) of the \( X \)-invariant \( n \)-tori is invariant under \( \Phi^* \tilde{X} \). It follows that the image \( \tilde{V} = \Phi(V) \) is a Cantor family of the \( \tilde{X} \)-invariant tori. Secondly, the above relation also implies that the symplectic normal linear part of \( N^\nu \tilde{X} \) at \( \tilde{V} \) has the same form as \( N^\nu X \) at \( V \); see (11).

A leading example is the case where the Floquet matrix \( \Omega_0 = \Omega(0, 0) \) has a double pair of purely imaginary eigenvalues with a nontrivial nilpotent part, i.e., when \( \Omega_0 \) is in generic (or non-semisimple) \( 1 : -1 \) resonance. This case allows a direct application of Theorem 1.6. This generically involves a Hamiltonian Hopf bifurcation [79]. For a nonlinear treatment of the quasi-periodic Hamiltonian Hopf bifurcation see [20]. The generic \( 1 : -1 \) resonance is an interesting phenomenon, since it occurs in many classical mechanical systems including the Lagrange top [45, 42], the restricted three body problem [1, 34], etc., but also in celestial mechanics and astronomy [41, 96].

1.3 Normal linear stability: nearly-integrable reversible case

We now formulate the corresponding reversible \( \mathbb{KAM} \) Theory, which runs similar to the symplectic case of the previous section, and which is completely covered
by the approach of this paper. For other proofs and further details compare with
[28, 18, 39], also see [17, 40].

Let the manifold \( M \) again be the phase space. Now we are given an involution
\( G : M \to M \), i.e., such that \( G^2 = \text{Id}_M \), which is assumed to commute with the
free \( \mathbb{T}^n \)-action. A vector field \( X \) on \( M \) is said to be \( G \)-reversible whenever
\[
G_*(X) = -X,
\]
which expresses that \( G \) takes integral curves of \( X \) to integral curves of \( X \), reversing the
time-direction. Given an integrable, reversible vector field \( X \) on \( M \); as before let
\( T \) be a \( T^n \)-orbit, which then also is an \( X \)-invariant \( n \)-torus
\( \text{with} \ G(T) = T \). By Bochner’s theorem [15, 84] we then find local coordinates
\((x, y, z) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^q \), such that \( T = \{(x, y, z) : (y, z) = (0, 0)\} \) and \( G \)
takes the local form \((x, y, z) \mapsto (-x, y, Rz) \). Here \( R \in \text{GL}(2q, \mathbb{R}) \) is a linear
involution, taking the eigenvalue +1 with multiplicity \( q \), which implies that the
fixed point manifold of \( G \) has dimension \( m + q \); this is also expressed by saying
that \( G \) is of type \((n + q, m + q)\). For a discussion compare with [30], §1.3.2. Thus
the \( G \)-reversible vector field \( X \) obtains the format
\[
X(x, y, z) = \sum_j f_j(x, y, z) \frac{\partial}{\partial x_j} + \sum_k g_k(x, y, z) \frac{\partial}{\partial y_k} + \sum \ell h_{\ell}(x, y, z) \frac{\partial}{\partial z_\ell},
\]
compare with (8), where now (12) translates to
\[
\begin{align*}
  f(-x, y, Rz) & \equiv f(x, y, z) \\
  g(-x, y, Rz) & \equiv -g(x, y, z) \\
  h(-x, y, Rz) & \equiv -Rh(x, y, z).
\end{align*}
\]
It directly follows from (14) that in the integrable case \( g(y, z_0) \equiv 0 \), for any \( z_0 \)
with \( Rz_0 = z_0 \). This means that for any \( y_0 \in \mathbb{R}^m \), with \( h(y_0, z_0) = 0 \), the \( n \)-torus
\( \mathbb{T}^n \times \{y_0\} \times \{z_0\} \) is \( X \)-invariant. Observe that this torus also is \( G \)-invariant and
an orbit of the above \( \mathbb{T}^n \)-action, while the dynamics on this torus is conditionally
periodic with frequency vector \( \omega = f(y_0, z_0) \). For simplicity restricting to the
above case \( z_0 = 0 \), we again consider the equation \( h(y, z) = 0 \) for \((y, z)\) near
\((y_0, 0)\). Under the generic condition \( \det(\partial h/\partial z)(0, 0) \neq 0 \), the Implicit Function
Theorem provides us with a continuum \( z = z(y) \) of solutions where \( z(0) = 0 \). Translating
as in the symplectic case puts normalizes this continuum locally to \( \{z = 0\} \).

Once more we turn to a \( p \)-parameter (real-analytic) family of integrable
reversible vector fields
\[
X_\mu(x, y, z) = f(y, z, \mu) \frac{\partial}{\partial x} + g(y, z, \mu) \frac{\partial}{\partial y} + h(y, z, \mu) \frac{\partial}{\partial z}
\]
(15)
where \( h(y_0, 0, \mu_0) = 0 \), investigating invariant \( n \)-tori of the form

\[
T_{y_0, \mu_0} = \mathbb{T}^n \times \{y_0\} \times \{0\}
\]

for their behaviour under nearly-integrable perturbations of the family (15). As in the symplectic case we normalize in a parameter dependent way, assuming

\[
\det(\partial h/\partial y(y_0, 0, \mu_0)) \neq 0,
\]

so obtaining a continuum of \( X \)-invariant \( n \)-tori \( \{z = 0\} \), locally parametrized by \((y, \mu)\). This means that \( h(y, 0, \mu) = 0 \), where \( \Omega(\mu, \nu) = \partial h/\partial y(y, 0, \mu) \) is (infinitesimally) \( R \)-reversible in the sense that \( \Omega R = -R \Omega \). The space of all such reversible matrices is denoted by \( \mathfrak{gl}_{-R}(2q, \mathbb{R}) \). From here simplify by setting \((y_0, \mu_0) = (0, 0)\).

We next localize as in the symplectic case, introducing the distinguished parameter \( \nu \in \mathbb{R}^m \) and putting \( y_{loc} = y - \nu \), compare with [67, 31]. This leads to the \( n \)-torus family

\[
T_{loc} = \{(x, y, z, \mu, \nu) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2q} \times \mathbb{R}^m \times \mathbb{R}^p \mid y = \nu, z = 0\} \quad (16)
\]

\[
= \{(x, y_{loc}, z, \mu, \nu) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2q} \times \mathbb{R}^m \times \mathbb{R}^p \mid y_{loc} = 0, z = 0\},
\]

where \((\mu, \nu)\) varies over a neighbourhood of \((0, 0) \in \mathbb{R}^p \times \mathbb{R}^m\), which is invariant for the extended integrable family \( X_{\mu, \nu}(x, y_{loc}, z) = X_\mu(x, y_{loc} + \nu, z) \). We so have

\[
X_{\mu, \nu}(x, y_{loc}, z) = f(y_{loc} + \nu, z, \mu) \frac{\partial}{\partial x} + g(y_{loc} + \nu, z, \mu) \frac{\partial}{\partial y_{loc}} + h(y_{loc} + \nu, z, \mu) \frac{\partial}{\partial z}
\]

In the localized coordinates \( G \) gets the form \((x, y_{loc}, z) \mapsto (-x, y_{loc}, Rz)\). Also the \( n \)-torus action directly carries over. Again we abbreviate \( \omega(\mu, \nu) = f(\nu, 0, \mu) \) and \( \Omega(\nu, 0, \mu) = \partial h/\partial y(y, 0, \mu) \).

The present analogue of the Theorems 1.4 and 1.6 now can be given for the extended integrable family \( X = X_{\mu, \nu}(x, y, z) \) as in (17), with respect to the invariant tori \( T_{loc} \) as in (16). The corresponding nondegeneracy condition is a variation on its symplectic counterpart. In particular we have to consider the adjoint action \( GL_R(2q, \mathbb{R}) \times gl_{-R}(2q, \mathbb{R}) \to gl_{-R}(2q, \mathbb{R}), (A, \Omega) \mapsto A\Omega A^{-1} \), where \( GL_R(2q, \mathbb{R}) \subset GL(2q, \mathbb{R}) \) is the Lie-subgroup of \( R \)-equivariant matrices, which is a well-defined, cf. [28]. The family \( X \) is nondegenerate at the \( n \)-torus \( T^{0,0}_{loc} = T_{loc} \cap \{ (\mu, \nu) = (0, 0) \} \), if the following holds for the product map

\[
(\mu, \nu) \in \mathbb{R}^q \times \mathbb{R}^m \to (\omega(\mu, \nu), \Omega(\mu, \nu)) \in \mathbb{R}^n \times \mathfrak{gl}_{-R}(2q, \mathbb{R}) :\]

(a) The Floquet matrix \( \Omega_0 = \Omega(0, 0) \) is invertible;
(b) The first component \((\mu, \nu) \mapsto \omega(\mu, \nu)\) is submersive at \((\mu, \nu) = (0, 0)\). Meanwhile, for the second component \((\mu, \nu) \mapsto \Omega(\mu, \nu)\) we further require the unfolding \((\mu, \nu) \mapsto \Omega(\mu, \nu)\) to be versal at \((\mu, \nu) = (0, 0)\), i.e., transversal to the \(GL_R(2q, \mathbb{R})\)-orbit of \(\Omega(0, 0)\), compare with [4, 28, 31, 39, 52, 17, 32].

Let \(\mathcal{X}^{-G}_{P_{loc}}(M)\) be the space of all real-analytic families of \(G\)-reversible vector fields on \(M = T^n \times \mathbb{R}^m \times \mathbb{R}^{2q} = \{x, y_{loc}, z\}\), parametrized by \((\mu, \nu) \in P_{loc} = \mathbb{R}^{m} \times \mathbb{R}^{p}\), varying over a neighbourhood of \((0, 0)\).

**Theorem 1.8** [18, 39] (NORMAL LINEAR STABILITY: NEARLY-INTEGRABLE REVERSIBLE CASE) Let \(X \in \mathcal{X}^{-G}_{P_{loc}}(M)\) be a real-analytic \((q + m)\)-parameter family of \(G\)-reversible vector fields given by (17). Assume that \(X\) is nondegenerate at the the invariant torus \(T_{loc}^{0,0} = \{(x, y_{loc}, \mu, \nu) \mid y_{loc} = 0, z = 0\}\) and \((\mu, \nu) = (0, 0)\). Then, for \(\gamma > 0\) sufficiently small and for any \(\hat{X} \in \mathcal{X}^{-G}_{P_{loc}}(M)\) sufficiently close to \(X\) in the compact-open topology on complex analytic extensions, there exists a domain \(U\) around \((0, 0)\) in \(P_{loc}\) and a map

\[
\Phi : M \times U \to M \times P_{loc},
\]

defined near the torus \(T_{loc}^{0,0}\), such that

i. The map \(\Phi\) is a \(C^\infty\) near-identity diffeomorphism onto its image;

ii. The image of the \(X\)-invariant Diophantine tori

\[
V = \bigcup_{(\mu, \nu) \in \Gamma_{r, \gamma}(U')} T_{loc}^{\mu, \nu} \times \{(\mu, \nu)\}
\]

under \(\Phi\) is \(\hat{X}\)-invariant and for the restriction \(\hat{\Phi} = \Phi|_V\) we have

\[
\hat{\Phi}_*(X) = \hat{X},
\]

that is, \(\hat{\Phi}\) conjugates \(X\) to \(\hat{X}\);

iii. The restriction \(\hat{\Phi}|_V\) is \(G\)-equivariant and preserves the normal linear part of \(X\).

The proof of Theorem 1.8 is a variation on the proof given in Appendix B, where both similarity and differences are comparable with those between [28] and [31]. For further details on Theorem 1.8 see [18, 39].

**Remarks 1.9**
1. As in the dissipative Theorems 1.4, the conclusion of Theorem 1.8 is called normal linear stability. It induces a similar conclusion for the non-extended family \( X = X_\mu(x, y, z) \) as in (15), namely via the projection \( M \times P \rightarrow M \times P \) given by

\[
(x, y_{\text{loc}}, z, \mu, \nu) \mapsto (x, y_{\text{loc}} + \nu, z, \mu);
\]

2. As in the symplectic case also here we consider the example where the Floquet matrix \( \Omega_0 = \Omega(0, 0) \) has a double pair of purely imaginary eigenvalues with nontrivial nilpotent part, where we can restrict to the case \( q = 2 \). We say that \( \Omega_0 \) is in \( 1 : 1 \) resonance, which in the reversible case happens to coincide with the \( 1 : -1 \) resonance. In the equilibrium or periodic case the nonlinear theory of this generically involves a reversible Hopf bifurcation [71, 70]. In the presently setting of invariant tori, we are dealing with a quasi-periodic reversible Hopf bifurcation, which is studied in detail in [17, 39].

3. It is instructive to compare the present reversible case with the symplectic case. That the linear details are already quite different has been shown in [65, 66]. Nonlinear aspects of the present situation have been studied by [17, 20], also see [71, 70, 16, 81].

In the case \( q = 2 \) the geometries, both in the linear and the nonlinear case are the same. This statement involves normal form reductions to \( \mathbb{R}^4 \), where a ramified circle bundle plays a role. Also the mathematics involves singularity theory. As a byproduct, this moreover shows that in the reversible supercritical Hopf case, there is nontrivial monodromy. However, it turns out that the reduced dynamics is slightly different, due to an additional drift in the fibre direction in the reversible case [17].

One may conclude that in qualitative terms the quasi-periodic Hopf bifurcation in the reversible and Hamiltonian cases are comparable to a large extent. For \( q \geq 3 \) Knobloch and Vanderbauwhede [71, 70] state similar conclusions, although the mathematical treatments of both cases may differ. Also compare with [16, 81].

4. Generally speaking reversible KAM theory, as this starts with Moser [87], to a great extent is parallel to its Hamiltonian counterpart, see, e.g., [88, 89, 91, 11, 7, 9, 28, 29, 30, 17, 18, 98, 106, 107, 109, 110, 114, 113]. In the case of reversible diffeomorphisms, however, some special effects show up [100]. For general references on reversible dynamical systems, see [74, 102]. For a similar discussion also see [32].
1.4 Plan of the paper

We briefly overview the plan for the remainder of this paper. In §2, we revisit the persistence of equilibria of vector fields in terms of linear stability. Here the concept of (uni-) versal matrix unfolding or deformation plays a central role. This study is related to the normal linear stability of the Theorems 1.4, 1.6 and 1.8, which can be most clearly seen in case the perturbations are assumed to be integrable as well. These are the contents of §3. Indeed, in that case linear stability almost directly applies to the relative equilibria obtained by factoring out the \( n \)-torus symmetry. This means that everything is proven by the Inverse Function Theorem and no KAM theory is needed. Also the surviving \( n \)-tori occur in continua, depending smoothly on the parameters, for this approach compare with [67, 31]. Next, §4 is devoted to a case study on the generic (or non-semisimple) \( 1 : -1 \) resonance, involving a corollary of Theorem 1.6. The generic nonlinear theory then leads to a quasi-periodic Hamiltonian Hopf bifurcation, which we sketch in §5, illustrated on the Lagrange top coupled with a quasi-periodic oscillator near gyroscopic stabilisation. This is a typical application of the theory, for details we refer to [20].

The Theorems 1.4, 1.6 and 1.8 all are within the axiomatic approach of §2. Among these three, the symplectic case of Theorem 1.6 undoubtedly is the most involved, which is why we have chosen to include a full proof of this in Appendix B. Proofs of the Theorems 1.8 and 1.4 can be obtained by verbatim transcriptions, though surely the latter case could be further simplified. For a similar proof in the reversible case we refer to [18, 39]. In Appendix A we deal with the Hölder condition that plays a technical role in Appendix B. Finally, in Appendix C we briefly describe the normal linear theory in the symplectic case, compare with [31], §6b.

2 Linear stability of equilibria

Persistence of equilibria of vector fields is closely related to normal linear stability. Compare with, e.g., the conclusions of the Theorems 1.4, 1.6 and 1.8 are all formulated in terms of smooth local conjugacies near a nondegenerate invariant torus. In view of this we now express the persistence of equilibria in terms of smooth local conjugacies. Existence of such a conjugacy implies similarity of the linear parts at the equilibria. Indeed, let \( X_1 \) and \( X_2 \) be two vector fields on \( \mathbb{R}^m \) locally conjugated by the diffeomorphism \( \phi \). Suppose that \( X_1(p) = 0 \) and \( q = \phi(p) \), then \( X_2(q) = 0 \) and \( D_pX_1 \) is similar to \( D_qX_2 \) as follows: \( (D_p\phi)D_pX_1(D_p\phi)^{-1} = D_qX_2 \). Similarity between linear parts of vector fields at equilibria defines an equivalence relation on the germs of such equilibria.
The linear part (up to similarity) thus becomes an intrinsic object.

Our aim in this section is to give sufficient conditions for linear stability of equilibria. This involves the concept of versal matrix unfoldings [4, 5, 6, 52], for which reason we first revisit the main ideas of (trans-) versality. This program is carried out, keeping a suitable preservation of structure into account, like a volume or a symplectic form, a symmetry or a reversing symmetry. For particular studies in this direction compare with, e.g., [50, 63, 108]

### 2.1 Stability in terms of versal unfoldings

Let \( G \neq \{I_m\} \) be a Lie subgroup of \( GL(m, \mathbb{R}) \) and \( V \neq \{0\} \) a linear subspace of \( \mathfrak{gl}(m, \mathbb{R}) \). Suppose that \( V \) is invariant under \( G \), i.e., that for any \( Q \in G \) and \( \Omega \in V \), one has \( Q\Omega Q^{-1} \in V \). Then the map \( \Psi : G \times V \to V \) given by \( \Psi(Q, \Omega) = Q\Omega Q^{-1} \), is a group action on \( V \), called the \( G \)-action on \( V \). The subset

\[
O(\Omega, G) = \{Q\Omega Q^{-1} : Q \in G\} \subseteq V
\]

is called the \( G \)-orbit of \( \Omega \). Suppose that the orbit \( O(\Omega, G) \) is a smooth submanifold, then the co-dimension of \( \Omega \), denoted by \( \text{cod} \ \Omega \), is defined as the co-dimension in \( V \) of the submanifold \( O(\Omega, G) \subseteq V \). Each orbit \( O(\Omega, G) \) is a smooth submanifold, provided, for instance, that the group \( G \) is semi-algebraic as a subset of \( \mathfrak{gl}(m, \mathbb{R}) \), see [52, 12]. Well-known examples are: \( GL(m, \mathbb{R}) \), \( SL(m, \mathbb{R}) \), \( SP(m, \mathbb{R}) \) and \( GL_+(m, \mathbb{R}) \),\(^3\) where \( R \in GL(m, \mathbb{R}) \) is a linear involution.

**Definition 2.1** The pair \((G, V)\) as above, where \( V \) is \( G \)-invariant, is a linear structure on \( \mathbb{R}^m \), if the following holds:

A1. \( G \) is a semi-algebraic subset of \( \mathfrak{gl}(m, \mathbb{R}) \);

A2. If \( \Omega \in V \), so is the transpose \( \Omega^T \) of \( \Omega \) with respect to the standard inner product of \( \mathbb{R}^m \).

Compare these conditions with set-ups of [89, 67, 31]. Condition (A1) ensures that the all orbits are smooth submanifolds; condition (A2) is used to compute the complements of the tangent spaces of the orbits, needed for constructing versal unfoldings, in a systematic way. Since these complements can be determined in other ways, condition (A2) is not necessary for the versal unfolding stability to be developed below.

**Example 2.2** The pairs formed by \( GL(m, \mathbb{R}) \), \( SL(m, \mathbb{R}) \), \( SP(m, \mathbb{R}) \) and \( GL_+(m, \mathbb{R}) \) together with their Lie algebras are linear structures. The same is true for the

\(^3\) \( GL_+(m, \mathbb{R}) \) denotes the group of \( R \)-equivariant matrices in \( GL(m, \mathbb{R}) \).
pair \((GL_R(m, \mathbb{R}), \mathfrak{gl}_R(m, \mathbb{R}))\), where \(\mathfrak{gl}_R(m, \mathbb{R})\) denotes the set of infinitesimally \(R\)-reversible matrices, i.e., \(\mathfrak{gl}_R(m, \mathbb{R}) = \{A \in \mathfrak{gl}(m, \mathbb{R}) : AR = -RA\}\). Observe that \(\mathfrak{gl}_R(m, \mathbb{R})\) is not the Lie algebra of \(GL_R(m, \mathbb{R})\), compare with [28, 18, 88, 89].

From now on, we assume that \((G, V)\) is a linear structure and let \(\Pi_p(V)\) be the set of all smooth \(p\)-parameter families on \(V\), that is, the set of germs \(\Omega : \mu \in \mathbb{R}^p \mapsto \Omega(\mu) \in V\) at \(\mu = 0\). These germs are also called matrix unfoldings or deformations of \(\Omega_0 = \Omega(0)\). We recall from [4, 6, 52] the definition of versality:

**Definition 2.3** An unfolding \(\Omega \in \Pi_p(V)\) of \(\Omega_0\) is versal with respect to the \(G\)-action, if for any \(B \in \Pi_q(V)\) with \(B(0) = \Omega_0\), there exists a smooth local map \((\rho, Q) : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0) \times (G, I_m)\) such that

\[
B(\mu) = Q(\mu)\Omega(\rho(\mu))Q^{-1}(\mu).
\] (18)


Again by [4, 6, 52], the family \(\Omega\) is versal at \(\mu = 0\) if and only if it is transversal at \(\Omega_0\), that is, if

\[
D_0\Omega(\mathbb{R}^p) + T_{\Omega_0}(O(\Omega_0, G)) = V,
\] (19)

where \(D_0\Omega\) is the derivative of \(\Omega\) at \(\mu = 0\) and \(T_{\Omega_0}(O(\Omega_0, G))\) is the tangent space of the submanifold \(O(\Omega_0, G)\) at \(\Omega_0\). By (19), the minimal number of parameters, needed for versality of the family \(\Omega\), equals the codimension \(\text{cod } \Omega_0\) (i.e., \(p \geq \text{cod } \Omega_0\)). A \(p\)-parameter versal matrix unfolding of \(\Omega_0\) is called universal if \(p = \text{cod } \Omega_0\).

**Remark 2.4** By invariance of eigenvalues under similarity, the co-dimension of any single matrix within \((GL(m, \mathbb{R}), \mathfrak{gl}(m, \mathbb{R}))\), seen as a constant unfolding, is non-zero. In particular, constant unfoldings can never be versal in \((GL(m, \mathbb{R}), \mathfrak{gl}(m, \mathbb{R}))\), or, in other words: individual matrices never are structurally stable.

To fulfill condition (19), we need to know the tangent space \(T_{\Omega_0}(O(\Omega_0, G))\) explicitly. To this end, we consider the map \(\Psi_{\Omega_0} : G \to O(\Omega_0, G) \subseteq V\) given by \(\Psi_{\Omega_0}(Q) = Q\Omega_0Q^{-1}\). Let \(g\) be the Lie algebra of \(G\). For any \(B \in g\), the map \(t \mapsto \exp(tB)\) is a smooth curve through the identity \(I_m\) in the group \(G\), where \(\exp : g \to G\) denotes the exponential map [97, 37, 1, 117]. Then, the map \(t \mapsto \Psi_{\Omega_0}(\exp(tB))\) defines a smooth curve in the orbit \(O(\Omega_0, G)\) passing through \(\Omega_0\). Since for any \(B \in g\), we have

\[
\frac{d}{dt} \bigg|_{t=0} \Psi_{\Omega_0}(\exp(tB)) = \Omega_0 B - B\Omega_0 = [\Omega_0, B] = \text{ad } \Omega_0(B),
\] (20)

and since \(\exp\) is a local diffeomorphism near \(0 \in g\), it follows that

\[
T_{\Omega_0}(O(\Omega_0, G)) = \text{ad } \Omega_0(g).
\] (21)
A special kind of universal unfoldings is given by the linear centralizer unfoldings (LCU’s).

**Definition 2.5** A linear unfolding \( \Omega \in \Pi_p(V) \) of \( \Omega_0 \) is called an LCU of \( \Omega_0 \) if \( p = \text{cod } \Omega_0 \) and \( D_0\Omega(\mathbb{R}^p) = \ker \text{ad } \Omega_0^T \subseteq V \).

The following is of importance for the linear stability results on equilibria of vector fields to be discussed in §2.2.

**Theorem 2.6** (VERSAL UNFOLDING STABILITY) Let \( \Omega \in \Pi_p(V) \) be versal within the linear structure \((G, V)\). Then, there exists a neighbourhood \( \mathcal{V} \) of \( \Omega \) in \( \Pi_p(V) \) in the \( C^1 \)-topology, and for all \( B \in \mathcal{V} \), we have smooth germs \((\rho, Q)\), such that

i. \( B(\mu) = Q(\mu)\Omega(\rho(\mu))Q(\mu)^{-1} \), for all small \( \mu \);

ii. The reparametrization \( \rho \) is \( C^1 \)-near the identity map.

**Remark 2.7** If \( B(0) = \Omega(0) \), then conclusion (i) directly follows from Definition 2.3. However, in Theorem 2.6, we do not require that \( B(0) = \Omega(0) \).

In the following we give a proof of Theorem 2.6. For \( \Omega_0 \in V \), let

\[ S(\Omega_0) = \{ P \in G : P\Omega_0 = \Omega_0 P \} \subseteq G, \]

be the isotropy subgroup of \( G \) of \( \Omega_0 \), compare with [61, 1, 117]. It is known that \( S(\Omega_0) \subseteq G \) is a Lie subgroup and that \( T_{1_m}S(\Omega_0) = \ker \text{ad } \Omega_0 \), in particular, we have \( \dim S(\Omega_0) = \dim G - \dim O(\Omega_0, G) \).

**Lemma 2.8** Given a linear structure \((G, V)\), suppose that \( \Omega \in \Pi_p(V) \) is a universal unfolding of \( \Omega_0 \). Let \( H \) be a smooth submanifold of \( G \), transversal to the isotropy group \( S(\Omega_0) \) at the identity \( 1_m \in G \). Suppose that \( \dim H = \text{cod } S(\Omega_0) \), then the map \( F : \mathbb{R}^p \times H \rightarrow V \) defined by

\[ F(\mu, P) = P\Omega(\mu)P^{-1}, \]

is a local \( C^\infty \)-diffeomorphism near \((0, 1_m) \in \mathbb{R}^p \times H \).

**Proof.** By assumption, \( T_{1_m}H \oplus T_{1_m}S(\Omega_0) = T_{1_m}G \). Since \( T_{1_m}S(\Omega_0) = \ker \text{ad } \Omega_0 \), we have

\[ \text{ad } \Omega_0(T_{1_m}H) = \text{ad } \Omega_0(T_{1_m}G). \quad (22) \]

By universality of \( \Omega \), \( p = \text{cod } \Omega_0 \). Since \( \dim H = \dim O(\Omega_0, G) = \dim V - \text{cod } \Omega_0 \), we have \( \dim(\mathbb{R}^p \times H) = \dim V \). The derivative \( D_{(0,1_m)}F \) of \( F \) at \((\mu, P) = (0, 1_m) \) is given by

\[ D_{(0,1_m)}F(\hat{\mu}, \hat{P}) = D_0\Omega(\hat{\mu}) + [\hat{P}, \Omega_0], \]
for \((\mu, \hat{P}) \in \mathbb{R}^p \times T_{0,m} H\). Now by (uni-) versality, \(D_{(0,1m)} F\) has maximal rank. Since \(\dim(\mathbb{R}^p \times H) = \dim V\), it follows that \(F\) is a local diffeomorphism. □

**Proof. (Theorem 2.6)** We first prove the claim for the case where \(\Omega\) is an universal unfolding. We take a submanifold \(H\) as given in Lemma 2.8. Then, the map \(F: \mathbb{R}^p \times H \to V\) given by

\[ F(\mu, P) = PA(\mu)P^{-1}. \]

is a local diffeomorphism near \((\mu, P) = (0, I_m)\). Suppose that the matrix \(B(0)\) is sufficiently close to \(0 = \Omega(0)\). Then, by the Inverse Function Theorem, for each sufficiently small \(\mu\), there exists a unique pair \(\rho(\mu), Q(\mu) \in \mathbb{R}^p \times G\) such that

\[ B(\mu) = F(\rho(\mu), Q(\mu)) = Q(\mu) \Omega(\rho(\mu)) Q(\mu)^{-1}. \tag{23} \]

Moreover, the pair \((\rho(\mu), Q(\mu))\) depends smoothly on \(\mu\) and \((\rho(0), Q(0)) = (0, I_m)\). Hence, the local product map \(\rho \times Q\) defined by

\[ \rho \times Q: \mu \in \mathbb{R}^p \to (\rho(\mu), Q(\mu)) \in \mathbb{R}^p \times G \tag{24} \]

gives the required germs for part (i).

Now we turn to part (ii). Let \(F^{-1}: V \to \mathbb{R}^p \times G\) be the local inverse of \(F\) near \(\Omega_0 = \Omega(0)\). Then by (23), for any \(B \in \Pi_p(V)\) sufficiently close to \(\Omega\) and for any small \(\mu\), we have

\[ (F^{-1} \circ B)(\mu) = F^{-1}(B(\mu)) = (\rho(\mu), Q(\mu)). \tag{25} \]

Let \(\pi: \mathbb{R}^p \times H \to \mathbb{R}^p\) be the natural projection. Then, by (25), the change of parameter \(\rho\) is given by

\[ \rho(\mu) = \pi \circ F^{-1} \circ B(\mu). \]

Observe that \((\pi \circ F^{-1} \circ \Omega)\) locally is the identity map. Hence, when \(B \in \Pi_p(V)\) is sufficiently \(C^1\)-close to \(\Omega\), the derivative of the map \(\rho = \pi \circ F^{-1} \circ B\) is near the identity in a neighbourhood of \(0 \in \mathbb{R}^p\). Hence, \(\rho\) is \(C^1\)-near the identity.

Finally, the case where \(\Omega\) is versal, but not necessarily universal, can be reduced to the above case. Indeed, by versality and the Inverse Function Theorem [1, 52], there exists a local reparametrization \(\mu \in \mathbb{R}^p \mapsto (\nu, \kappa) \in \mathbb{R}^c \times \mathbb{R}^{p-c}\) such that \(\Omega\) restricted to the \(\nu\)-direction is an universal unfolding of \(\Omega_0\). Rewrite \(B(\mu) = B(\nu, \kappa)\). Then, we can apply the above arguments to the \(\nu\)-direction leaving the \(\kappa\)-direction unchanged. □

**Remark 2.9** In [10] V.I. Arnold proposes problem 1970–1 as “Construct all versal unfoldings of endomorphisms (of vector spaces and groups).” M.B. Sevryuk comments on this problem in [112], by discussing it from a general view point,
with many references. The present section can be seen as a contribution to this problem, in the footsteps of [89, 31, 67, 64] and motivated by the normal linear stability problem in KAM theory. For another general contribution see I. Hoveijn [65].

2.2 Linear stability of equilibria: dissipative case

Here we formulate our first linear stability theorem for equilibria in the dissipative setting. Analogous to (1), we consider a $p$-parameter family $X \in \mathcal{X}_P(\mathbb{R}^m)$ of the form

$$X_\mu(y) = \left[\Omega(\mu)y + O(|y|^2)\right] \frac{\partial}{\partial y},$$  \hspace{1cm} (26)

where $\mu \in P$ and $\Omega(\mu) \in \text{gl}(m, \mathbb{R})$. We say that $X$ is nondegenerate at $(y, \mu) = (0, 0)$, if the matrix $\Omega_0 = \Omega(0)$ is invertible and the unfolding $\Omega$ of $\Omega_0$ is versal within the linear structure $(GL(m, \mathbb{R}), \text{gl}(m, \mathbb{R}))$, compare with Definition 1.1.

Our present concern is with the persistence of the zero set $\mathcal{Z}(X)$ of the family $X$, where

$$\mathcal{Z}(X) = \{y = 0, \mu \text{ in a full neighbourhood of } 0 \in P\} \subset \mathbb{R}^m \times P,$$

and the linear behavior of the family $X$.

**Theorem 2.10 (Linear stability: dissipative case)** Let the family $X$ be of the form (26). If $X$ is nondegenerate at $(y, \mu) = (0, 0)$, then, for any perturbation $\tilde{X} \in \mathcal{X}_P(\mathbb{R}^m)$ with $X - \tilde{X}$ sufficient small in the $C^2$-topology the following holds. There exists a local diffeomorphism

$$\Phi : \mathbb{R}^m \times P \rightarrow \mathbb{R}^m \times P$$

of the form $\Phi(y, \mu) = (\phi_\mu(y), \rho(\mu))$, defined near $(y, \mu) = (0, 0)$, such that

i. $\Phi$ is $C^1$-near the identity map;

ii. The image $\Phi(\mathcal{Z}(X))$ is a zero set of the family $\tilde{X}$, that is $\Phi(\mathcal{Z}(X)) = \mathcal{Z}(\tilde{X})$;

iii. $\Phi$ preserves the linear behaviour of $X$.

The conclusion of Theorem 2.10 is referred to as the linear stability of the family $X$ at the equilibrium $(y, \mu) = (0, 0)$.

**Remarks 2.11**
1. Observe that no single vector field $X_0$ is linearly stable. Indeed, a small perturbation of $X_0$ in general leads to a change in eigenvalues of the linear part of $X_0$, compare with Remark 2.4.

2. By the Inverse Function Theorem, the zero set $\mathcal{Z}(\tilde{X})$ of the perturbation $\tilde{X}$ is a smooth graph $(y(\mu), \mu)$ in $\mathbb{R}^m \times P$.

**Proof.** (Theorem 2.10) By the Inverse Function Theorem, there is a smooth map $\mu \in P \mapsto y(\mu) \in \mathbb{R}^m$, such that, $\tilde{X}_\mu(y(\mu)) = 0$ for each small $\mu$. After the translation $y \mapsto y - y(\mu)$, we may assume that $\tilde{X}_\mu(y) = [B(\mu)y + O(|y|^2)] \frac{\partial}{\partial y}$. By assumption, the unfolding $B \in \Pi_p(V)$ is $C^1$-close to $\Omega$. By Theorem 2.6, there exists a $\mu$-dependent transformation $\psi_\mu : y \mapsto Q(\mu)y$ with $Q(\mu) \in GL(m, \mathbb{R})$ and a reparametrization $\rho : P \rightarrow P$, that is $C^1$-near the identity, such that

$$\psi_\mu^* \tilde{X}_\rho(y) = [\Omega(\mu)y + O(|y|^2)] \frac{\partial}{\partial y}. \quad (27)$$

Let $\phi_\mu(y) = y(\mu) + Q(\mu)y$ and $\Phi(y, \mu) = (\phi_\rho(y), \mu)$. By construction, the map $\Phi$ is a $C^1$-near-identity-diffeomorphism, which shows (i). Moreover, by (27), $\Phi$ preserves the linear part of the family $X$, which justifies conclusion (iii). Finally, to show (ii), we notice that for all small $\mu$

$$\tilde{X}_\rho(\phi_\rho(0)) = \tilde{X}_\rho(y(\rho(\mu))) = 0, \quad (28)$$

which implies that $\Phi$ maps zeros of the family $X$ to those of $\tilde{X}$. $\square$

### 2.3 Linear stability of equilibria: symplectic case

Next we consider linear stability of Hamiltonian equilibria in the standard symplectic space $\mathbb{R}^{2q} = \{z_1, \ldots, z_{2q}\}$ endowed with the canonical 2-form

$$\sigma = \sum_{i=1}^{q} dz_i \wedge dz_{q+i} = dz^2.$$ 

We denote by $\mathcal{X}^\sigma(\mathbb{R}^{2q})$ the set of all Hamiltonian vector fields on $(\mathbb{R}^{2q}, \sigma)$ and $\mathcal{X}^\sigma_P(\mathbb{R}^{2q})$ the set of all $p$-parameter families of such Hamiltonian vector fields, with parameter space $P$. Let $Y \in \mathcal{X}^\sigma(\mathbb{R}^{2q})$ with $Y(0) = 0$ be given, then $Y$ has the form $Y(z) = [\Omega_0z + O(|z|^2)] \frac{\partial}{\partial z}$, where $\Omega_0 \in \mathfrak{sp}(2q, \mathbb{R})$. As an analogue to the family (26) in the dissipative setting, we here consider a Hamiltonian family $X = X_\mu \in \mathcal{X}^\sigma_P(\mathbb{R}^{2q})$ of the form:

$$X_\mu(z) = [\Omega(\mu)z + O(|z|^2)] \frac{\partial}{\partial z}. \quad (29)$$
where $\Omega(\mu) \in \mathfrak{sp}(2q, \mathbb{R})$ for each value $\mu$. Similar to the dissipative case, the Hamiltonian family $X$ given by (29) is linearly stable at $(z, \mu) = (0, 0)$, provided that $X$ is nondegenerate at $(z, \mu) = (0, 0)$ within the linear structure $(\mathcal{SP}(2q, \mathbb{R}), \mathfrak{sp}(2q, \mathbb{R}))$. Nondegeneracy means here that $\Omega_0 = \Omega(0)$ is non-singular, while the unfolding $\Omega = \Omega(\mu)$ is versal at $\mu = 0$ within $(\mathcal{SP}(2q, \mathbb{R}), \mathfrak{sp}(2q, \mathbb{R}))$. More precisely, we have

**Theorem 2.12 (Linear Stability: Symplectic Case)** Let the Hamiltonian family $X \in \mathcal{X}_P^R(\mathbb{R}^m)$ be given by (29). Assume that $X$ is nondegenerate at $(z, \mu) = (0, 0)$. Then, for any perturbation $\tilde{X} \in \mathcal{X}_P^R(\mathbb{R}^m)$ with $X - \tilde{X}$ sufficiently small in the $C^2$-topology, there exists a local map

$$\Phi : (z, \mu) \in \mathbb{R}^{2q} \times P \mapsto (\phi_\mu(z), \rho(\mu)) \in \mathbb{R}^{2q} \times P$$

defined near $(z, \mu) = (0, 0)$, such that,

i. $\Phi$ is $C^1$-near the identity map;

ii. The image $\Phi(Z(X))$ is a zero set of the family $\tilde{X}$, that is $\Phi(Z(X)) = Z(\tilde{X})$;

iii. $\Phi$ is symplectic and preserves the linear behavior.

The proof of Theorem 2.13 is completely analogous to that of Theorem 2.10.

### 2.4 Linear stability of equilibria: reversible case

The reversible linear stability theorem sounds runs similar to its symplectic analogue of the previous subsection. Indeed, we now consider $\mathbb{R}^{2q}$ with the linear involution $R \in GL(2q, \mathbb{R})$ as before, see §1.3, so where the eigenvalue 1 occurs with multiplicity $q$. In this case we denote by $\mathcal{X}_P^{-R}(\mathbb{R}^{2q})$ the space of all $R$-reversible vector fields on $\mathbb{R}^{2q}$, i.e., with $R_* X = -X$, while the space of $R$-reversible families parametrized over $P \in \mathbb{R}^q$ is denoted $\mathcal{X}_P^{-R}(\mathbb{R}^{2q})$. Any $Y \in \mathcal{X}_P^{-R}(\mathbb{R}^{2q})$ with $Y(0) = 0$, has the form $Y(z) = [\Omega_0 z + O(|z|^2)] \frac{\partial}{\partial z}$, with $\Omega_0 \in \mathfrak{gl}_{-R}(2q, \mathbb{R})$. As an analogue to (26) and (29), we now consider a reversible family $X \in \mathcal{X}_P^{-R}(\mathbb{R}^{2q})$, of the form

$$X_\mu(z) = [\Omega(\mu) z + O(|z|^2)] \frac{\partial}{\partial z},$$

with $\Omega(\mu) \in \mathfrak{gl}_{-R}(2q, \mathbb{R})$, for all $\mu$. Similar to the above two cases, the reversible family (30) is linearly stable at $(z, \mu) = (0, 0)$, provided that $X$ is nondegenerate at $(z, \mu) = (0, 0)$ within the linear structure $(GL_R(2q, \mathbb{R}), \mathfrak{gl}_{-R}(2q, \mathbb{R}))$, see §2.1, Example 2.2. Here non-degeneracy means that $\Omega_0 = \Omega(0)$ is non-singular, while the unfolding $\Omega = \Omega(\mu)$ is versal at $\mu = 0$ within $(GL_R(2q, \mathbb{R}), \mathfrak{gl}_{-R}(2q, \mathbb{R}))$. To be precise, we have
Theorem 2.13 (Linear Stability: Reversible Case) Let the reversible family \( X \in \mathcal{X}_P^{-R}(\mathbb{R}^{2q}) \) be given by (30). Assume that \( X \) is non-degenerate at \((z, \mu) = (0, 0)\). Then, for any perturbation \( \tilde{X} \in \mathcal{X}_P^{-R}(\mathbb{R}^{2q}) \) with \( X - \tilde{X} \) sufficiently small in the \( C^2 \)-topology, there exists a local map

\[
\Phi : (z, \mu) \in \mathbb{R}^{2q} \times P \mapsto (\phi_\mu(z), \rho(\mu)) \in \mathbb{R}^{2q} \times P
\]

defined near \((z, \mu) = (0, 0)\), such that,

i. \( \Phi \) is \( C^1 \)-near the identity map;

ii. The image \( \Phi(Z(X)) \) is a zero set of the family \( \tilde{X} \), that is \( \Phi(Z(X)) = Z(\tilde{X}) \);

iii. \( \Phi \) is \( R \)-equivariant and preserves the linear behaviour.

The proof of Theorem 2.13 again is completely analogous to that of the Theorem 2.10.

2.5 Examples of linearly stable systems in resonant cases

In this section, we give a few examples of linearly stable families of Hamiltonian vector fields, which are relevant for applications of Theorem 1.6. Our interest is with local Hamiltonian systems the linear parts of which are in \( 1:1 \) resonance, compare with [79, 65, 69]. We also shall describe this resonance in the reversible setting, compare with [39, 18].

a. Preliminaries

Consider the standard symplectic space \( \mathbb{R}^4 = \{ z_1, z_2, z_3, z_4 \} \) with symplectic 2-form \( \sigma = dz_1 \wedge dz_3 + dz_2 \wedge dz_4 \). To study linear stability of Hamiltonian equilibria, by Theorem 2.13, it is sufficient to consider linear Hamiltonian systems. Let \( Y \) be a Hamiltonian vector field given by

\[
Y(z) = \Omega_0 z \frac{\partial}{\partial z} \in \mathcal{X}^\sigma(\mathbb{R}^4),
\]

where \( \Omega_0 \in \mathfrak{sp}(4, \mathbb{R}) \). We say that the linear system (31) is in \( 1:1 \) resonance, if the matrix \( \Omega_0 \) has a double pair of purely imaginary eigenvalues, say \( \pm i \lambda_0 \), and if \( \Omega_0 \) can be transformed (by a symplectic linear map) to the following real normal form:

\[
N_z = \begin{pmatrix}
0 & -\lambda_0 & 0 & 0 \\
\lambda_0 & 0 & 0 & 0 \\
\varepsilon & 0 & 0 & -\lambda_0 \\
0 & \varepsilon & \lambda_0 & 0
\end{pmatrix},
\]

(32)
where \( \lambda_0 \neq 0 \) and \( \varepsilon = 0, \pm 1 \). Depending on the value of \( \varepsilon \), we distinguish between two cases: the semisimple (\( \varepsilon = 0 \)) and the non-semisimple or generic case (\( \varepsilon = \pm 1 \)). For computation of real normal forms of \( \Omega_0 \), such as \( N_\varepsilon \), we refer to [35, 125].

Let us recall certain notations from \( \S 2.1 \). The co-dimension of \( \Omega_0 \) is given by 
\[
\text{co-dim } \Omega_0 = \text{dim } \mathfrak{sp}(4, \mathbb{R}) - \text{dim } O(\Omega_0),
\]
where \( O(\Omega_0) \) denotes the \( \mathfrak{sp}(4, \mathbb{R}) \)-orbit of \( \Omega_0 \). A \( p \)-parameter linear unfolding \( \Omega(\mu) \) is an LCU of \( \Omega_0 \) if the derivative \( D_0 \Omega : \mathbb{R}^p \to \ker \text{ad } \Omega_0^T \) is an isomorphism. In particular, for LCU’s one always has \( p = \text{co-dim } \Omega_0 = \dim \ker \text{ad } \Omega_0^T \).

b. Hamiltonian \( 1 : -1 \) resonance

Consider the linear Hamiltonian vector field \( Y \) as in (31). In the \( 1 : -1 \) resonant case, we can replace the linear part \( \Omega_0 \) by the real normal form \( N_\varepsilon \) as in (32). As said before, the normal forms \( N_{\pm1} \) correspond to the generic (or non-semisimple) case and \( N_0 \) to the semisimple case.

**Proposition 2.14** Depending on the value \( \varepsilon \), we have the following.

i. For \( \varepsilon = \pm 1 \) (i.e., the generic case), an LCU of \( N_\varepsilon \) as in (32) is given by the matrix family

\[
\Omega = \Omega(\mu) = \begin{pmatrix}
0 & -\lambda_0 + \mu_1 & \mu_2 & 0 \\
\lambda_0 - \mu_1 & 0 & 0 & \mu_2 \\
\varepsilon & 0 & 0 & -\lambda_0 + \mu_1 \\
0 & \varepsilon & \lambda_0 - \mu_1 & 0
\end{pmatrix}, \quad (33)
\]

where \( \mu_1, \mu_2 \in \mathbb{R} \).

ii. For \( \varepsilon = 0 \) (i.e., the semisimple case), an LCU \( \Omega \) of \( N_\varepsilon \) as in (32) takes the form

\[
\Omega = \Omega(\mu) = \begin{pmatrix}
\mu_3 & -\lambda_0 + \mu_1 & \mu_2 & 0 \\
\lambda_0 - \mu_1 & \mu_3 & \mu_2 & 0 \\
\mu_4 & 0 & -\mu_3 & -\lambda_0 + \mu_1 \\
0 & \mu_4 & \lambda_0 - \mu_1 & -\mu_3
\end{pmatrix}, \quad (34)
\]

where \( \mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{R} \).

In particular, in the generic case we have \( \text{co-dim } N_{\pm1} = 2 \), while in the semisimple case, we have \( \text{co-dim } N_0 = 4 \).
The eigenvalues of $\Omega(\mu)$ as in (33) and (34) are of the form
\[ \pm i(\lambda_0 - \mu_1) \pm \sqrt{\nu}, \]
where $\nu = \mu_2$ for the generic case and $\nu = \mu_3^2 + \mu_2 \mu_4$ for the semisimple case. In both cases, the eigenvalues behave as follows (see Figure 1):

(a) Elliptic case ($\nu < 0$): $\Omega(\mu)$ has two pairs of purely imaginary eigenvalues $\pm i(\lambda_0 - \mu_1) \pm \sqrt{-\nu}$.

(b) Parabolic\(^4\) case ($\nu = 0$): $\Omega(\mu)$ has a pair of double purely imaginary eigenvalues, namely $i(\lambda_0 - \mu_1)$, $-i(\lambda_0 - \mu_1)$.

(c) Hyperbolic case ($\nu > 0$): $\Omega(\mu)$ has four complex eigenvalues with non-zero real parts, namely, $\pm i(\lambda_0 - \mu_1) \pm \sqrt{\nu}$.

\[ \begin{array}{ccc}
\circ & \bullet & \bullet \\
\bullet & \circ & \bullet \\
\end{array} \]

Hyperbolic: $\nu < 0$ \hspace{1cm} Parabolic: $\nu = 0$ \hspace{1cm} Elliptic: $\nu > 0$

Figure 1: Eigenvalues in both generic and semisimple $1 : -1$ resonance.

c. Reversible $1 : 1$ resonance

We briefly discuss the strong resonance in the reversible setting, see [65, 39, 18, 17], which is similar to the Hamiltonian one, also compare the remarks and references at the end of §1.3. In both settings, the semisimple case requires (at least) four parameters to ensure the linear stability, while the generic case needs two parameters. In contrast to the Hamiltonian setting, we here do not have to distinguish between the $1 : -1$ and the $1 : 1$ resonance, for details see [65, 18].

Next we construct LCU’s for the reversible $1 : 1$ resonance. Letting $R$ be a linear involution on $\mathbb{R}^4 = \{z_1, z_2, z_3, z_4\}$, we denote by $\mathcal{X}_{-R}(\mathbb{R}^4)$ the set of all $R$-reversible vector fields on $\mathbb{R}^4$, i.e.,
\[ \mathcal{X}^{-R} = \{ X \in \mathcal{X}(\mathbb{R}^4) : R_x X = -X \}. \]

We consider the linear system $Y(z) = \Omega_0 z \frac{\partial}{\partial z}$, where $\Omega_0 \in \mathfrak{gl}(4, \mathbb{R})$. Then, $Y$ is $R$-reversible if and only if $\Omega_0$ is a $R$-reversible matrix, i.e., $\Omega_0 \in \mathfrak{gl}_{-R}(4, \mathbb{R})$.

\(^4\)A $4 \times 4$-matrix is called parabolic if it has a double pair of purely imaginary eigenvalues.
Reversible 1:1 resonance reflects the fact that $\Omega_0$ has a double pair of purely imaginary eigenvalues, say $\pm i\lambda_0$. In this resonant case, following [18], one can normalize the involution $R$ into $R = \text{diag}\{1, -1, 1, -1\}$, while the linear part $\Omega_0$ into

$$\Omega_0 = \begin{pmatrix} \alpha J_2 & \varepsilon J_2 \\ \varepsilon J_2 & \alpha J_2 \end{pmatrix}$$

where $\varepsilon = 0$ in the semisimple case and $\varepsilon = 1$ in the non-semisimple case; the matrix $J_2$ is given by

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For a fixed choice of the involution $R = \text{diag}\{1, -1, 1, -1\}$, the matrix $\Omega_0$ as in (35) has two different LCU’s $\Omega(\mu)$ depending on the value $\varepsilon$, compare with Proposition 2.14.

i. The generic case ($\varepsilon = 1$):

$$\Omega(\mu) = \begin{pmatrix} (\lambda_0 + \mu_1)J_2 & J_2 \\ \mu_2 J_2 & (\lambda_0 + \mu_1)J_2 \end{pmatrix};$$

ii. the semisimple case ($\varepsilon = 0$):

$$\Omega(\mu) = \begin{pmatrix} (\lambda_0 + \mu_1)J_2 & \mu_2 J_2 \\ \mu_3 J_2 & (\mu_0 + \mu_4)J_2 \end{pmatrix}.$$

The eigenvalue configurations again are as in Figure 1.

3 Normal linear stability of invariant tori: the integrable case

Returning to the setting of §1, we start with an $(n+m)$-dimensional phase space $M$, a $p$-dimensional parameter space $P$ and a free $\mathbb{T}^n$-action on $M$. We are given a $p$-parameter family $X$ of integrable (i.e., $\mathbb{T}^n$-symmetric) vector fields on $M$ with an $X$-invariant submanifold $T$ of the form $T = \bigcup_{\mu \in U} T_\mu \times \{\mu\} \subset M \times P$, where $U$ is a domain containing $0 \in P \subset \mathbb{R}^p$ and the submanifolds $T_\mu \subseteq M$ are $X_\mu$-invariant $\mathbb{T}^n$-orbits. Recall that each orbit $T_\mu$ is diffeomorphic to the standard $n$-torus $\mathbb{T}^n$ carrying conditionally periodic (or parallel) dynamics, see §1.1. In the present section, we study persistence of the invariant torus family $T$ of the integrable family $X$, in cases where the perturbations are integrable as well. This persistence is very similar to the linear stability of vector fields at equilibria (as discussed in §2) and is a direct consequence of the Inverse Function
Theorem, compare with [67, 31]. As before, the persistence is formulated both for the dissipative, the Hamiltonian and the reversible settings. Following [67, 31], one can simplify the perturbation problem by transferring it to the so-called normal bundle of $T$, compare with a rescaling argument given in [88].

3.1 Transfer to normal bundle

Here we briefly revisit the ideas of the simplification of our perturbation problems, as explained in [67, 31]. For the sake of completeness, we recall the definition of the normal bundle: for a submanifold $V$ of $M$, the normal bundle $NV$ of $V$ is the quotient $NV = TV / TV$ of tangent bundles, the fibers of which are the vector spaces $T_x M / T_x V$ for $x \in V$, compare, e.g., [62]. For the $\mathbb{T}^n$-orbits $T_\mu \subseteq M$, by §1.1, we have the identification $NT_\mu \cong \mathbb{T}^n \times \mathbb{R}^m$ for each $\mu \in \mathcal{U}$. The vector field $X_\mu$ induces a normal linear vector field $NX_\mu$ on $NT_\mu$ as follows. For each $\varepsilon > 0$, we define a scaling operator

$$D_\varepsilon : (x, y) \in (NT_\mu \cong \mathbb{T}^n \times \mathbb{R}^m) \mapsto (x, \varepsilon y) \in M,$$

in a neighbourhood of the zero-section of $NT_\mu$. Now the normal linear part $NX_\mu$ of $X_\mu$ is defined as $\hat{NX}_\mu = \lim_{\varepsilon \to 0} D_\varepsilon^\ast (X_\mu)$. Hence, for the family $X$ as in (1), the corresponding normal linear part $NX$ in the coordinates $(x, y) \in \mathbb{T}^n \times \mathbb{R}^m$ obtains the form:

$$NX_\mu(x, y) = \omega(\mu) \frac{\partial}{\partial x} + \Omega(\mu)y \frac{\partial}{\partial y}.\quad (37)$$

To show normal linear stability for the family $X$ in the set $\mathcal{X}_P(M)$, see the conclusions of Theorem 1.4, it is sufficient to verify this stability for the normal linear part $NX$ in the space $\mathcal{X}_P(\mathbb{T}^n \times \mathbb{R}^m)$. Indeed, suppose that $NX$ is normally linearly stable. Then, there is a neighbourhood $\mathcal{V} \subset \mathcal{X}_P(\mathbb{T}^n \times \mathbb{R}^m)$ of $NX$ such that for any family $\tilde{X} \in \mathcal{V}$, there is a smooth conjugacy which sends $\tilde{X}$ to a family with $NX$ as the normal linear part. If $\varepsilon > 0$ is sufficiently small, then we can find an open neighbourhood $\mathcal{W} \subset \mathcal{X}_P(M)$ of $X$ such that the open set $D_\varepsilon^\ast \mathcal{W}$ is contained in $\mathcal{V}$, i.e., for any $\tilde{X} \in \mathcal{W}$, we have $D_\varepsilon^\ast \tilde{X} \in \mathcal{V}$. This implies that the family $D_\varepsilon^\ast X \in \mathcal{V}$, and therefore the family $X$, is normally linearly stable, compare with Figure 2.

3.2 Normal linear stability: integrable dissipative case

In the dissipative case, by transferring the perturbation problem to the normal bundle, we may replace $M$ by $\mathbb{T}^n \times \mathbb{R}^m$ and the family (1) by its normal linear part. In the following, we simply put $M = \mathbb{T}^n \times \mathbb{R}^m$ and the $p$-parameter integrable
family $X \in \mathcal{X}_P(M)$ as

$$X_\mu(x, y) = \omega(\mu) \frac{\partial}{\partial x} + \Omega(\mu)y \frac{\partial}{\partial y},$$

(38)

in vectorial notation. In the case where $X = X_\mu(x, y)$ is non-degenerate at the invariant torus $T_0 = \{(x, y, \mu) : (y, \mu) = (0, 0)\}$, the following lemma provides a simplification of the parameter dependence.

**Lemma 3.1** Let the $p$-parameter family $Y \in \mathcal{X}_P(M)$ be given by

$$Y_\mu(x, y) = [\omega(\mu) + O(|y|)] \frac{\partial}{\partial x} + [\Omega(\mu)y + O(|y|^2)] \frac{\partial}{\partial y},$$

(39)

where $\mu \in P$. Assume that $Y$ is non-degenerate at the torus $(y, \mu) = (0, 0)$ and that $	ext{cod} \Omega(0) = c$. Then, there exists a local reparametrization

$$\mu \in \mathbb{R}^p \mapsto (\alpha, \beta, \lambda) \in \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^{p-n-c}$$

such that the family $Y$ (after reparametrization) obtains the form

$$Y_{\alpha, \beta, \lambda}(x, y) = [(\omega(0) + \alpha) + O(|y|)] \frac{\partial}{\partial x} + [B(\beta)y + O(|y|^2)] \frac{\partial}{\partial y},$$

(40)

where $B(\beta)$ is an LCU of $\Omega_0 = \Omega(0)$.

**Proof.** Let $\tilde{\Omega} : (\mathbb{R}^c, 0) \mapsto (\mathfrak{gl}(m, \mathbb{R}), \Omega_0)$ be an LCU of $\Omega_0$. By versality, we have $\Omega(\mu) = Q(\mu) \tilde{\Omega}(\rho(\mu)) Q^{-1}(\mu)$, where $Q : (\mathbb{R}^p, 0) \mapsto (GL(m, \mathbb{R}), I_m)$ and $\rho : (\mathbb{R}^p, 0) \mapsto (\mathbb{R}^c, 0)$, compare with Definition 2.3. Due to universality of $\tilde{\Omega}$ and versality of $\Omega$, the map $\rho : \mathbb{R}^p \mapsto \mathbb{R}^c$ is a submersion at $\mu = 0$. Let $\tilde{\pi} : \mathfrak{gl}(m, \mathbb{R}) \mapsto \mathbb{R}^p$ be the local projection given by $\tilde{\pi}(\Omega(\mu)) = \mu$. Define the map $\pi = \rho \circ \tilde{\pi} : (\mathfrak{gl}(m, \mathbb{R}), \Omega_0) \mapsto (\mathbb{R}^c, 0)$. Notice that $\pi \circ \tilde{\Omega} = \rho$. Hence, $\pi : \mathbb{R}^p \mapsto \mathbb{R}^c$ is submersive at $\mu = 0$. By non-degeneracy, the germ

$$(\omega - \omega(0), \pi \circ \tilde{\Omega}) : (\mathbb{R}^p, 0) \mapsto (\mathbb{R}^n \times \mathbb{R}^c, (0, 0))$$
is submersive at \( \mu = 0 \). By the Inverse Function Theorem [52, 1], there exists a local diffeomorphism \( \phi : \mathbb{R}^p \to \mathbb{R}^p \) with \( \phi(0) = 0 \) such that

\[
(\omega - \omega(0), \pi \circ \Omega)(\phi(\mu)) = (\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^c.
\]

This implies that

\[
Y_{\phi(\mu)}(x, y) = [(\omega(0) + \alpha) + O(|y|)] \frac{\partial}{\partial x} + [\pi^{-1}(\beta) + O(|y|^2)] \frac{\partial}{\partial y},
\]

where \( B(\beta) = \pi^{-1}(\beta) \) is any universal unfolding of \( \Omega_0 \). In particular, we can choose \( B \) as an LCU of \( \Omega_0 \). \( \square \)

Now we formulate a normal linear stability for the integrable family \( X \) given by (38), regarding persistence of the \( X \)-invariant \( n \)-torus family

\[
T = \{(x, y, \mu) \in \mathbb{T}^n \times \mathbb{R}^n \times P : y = 0\} \subset M \times P
\]

under integrable perturbations. This integrable version of Theorem 1.4 is a consequence of the Inverse Function Theorem, also see Theorem 3.1 in [31].

**Theorem 3.2** (Normal Linear Stability: Integrable Dissipative Case)

*Let* \( X \in \mathcal{X}_P(M) \) *be the family of the vector fields given by* (38). *Assume that* \( X \) *is non-degenerate at the torus* \( T_0 = \{(x, y, \mu) : (y, \mu) = (0, 0)\} \). *Then, for any integrable family* \( \tilde{X} \in \mathcal{X}_P(M) \) *that is \( C^2 \)-close to* \( X \), *there exists a local diffeomorphism

\[
\Phi : M \times P \to M \times P
\]

*of the form* \( \Phi(x, y, \mu) = (x, \phi_{\mu}(y), \rho(\mu)) \) *defined near the torus* \( T_0 \) *such that*

i. \( \Phi \) *is* \( C^1 \)-near the identity map;

ii. *The image of the torus family* \( T \) *under* \( \Phi \) *is* \( \tilde{X} \)-invariant and for the restriction of* \( \tilde{\Phi} = \Phi|T \) *one has

\[
\tilde{\Phi}_*X = \tilde{X},
\]

*that is* \( \tilde{\Phi} \) *conjugates* \( X \) *to* \( \tilde{X} \);

iii. \( \Phi \) *preserves the normal linear behavior.*

**Proof.** Suppose that the perturbation \( \tilde{X} = \tilde{X}_\mu(y) \) is given by

\[
\tilde{X}_\mu(y) = f(y, \mu) \frac{\partial}{\partial x} + g(y, \mu) \frac{\partial}{\partial y}
\]
and that the required map $\phi_\mu$ is of the form $\phi_\mu(y) = q(\mu) + Q(\mu)y$. The map $q(\mu)$ can be obtained by an application of the Inverse Function Theorem. Indeed, since the family $X$ is non-degenerate at the torus $T_0$, the Inverse Function Theorem gives a smooth $\mu$-dependent map $g(\mu)$ such that $g(q(\mu), \mu) \equiv 0$. This implies that \{$y = q(\mu)$\} is an invariant torus family of the perturbation $\tilde{X}$. Next we need to find the map $Q$ and a suitable reparametrization $\rho$. By Lemma 3.1, there exists a reparametrization $\mu \in \mathbb{R}^p \mapsto (\alpha(\mu), \beta(\mu), \lambda(\mu)) \in \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^{p-n-c}$ such that $X$ (after reparametrization) takes the form:

$$X_{\alpha,\beta,\lambda}(x, y) = [\omega(0) + \alpha] \frac{\partial}{\partial x} + \tilde{\Omega}(\beta) y \frac{\partial}{\partial y},$$

where $\tilde{\Omega}(\beta)$ is an LCU of $\Omega_0$. The reparametrization $\rho$ will be chosen of the form $\rho(\alpha, \beta, \lambda) = (\rho_1, \rho_2, \lambda) \in \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^{p-n-c}$, where $\rho_1$ and $\rho_2$ depend on $(\alpha, \beta, \lambda)$, respectively. Now the map $\rho_1$ is determined by the parameter shift in the $\alpha$-direction, preserving the internal frequencies of the unperturbed family $X$. The maps $\rho_2$ in the $\beta$-direction and $Q$ can be obtained by an application of Theorem 2.6, also see the proof of Theorem 2.10. \hfill $\square$

3.3 Normal linear stability: integrable symplectic case

Summarizing from §1.2, we are given the $(2n + 2q)$-dimensional symplectic manifold $(M, \sigma)$ with a free (locally Hamiltonian) $\mathbb{T}^n$-action. By the generalized Darboux Theorem [1], near each $\mathbb{T}^n$-orbit, there are local coordinates $(x, y, z) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q}$ such that the orbit is given by the set \{(x, y, z) : (y, z) = (0, 0)\} and the 2-form $\sigma = \sum_{i=1}^n dx_i \wedge dy_i + \sum_{j=1}^q dz_j \wedge dz_{j+q} = dx \wedge dy + dz^2$. For a $p$-parameter family $X \in \mathcal{X}_\mathbb{T}^n(M)$ of integrable (i.e. $\mathbb{T}^n$-symmetric) Hamiltonian vector fields with $\mathbb{T}^n$-orbits as invariant submanifolds, we locally can write

$$X_\mu(x, y, z) = f(y, z, \mu) \frac{\partial}{\partial x} + h(y, z, \mu) \frac{\partial}{\partial z},$$

where $h(0, 0, \mu) \equiv 0$. Assuming that $\det(\partial h/\partial z)(0, 0, 0) \neq 0$, we again obtain a continuum of $X$-invariant $n$-tori, normalized to \{z = 0\}, locally parametrized by both $y$ and $\mu$. This means that $h(y, 0, \mu) \equiv 0$, where $\partial h/\partial z (y, 0, \mu) \in \mathfrak{sp}(2q, \mathbb{R})$ for each $(y, \mu)$. Moreover, we use the localized setting with the distinguished parameter $\nu$, varying over a neighbourhood of $0 \in \mathbb{R}^n$ and the localized coordinate $y_{loc} = y - \nu$, again see [67, 31]. This leads to the $n$-torus family $T_{loc}$ as in (9).

$$T_{loc} = \{(x, y_{loc}, z, \mu, \nu) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times \mathbb{R}^p \times \mathbb{R}^n \mid y_{loc} = 0, z = 0\},$$

where $(\mu, \nu)$ varies over a neighbourhood of $(0, 0) \in P \subset \mathbb{R}^p \times \mathbb{R}^n$, which is invariant for the extended integrable family $X_{\mu,\nu}(x, y_{loc}, z) = X_\mu(x, y_{loc} + \nu, z)$,
which has the form (10)

\[ X_{\mu,\nu}(x, y_{\text{loc}}, z) = f(y_{\text{loc}} + \nu, z, \mu) \frac{\partial}{\partial x} + h(y_{\text{loc}} + \nu, z, \mu) \frac{\partial}{\partial z}. \]

We investigate the normal linear stability of this (extended) family \( X = X_{\mu,\nu}(x, y_{\text{loc}}, z) \), regarding the persistence of the invariant tori family \( T_{\text{loc}} = \{ (y_{\text{loc}}, z) = (0, 0) \} \) under an integrable Hamiltonian perturbation. Non-degeneracy for the family \( X \) means: the Floquet matrix \( \frac{\partial}{\partial z}(\mu, \nu) \) has maximal rank at \((\mu, \nu) = (0, 0)\); the map \( \omega : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n \) given by \( \omega(\mu, \nu) = f(\nu, 0, \mu) \) is a submersion at \((\mu, \nu) = (0, 0)\), while ‘simultaneously’ the matrix family \( \Omega(\mu, \nu) \) is a versal unfolding of \( \Omega(0, 0) \) within the linear structure \((SP(2q, \mathbb{R}), \mathfrak{sp}(2q, \mathbb{R}))\).

For each fixed parameter value \((\mu, \nu)\), we denote by \( T_{\text{loc}} ;\) the \( n \)-torus \( T_{\text{loc}} = f(y_{\text{loc}}; z) = (0; 0) \) \( g \). Similar to the dissipative case, we use the normal bundle \( NT_{\text{loc}} \) of \( T_{\text{loc}} \), where \( NT_{\text{loc}} = T_{\text{loc}} \mathbb{R}^n \mathbb{R}^{2q} \) for each \((\mu, \nu)\), to simplify the problem, see §3.1. However, the scaling operator \( D_e \) which defines normal linearization, slightly differs:

\[ D_e (x, y_{\text{loc}}, z) = (x, \varepsilon^2 y_{\text{loc}}, \varepsilon z), \text{ for } \varepsilon > 0, \quad (41) \]

see [31] (§2a, p9). Since \( D_e \sigma = \varepsilon^2 N(\sigma) \), where \( N(\sigma) = dx \wedge dy_{\text{loc}} + dz^2 \) is the standard 2-form on the normal bundle \( NT_{\text{loc}} \), the pull-back \( D_e^* \) is a symplectic diffeomorphism for any \( \varepsilon > 0 \). The normal linear part \( NX \) of the family \( X \) is given by

\[ NX_{\mu,\nu}(x, y_{\text{loc}}, z) = \lim_{\varepsilon \to 0} D_e^* X_{\mu,\nu} = \omega(\mu, \nu) \frac{\partial}{\partial x} + \Omega(\mu, \nu) z \frac{\partial}{\partial z}. \quad (42) \]

which is Hamiltonian with respect to \( N(\sigma) \).

**Remark 3.3** For a non-integrable Hamiltonian vector field \( Y = Y(x, y, z) \) on \((M, \sigma)\), the scaling (41) does not induce a normal linearization as the form (42). Indeed, in the non-integrable case, the flow in the \( y_{\text{loc}} \)-direction in general is not constant, giving rise to a quadratic term in \( z \) for the vector field \( \lim_{\varepsilon \to 0} D_e Y \). To overcome this problem, one can use the (smaller) symplectic normal bundle of \( T \), for a discussion see [31] (§6b, p30) or Appendix C below.

Similar to the dissipative case, we now may assume that the phase space has the form \( M = \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} = \{ x, y, z \} \) with symplectic 2-form \( \sigma = dx \wedge dy + dz^2 \). The (extended) family \( X = X_{\mu,\nu}(x, y, z) \) may be taken as \( X_{\mu,\nu}(x, y, z) = \omega(\mu, \nu) \frac{\partial}{\partial z} + \Omega(\mu, \nu) z \frac{\partial}{\partial z} \). Since any integrable Hamiltonian vector field \( Y \in \mathcal{X}^\sigma(M) \) takes the form

\[ Y(x, y, z) = f(y, z) \frac{\partial}{\partial x} + g(y, z) \frac{\partial}{\partial z}, \]

33
the present symplectic situation is the same as that of the dissipative case, except that Floquet matrices now belong to the matrix space $\mathfrak{sp}(2q, \mathbb{R})$. Analogous to Theorem 3.2, we have the integrable version of Theorem 1.6.

**Theorem 3.4 (Normal Linear Stability: Integrable Symplectic Case)**

Let $X \in \mathcal{X}_{P}^{r}(M)$ be the family of Hamiltonian vector fields given by (10). Assume that $X$ is non-degenerate at the invariant torus $T_{loc}^{0,0} = T_{loc}\cap\{(\mu, \nu) = (0, 0)\}$. Then, for any integrable family $\tilde{X} \in \mathcal{X}_{P}^{r}(M)$ that is $C^2$-close to $X$, there exists a local diffeomorphism

$$\Phi : M \times P_{loc} \rightarrow M \times P_{loc}$$

of the form $\Phi(x, y, z, \mu) = (x, y, \phi_{\mu}(z), \rho(\mu))$ defined near the torus $T_{loc}^{0,0}$ such that

i. $\Phi$ is a $C^1$-near the identity map;

ii. The image of the torus family $T$ under $\Phi$ is $\tilde{X}$-invariant and for the restriction of $\Phi = \Phi|T$ one has

$$\hat{\Phi}_{*}X = \tilde{X},$$

that is $\hat{\Phi}$ conjugates $X$ to $\tilde{X}$;

iii. $\Phi$ is symplectic and preserves the normal linear behavior.

### 3.4 Normal Linear Stability: Integrable Reversible Case

We briefly recall elements from §1.3, where on the phase space $M$ a free $\mathbb{T}^n$-action is given, as well as an involution $G$, that commute. Using Bochner’s theorem as before [15, 84], also compare with [30], we may assume that the phase space is $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2q} = \{x, y, z\}$, where $G$ has the form $G(x, y, z) = (-x, y, Rz)$. Here $R \in GL(2q, \mathbb{R})$ is a linear evolution, where we assume that the eigenvalue occurs with multiplicity $p$. Also a free $A$ vector field $X$ on $M$ $G$-reversible whenever

$$G_{*}(X) = -X.$$  

We recall that integrability of $X$ amounts to equivariance under the $\mathbb{T}^n$-action. We consider the space $\mathcal{X}_{P}^{-G}(\mathbb{R}^{2q})$ of all $G$-reversible families of vector fields, parametrized over a domain $P \subset \mathbb{R}^p$. For an integrable family $X \in \mathcal{X}_{P}^{-G}(\mathbb{R}^{2q})$ we consider invariant tori, that are both $G$-invariant and orbits under the $n$-torus action, for their persistence under small perturbation. In the present section we apply only integrable perturbations.

We recall that under generic circumstances such invariant tori come in continua, locally parametrized by $(y, \mu)$, say in a neighbourhood of $(y_0, \mu_0)$, that can
safely be put at \((0, 0)\), and that by a \(G\)-equivariant change of coordinates coincide with the submanifold \(\{z = 0\}\). This leads to the normalized format (15)

\[
X_\mu(x, y, z) = f(y, z, \mu) \frac{\partial}{\partial x} + g(y, z, \mu) \frac{\partial}{\partial y} + h(y, z, \mu) \frac{\partial}{\partial z},
\]

with \(g(y, 0, \mu) = 0\) and \(h(y, 0, \mu) = 0\). Introducing again the localizing parameter \(\nu\) and coordinate \(y_{loc} = y - \nu\), we obtain an extended integrable family \(X_{\mu, \nu}(x, y_{loc}, z) = X_\mu(x, y_{loc} + \nu, z)\) of the form (17)

\[
X_{\mu, \nu}(x, y_{loc}, z) = f(y_{loc} + \nu, z, \mu) \frac{\partial}{\partial x} + g(y_{loc} + \nu, z, \mu) \frac{\partial}{\partial y_{loc}} + h(y_{loc} + \nu, z, \mu) \frac{\partial}{\partial z}
\]

with invariant tori (16)

\[
T_{loc} = \{(x, y_{loc}, z, \mu, \nu) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^2 | y_{loc} = 0, z = 0\},
\]

where \((\mu, \nu)\) varies over a neighbourhood of \((0, 0) \in P_{loc} \subset \mathbb{R} \times \mathbb{R}^m\). Both the reversor \(G\) and the commuting \(n\)-torus action again directly carry over to this new setting.

One last reduction is a scaling operator, that reduces the perturbation problem to the normal bundle \(NT_{loc}\). Similar to the dissipative and the symplectic case, we have that \(NT_{loc} \cong \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^2\), where now a scaling operator

\[
\mathcal{D}_\varepsilon(x, y, z) = (x, \varepsilon y, \varepsilon^2 z), \text{ for } \varepsilon > 0
\]

has to be applied, see [28]. The pull back vector field \(\mathcal{D}_\varepsilon^* X_{\mu, \nu}\) is well-defined for all \(\varepsilon > 0\), and so is the limit

\[
NX_{\mu, \nu}(x, y_{loc}, z) = \lim_{\varepsilon \to 0} \mathcal{D}_\varepsilon^* X_{\mu, \nu} = \omega(\mu, \nu) \frac{\partial}{\partial x} + \Omega(\mu, \nu) z \frac{\partial}{\partial z}
\]

with \(\omega(\mu, \nu) = f(\nu, 0, \mu)\) and \(\Omega(\mu, \nu) = h_z(\nu, 0, \mu)\). This is the normal linear part, properly defined on the normal bundle \(NT_{loc}\), to which both the reversor \(G\) and the commuting \(n\)-torus action again directly carry over.

We recall the non-degeneracy condition imposed on the product map \(\omega \times \Omega : \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^n \times gl_{-R}(2q, \mathbb{R})\). The matrix \(\Omega(0, 0)\) has maximal rank, while ‘simultaneously’ the map \(\omega\) has to be a submersion at \((\mu, \nu) = (0, 0)\), the component \(\Omega(\mu, \nu)\) is a versal unfolding of \(\Omega(0, 0)\) within the linear structure \((GL_R(2q, \mathbb{R}), gl_{-R}(2q, \mathbb{R}))\), see §2.1, Example 2.2 as well as §2.4. Analogously to Theorems 3.2 and 3.4, we now formulate the integrable version of Theorem 1.8.
Theorem 3.5 (Normal linear stability: integrable reversible case)
Let the reversible family $X \in \mathcal{X}_P^G(M)$ be given by (17). Assume that $X$ is non-degenerate at the invariant $n$-torus $T^0_{\text{loc}} = T_{\text{loc}} \cap \{(\mu, \nu) = (0,0)\}$. Then for any integrable family $X \in \mathcal{X}_P^G(M)$ that is sufficiently $C^2$ close to $X$, there exists a local diffeomorphism

$$\Phi : M \times P_{\text{loc}} \mapsto (\phi_\mu(z), \rho(\mu)) \in M \times P_{\text{loc}}$$

of the form $(x, y_{\text{loc}}, z, \mu, \nu) \mapsto (x, y_{\text{loc}}, \Phi_\mu(z), \rho(\mu), \nu)$, defined near the torus $T^0_{\text{loc}}$, such that,

i. $\Phi$ is $C^1$-near the identity map;

ii. The image of the tori $T_{\text{loc}}$ under $\Phi$ is $\tilde{X}$-invariant and for the restriction $\tilde{\Phi}|_{T_{\text{loc}}}$ we have

$$\tilde{\Phi}|_{T_{\text{loc}}} = \tilde{X},$$

that is, $\Phi$ conjugates $X$ to $\tilde{X}$;

iii. $\Phi$ is $G$-equivariant and preserves the normal linear behaviour.

4 Case study: generic $1 : -1$ resonance

Normal linear stability of vector fields with invariant tori for the nearly-integrable case is discussed in §1.1 (the dissipative setting), 1.2 (the symplectic setting) and 1.3 (the reversible setting). As we already pointed out in §1.2, a special case of Theorem 1.6 occurs when the Floquet matrix of the central torus is in non-semisimple or generic $1 : -1$ resonance. This generically involves a quasi-periodic Hamiltonian Hopf bifurcation [20, 79]. The quasi-periodic reversible Hopf bifurcation is dealt with in [17] and will not be discussed here. The Hamiltonian $1 : -1$ resonance is of interest, since it is observed in many mechanical systems including the Lagrange top [45, 42], the double spherical pendulum [78], the restricted three body problem [79] and the $3D$ Hénon-Heiles family as this models the movement of galaxies [49, 58, 59].

Let us briefly recall the setting from §1.2. According to §3.3, we take the phase space $M = T^n \times \mathbb{R}^n \times \mathbb{R}^4 = \{x, y, z\}$ with the symplectic 2-form $\sigma = dx \wedge dy + dz^2$. We consider the integrable Hamiltonian family $X = X_{\mu,\nu}$ of the form

$$X_{\mu,\nu}(x, y, z) = \omega(\mu, \nu) \frac{\partial}{\partial x} + \Omega(\mu, \nu)z \frac{\partial}{\partial z},$$

where $\Omega(\mu, \nu) \in \mathfrak{sp}(4, \mathbb{R})$, with the invariant tori $\{(x, y, z) : (y, z) = (0,0)\}$. To simplify notation, we include $\nu$ into a general $\mu$ (i.e., we use $\mu$ instead of $(\mu, \nu)$).
The Floquet matrix $\Omega_0 = \Omega(0)$ is assumed to be in generic $1: -1$ resonance, that is, it has a double pair of purely imaginary eigenvalues with a non-trivial nilpotent part. Then, the matrix $\Omega_0$ is symplectically conjugate to the real normal form
\[
\begin{pmatrix}
0 & -\lambda_0 & 0 & 0 \\
\lambda_0 & 0 & 0 & 0 \\
\varepsilon & 0 & 0 & -\lambda_0 \\
0 & \varepsilon & \lambda_0 & 0 \\
\end{pmatrix},
\]
where $\lambda_0 \neq 0$ and $\varepsilon = \pm 1$, see 2.5. As before, we assume that $X$ is non-degenerate at the torus $(z, \mu) = (0, 0)$, that is, the map $\omega : \mathbb{R}^p \to \mathbb{R}^n$ is submersive at $\mu = 0$, while $\Omega$ is versal at $\mu = 0$ within the linear structure $(\text{SP}(4, \mathbb{R}), \text{sp}(4, \mathbb{R}))$, see §2.1. We recall that by Proposition 2.14 the co-dimension of $\Omega_0$ is equal to 2. By Lemma 3.1, after a suitable reparametrization $\mu \in \mathbb{R}^p \to (\tilde{\omega}, \tilde{\mu}, \tilde{\nu}) \in \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{p-n-2}$, the family $X$ we get the following:
\[
X_{\tilde{\omega},\tilde{\mu},\tilde{\nu}}(x, y, z) = \frac{\partial}{\partial x} + \tilde{\Omega}(\tilde{\mu}) z \frac{\partial}{\partial z},
\]
where $\tilde{\Omega}$ is an LCU of $\Omega_0$ within $(\text{SP}(4, \mathbb{R}), \text{sp}(4, \mathbb{R}))$. In what follows, for simplicity we suppress the parameter $\tilde{\nu}$, since it can easily be incorporated again in our discussion below. Also in (46) we drop all tildes. Furthermore, again by Proposition 2.14, the LCU $\Omega(\mu)$ can be taken of the form
\[
\Omega(\mu) = \begin{pmatrix}
0 & -\lambda_0 - \mu_1 & -\mu_2 & 0 \\
\lambda_0 + \mu_1 & 0 & 0 & -\mu_2 \\
\varepsilon & 0 & 0 & -\lambda_0 - \mu_1 \\
0 & \varepsilon & \lambda_0 + \mu_1 & 0 \\
\end{pmatrix}.
\]
The eigenvalues of $\Omega(\mu)$ read
\[
\pm i(\lambda_0 + \mu_1 \pm \sqrt{\varepsilon \mu_2}).
\]
In what follows, let us stick to the integrable family $X$ as in (46), where $\Omega(\mu)$ is given by the LCU (47). With these choices, the normal frequency vector $\omega^N(\mu)$ of the family $X$ is given by
\[
- \varepsilon \mu_2 \leq 0 : \quad \omega^N(\mu) = (\lambda_0 + \mu_1, \lambda_0 + \mu_1); \quad (48)
\]
\[
- \varepsilon \mu_2 > 0 : \quad \omega^N(\mu) = (\lambda_0 + \mu_1 + \sqrt{\varepsilon \mu_2}, \lambda_0 + \mu_1 - \sqrt{\varepsilon \mu_2}). \quad (49)
\]
Let $\mathcal{U}$ be a convex domain around the origin in the parameter space $\mathbb{R}^n \times \mathbb{R}^2$. Denote $\mathcal{U}^- = \{ (\omega, \mu) \in \mathcal{U} : \varepsilon \mu_2 \leq 0 \}$ and $\mathcal{U}^+ = \{ (\omega, \mu) \in \mathcal{U} : \varepsilon \mu_2 > 0 \}$. Following §1.1, the (generalized) frequency map $\mathcal{F} : \mathcal{U} \to \mathbb{R}^n \times \mathbb{R}^2$ here is given by
\[
\mathcal{F}(\omega, \mu) = \begin{cases} 
(\omega, \lambda_0 + \mu_1, \lambda_0 + \mu_1), \text{ for } \varepsilon \mu_2 \leq 0 \\
(\omega, \lambda_0 + \mu_1 + \sqrt{\varepsilon \mu_2}, \lambda_0 + \mu_1 - \sqrt{\varepsilon \mu_2}), \text{ for } \varepsilon \mu_2 > 0.
\end{cases}
\]

Recall from Remark 1.3 that the set \((\mathbb{R}^n \times \mathbb{R}^2)_{\tau, \gamma}\) consists of all pairs \((\omega, \omega^N) \in \mathbb{R}^n \times \mathbb{R}^2\) satisfying the \((\tau, \gamma)\)-Diophantine conditions (2). The ‘Cantor set’ \(\Gamma_{\tau, \gamma}(U)\) is the union \(\Gamma_{\tau, \gamma}(U) = \Gamma_{\tau, \mu}(U^-) \cup \Gamma_{\tau, \mu}(U^+),\) where

\[
\Gamma_{\tau, \gamma}(U^-) = \{(\omega, \mu) \in U^- : \mathcal{F}(\omega, \mu) \in (\mathbb{R}^n \times \mathbb{R}^2)_{\tau, \gamma}\}.
\]

\[
\Gamma_{\tau, \gamma}(U^+) = \{(\omega, \mu) \in U^+ : \mathcal{F}(\omega, \mu) \in (\mathbb{R}^n \times \mathbb{R}^2)_{\tau, \gamma}\}.
\]

**Remark 4.1** Recall that for \((\omega, \mu) \in \mathbb{R}^n \times \mathbb{R}^2\) we have \((\omega, \mu) \in \Gamma_{\tau, \gamma}(U^+)\) if and only if

\[
|\langle \omega, k \rangle + \ell_1(\lambda_0 + \mu_1) + \ell_2 \sqrt{|\mu_2|}| \geq \gamma |k|^{-\tau},
\]

for all \(k \in \mathbb{Z}^n \setminus \{0\}\) and for integers \(|\ell_1|, |\ell_2| \leq 2\), where \((\ell_1 \pm \ell_2)\) is even. On the other hand, the set \(\Gamma_{\tau, \gamma}(U^-)\) consists of \((\omega, \mu) \in \mathbb{R}^n \times \mathbb{R}^2\) satisfying

\[
|\langle \omega, k \rangle + \ell(\lambda_0 + \mu_1)| \geq \gamma |k|^{-\tau},
\]

for all \(k \in \mathbb{Z}^n \setminus \{0\}\) and for all \(|\ell| = 0, 1, 2\).

Let us briefly describe the geometric of the ‘Cantor sets’ \(\Gamma_{\tau, \gamma}(U^-)\) and \(\Gamma_{\tau, \gamma}(U^+)\) for \(\varepsilon = 1\). In \(\Gamma_{\tau, \gamma}(U^+)\) there no Cantor gaps occur in the \(\mu_2\)-direction, since the parameter \(\mu_2\) does not occur in the normal frequencies, see (48). The intersections of planes \(\{\mu_2 = \text{const.}\}\) with \(\Gamma_{\tau, \gamma}(U^-)\) project to subsets of the \((\omega, \mu_1)\)-plane satisfying the Diophantine conditions (52). By Remark 1.3, this subset of \((\omega, \mu_1)\) is a Cantor family of closed half lines, compare with Figure 3(a). Taking the (continuous) \(\mu_2\)-direction into account, it follows that \(\Gamma_{\tau, \gamma}(U^-)\) is a Cantor family of closed half planes of the form \(\{(t\omega, t(\lambda_0 + \mu_1)) : t \geq 1\}\), where evidently all takes place inside the set \(U^-\). Regarding the set \(\Gamma_{\tau, \gamma}(U^+)\) we observe that if \((\omega, \lambda_0 + \mu_1, \mu_2)\) satisfies the Diophantine conditions (51), so do all \((t\omega, t(\lambda_0 + \mu_1), t^2 \mu_2)\) for \(t \geq 1\). Therefore, the set \(\Gamma_{\tau, \gamma}(U^+)\) is a Cantor family of closed half parabole of the form \(\{(t\omega, t(\lambda_0 + \mu_1), t^2 \mu_2) : t \geq 1\}\), see Figure 3(b).

As a special case of Theorem 1.6, we now have the following KAM stability statement for the generic (or non-semisimple) \(1 : -1\) resonance.

**Corollary 4.2** (NORMAL LINEAR STABILITY: GENERIC \(1 : -1\) RESONANT CASE) Let \(X\) be a \(p\)-parameter real-analytic family of Hamiltonian vector fields defined on the phase space \(M = \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^4 = \{x, y, z\}\) with the symplectic 2-form \(\sigma = dx \wedge dy + dz^2\), where \(X_\mu(x, y, z) = \omega(\mu)\frac{\partial}{\partial z} + \Omega(\mu)z\frac{\partial}{\partial z}\). Assume that

- \(X\) is nondegenerate at the \(n\)-torus \((y, z, \mu) = (0, 0, 0)\);
- The matrix \(\Omega(0) \in \mathfrak{sp}(4, \mathbb{R})\) is in generic \(1 : -1\) resonance.
Then, for \( \gamma > 0 \) sufficiently small and for any \( \tilde{X} \in X_{\mathbb{R}^p}(M) \) sufficiently close to \( X \) in the compact-open topology on complex analytic extensions, there exists a domain \( U \) around \( 0 \in \mathbb{R}^p \) and a map
\[
\Phi : M \times U \to M \times \mathbb{R}^p,
\]
defined near the torus \((y, z, \mu) = (0, 0, 0)\) such that

i. \( \Phi \) is a \( C^\infty \)-near-identity diffeomorphism onto its image;

ii. The image of the Diophantine tori
\[
V = \bigcup_{\mu \in \Gamma_{\gamma}(U)} \mathbb{T}^n \times \{y = 0\} \times \{z = 0\} \times \{\mu\}
\]
under \( \Phi \) is \( \tilde{X} \)-invariant, and for the restriction \( \hat{\Phi}|_{T_{loc}} \) we have
\[
\hat{\Phi}_*(X) = \tilde{X},
\]
that is, \( \hat{\Phi} \) conjugates \( X \) to \( \tilde{X} \);

iii. The restriction \( \Phi|_V \) is symplectic and preserves the (symplectic) normal linear part of \( X \).

5 Application: nearly-integrable perturbations of the Lagrange top

In the present section, we illustrate an application of Corollary 4.2 to the Lagrange top coupled with a quasi-periodic oscillator. We consider a local perturbation problem concerning the persistence of the quasi-periodic invariant tori associated with the upward spinning of the top and the quasi-periodic motion of the forcing. Our main interest is with the situation where the Lagrangian top becomes gyroscopically stabilized by Hamiltonian Hopf bifurcation [45]. For a detailed discussion on this problem see [20]. The Lagrange top is an axially symmetric rigid body, situated in a constant gravitational field pointing downwards vertically, with its base point fixed in space, see e.g. [54, 75, 42]. The configuration space of the top is the 3-dimensional Lie group \( SO(3) \) of all (orientation-preserving) rotations of \( \mathbb{R}^3 \) and the phase space is the tangent bundle \( TSO(3) \cong SO(3) \times \mathfrak{so}(3) \). The Lagrange top is described by a Hamiltonian system in the phase space \( M = (SO(3) \times \mathfrak{so}(3), \sigma) \), where the symplectic form \( \sigma \) is inherited from the canonical 2-form of the bundle \( T^*SO(3) \), compare with [1, 101, 42]. The Hamiltonian
Figure 3: Sketch of the ‘Cantor sets’ $\Gamma_{\tau,\gamma}(U^{-})$ and $\Gamma_{\tau,\gamma}(U^{+})$ corresponding to $\mu_{2} \geq 0$ and $\mu_{2} < 0$, respectively, in the $(\omega, \mu_{1}, \mu_{2})$-space. The union $\Gamma_{\tau,\gamma}(U) = \Gamma_{\tau,\gamma}(U^{-}) \cup \Gamma_{\tau,\gamma}(U^{+})$ is sketched in (c). The half planes in (b) and (c) give continua of invariant $m$-tori. In (d) a section of the ‘Cantor set’ $\Gamma_{\tau,\gamma}(U)$, along the $\mu_{2}$-axis, is singled out: the grey region above corresponds to a half plane given in (c): this is a continuum of $n$-tori.
is defined as the total energy of the system, i.e., as the sum of the kinetic and potential energy of the top. The Lagrange top has two rotational symmetries: the rotations about the body-symmetric (or figure) axis and about the vertical axis. Let $\mathcal{S} \subset SO(3)$ denote the subgroup of rotations preserving the vertical axis. Then, for a suitable choice of the fixed space-coordinate system, the two symmetries correspond to a symplectic right action $\Phi^r$ and to a symplectic left action $\Phi^l$ of the Lie subgroup $\mathcal{S}$ on $M$. By the Noether Theorem [1, 8], these Hamiltonian symmetries give rise to integrals $\mathcal{M}^r$ and $\mathcal{M}^l$ of the Hamiltonian $H$: the angular momenta along the figure axis and along the vertical axis, respectively. These integrals induce the so-called energy-momentum map $\mathcal{EM} = (H, \mathcal{M}^r, \mathcal{M}^l) : M \to \mathbb{R}^3$. The inverse images of the map $\mathcal{EM}$ give a singular foliation of the phase space consisting of $X_H$-invariant tori, see [42], which provides a qualitative picture of the dynamics of the top.

Next, we consider a perturbation problem where the Lagrange top is weakly coupled to a quasi-periodic oscillator, e.g., the base point of the top is coupled to a vibrating table-surface by a massless spring, see Figure 4. As said before, in this case the spring constant $\varepsilon$ acts as a perturbation. More generally, we consider the perturbed Hamiltonian $H_\varepsilon$ of the form

$$H_\varepsilon = H + G + \varepsilon F,$$  \hfill (53)

where $H$ and $G$ are the Hamiltonians of the Lagrange top and of the oscillator, respectively. The function $F$ depends on the coupling between the top and the oscillator. We assume that the oscillator is Liouville-integrable and has $n \geq 1$ frequencies. Gyroscopic stabilisation corresponds to a normally resonant invariant torus in the phase space. Near this torus the new phase space $\tilde{M} \times \mathbb{T}^n \times \mathbb{R}^n$ has a singular foliation consisting of invariant $(n + 1)$-, $(n + 2)$- and $(n + 3)$-tori. This local foliation gives a stratification by tori in the $(a, b, h)$-space, where $(a, b, h)$ are the values of $(\mathcal{M}^r, \mathcal{M}^l, H)$, as sketched in Figure 5. Our present concern is

Figure 4: The Lagrange top coupled with a vibrating table-surface by a spring.
with the fate of these invariant \((n + 1)\)-tori for small but non-zero \(\varepsilon\). In particular, we address this persistence problem in the case when the Lagrange top is close to gyroscopic stabilisation. We speak of gyroscopic stabilisation when the unstable spinning motion of a vertical top becomes stable as the angular-momentum value \(a\) increases and passes through a critical value \(a_0\). This physical phenomenon is explained by a generic Hamiltonian Hopf bifurcation, see [79, 45, 42].

Before addressing the perturbation problem, let us first consider the local dynamics of the (unperturbed) Lagrange top close to gyroscopic stabilisation. To this end, one applies a regular reduction [77, 1] by the right symmetry \(r\) to the three-degrees-of-freedom Hamiltonian \(H\) as follows. For a fixed value \(a\), we deduce from \(H\) a two-degrees-of-freedom Hamiltonian \(H_a\) on the four-dimensional orbit space \(M_a = (M^r_r)^{-1}(a)/S\) under the \(r\)-action. This space \(M_a\) can be identified with the four-dimensional submanifold \(R_a = \{(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 : u \cdot u = 1, u \cdot v = a\}\) with the induced symplectic form \(\omega_a\) given by

\[
\omega_a(u, v)((x, y), (p, q)) = \hat{x} \cdot \hat{q} - \hat{y} \cdot \hat{p} + v \cdot (\hat{v} \times \hat{p}),
\]

where \((x, y) = (\hat{x} \times u, \hat{x} \times v + \hat{y}) \in T_{(u,v)}R_a, (p, q) = (\hat{p} \times u, \hat{p} \times v + \hat{q}) \in T_{(u,v)}R_a\). Here \(\cdot\) and \(\times\) denote the standard inner and cross product of \(\mathbb{R}^3\), respectively. The reduced Hamiltonian \(H_a : R_a \to \mathbb{R}\) thereby obtains the form

\[
H_a(u, v) = \frac{1}{2}v \cdot v + cu_3 + \frac{1}{2}(\kappa - 1)a^2,
\]

where \(\cdot\) denotes the standard inner product and \(c, \kappa > 0\). Observe that \(p_a = (0, 0, 1, 0, 0, a) \in R_a\) is an isolated (relative) equilibrium, corresponding to a periodic solution of the full Hamiltonian system.

In suitable local Darboux coordinates \(z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4\) near the critical point \(p_a\), the reduced Hamiltonian \(H_a\) reads

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_2 + O(|z|^4),
\]

where \(\mathcal{H}_0 = H_a(p_a)\) is the constant part and \(\mathcal{H}_2\) is the quadratic terms given by

\[
\mathcal{H}_2 = \frac{1}{8}(a^2 - 4c)(z_1^2 + z_2^2) + \frac{1}{2}(z_3^2 + z_4^2) + \frac{1}{2}a(z_2z_3 - z_1z_4),
\]

see [42]. From this quadratic part, we see that gyroscopic stabilisation takes place as the parameter \(\mu_2 = \frac{1}{4}a^2 - c\) passes through the critical value \(\mu_2 = 0\). Re-including the rotation about the figure axis in the reduced local model \(\mathcal{H}\), we recover a full local Hamiltonian model \(\mathcal{H}_{loc}\) for motions near the gyroscopic stabilisation as follows. Let \(q_1\) and \(p_1\) denote the rotation and the angular momentum about the figure axis, respectively. Following [95], the variables \((q_1, p_1) \in S^1 \times \mathbb{R},\)
together with the local coordinates \((z_1, z_2, z_3, z_4) \in \mathbb{R}^4\), provide local symplectic coordinates in a neighbourhood of the singularity \(p_a\), such that, the symplectic 2-form is given by \(dq_1 \wedge dp_1 + dz_1 \wedge dz_3 + dz_2 \wedge dz_4\) and the Hamiltonian \(H\) of the Lagrange top locally takes the form:

\[
H(q_1, p_1, z) = \frac{1}{2} \kappa (a + p_1)^2 + c + \frac{1}{8} \left((a + p_1)^2 - 4c\right)(z_1^2 + z_2^2) + \frac{1}{2}(a + p_1)(z_2 z_3 - z_1 z_4) + \frac{1}{2}(z_3^2 + z_4^2) + O(|z|^4). \tag{55}
\]

Now let us turn back to our perturbation problem \((53)\) for \(\varepsilon \neq 0\). Since our interest is with perturbations of the Lagrange top near gyroscopic stabilisation, we use the local model \((55)\) for the unperturbed Hamiltonian of the Lagrangian top. Then, the perturbed Hamiltonian system \(H_\varepsilon\) reads

\[
H_\varepsilon(p_1, z, \xi, \eta) = H(q_1, p_1, z) + G(\eta) + \varepsilon F(q_1, p_1, z, \xi, \eta, \varepsilon), \tag{56}
\]

where \((q_1, p_1, z) \in \mathbb{S} \times \mathbb{R} \times \mathbb{R}^4\) and where \((\xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n\) denote angle-action coordinates of the oscillator. The Hamiltonian vector field\(^5\) \(X_\varepsilon\) associated to \(H_\varepsilon\) is given by

\[
X_\varepsilon : \begin{cases} 
\dot{q}_1 = \kappa (a + p_1) + O(|z|^2) + O(\varepsilon) \\
\dot{p}_1 = O(\varepsilon) \\
\dot{\xi}_1 = \sigma_1(\eta) + O(\varepsilon) \\
\dot{\eta}_1 = O(\varepsilon) \\
\vdots \\
\dot{\xi}_n = \sigma_n(\eta) + O(\varepsilon) \\
\dot{\eta}_n = O(\varepsilon) \\
\dot{z} = \Omega_{a,c} z + O(|p_1|, |z|^3) + O(\varepsilon),
\end{cases} \tag{57}
\]

where \(\sigma(\theta)\) gives the frequencies of the quasi-periodic motion of the forcing restricted to the invariant torus \(\eta = \theta\), and where the Floquet matrix \(\Omega_{a,c} \in \text{sp}(4, \mathbb{R})\) is of the form

\[
\Omega_{a,c} = \begin{pmatrix}
0 & \mu_1 & 1 & 0 \\
-\mu_1 & 0 & 0 & 1 \\
-\mu_2 & 0 & 0 & \mu_1 \\
0 & -\mu_2 & -\mu_1 & 0
\end{pmatrix}, \tag{58}
\]

where \(\mu_1 = \frac{1}{2}a\) and \(\mu_2 = \frac{1}{4}a^2 - c\). Near the invariant quasi-periodic \(n\)-torus \(\{\eta = \theta\}\) of the forcing, we introduce the local coordinate \(\tilde{\eta} = \eta - \theta\). Our goal finally is to study the persistence of the invariant \((n+1)\)-torus \(\{p_1 = 0, \tilde{\eta} = 0, z = 0\}\).

\(^5\)With respect to the symplectic form \(dq_1 \wedge dp_1 + d\xi \wedge d\eta + dz^2\).
To this end, we let
\[
x = (q_1, \xi) = (q_1, \xi_1, \ldots, \xi_n) \in \mathbb{T}^1 \times \mathbb{T}^n
\]
(59)
\[
y = (p_1, \eta) = (p_1, \eta_1, \ldots, \eta_n) \in \mathbb{R}^1 \times \mathbb{R}^n
\]
(60)
Also, we write
\[
\omega = \omega(a, \theta) = (\kappa a, \sigma_1(\theta), \ldots, \sigma_n(\theta)) \in \mathbb{R}^{n+1}
\]
(61)
for the internal frequencies of the invariant \((n + 1)\)-torus
\[
\{(x, y, z) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^4 : (y, z) = (0, 0)\}
\]
of the unperturbed Hamiltonian system \(X^0\). In the new coordinates \((x, y, z)\), the perturbed Hamiltonian \(X^\varepsilon = X^\varepsilon(x, y, z)\) reads
\[
X^\varepsilon(x, y, z) = \left[ \omega(a, \theta) + O(|y|, |z|^2) + O(\varepsilon) \right] \frac{\partial}{\partial x} + O(\varepsilon) \frac{\partial}{\partial y} + \left[ \Omega_{a,c} z + O(|y|, |z|^2) + O(\varepsilon) \right] \frac{\partial}{\partial z},
\]
(62)
where the matrix \(\Omega_{a,c}\) is given by (58). By considering the conserved quantities \((a, \theta)\) and the physical quantities \((c, \kappa)\) as parameters, the unperturbed system \(X^0\) becomes an \((n + 3)\)-parameter family of vector fields on \(\mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^4\) parameterized by
\[
\nu = (\theta, a, c, \kappa) \in \mathbb{R}^{n+1} \times \mathbb{R}^2.
\]
Given all this, application of Corollary 4.2 gives the following stability theorem regarding the invariant \((n + 1)\)-tori \(\{(y, z) = (0, 0)\}\):

**Theorem 5.1 (Forced Lagrange Top: Persistence of Diophantine \((n+1)\)-Tori)** Let \(\nu_0 = (\theta_0, \pm 2 \sqrt{c_0}, c_0, \kappa_0) \in \mathbb{R}^{n+3}\), where \(\theta_0, c_0, \kappa_0\) are fixed with \(c_0, \kappa_0 > 0\). Assume that the map
\[
\eta \in \mathbb{R}^n \mapsto (\sigma_1(\eta), \ldots, \sigma_n(\eta)) \in \mathbb{R}^n
\]
(63)
is a local diffeomorphism at \(\eta = \theta_0\). Then, for sufficiently small \(|\varepsilon|\), there exists a neighbourhood \(U\) of \(\nu_0\) in the parameter space \(\mathbb{R}^{n+3}\), and a local map
\[
\Phi : \mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^4 \times U \to \mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^4 \times \mathbb{R}^{n+3}
\]
defined near the torus \(\mathbb{T}^{n+1} \times \{y = 0\} \times \{z = 0\} \times \{\nu_0\}\), such that,

i. \(\Phi\) is a \(C^\infty\)-smooth diffeomorphism onto its image and is \(C^\infty\)-near the identity map;
ii. The image $\tilde{V}$ of invariant Diophantine $(n + 1)$-tori

$$V = \bigcup_{\nu \in \Gamma_{\tau,\gamma}(U')} \left( \mathbb{T}^{n+1} \times \{ y = 0 \} \times \{ z = 0 \} \times \{ \nu \} \right)$$

of the unperturbed system $X^0$ under the map $\Phi$ is $X^\varepsilon$-invariant, and the restriction of $\Phi$ to $V$ induces a conjugacy between $X^0$ and $X^\varepsilon$.

iii. The normal linear part of the perturbed family $X^\varepsilon$ on $\tilde{V}$ is (symplectically) conjugate to $NX^0 = \omega \frac{\partial}{\partial z} + \Omega z \frac{\partial}{\partial z}$, where $\nu = (\theta, \alpha, c, \kappa)$ is restricted to the ‘Cantor set’ $\Gamma_{\tau,\gamma}(U')$.

Theorem 5.1 roughly says that the quasi-periodic motions of a weakly forced Lagrange top can be predicted, based on the uncoupled motion of the top and the forcing. More precisely, when $\varepsilon = 0$, we have a family of invariant $(n + 1)$-tori carrying quasi-periodic flow generated by the vertical upwards spinning of the top and the quasi-periodic motion of the forcing in the phase space. Theorem 5.1 says that the ‘majority’ of these quasi-periodic invariant $(n + 1)$-tori survives when $\varepsilon$ is sufficiently small. Moreover, by conclusion (iii), the local dynamics of the unperturbed system $X^0$, near the surviving invariant $(n + 1)$-tori, also is preserved.

For completeness, we mention that invariant $(n + 1)$-tori of $X^0$ in the phase space are ‘surrounded’ by invariant elliptic $(n + 2)$- and Lagrangian $(n + 3)$-dimensional tori. The local Hamiltonian $H$ for the Lagrange top, see (55), undergoes a generic Hamiltonian Hopf bifurcation at $\mu_2 = 0$. Following [79, 20], the projection of this foliation by tori into the $(a, b, h)$-space, near the Hamiltonian Hopf bifurcation point, is a piece of swallowtail catastrophe set [45], see Figure 5. Now the crease (i.e., the singular part of the surface) and the thread correspond to invariant $(n + 1)$-tori; the smooth part of the surface is associated with the elliptic $(n + 2)$-tori; the open region above the surface gives rise to invariant Lagrangian $(n + 3)$-tori. Persistence of these higher dimensional isotropic tori can be obtained by using ‘standard’ KAM theory [3, 89, 98, 67, 31], for a detailed treatment of this see [20].

Remarks 5.2

1. The total union of perturbed $(n + 3)$-, $(n + 2)$ and $(n + 1)$-tori has been called a Cantor stratification, see [32] and references given there. The relative measure of the parameter regimes with invariant $(n + 3)$-, $(n + 2)$ and $n$-tori tends to full measure when approaching the the ‘strata’ of codimension 1, 2 and 3, respectively, that here occur as (secondary) bifurcation sets. Here we take the Hausdorff measure of the appropriate dimension. Compare with Figure 5. This can be shown by repeated normalization (averaging)
and ‘playing’ with the Diophantine parameter $\gamma > 0$ in an appropriate way, compare with Remark 1.5 (2). In [30] such phenomena have been described in terms of density points of $j$-quasi-periodicity $j = n + 1, n + 2, n + 3$. Exploiting the real analyticity, it even can be shown that the full relative measure is attained in an exponentially fast way, in which case the phenomenon also is described in terms of exponential condensation. For a discussion of these notions and many references compare with [32].

2. The total number of parameters needed for the occurrence of a generic quasi-periodic Hamiltonian Hopf bifurcation is $n + 3 = (n + 1) + 2$, which in this example exactly is the number of degrees of freedom.

On the one hand one may wish to view the $n$ forcing frequencies as external parameters. On the other hand they may be ‘compensated’ by action variables conjugate to the corresponding angles; this would be a typical example of distinguished parameters. In both set-ups one needs three extra parameters, which might be provided by the triple $(a, b, h)$. In any case, in individual systems of at least 5 degrees of freedom, the quasi-periodic Hamiltonian Hopf bifurcation occurs as a typical phenomenon.

Acknowledgments

The authors want to thank Boele Braaksma, Cristina Ciocci, Heinz Hanßmann, Mikhail Sevryuk, Floris Takens and Florian Wagener for helpful discussions.
References


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[71] Hopf bifurcation at k-fold resonances in reversible systems, Preprint N, M16/95, Technische Universitat Ilmenau, 1996.


**A  Hölder condition on spectra of Floquet matrices**

In the proofs of the main Theorems 1.4, 1.6 and 1.8 we use a Hölder condition on the spectrum $\text{Spec } \Omega(\mu)$ of the unfolding $\mu \in \mathbb{R}^p \mapsto \Omega(\mu) \in \mathfrak{gl}(m, \mathbb{R})$, as this occurs in the leading, normal linear part
\[
N_\mu(X) = \omega(\mu) \frac{\partial}{\partial x} + \Omega(\mu) y \frac{\partial}{\partial y}
\]
of (1). Also compare with the symplectic and reversible counterparts in §§1.2 and 1.3, where in both cases $m$ just has to be replaced by $2q$.

Since the family $X = X_\mu(x, y)$, $x \in \mathbb{T}^n$, $y \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$ of vector fields is real analytic in all variables, the map $\mu \mapsto \Omega(\mu)$ has a holomorphic extension to a domain $G \subset \mathbb{C}^p$. 

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Theorem A.1 (Hölder condition on Floquet spectrum) For any $\tilde{\mu} \in G$ and $\mu \in G \cap \mathbb{R}^p$ and for any $\tilde{\lambda} \in \text{Spec} \Omega(\tilde{\mu})$, there exists an eigenvalue $\lambda \in \text{Spec} \Omega(\mu)$ such that
\[ |\lambda - \tilde{\lambda}| \leq L|\mu - \tilde{\mu}|^{1/m}, \tag{64} \]
for a positive constant $L$.

In the KAM proof in Appendix B, see below, we shall use the weaker statement that
\[ |\text{Im} \lambda - \text{Im} \tilde{\lambda}| \leq L|\mu - \tilde{\mu}|^{1/m}, \]
see (5). A proof of Theorem A.1 is given now. It is closely related to [53] and based on application of the Rouché Lemma to the characteristic polynomial of $\Omega(\mu)$.

We start with the statement of the Rouché Lemma, that is quite standard in Complex analysis.

Lemma A.2 (E. Rouché, 1862) Let $G \subset \mathbb{C}$ be a domain bounded by a closed curve $\partial G$ (not necessarily connected), and let $F, f : \overline{G} \to \mathbb{C}$ be two holomorphic functions. If $|F(z)| > |f(z) - F(z)|$ for every $z \in \partial G$ then the functions $F$ and $f$ possess the same number of zeros inside $G$ (counting multiplicities).

Theorem A.3 Let $G$ be a domain in $\mathbb{C}$ and let $p$ be a polynomial of degree $n$ with leading coefficient $s \neq 0$. Suppose that all the roots of $p$ lie inside $G$ at distances no less than $a^{1/n}$ from $\partial G$, where $a > 0$ is a certain number. Let $f : \overline{G} \to \mathbb{C}$ be a holomorphic function such that $|f(z) - p(z)| < a|s|$ for every $z \in \partial G$. Then for each root $z_0$ of the polynomial $p$, there exists a zero $z'_0$ of the function $f$ such that $|z'_0 - z_0| < (2n - 1)a^{1/n}$.

Proof. Denote by $D \subset G$ the union of open disks of radius $a^{1/n}$ centered at the roots of the polynomial $p$, and let $D_0$ be the connected component of $D$ containing $z_0$. Each point $z'$ of the curve $\partial D_0$ lies at a distance of $a^{1/n}$ from one of the roots of the polynomial $p$ and at distances no less than $a^{1/n}$ from the remaining roots. Thus, $|p(z')| \geq a|s|$. According to Lemma A.2, the number of the zeros of the function $f$ inside $D_0$ is equal to the number of the roots of the polynomial $p$ inside $D_0$. In particular, this number is positive. Let $z'_0$ be one of the zeros of the function $f$ inside $D_0$.

The point $z'_0$ belongs to the same connected component of the set $D$ as $z_0$ does. Hence, there exists a sequence of roots $z_0, z_1, \ldots, z_k$ of the polynomial $p$ such that $|z_1 - z_0| < 2a^{1/n}, |z_2 - z_1| < 2a^{1/n}, \ldots, |z_k - z_{k-1}| < 2a^{1/n}$, and $|z'_0 - z_k| < a^{1/n}$. Since $k \leq n - 1$, one obtains $|z'_0 - z_0| < (2k + 1)a^{1/n} \leq (2n - 1)a^{1/n}$. \[\Box\]
Theorem A.4 Let $K$ be a compact metric space, and let

$$p_t(z) = z^n + q_1(t)z^{n-1} + \cdots + q_{n-1}(t)z + q_n(t)$$

be a polynomial in $z$ whose coefficients $q_j : K \to \mathbb{C}$ ($1 \leq j \leq n$) are Lipschitz continuous. Then there exists a number $C > 0$ such that for any $t_1, t_2 \in K$ ($t_1 \neq t_2$) and for each root $z_0$ of the polynomial $p_{t_1}$, there is a root $z'_0$ of the polynomial $p_{t_2}$ for which $|z'_0 - z_0| < C[\rho(t_1, t_2)]^{1/n}$, where $\rho(t_1, t_2)$ denotes the distance between $t_1$ and $t_2$ in $K$.

**Proof.** Let $G$ be a domain in $\mathbb{C}$ such that for any $t \in K$, all the roots of the polynomial $p_t$ lie inside $G$ at distances no less than 1 from $\partial G$. There exists a number $c > 0$ such that for any $t_1, t_2 \in K$ ($t_1 \neq t_2$) and for each $z \in \overline{G}$, the inequality $|p_{t_1}(z) - p_{t_2}(z)| < cp(t_1, t_2)$ holds. Obviously, it suffices to verify the conclusion of the theorem for $t_1, t_2 \in K$ such that $\rho(t_1, t_2) \leq c^{-1}$.

Thus, suppose that $t_1, t_2 \in K$ ($t_1 \neq t_2$) and $\rho(t_1, t_2) \leq c^{-1}$. Set $a = cp(t_1, t_2) \leq 1$. All the roots of the polynomial $p_{t_1}$ lie inside $G$ at distances no less than $1 \geq a^{1/n}$ from $\partial G$. On the other hand, $|p_{t_2}(z) - p_{t_1}(z)| < cp(t_1, t_2) = a$ for every $z \in \overline{G}$. According to Theorem A.3, for each root $z_0$ of the polynomial $p_{t_1}$ there is a root $z'_0$ of the polynomial $p_{t_2}$ such that

$$|z'_0 - z_0| < (2n - 1)a^{1/n} = (2n - 1)[cp(t_1, t_2)]^{1/n}.$$ 

$\square$

Theorem A.1 is a direct corollary of Theorem A.4, applied to the characteristic polynomial of $\Omega$.

**B Proof of Theorem 1.6**

We now present a proof of Theorem 1.6 in the Hamiltonian setting. A direct transcription gives proofs of the reversible and dissipative counterpart Theorems 1.8 and 1.4. For further details in the reversible case also see [39, 18].

The section will be split into four parts. In the first part we sketch the idea behind the proof and its frame. In the second part, for a Newton-like iteration process, we define a sequence of approximations $\{\Phi_j\}_{j \geq 0}$ of the desired map $\Phi$ in Theorem 1.6. Here the linear versal unfolding theory as developed in §2 will be used; indeed, the linear centralizer unfoldings (the LCU’s) enable us to solve the so-called homological equations, used for the definition of the $\{\Phi_j\}_{j \geq 0}$'s. The maps $\Phi_j$ are defined on complex domains and will be constructed in such a way that the Inverse Approximation Lemma [126, 127, 98] applies, yielding that the limit $\Phi_\infty$ is a Whitney-smooth map\(^6\). The third part of this section concerns with the control

\(^6\)For background on Whitney-smoothness, see [124, 115]
of errors per iteration step, where in view of the Inverse Approximation Lemma, we have to make sure that the differences $|\Phi_j - \Phi_{j+1}|$ decrease in an exponential way. In this part, the Diophantine conditions on frequencies come into play. In the fourth and final part we discuss the convergence of the iteration process, where a proper choice will be made for the complex domains, introduced in the second part. Our proof closely follows that of [89, 99, 67, 31].

B.1 Preliminaries

a. Reduction to a special case

As a starting point for the proof for Theorem 1.6, we consider the symplectic manifold $(M, \sigma) = (\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} = \{x, y, z\}, dx \wedge dy + dz^2)$, assuming that the integrable family $X = X_{\omega, \mu} \in \mathcal{X}^P(M)$ is given by

$$X_{\omega, \mu}(x, y, z) = \omega \frac{\partial}{\partial x} + \Omega(\mu)z \frac{\partial}{\partial z}, \quad (65)$$

where $\Omega(\mu)$ is a LCU of $\Omega_0 = \Omega(0)$ and where $(\omega, \mu) \in P \subset \mathbb{R}^n \times \mathbb{R}^c$ (as an open subset) with $c = \text{cod} \Omega_0$. In view of §3.1 and Lemma 3.1, these special choices do not limit the generality of our proof for Theorem 1.6. Furthermore, we observe that it is sufficient to prove Theorem 1.6 for $\gamma = \gamma_0$, for a positive constant $\gamma_0$. This can be realized by rescaling the time $t$ to $\frac{2\mu t}{\gamma}$ and by stretching the parameters $(\omega, \mu)$ to $(\frac{2\mu}{\gamma} \omega, \frac{2\mu}{\gamma} \mu)$, compare with [89, 98, 67, 31]. We will come back to this later on.

b. Compact-open neighbourhoods

Here we specify the compact-open neighbourhoods of the $(n + c)$-parameter integrable family $X \in \mathcal{X}^P(M)$ given by (65). To this end, for any subset $I \subseteq \mathbb{R}^k$ and $\rho > 0$, we introduce the complex ‘strip’ $I + \rho$ defined as

$$I + \rho = \{z \in \mathbb{C}^k : \exists x \in I \text{ such that } |x_j - z_j| \leq \rho, \text{ for all } 0 \leq j \leq k\}.$$ 

Similarly we introduce $\mathbb{T}^n + \kappa \subset \mathbb{C}^n/(2\pi \mathbb{Z})^n$, where $\kappa > 0$. Since the family $X$ is real analytic, it has a holomorphic extension to a compact neighbourhood $\mathcal{N}$ of the form

$$\mathcal{N} = (\mathbb{T}^n + \kappa) \times \triangle_y \times \triangle_z \times (U + \rho) \subset \mathbb{C}^n/(2\pi \mathbb{Z})^n \times \mathbb{C}^n \times \mathbb{C}^4 \times \mathbb{C}^{n+c} \quad (66)$$

with $\kappa$ and $\rho$ small positive constants. Here the set $\triangle_y \times \triangle_z$ is a compact neighbourhood of $(y, z) = (0, 0) \in \mathbb{C}^n \times \mathbb{C}^{2q}$ and $U$ a compact neighbourhood of $\mu = 0 \in P \subset \mathbb{R}^n \times \mathbb{R}^c$ (as an open subset). Now a compact-open neighbourhood
A of X in $\mathcal{X}_p^\sigma(M)$ is specified as follows. For a positive constant $\delta$ given by the proof below and the parameter $\gamma > 0$ of the Diophantine conditions (2), the set $\mathcal{A} = \mathcal{A}_{\gamma, \delta}$ consists of families of the forms

$$(\omega + f) \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + (\Omega(\mu) z + h) \frac{\partial}{\partial z} \in \mathcal{X}_p^\sigma(M),$$

where the functions $f, g$ and $h$ of $(x, y, z, \omega, \mu)$ have holomorphic extensions to the compact domain $\mathcal{N}$, such that,

$$|f|_{\mathcal{N}} < \gamma \delta, \quad |g|_{\mathcal{N}} < \gamma \delta^2 \quad \text{and} \quad |h|_{\mathcal{N}} < \gamma \delta^2. \quad (67)$$

Here $|\cdot|_{\mathcal{N}}$ denotes the supremum norm on the complex domain $\mathcal{N}$. We shall show that Theorem 1.6 holds for perturbations $\tilde{X}$ of $\mathcal{A}$ of $X$.

c. Taylor- and Fourier-truncation

Suppose that the Hamiltonian vector field $Y = Y(x, y, z) \in \mathcal{X}^\sigma(M)$ has the form

$$Y(x, y, z) = f(x, y, z) \frac{\partial}{\partial x} + g(x, y, z) \frac{\partial}{\partial y} + h(x, y, z) \frac{\partial}{\partial z}.$$

For simplicity we here suppress the parameter. Associated to $Y$, we introduce the following Taylor- and Fourier-truncations. For arbitrary non-negative integer $d$

$$Y_{\text{lin}}(x, y, z) = f(x, 0, 0) \frac{\partial}{\partial x} + \left[ g(x, 0, 0) + \frac{\partial g}{\partial y}(x, 0, 0)y + \frac{\partial g}{\partial z}(x, 0, 0)z \right] \frac{\partial}{\partial y} + \left[ h(x, 0, 0) + \frac{\partial h}{\partial z}(x, 0, 0)z \right] \frac{\partial}{\partial z}$$

$$(Y)_d(x, y, z) = \sum_{|k| \leq d} e^{i(x,k)} \left( f_k(y, z) \frac{\partial}{\partial x} + g_k(y, z) \frac{\partial}{\partial y} + h_k(y, z) \frac{\partial}{\partial z} \right), \quad (68)$$

where

$$\frac{\partial^2 g}{\partial z^2} z^2 = \left( \frac{\partial^2 g_1}{\partial z^2}, \ldots, \frac{\partial^2 g_n}{\partial z^2} \right) \in \mathbb{R}^n$$

and where $f_k, g_k$ and $h_k$ are the $k$th Fourier coefficients of the $f, g$ and $h$ with respect to the angle variable $x$. Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^n$. We denote by $\mathcal{X}_p^\sigma$ and $\mathcal{X}_d^\sigma$ the sets of these truncations. One directly shows that $Y_{\text{lin}}, (Y)_d \in \mathcal{X}^\sigma(M)$, whenever $Y \in \mathcal{X}^\sigma(M)$, compare with [67, 31].

Remark B.1 In the reversible case the similar property is that $Y_{\text{lin}}, (Y)_d \in \mathcal{X}^{-G}(M)$, whenever $Y \in \mathcal{X}^{-G}(M)$, compare [28]. For this use of notation see §3.4. In general this property is another axiom of the theory [67, 31], which amounts to homogeneity of the structure that has to preserved (symplectic or volume form, symmetry, etc.) with respect to well-chosen coordinates.
B.2 Idea behind the proof

Here we briefly discuss the idea of the proof. The unperturbed integrable family $X = X_{\omega,\mu}(y, z)$ is given by

$$X_{\omega,\mu}(y, z) = \omega \frac{\partial}{\partial x} + \Omega(\mu)z \frac{\partial}{\partial z}$$

where $\Omega(\mu) \in \mathfrak{sp}(2q, \mathbb{R})$ is a LCU of $\Omega_0$ and where $(\omega, \mu) \in \mathbb{R}^n \times \mathbb{R}^c$ with $c = \text{cod } \Omega_0$, see A.1 (a). Let the family $\tilde{X} \in \mathcal{X}_p^\nu(M)$ be non-integrable small perturbation of the family $X$. To prove Theorem 1.6, we need to find a $C^\infty$-near-identity symplectic map $\Phi : M \times P \to M \times P$, such that,

$$\Phi^* \tilde{X} = X + \left[ \tilde{f} \frac{\partial}{\partial x} + \tilde{g} \frac{\partial}{\partial x} + \tilde{h} \frac{\partial}{\partial z} \right]$$

(69)

where $\tilde{f} = O(|y|, |z|)$, $\tilde{g} = O(|y|, |z|)$ and $\tilde{h} = O(|y|, |z|^2)$. Indeed, the conjugacy relation (69) implies that the perturbation $\tilde{X}$ possesses a family of invariant tori, which is the image $\tilde{V}$ of the $X$-invariant tori $(y, z) = (0, 0)$ under the map $\Phi$, compare with conclusion (ii) of Theorem 1.6. The family of $\tilde{X}$-invariant tori is, however, parametrized over a ‘Cantor set’ of parameters, determined by the Diophantine conditions (2). Moreover, the relation (69) also means that normal linear part of the perturbation $\tilde{X}$ on the invariant tori $\tilde{V}$ is conjugate to $X$, compare with conclusion (iii) of Theorem 1.6.

So our task is to construct a map $\Phi$ satisfying (69). Suppose that the map $\Phi = \Phi(x, y, z, \omega, \mu)$ is of the form

$$\Phi(x, y, z, \omega, \mu) = \left( x + U(x, \omega, \mu), y + V(x, y, z, \omega, \mu), z + W(x, z, \omega, \mu), \omega + \Lambda(x, \omega, \mu), \mu + \Lambda_2(\omega, \mu) \right)$$

(70)

and that the perturbation $\tilde{X}$, in coordinates $(\xi, \eta, \zeta, \sigma, \nu) = \Phi^{-1}(x, y, z, \omega, \mu)$, can be written as

$$\tilde{X}_{\sigma,\nu}(\xi, \eta, \zeta) = \sigma + f(\xi, \eta, \zeta, \sigma, \nu) \frac{\partial}{\partial x} + g(\xi, \eta, \zeta, \sigma, \nu) \frac{\partial}{\partial y} + [\Omega(\nu)\zeta + h(\xi, \eta, \zeta, \sigma, \nu)] \frac{\partial}{\partial z}.$$

The conjugacy relation (69) implies that

$$\begin{cases}
\frac{\partial \Omega}{\partial x}(\omega + \tilde{f}) + \tilde{f} = \Lambda_1 + f \\
(1 + \frac{\partial \nu}{\partial y})\tilde{g} + \frac{\partial \nu}{\partial x}(\omega + \tilde{f}) + \frac{\partial \nu}{\partial z}(\Omega(\mu)z + \tilde{h}) = g \\
(1 + \frac{\partial \nu}{\partial z})(\Omega(\mu)z + \tilde{h}) + \frac{\partial \nu}{\partial x}(\omega + \tilde{f}) = h + \Omega(\mu + \Lambda_2)(z + W),
\end{cases}$$

(71)

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where everything is expressed in the coordinates \((x, y, z, \omega, \mu)\). This nonlinear equation is to be solved in \(U, V, W, \Lambda_1\) and \(\Lambda_2\) by an Newtonian iteration process, proposed by [72, 3]. The corresponding Newtonian linearization of equation (71) is to be solved in \(V, W, \mu\) again when necessary.

The idea of the proof is to inductively construct a sequence \(\Phi_j\) for all approximations of the Diophantine conditions (2) is fixed to the constant \(\gamma_0\). For the present proof, we use the scaled families \(X, \tilde{X}\) as the unperturbed family \(X\) and perturbation \(\tilde{X}\), respectively, where \(c_0 = \frac{c}{\gamma_0}\) with \(\gamma_0\) a positive constant. Under such circumstances, the ‘parameter’ \(\gamma\) of the Diophantine conditions (2) is fixed to the constant \(\gamma_0\). For simplicity, in what follows we will suppress \(c_0\) in the proof and incorporate it again when necessary.

The idea of the proof is to inductively construct a sequence \(\{\Phi_j\}_{j \geq 0}\) of approximations of \(\Phi\) based on the Newtonian linearizations (72). The solution \(\Phi\) for (69) will be obtained as a Whitney-smooth limit of this sequence by applying the Inverse Approximation Theorem [126, 127, 98, 30, 31]. All approximations \(\Phi_j\) will be real analytic in the \(\mathbb{T}^n\)-direction and be \(C^\infty\)-near-identity symplectic diffeomorphisms. To describe the construction of the sequence \(\{\Phi_j\}\), we introduce the following notation. For any \(j \geq 0\), whenever \(\Phi_j\) is well-defined, we denote by \((x_j, y_j, z_j, \omega_j, \mu_j) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times \mathbb{R}^n \times \mathbb{R}^c\) the coordinates such that \(\Phi_j(x_j, y_j, z_j, \omega_j, \mu_j) = (x, y, z, \omega, \mu)\). We initialize the sequence \(\{\Phi_j\}\) by setting \(\Phi_0 = \text{Id}\). Assume that all \(\Phi_j\) \((j \geq 1)\) are normally affine, that is, of the form

\[
\Phi_j(x_j, y_j, z_j, \omega_j, \mu_j) = (x_j + \tilde{U}^j(x_j, \omega_j, \mu_j), y_j + \tilde{V}^j(x_j, y_j, z_j, \omega_j, \mu_j), z_j + \tilde{W}^j(x_j, z_j, \omega_j, \mu_j, \omega_j + \Lambda_1(\omega_j, \mu_j), \mu_j + \Lambda_2(\omega_j, \mu_j)),
\]

For more details on this kind of Newtonian linearization see, e.g., [3, 89, 103].

Remark B.2 In order to prove Theorem 1.6, by the conjugacy relation (71), one can replace the family \(X\) by any family \(c_0X = c_0\omega \frac{\partial}{\partial x} + c_0\Omega(\mu)z \frac{\partial}{\partial z}\) with a non-zero constant \(c_0\). Indeed, suppose that \(c_0X\) satisfies (71). Then, after replacing \(\tilde{f}\) by \(c_0\tilde{f}\) etc., the family \(X\) fulfills (71) as well. For the present proof, we use the scaled families \(c_0X\) and \(c_0\tilde{X}\) as the unperturbed family \(X\) and perturbation \(\tilde{X}\), respectively.
Remark B.3 In the following, for simplicity we suppress the occurrence of parameters whenever they are not essential for our arguments.

B.3 Construction of the sequence \( \{ \Phi_j \} \)

In this part we construct the sequence \( \{ \Phi_j \} \) as mentioned above by a Newtonian iteration process. We first define the iteration relation by putting \( \Phi_{j+1} = \Phi_j \circ \Psi_j \), where \( \Psi_j : D_j^e \to D_j^e \). It follows that \( \Phi_{j+1} = \Psi_0 \circ \cdots \circ \Psi_j \) and that \( \bar{X}_{j+1} = \Psi_j \bar{X}_j \). Notice that \( \Phi_0 = \text{Id} \) and \( \bar{X}_0 = \bar{X} \). Our aim is to construct the sequence \( \{ \Psi_j \} \) by using linearizations of the type (72). In the following, we simplify notation as follows. As long as we are concerned with one
single iteration step, we suppress the occurrence of the index \( j \). For instance, we write \( f \) for \( f_j \) etc. The index \((j + 1)\) will be replaced by the plus-sign. So we replace the expressions \( f_{j+1}, D^e_{j+1} \) by \( f^+, D^e_+ \) etc. Moreover we abbreviate \((\xi, \eta, \zeta, \sigma, \nu) = (x_{j+1}, y_{j+1}, z_{j+1}, \omega_{j+1}, \mu_{j+1})\). Now let \( \Psi : D^e_+ \rightarrow D^e \) be of the form

\[
\Psi(\xi, \eta, \zeta, \sigma, \nu) = \left( \exp \left( \Psi_{\sigma, \nu} \right) (\xi, \eta, \zeta), \sigma + \Lambda_1(\sigma, \nu), \nu + \Lambda_2(\sigma, \nu) \right),
\]

where the vector field \( \Psi_{\sigma, \nu} \in \mathcal{X}^\sigma_\text{lin} \) is to be determined. Assume that the time-1 map \( \exp \Psi_{\sigma, \nu} \) of \( \Psi_{\sigma, \nu} \) has the form

\[
\exp \Psi_{\sigma, \nu}(\xi, \eta, \zeta) = \left( \xi + U(\xi, \eta, \zeta, \sigma, \nu), \eta + V(\xi, \eta, \zeta, \sigma, \nu), \zeta + W(\xi, \eta, \zeta, \sigma, \nu) \right),
\]

where the maps \( V \) and \( W \) are as in (73) and (74), respectively. Notice that the left side of (78) consists of the components of the vector field \( \exp \Psi_{\sigma, \nu} \left( \right) \). The right side of (78) consists of the components of the vector field \( \exp \Psi_{\sigma, \nu} \left( \right) \). The truncation order \( d \) represents the Fourier-truncations of \( f \) (with respect to the \( \xi \)-variable) up to the order \( d \) etc. The truncation order \( d \) will be chosen appropriately later on. Notice that the left side of (78) consists of the components of the vector field \( \exp \Psi_{\sigma, \nu} \left( \right) \), where \( X = X_{\sigma, \nu}(\xi, \eta, \zeta) = \sigma \frac{\partial}{\partial \xi} + \Omega(\nu)\zeta \frac{\partial}{\partial \xi} \), compare with (65). Therefore, we can rewrite (78) as follows

\[
\text{ad} \ X_{\Psi_{\sigma, \nu}} = S + N,
\]

usually called the \textit{homological equation}, where

\[
S_{\sigma, \nu}(\xi, \eta, \zeta) = \left\{ f(\xi, 0, 0, \sigma, \nu) \right\}_d \frac{\partial}{\partial \xi} + \left\{ g(\xi, 0, 0, \sigma, \nu) + \frac{\partial}{\partial \eta}(\xi, 0, 0, \sigma, \nu) \right\}_d \frac{\partial}{\partial \eta} + \left\{ h(\xi, 0, 0, \sigma, \nu) \right\}_d \frac{\partial}{\partial \eta} + \Omega(\Lambda_2(\sigma, \nu))\zeta \frac{\partial}{\partial \xi},
\]

\[
N_{\sigma, \nu}(\xi, \eta, \zeta) = \Lambda_1(\sigma, \nu) \frac{\partial}{\partial \xi} + \Omega(\Lambda_2(\sigma, \nu))\zeta \frac{\partial}{\partial \xi}.
\]
Observe that $S$ is determined by $\tilde{X}$, see (75). Our concern is to find solutions $\Psi$ and $N$ for the homological equation (79) for a given $\tilde{X} \in \mathcal{X}^\sigma_{lin}(M)$ and hence for a given $S$. Suppose that we can solve the unknowns $\Psi$ and $N$ in (79) for any given $\tilde{X}$, then, by (76), we obtain the map $\Psi$. Since $\tilde{X}_+ = \Psi^*\tilde{X}$, the perturbation $X_+$ and hence $S_+$ are determined. So, for the moment assuming solvability of the equation, (79), the vector field $S_+$ in turn gives the vector fields $\Psi_+$ and $N_+$. In this way we obtain the sequence $(\Psi_j)_{j \geq 0}$ and hence the sequence $(\Phi_j)_{j \geq 0}$. Therefore we see that the construction of the sequence of maps $\Phi_j$ is reduced to solving the homological equation (79).

**Solvability of the homological equation**

Here we study the solvability of (79) in the unknowns $\Psi$ and $N$ for a given $S \in \mathcal{X}^\sigma_{lin}(M)$, which is determined by $\tilde{X}$. We consider a fixed value of $(\sigma, \nu) \in \Gamma_\gamma(U')$. Observe that $S \in \mathcal{X}^\sigma_{lin} \cap \mathcal{X}^\nu_{lin}$. Our present aim is to find $\Psi, N \in \mathcal{X}_{lin}^\sigma$ such that (78) holds. We write

$$\Psi(\xi, \eta, \zeta) = \pi(\xi) \frac{\partial}{\partial \xi} + \pi(\xi, \eta, \zeta) \frac{\partial}{\partial \eta} + \pi(\xi, \zeta) \frac{\partial}{\partial \zeta},$$

where the maps $\pi$ and $\pi'$ are of the form $\pi(\xi, \eta, \zeta) = \pi_0(\xi) + \pi_1(\xi) \eta + \pi_2(\xi) \zeta + \frac{1}{2} \pi_3(\xi) \zeta^2$ and $\pi(\xi, \zeta) = \pi_0(\xi) + \pi_1(\xi) \righttoketzeta$. Thus we have

$$\text{ad} X_{\sigma,\nu}(\Psi)(\xi, \eta, \zeta) = \left( \frac{\partial \pi}{\partial \xi} \sigma \right) \frac{\partial}{\partial \xi} + \left( \frac{\partial \pi}{\partial \eta} \sigma + \frac{\partial \pi}{\partial \zeta} \Omega(\nu) \zeta \right) \frac{\partial}{\partial \eta} + \left( \frac{\partial \pi}{\partial \xi} \sigma + \frac{\partial \pi}{\partial \zeta} \Omega(\nu) \zeta - \Omega(\nu) \pi \right) \frac{\partial}{\partial \zeta},$$

which implies that linear map $\text{ad} X_{\sigma,\nu}$ leaves the linear spaces $\mathcal{X}^\sigma_{lin}$ and $\mathcal{X}^\nu_{lin}$ invariant. Denoting by $\mathcal{X}^\sigma_{lin,d}$ the intersection of the linear spaces $\mathcal{X}^\sigma_{lin}$ and $\mathcal{X}^\nu_{lin}$, we have

**Proposition B.4** Consider the linear map $\text{ad} X_{\sigma,\nu} : \mathcal{X}^\sigma_{lin,d} \to \mathcal{X}^\sigma_{lin,d}$ where $(\sigma, \nu) \in \Gamma_\gamma(U')$. Then, for any $d \in \mathbb{N} \cup \{0\}$,

$$\mathcal{X}^\sigma_{lin,d} = \text{im} \text{ad} X_{\sigma,\nu} \oplus \ker \text{ad} X_{\sigma,\nu}^T,$$

where $X_{\sigma,\nu}^T(\xi, \eta, \zeta) = -\sigma \frac{\partial}{\partial \xi} + \Omega^T(\nu) \zeta \frac{\partial}{\partial \zeta}$.\footnote{Here $\Omega(\mu)^T$ denotes the usual matrix transpose of $\Omega(\mu)$.}

**Proof.** Consider the vector field $\Psi$ as given above. By comparing the Fourier coefficients of $\Psi$ and of $\text{ad} X_{\sigma,\nu} \Psi$, we see that the eigenvalues of $\text{ad} X_{\sigma,\nu}(\Psi)$ are of the form.

$$i(\sigma, k), \quad i(\sigma, k) - \lambda_j(\nu), \quad i(\sigma, k) - \lambda_i(\nu) + \lambda_j(\nu),$$
where \( \lambda_j(\nu) \) (1 \( \leq j \leq 2q \)) are the eigenvalues of \( \Omega(\mu) \). Since \((\sigma, \nu) \in \Gamma'_{\nu}(U)\) satisfies the Diophantine conditions, it follows that for \( k \neq 0 \) none of the eigenvalues is equal to zero. Hence, only the zero-th Fourier coefficients of \( \overline{\Psi} \) belong to \( \ker ad X^T_{\sigma,\nu} \), which implies that

\[
\ker ad X^T_{\sigma,\nu} = \left\{ a \frac{\partial}{\partial \xi} + A \zeta \frac{\partial}{\partial \zeta} : a \in \mathbb{R}^n \quad \text{and} \quad A \in \ker ad \Omega^T(\nu) \right\}.
\]  

(81)

On the other hand, the linear subspace \( \text{im} ad X_{\sigma,\nu} \) consists of elements in the forms: \( \{ u(\xi) \frac{\partial}{\partial \xi} + v(\xi, \eta, \zeta) \frac{\partial}{\partial \eta} + w_0(\xi) + w_1(\xi) \zeta \frac{\partial}{\partial \zeta} \}_d \), where \( \{ u \}_0 = 0, \{ w_1 \}_0 \in \text{im} ad \Omega(\nu) \).

**Remark B.5** It can be shown, with help of the Riesz Representation Theorem, cf. [46], that the linear map \( ad X^T_{\sigma,\nu} \) is the transpose of \( ad X_{\sigma,\nu} \) with respect to a suitable inner product of the space \( \mathcal{X}_{\text{lin};d}^\sigma \). This once again establishes the decomposition (80).

Let us return to the homological equation (79). The following lemma tells us how (79) is solvable in \( \overline{\Psi} \) and \( N \) for any given \( \overline{X} \in \mathcal{X}^\sigma_{\text{lin};d}(M) \).

**Lemma B.6** For any given \( S \in \mathcal{X}_{\text{lin};d}^\sigma \) and for each \((\sigma, \nu) \in \Gamma_{\nu}(U')\), there exist (unique) real-analytic vector fields \( \overline{\Psi}_{\sigma,\mu} \in \mathcal{X}_{\text{lin};d}^\sigma \) and \( N_{\sigma,\nu} \in \ker ad X^T_{\sigma,\nu} \) satisfying (79).

**Proof.** By Proposition B.4, we have that \( \mathcal{X}_{\text{lin};d}^\sigma = \text{im} ad X_0 \oplus \ker ad X^T_0 \). Hence, the vector field \( S \) has the unique decomposition \( S = S^1 + S^0 \), where \( S^1 \in \text{im} ad X_0 \) and \( S^0 \in \ker ad X^T_0 \). Take a subspace \( V \subset \mathcal{X}_{\text{lin};d}^\sigma \) such that \( V \oplus \ker ad X_0 = \mathcal{X}_{\text{lin};d}^\sigma \). Then, the restriction \( ad X_0|_V : V \to \text{im} ad X_0 \) is an isomorphism. By the Inverse Function Theorem, for each small \((\sigma, \nu)\), there exists \( \overline{\Psi}_{\sigma,\nu} \in V \) such that \( ad X_{\sigma,\nu}(\overline{\Psi}_{\sigma,\nu}) = S^1_{\sigma,\nu} \). It remains to find the unknown \( N \) satisfying \( N + S^0 = 0 \), where \( N = N_{\sigma,\nu} \) is of the form

\[
N_{\sigma,\nu}(\xi, \eta, \zeta) = \Lambda_1(\sigma, \nu) \frac{\partial}{\partial \xi} + \Omega(\Lambda_2(\sigma, \nu)) \zeta \frac{\partial}{\partial \zeta}.
\]

By (81), the vector field \( S^0 \) is the zero-th Fourier truncation of \( S \) and is of the form

\[
S^0_{\sigma,\nu} = a(\sigma, \nu) \frac{\partial}{\partial \xi} + A(\sigma, \nu) \zeta \frac{\partial}{\partial \zeta}
\]

for \( a \in \mathbb{R}^n \) and \( A \in \ker ad \Omega^T_0 \). Hence, the vector field \( N_{\sigma,\nu} \) is determined by the following equations.

\[
\Lambda_1(\sigma, \nu) = -a(\sigma, \nu) \quad \text{and} \quad \Omega(\Lambda_2(\sigma, \nu)) = -A(\sigma, \nu).
\]
Since $\Omega$ is an LCU of $\Omega_0$, the latter equation admits the solution $\Lambda_2(\sigma, \nu) = \Omega^{-1}(-A(\sigma, \nu))$. \hfill \Box

We can also construct the vector field $\Psi_{\sigma, \nu} \in \mathcal{X}_{\text{lin}, d}$ explicitly from the homological equation (78) as follows. Suppose that

$$
\Psi_{\sigma, \nu}(\xi, \eta, \zeta) = \overline{\psi}_0 + [v_0 + v_1 \eta + v_2 \zeta + \frac{1}{2} v_3 \zeta^2] \frac{\partial}{\partial \eta} + [w_0 + w_1 \zeta] \frac{\partial}{\partial \zeta},
$$

(82)

where the functions $v_j$ ($j = 0, 1, 2, 3$), $w_0$ and $w_1$ depend on $\xi$ and on the multi-parameter $(\sigma, \nu)$. Since $\Psi_{\sigma, \nu} \in \mathcal{X}_{\text{lin}}$, the functions $v_j$ ($j = 1, 2, 3$) are determined by $\overline{\psi}_0, w_0$ and $w_1$. Indeed, we have

$$
v_1 = -\left(\frac{\partial}{\partial \xi}\right)^T, \quad v_2 = \left(\frac{\partial}{\partial \eta}\right)^T J_{2\eta} \quad \text{and} \quad v_3 = \left(\frac{\partial}{\partial \zeta}\right)^T J_{2\eta}.
$$

Hence, in order to find $\Psi$ explicitly, we only need to construct $\overline{\psi}_0, v_0, w_0$ and $w_1$. By comparing the Fourier coefficients of the functions in (78), we obtain

$$
\overline{\psi}_0 = \sum_{0 < |k| \leq d} \frac{f_k}{i \langle \sigma, k \rangle} e^{i \langle \xi, k \rangle}
$$

$$
v_0 = \sum_{0 < |k| \leq d} \frac{g_k}{i \langle \sigma, k \rangle} e^{i \langle \xi, k \rangle}
$$

$$
w_0 = -\Omega(\nu)^{-1}(g_0(\sigma, \nu)) + \sum_{0 < |k| \leq d} (i \langle \sigma, k \rangle - \Omega(\nu))^{-1} h_k e^{i \langle \xi, k \rangle}
$$

$$
w_1 = \{w_1\}_0 + \sum_{0 < |k| \leq d} (i \langle \sigma, k \rangle - \text{ad} \Omega(\nu))^{-1} \left\{ \frac{\partial h}{\partial \zeta} \right\}_k e^{i \langle \xi, k \rangle},
$$

(83)

where $\text{ad} \Omega(\mu) \{w_1\}_0$ is the restriction of $\left\{ -\frac{\partial h}{\partial \xi} \right\}_0 \in \text{sp}(2q, \mathbb{R})$ to the subspace $\text{im} \text{ad} \Omega(\mu)$, according to the splitting $\text{sp}(2q, \mathbb{R}) = \ker \text{ad} \Omega^T(\mu) \oplus \text{im} \text{ad} \Omega(\mu)$.

**B.4 Estimates of errors per iteration step**

In this section, we specify the complex domains $D^e = D_j^e$ and the choice of the order $d = d_j$ of the Fourier-truncations, see §B.3. Based on these choices, we estimate the errors $|f^+|_{D_j^e}, |g^+|_{D_j^e}, |h^+|_{D_j^e}$ and $|\Phi^+ - \Phi|_{D_j^e}$ in terms of $|f|_{D_j^e}, |g|_{D_j^e}, |h|_{D_j^e}$. Here we take care that the sequence of maps $\{\Phi_j\}$ satisfies the conditions of the Inverse Approximation Lemma. Basic tools to be used in this section for estimates are the Cauchy Integral Formula, Gronwall’s inequality and the Mean Value Theorem.
Specification of complex domains and truncation orders

We first recall the Hölder condition (5)

$$\left| \text{Im} \lambda - \text{Im} \tilde{\lambda} \right| \leq L|\mu - \tilde{\mu}|^\theta,$$

where now $\theta = 1/2q$, as this follows from Theorem A.1. Then let $\{s_j\}, \{\rho_j\}$ and $\{\varepsilon_j\} (j \geq 0)$ be geometric sequences of positive numbers with ratios smaller than $\frac{1}{2}$. We define the sequence $\{r_j\}_{j \geq 0}$ by

$$r_j = s_j^{2\theta+2}, \quad \text{(84)}$$

where the constant $\theta \in (0, 1)$ comes from the above Hölder condition. We specify the following complex domains

$$D_j = \left( \mathbb{T}^n + \frac{s}{2} + s_j \right) \times (\Gamma_\gamma(U) + r_j)$$

$$D_j' = \left( \mathbb{T}^n + \frac{s}{2} + s_j \right) \times (\Delta_y)_j \times (\Delta_z) \times \Gamma_\gamma(U),$$

where $(\Delta_y)_j = \{ y \in \mathbb{C}^n : |y| \leq \rho_j \}$ and $(\Delta_z)_j = \{ z \in \mathbb{C}^n : |z| \leq \varepsilon_j \}$. We require that $0 < s_0 < \min \left\{ \frac{s}{2}, \frac{1}{2n+1} \right\}$ and that both $\rho_0$ and $\varepsilon_0$ are sufficiently small. These conditions ensure that $D_j' \subseteq \mathcal{N}$ for all $j \geq 0$, where $\mathcal{N}$ is a holomorphic extension domain of the unperturbed real analytic family $X$, see (66). The sequences $\{s_j\}, \{\rho_j\}$ and $\{\varepsilon_j\}$ later on will be chosen in such a way that the iteration process converges. From now on, we apply the plus-sign convention again. In order to estimate the errors $|f^+|_{D^+_*}$, etc., we introduce the intermediate sets $D_*, D_*, D^e_*, D^e_*$ and the positive numbers $s_* = \frac{1}{2}(s + s_+)$ and $s^{**} = \frac{1}{2}(2s + s_+)$, compare with [67, 31]. The numbers $\rho_*, \rho^{**}, \varepsilon_*, \varepsilon^{**}, r_*$ and $r^{**}$ are defined accordingly. The complex domain $D_*$ is defined as

$$D_* = \left( \mathbb{T}^n + \frac{s}{2} + s_* \right) \times (\Gamma_\gamma(U') + r_*),$$

and the sets $D_*, D^e_*$ and $D^{**}_*$ are defined similarly. Notice that $D^e_+ \subseteq D^e_* \subseteq D^{**}_*$.

We return to the vector field $\overline{\Psi}$ as in (79), which is solved for a truncation order $d$, specified as

$$d = \text{int}(s^{-2}), \quad \text{(85)}$$

that is, the integral part of $s^{-2}$. By choosing a sufficiently small $s_0$, we may assume that the eigenvalues $\lambda_1, \ldots, \lambda_{2q}$ of the matrix $\Omega(\mu)$ satisfy the Hölder condition (5) on the complex domain $U + r_0$ and hence on all domains $U + r_j$ for $j \geq 1$. Referring to Remark B.2, we now fix the parameter $\gamma = \gamma_0 = 2(L + 1)$ for the Diophantine conditions (2), where $L$ is the constant given by Hölder condition (5).
Lemma B.7 Suppose that $(\omega, \mu) \in \Gamma_{\tau, \gamma}(U') + r$. Then, for all $k \in \mathbb{Z}^n$ with $0 < |k| \leq d$ and any eigenvalues $\alpha(\mu), \beta(\mu)$ of $\Omega(\mu)$, we have

$$|i\langle k, \omega \rangle|, |i\langle k, \omega \rangle - \alpha(\mu)|, |i\langle k, \omega \rangle + \alpha(\mu) - \beta(\mu)| \geq |k|^{-\tau}. \tag{86}$$

Proof. Let $I = |i\langle k, \omega \rangle + \ell_1\alpha(\mu) + \ell_2\beta(\mu)|$, where $\ell_1, \ell_2 = 0, \pm 1$. To prove Lemma B.7, we need to show that $I \geq |k|^{-\tau}$. By assumption, there is a pair $(\tilde{\omega}, \tilde{\mu}) \in \Gamma_{\tau, \gamma}(U') \subset \mathbb{R}^n \times \mathbb{R}^c$ such that $|\tilde{\omega} - \omega| \leq r$ and $|\tilde{\mu} - \mu| \leq r$. Now

$$I \geq \left| \langle k, \Re \omega \rangle + \ell_1 \Im \alpha(\mu) + \ell_2 \Im \beta(\mu) \right|$$

$$= \left| \langle k, \tilde{\omega} \rangle + \ell_1 \Im \tilde{\alpha}(\tilde{\mu}) + \ell_2 \Im \tilde{\beta}(\tilde{\mu}) + \langle k, \Re \omega - \tilde{\omega} \rangle + \ell_1 \Im \left( \alpha(\mu) - \tilde{\alpha}(\tilde{\mu}) \right) + \ell_2 \Im \left( \beta(\mu) - \tilde{\beta}(\tilde{\mu}) \right) \right|$$

$$\geq \left| \langle k, \tilde{\omega} \rangle + \ell_1 \Im \tilde{\alpha}(\tilde{\mu}) + \ell_2 \Im \tilde{\beta}(\tilde{\mu}) \right| - |k||\omega - \tilde{\omega}| - \left| \Im \alpha(\mu) - \Im \tilde{\alpha}(\tilde{\mu}) \right| - \left| \Im \beta(\mu) - \Im \tilde{\beta}(\tilde{\mu}) \right|.$$ 

Here $\tilde{\alpha}(\tilde{\mu})$ and $\tilde{\beta}(\tilde{\mu})$ are two eigenvalues of $\Omega(\tilde{\mu})$. Now by the Diophantine conditions (2) and the Hölder condition (5), these eigenvalues can be chosen such that

$$I \geq |\gamma| |k|^{-\tau} - |k||r - 2Lr^\theta|.$$ 

We claim that both $|k||r$ and $r^\theta$ are bounded from above by $|k|^{-\tau}$. Indeed, we have that $|k|r \leq s^{2+\tau} \leq s^{2\tau} \leq |k|^{-\tau}$ and that $r^\theta = s^{2\tau+2\theta} \leq |k|^{-\tau}$. Hence, $I \geq |\gamma - 2L - 1||k|^{-\tau} = |k|^{-\tau}$. \hfill \Box

Now we are ready to perform the estimates for the vector field $\overline{\Psi}$ and some of its derivatives on the domain $D_{ss}$. Recall that $\overline{\Psi}(\xi, \eta, \zeta) = \overline{\pi}(\xi) \frac{\partial}{\partial \xi} + \overline{\pi}(\eta) \frac{\partial}{\partial \eta} + \overline{\pi}(\zeta) \frac{\partial}{\partial \zeta}$, where $\overline{\pi} = \overline{\pi}_0(\xi) + \overline{\pi}_1(\xi)\eta + \overline{\pi}_2(\xi)\zeta + \frac{1}{2}\overline{\pi}_3(\xi^2 \zeta^2)$ and where $\overline{\pi} = + \overline{\pi}_0(\xi) + \overline{\pi}_1(\xi)\eta$. The explicit forms of the components of $\overline{\Psi}$ are given by (83).

From now on, constants which appear in our estimations will only depend on $n, q, \tau$ and $\kappa$, to be denoted by $C_0, C_1$ etc. Whenever there is no need to remembered these constants, we will simply put a dot sign in the corresponding inequality.

**Proposition B.8** The vector field $\overline{\Psi}$ is real analytic in $\xi$ on the domain $D_{ss}$. Moreover, there exists a constant $C_0 = C_0(n, q, \tau, \kappa)$, such that the following estimates hold.

i. $s^{2\tau+1} |\overline{\pi}|_{D_{ss}}, s^{2\tau+2} \left| \frac{\partial \overline{\pi}}{\partial \xi} \right|_{D_{ss}} \leq C_0 |f|_{D^s};$

ii. $s^{2\tau+1} |\overline{\pi}_0|_{D_{ss}}, s^{2\tau+2} \left| \frac{\partial \overline{\pi}_0}{\partial \xi} \right|_{D_{ss}} \leq C_0 |g|_{D^s};$
iii. \( \rho s^{\tau+1} |\overline{\varphi}_1|_{D_s^*}, \rho s^{\tau+2} |\frac{\partial \overline{\varphi}_1}{\partial \xi}|_{D_s^*} \leq C_0 |g|_{D^e} \)

iv. \( \varepsilon s^{\tau+1} |\overline{\varphi}_2|_{D_s^*}, \varepsilon s^{\tau+2} |\frac{\partial \overline{\varphi}_2}{\partial \xi}|_{D_s^*} \leq C_0 |g|_{D^e} \)

v. \( \varepsilon^2 s^{\tau+1} |\overline{\varphi}_3|_{D_s^*}, \varepsilon^2 s^{\tau+2} |\frac{\partial \overline{\varphi}_3}{\partial \xi}|_{D_s^*} \leq 2C_0 |g|_{D^e} \)

vi. \( s^{\tau+1} |\overline{w}_0|_{D_s^*}, s^{\tau+2} |\frac{\partial \overline{w}_0}{\partial \xi}|_{D_s^*} \leq C_0 |h|_{D^e} \)

vii. \( s^{\tau+1} |\overline{w}_1|_{D_s^*}, s^{\tau+2} |\frac{\partial \overline{w}_1}{\partial \xi}|_{D_s^*} \leq C_0 |h|_{D^e} \)

**Proof.** We only prove item (i), since all other items can be shown similarly. Our proof follows [67, 31, 30]. By (83), we have that

\[
|\overline{\varphi}|_{D_s^*} \leq \sum_{|k| \neq 0} \left| \frac{f_k}{\langle \sigma, k \rangle} \right| e^{i\varepsilon \langle \sigma, \xi \rangle} \leq |f|_{D^e} \sum_{|k| \neq 0} |k|^\tau e^{\varepsilon |\langle \xi, k \rangle| - \varepsilon s^\gamma},
\]

where we used the Paley-Wiener Lemma, see [30]. It remains to estimate the infinite summation at the right side. By assumption, \( |\langle \xi, k \rangle| \leq \frac{s}{2} + s_{ss} \) and \( s_{ss} < \frac{s}{6} \), it follows that

\[
\sum_{|k| \neq 0} |k|^\tau e^{[\langle \xi, k \rangle - \frac{s}{2} - \varepsilon s^\gamma]} \leq \sum_{|k| \neq 0} |k|^\tau e^{- \frac{s}{6} |k|}.
\]

But the latter is bounded from above by \( s^{\tau+1} \), which gives the first estimate of item (i). By the Cauchy Integral Formula, for \((\tilde{x}, \tilde{\sigma}) \in D_{ss}^*\), we have

\[
\frac{\partial \overline{\varphi}(\tilde{x}, \tilde{\sigma})}{\partial \tilde{x}_k} = \frac{1}{2\pi i} \oint \frac{\overline{u}(\tilde{x}, \tilde{\sigma})}{(z - \tilde{x}_k)^2} dz, \quad \text{for } 1 \leq k \leq n,
\]

where the integration contour is the circle centered at \( \tilde{x}_k \) with the radius \( s - s_{ss} \). By definition, \( s - s_{ss} \geq \frac{1}{6} s \), we have that

\[
\left| \frac{\partial \overline{\varphi}(\tilde{x}, \tilde{\sigma})}{\partial \tilde{x}} \right| \leq 6s^{-1} |\overline{\varphi}|_{D^e},
\]

which shows the second estimate of item (i), again compare with [30]. □

**Corollary B.9** Let \( \overline{\varphi} \) and \( \overline{w} \) be as above. Then,

\[
s^{\tau+1} |\overline{\varphi}|_{D_s^*} \leq 5C_0 |g|_{D^e} \quad \text{and} \quad s^{\tau+1} |\overline{w}|_{D_s^*} \leq 2C_0 |h|_{D^e}.
\]

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Estimates on $|f^+|_{D^+_e}, |g^+|_{D^+_e}$ and $|h^+|_{D^+_e}$

Here we estimate the errors $|f^+|_{D^+_e}, |g^+|_{D^+_e}, |h^+|_{D^+_e}$ and also the transformation $\Psi = \exp \overline{\Psi}$ of the form (76) and some of its derivatives. These quantities will be given in terms of $|f|_{D^+_e}, |g|_{D^+_e}$ and $|h|_{D^+_e}$. Most of proofs given below are close to [67, 31]. For $t \in \mathbb{R}$ and $(\xi, \eta, \zeta) \in D^+_e$, we define $(x(t, \xi), y(t, \xi, \eta, \zeta), z(t, \xi, \eta, \zeta)) = \exp t \overline{\Psi}(\xi, \eta, \zeta)$, where $\overline{\Psi}$ is of the form (82). We write

\[
\begin{align*}
x(t, \xi) &= \xi + u(t, \xi) \\
y(t, \xi, \eta, \zeta) &= \eta + v_0(t, \xi) + v_1(t, \xi)\eta + v_2(t, \xi)\zeta + \frac{1}{2}v_3(t, \xi)\zeta^2 \\
z(t, \xi, \zeta) &= \zeta + w_0(t, \xi) + w_1(t, \xi)\zeta.
\end{align*}
\]

Recall from §B.3 that the time-1 map $\exp \overline{\Psi}$ has the form

\[
\exp \overline{\Psi}(\xi, \eta, \zeta) = (\xi + U(\xi), \eta + V(\xi, \eta, \zeta), \zeta + W(\xi, \zeta)),
\]

see (77). Hence, we have $U(\xi) = x(1, \xi) - \xi$, $V(\xi, \eta, \zeta) = y(1, \xi, \eta, \zeta) - \eta$ and $W(\xi, \zeta) = z(1, \xi, \zeta) - \zeta$.

**Lemma B.10** Suppose that

\[
|f|_{D^+_e} \leq \frac{1}{12C_0} s^{2r+2}, |g|_{D^+_e} \leq \frac{1}{48C_0} \rho s^{2r+1}, \text{ and } |h|_{D^+_e} \leq \frac{1}{48C_0} \varepsilon s^{2r+1},
\]

where $C_0$ is the same constant as from Proposition B.8. Then, for $(\xi, \eta, \zeta) \in D^+_e$ and for $t \in [0, 1]$, we have that $(x(t, \xi, \eta, \zeta), y(t, \xi, \eta, \zeta), z(t, \xi, \eta, \zeta)) \in D^*_{x_t}$.

Moreover, there exists a constant $C_1 > C_0$ such that the following estimates hold.

i. $s^{2r+1} |U|_{D^+_e} \leq C_1 |f|_{D^+_e}$;

ii. $s^{2r+1} |V|_{D^+_e} \leq C_1 |g|_{D^+_e}$;

iii. $s^{2r+1} |W|_{D^+_e} \leq C_1 |h|_{D^+_e}$.

**Proof.** First we notice that the maps $(x(t, \xi), y(t, \xi, \eta, \zeta), z(t, \xi, \eta, \zeta))$ satisfy the following differential equations,

\[
\begin{align*}
\dot{x} &= \overline{\Psi}(x) \\
\dot{y} &= \overline{\Psi}_0(x) + \overline{\Psi}_1(x)y + \overline{\Psi}_2(x)z + \frac{1}{2}\overline{\Psi}_3(x)z^2 \\
\dot{z} &= \overline{\Psi}_0(x) + \overline{\Psi}_1(x)z
\end{align*}
\]

Let $(\xi, \eta, \zeta) \in D^*_e$ and $t \in [0, 1]$. By (88), we have $|x(t, \xi) - \xi| \leq |\overline{\Psi}|_{D^*_e} \leq \frac{1}{12}s \leq s_{x} - s_{e}$. Hence, for all $t \in [0, 1]$ and $\xi \in T^n + \frac{\sigma}{2} + s_{x}$, we have $x(t, \xi) \in T^n + \frac{\sigma}{2} + s_{x}$. The same approach gives $y(t, \xi, \eta, \zeta) \in \Delta_{y}^*$ and $z(t, \xi, \zeta) \in \Delta_{z}^*$. 

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Next we estimate $|U|_{D^c_x}$, $|V|_{D^c_x}$ and $|W|_{D^c_x}$. For all $t \in [0, 1]$, one has $|u(t, x)| \leq |w(t, x)|$, which implies that $s^{2r+1} |U|_{D^c_x} \leq s^{2r+1} |w|_{D^c_x} \leq C_0 |f|_{D^c_x}$. We use Gronwall's inequality to estimate $|V|_{D^c_x}$ and $|W|_{D^c_x}$. By (90) and (87) we have $w_0 = \varpi_0(x(t, \xi)) + \varpi_1(x(t, \xi))w_0$ and $w_1 = \varpi_0(x(t, \xi)) + \varpi_1(x(t, \xi))w_1$. By the first differential equation,

$$|w_0| \leq |\varpi_0(t, \xi)|_{D^c_x} + \int_0^t |\varpi_1(s, \xi)| |w_0(s, \xi)| \, ds.$$  

By Gronwall's inequality $|w_0(t, \xi)| \leq |\varpi_0|_{D^c_x} e^{\varpi_1|_{D^c_x}} \leq 2 |\varpi_0|_{D^c_x}$. Similarly, we have $|w_1(t, \xi)| \leq 2 |\varpi_1|_{D^c_x}$ for all $t \in [0, 1]$. In particular,

$$s^{2r+1} |W|_{D^c_x} \leq 2s^{2r+1}(|\varpi_0|_{D^c_x} + \varepsilon |\varpi_1|_{D^c_x}) \leq 4C_0 |f|_{D^c_x},$$

which proves (iii).

Finally, we show the inequality (ii). By (89) and (87), the map $v_0(t, \xi)$ satisfies the differential equation

$$\dot{v}_0 = \varpi_0(x(t, \xi)) + \varpi_2(x(t, \xi))w_0 + \frac{1}{2} \varpi_3(x(t, \xi))w_0^2 + \varpi_4(x(t, \xi))v_0.$$  

Again by Gronwall's inequality we have

$$|v_0(t, \xi)| \leq e^{\varpi_1|_{D^c_x}} \left( |\varpi_0|_{D^c_x} + |\varpi_2|_{D^c_x} |w_0(t, \xi)| + \frac{1}{2} |\varpi_3|_{D^c_x} |w_0(t, \xi)|^2 \right) \leq 2 \left( |\varpi_0|_{D^c_x} + \varepsilon |\varpi_2|_{D^c_x} + \varepsilon^2 |\varpi_3|_{D^c_x} \right),$$

which implies that $s^{2r+1} |v_0(t, \xi)| \leq 6C_0 |g|_{D^c_x}$. Similarly, one can show that

$$s^{2r+1} |v_1(t, \xi)|, s^{2r+1} |v_2(t, \xi)|, s^{2r+1} |v_3(t, \xi)| \leq 6C_0 |g|_{D^c_x}.$$  

Therefore, $s^{2r+1} |V|_{D^c_x} \leq 21C_0 |g|_{D^c_x}$.

Taking the parameter $(\sigma, \nu)$ into account, the map $\Psi : D^c_x \mapsto D^c_x$ is written in the form $\Psi(\xi, \eta, \zeta, \sigma, \nu) = (\xi + U(\xi), \eta + V(\xi, \eta, \zeta), \zeta + W(\xi, \zeta), \sigma + \Lambda_1(\sigma, \nu), \nu + \Lambda_2(\sigma, \nu))$. We observe that from the construction of $\Lambda_1$ and $\Lambda_2$ (see the proof of Lemma B.6), we have

$$|\Lambda_1|_{D_x} \leq C_1 |f|_{D^c_x} \text{ and } \varepsilon |\Lambda_2|_{D_x} \leq C_1 |h|_{D^c_x}.$$  

As a consequence of Lemma B.10 and the Cauchy Integral Formula, we have the following.

**Corollary B.11** Under the assumptions of Lemma B.10, we have

$$\text{91}$$
i. \(|\Psi - \text{Id}|_{D_x^+} \leq C_2 \max \{s^{-2\tau - 1} |f|_{D^\tau}, s^{-2\tau - 1} |g|_{D^\tau}, \varepsilon^{-1}s^{-2\tau - 1} |h|_{D^\tau}\}\)

ii. \(\|D(\Psi - \text{Id})\|_{D_x^+} \leq C_2 \max \{s^{-(2+\frac{2}{3})\tau - 3} |f|_{D^\tau}, s^{-(2+\frac{2}{3})\tau - 3} |g|_{D^\tau}, \rho^{-1}s^{-2\tau - 1} |g|_{D^\tau}, s^{-(2+\frac{2}{3})\tau - 3} |h|_{D^\tau}, \varepsilon^{-1}s^{-\frac{2}{3}\tau - 2} |h|_{D^\tau}\},\)

where \(D(\Psi - \text{Id})\) is the derivative of \((\Psi - \text{Id})\).

Next we estimate the terms \(|f^+|_{D_x^+}, |g^+|_{D_x^+}\) and \(|h^+|_{D_x^+}\). To this end, we use the iteration relation \(\tilde{X}_+ = \Psi^*\tilde{X}\) and the linearization (78), which gives the following equations.

\[
\begin{cases}
(1 + \frac{\partial f}{\partial \sigma})f^+ = R_1 \\
\frac{\partial V}{\partial \xi} f^+ + (1 + \frac{\partial V}{\partial \eta})g^+ + \frac{\partial V}{\partial \zeta} h^+ = R_2 \\
\frac{\partial V}{\partial \xi} f^+ + (1 + \frac{\partial V}{\partial \eta})h^+ = R_3
\end{cases}
\]

(92)

where

\[
R_1 = f(\xi + U, \eta + V, \zeta + W, \sigma + \Lambda_1, \nu + \Lambda_2) - [f(\xi, 0, 0, \sigma, \nu)]_{d} + D_1(\overline{\nu} - U)(\xi, \sigma, \nu)
\]

\[
R_2 = g(\xi + U, \eta + V, \zeta + W, \sigma + \Lambda_1, \nu + \Lambda_2) - [g(\xi, 0, 0, \sigma, \nu)]_{d} + \frac{\partial g}{\partial \sigma}(\xi, 0, 0, \sigma, \nu) + \frac{\partial g}{\partial \eta}(\xi, 0, 0, \sigma, \nu)\zeta + \frac{1}{2}\frac{\partial^2 g}{\partial \sigma^2}(\xi, 0, 0, \sigma, \nu)\zeta^2
\]

\[
R_3 = h(\xi + U, \eta + V, \zeta + W, \sigma + \Lambda_1, \nu + \Lambda_2) - [h(\xi, 0, 0, \sigma, \nu)]_{d} + \frac{\partial h}{\partial \sigma}(\xi, 0, 0, \sigma, \nu)\zeta + \frac{1}{2}\frac{\partial^2 h}{\partial \sigma^2}(\xi, 0, 0, \sigma, \nu)\zeta^2
\]

Here \(D_1, D_2\) and \(D_3\) are the following linear operators. For suitable maps \(f_1, f_2\) and \(f_3\), \(D_1f_1(\xi, \sigma, \nu) = \frac{\partial f_1}{\partial \xi} \sigma, D_2f_2(\xi, \eta, \zeta, \sigma, \nu) = \frac{\partial f_2}{\partial \xi} \sigma + \frac{\partial f_2}{\partial \zeta} \Omega(\nu) \zeta\) and \(D_3f_3(\xi, \zeta, \sigma, \nu) = \frac{\partial f_3}{\partial \xi} \sigma + \frac{\partial f_3}{\partial \zeta} \Omega(\nu) \zeta - \Omega(\nu)f_3\).

**Lemma B.12** Under the assumptions of Proposition B.8, we have

i. \(|D_1\overline{\nu}|_{D_x^+}, |D_1U|_{D_x^+} \leq |\psi|_{D^\tau}\) and \(|D_1(\overline{\nu} - U)|_{D_x^+} \leq s^{-2\tau - 2} |f|_{D^\tau}\)

ii. \(|D_2\overline{\nu}|_{D_x^+}, |D_2V|_{D_x^+} \leq |g|_{D^\tau}\) and

\(|D_2(\overline{\nu} - V)|_{D_x^+} \leq s^{-2\tau - 1} |g|_{D^\tau} \max \{s^{-1} |f|_{D^\tau}, \rho^{-1} |g|_{D^\tau}, \varepsilon^{-1} |h|_{D^\tau}\}\)

iii. \(|D_3\overline{\nu}|_{D_x^+}, |D_3W|_{D_x^+} \leq |\psi|_{D^\tau}\) and

\(|D_3(\overline{\nu} - W)|_{D_x^+} \leq s^{-2\tau - 1} |h|_{D^\tau} \max \{s^{-1} |f|_{D^\tau}, \varepsilon^{-1} |h|_{D^\tau}\}\)
Proof. For a proof of item (i) see [31, 30]. We only prove item (iii), since item (ii) can be shown in the same way. Since
\[
D_3 \overline{w} = \left\{ h(\xi, 0, 0, \sigma, \nu) + \frac{\partial h}{\partial \xi}(\xi, 0, 0, \sigma, \nu) \right\}_d,
\]
it follows that \( |D_2 \overline{w}|_{D^e} \leq |h|_{D^e} \). Observe that
\[
\frac{d}{dt} w(t, \xi, \zeta) = (D_3 \overline{w})(x(t, \xi), z(t, \xi, \zeta)) + \frac{\partial w}{\partial \xi}(D_1 u)(t, \xi)
+ \frac{\partial w}{\partial \zeta}(D_3 w)(t, \xi, \zeta).
\]
By Gronwall’s inequality, for all \( t \in [0, 1] \), we have
\[
|w(t, \xi, \eta, \zeta)| \leq \max \left\{ |D_3 \overline{w}|_{D^e}, \left| \frac{\partial w}{\partial \xi} \right|_{D^e}, |D_1 u|_{D^e} \right\} e^{\left| \frac{\partial w}{\partial \xi} \right|_{D^e}} \leq |h|_{D^e}.
\]
Since \( W(\xi, \zeta) = w(1, \xi, \zeta) \), it follows that \( |W|_{D^e} \leq |h|_{D^e} \). In order to estimate \( |D_2 (\overline{w} - W)|_{D^e} \), we write
\[
D_3 (\overline{w} - W)(\xi, \zeta) = \int_0^1 D_3 (\overline{w}(x(t, \xi), z(t, \xi, \zeta) - \overline{w}(\xi, \zeta))
+ \int_0^1 \frac{\partial w}{\partial \xi}(x(t, \xi), z(t, \xi, \zeta))(D_1 U)(t, \xi) dt
+ \int_0^1 \frac{\partial w}{\partial \zeta}(x(t, \xi), z(t, \xi, \zeta))(D_3 W)(t, \xi, \zeta) dt,
\]
where we used (93). Now by estimating each of the integrals, we obtain the desired estimate for \( |D_3 (\overline{w} - W)|_{D^e} \).

Proposition B.13 Assume that
\[
|f|_{D^e} \leq \frac{1}{12 C_1} s^{2 \nu + 2}, \quad |g|_{D^e} \leq \frac{1}{48 C_1} \rho s^{2 \nu + 1} \quad \text{and} \quad |h|_{D^e} \leq \frac{1}{48 C_1} \varepsilon s^{2 \nu + 1}.
\]
Then \( \Psi(D^e) \subseteq D^e \). Moreover, if we assume that \( ds > 2n \), then the following holds for a constant \( C_3 \geq C_1 \).
\begin{itemize}
  \item[i.] \( |f^+|_{D^e} \leq C_3 |f|_{D^e} (\mathcal{M} + \max\{\frac{\rho}{\rho}, \frac{\varepsilon}{\varepsilon}, d^n e^{-\frac{\rho_0}{2}}\}) \);
  \item[ii.] \( |g^+|_{D^e} \leq C_3 |g|_{D^e} (\mathcal{M} + \max\{\frac{\rho_0}{\rho}, \frac{\varepsilon}{\varepsilon}, d^n e^{-\frac{\rho_0}{2}}\} + \mathcal{N}_1) \);
  \item[iii.] \( |h^+|_{D^e} \leq C_3 |h|_{D^e} (\mathcal{M} + \max\{\frac{\rho_0}{\rho}, \frac{\varepsilon}{\varepsilon}, d^n e^{-\frac{\rho_0}{2}}\} + \mathcal{N}_2) \)
\end{itemize}
where \( \mathcal{M} = \max\{s^{-4 \nu - 4} |f|_{D^e}, \rho^{-1} s^{-2 \nu - 1} |g|_{D^e}, \varepsilon^{-1} s^{-4 \nu - 4} |h|_{D^e}\} \) and where \( \mathcal{N}_1 = s^{-2 \nu - 1}(s^{-1} |f^+|_{D^e} + \varepsilon^{-1} |h^+|_{D^e}) \) and \( \mathcal{N}_2 = s^{-2 \nu - 2} |f^+|_{D^e} \).
Proof. Since \( s_\ast - s_+ > \frac{\varepsilon}{4} \) etc., the first claim follows from the fact that \(|U|_{D_\varepsilon}, |V|_{D_\varepsilon}, |W|_{D_\varepsilon}, |\Lambda_1|_{D_\varepsilon} \)
and \(|\Lambda_2|_{D_\varepsilon} \) are smaller than \( \frac{s}{4}, \frac{\rho}{4}, \frac{\varepsilon}{4}, \frac{\rho - \varepsilon}{4} \) respectively. Let \( ds > 2n \). We
show (iii). The other two inequalities can be shown similarly. By (92), we have
\[
|h^+|_{D_\varepsilon} \leq |R_3|_{D_\varepsilon} + \left| \frac{\partial W}{\partial \xi} \right|_{D_\varepsilon} |f^+|_{D_\varepsilon}.
\]
We notice that
\[
|R_3|_{D_\varepsilon} \leq |h(\xi + U, \eta + V, \zeta + W, \sigma + \Lambda_1, \nu + \Lambda_2) - h(\xi, \eta, \zeta, \sigma, \nu)|_{D_\varepsilon}
+ |h(\xi, \eta, \zeta, \sigma, \nu) - h(\xi, \eta, \zeta, \sigma, \nu) - \frac{\partial h}{\partial \xi}(\xi, 0, 0, \sigma, \nu)\eta|_{D_\varepsilon}
+ |h(\xi, 0, 0, \sigma, \nu) - \{h(\xi, 0, 0, \sigma, \nu)\} d|_{D_\varepsilon}
+ \left| \frac{\partial h}{\partial \eta}(\xi, 0, 0, \sigma, \nu)\eta - \left\{ \frac{\partial h}{\partial \eta}(\xi, 0, 0, \sigma, \nu)\right\} d \right|_{D_\varepsilon} + |D_3(\overline{W} - W)|_{D_\varepsilon}.
\]
By the Mean Value Theorem, the sum of the first two terms are bounded from
above by
\[
|h|_{D_\varepsilon} \left( \mathcal{M} + \max \left\{ \left( \frac{\xi}{\varepsilon} \right)^2, \left( \frac{\rho + \varepsilon}{\rho} \right)^2 \right\} \right).
\]
It can be shown, see [30] p150, that the third and fourth terms are upper bounded
by \( d^\rho e^{-\frac{\rho}{\varepsilon}} |h|_{D_\varepsilon} \). These estimates together with estimates from Lemma B.12 and
Lemma B.10 give the desired inequality (iii). \( \square \)

Finally we give estimate \(|\tilde{U} - \tilde{U}^+|_{D_\varepsilon}, |\tilde{V} - \tilde{V}^+|_{D_\varepsilon}\) and \(|\tilde{W} - \tilde{W}^+|_{D_\varepsilon}\). To this end, we
denote by \(|\Psi_j|_{1,D_\varepsilon}\) the \(C^1\)-norm on the domain \(D_\varepsilon\), i.e.,
\[
|\Phi_j|_{1,D_\varepsilon} = \max \left\{ |\Phi_j|_{D_\varepsilon}, |D\Phi_j|_{D_\varepsilon} \right\}.
\]
By the Mean Value Theorem and the iteration relation \( \Phi_+ = \Phi \circ \Psi \), it follows,

**Proposition B.14** Under the assumptions of Proposition B.13, we have

i. \(|\Phi_j|_{1,D_\varepsilon} \leq |\Phi_j|_{1,D_\varepsilon} \max \left\{ 1 + |\psi|_{D_\varepsilon}, 1 + |D\psi|_{D_\varepsilon} \right\} \)

ii. \(|\tilde{U} - \tilde{U}^+|_{D_\varepsilon}, |\tilde{V} - \tilde{V}^+|_{D_\varepsilon}, |\tilde{W} - \tilde{W}^+|_{D_\varepsilon}, |\tilde{\Lambda}_1 - \tilde{\Lambda}_1^+|_{D_\varepsilon}, |\tilde{\Lambda}_2 - \tilde{\Lambda}_2^+|_{D_\varepsilon} \leq |\Phi|_{1,D_\varepsilon} |\psi|_{D_\varepsilon} \)

**B.5 Convergence of iteration process**

In order to ensure that the iteration process converges, we need to specify the
sequences \(\{s_j\}, \{\rho_j\}\) and \(\{\varepsilon_j\}\) in a proper way. We first define a sequence \(\{\delta_j\}_{j \geq 0}\)
of positive numbers by the relation $\delta_{j+1} = \delta_j^{p+1}$, where $p > 0$. Next we choose the sequences $\{s_j\}, \{\rho_j\}$ and $\{\varepsilon_j\}$ as follows.

\[ s_j = s_0 a^j, \quad \rho_j = \delta_j^q \quad \text{and} \quad \varepsilon_j = \delta_j^q \quad (j \geq 0), \]

where $a \in (0, \frac{1}{2})$ and $q > 0$. In the following, we fix the values $p$ and $q$ in such a way that $q \in (0, 2)$ and $p \in (0, 1 - \frac{q}{2})$. Furthermore we assume that

\[ 0 < s_0 < \min \left\{ \frac{\kappa}{2}, \frac{1}{2n + 1} \right\}. \quad (94) \]

Recall that the truncation orders $d_j = \text{int}(s^{-2})$, see (85). Now the exponential sequence $\{\delta_j\}$ will be used to control the errors $|f_j|_{D_j^2}, |g_j|_{D_j^2}$ and $|h_j|_{D_j^2}$. The following proposition says that we can fix the initial values $s_0$, $s_0$ in such a way that the maps $\Phi_j$ converge to a Whitney-smooth map $\Phi = \Phi_\infty$, for $j \to \infty$.

**Proposition B.15** For a sufficiently small $\delta_0 > 0$. There exists $s_0 \in (0, \min \left\{ \frac{s}{2}, \frac{1}{2n+1} \right\})$ such that for all $j \geq 0$, the following holds:

i. the assumptions of Proposition B.13 are satisfied;

ii. $|f_j|_{D_j^2} \leq \delta_j$, $|g_j|_{D_j^2} \leq \delta_j^2$, $|h_j|_{D_j^2} \leq \delta_j^2$;

iii. there exists a number $b \in (0, 1)$ such that on the domain $D_{j+1}$,

\[ \left| \tilde{U} - \tilde{U}_j^+ \right|, \left| \tilde{V} - \tilde{V}_j^+ \right|, \left| \tilde{W} - \tilde{W}_j^+ \right|, \left| \tilde{\lambda}_1 - \tilde{\lambda}_j^+ \right|, \left| \tilde{\lambda}_2 - \tilde{\lambda}_j^+ \right| \leq C_4 \delta_j^b \]

and that

\[ \left| \frac{\partial h_j}{\partial z} \right|_{D_j^2} \leq C_4 \delta_j^b. \]

Proposition B.15 will be proved with help of the lemmas given below, compare with the proof of [13] (Prop. 5.9).

**Remark B.16** We should keep in mind that the present proof for Theorem 1.6 is based on the scaled families $\frac{\pi}{\gamma} X$ and $\frac{\pi}{\gamma} \tilde{X}$, respectively, as the perturbed and unperturbed family, compare with Remark B.2. For the un-scaled families $X$ and $\tilde{X}$, we have to multiply the right hand of the inequalities from item (ii) by the factor $\gamma / \gamma_0$.

**Lemma B.17** Assume that $\delta_0 \in (0, 1)$ satisfy the following conditions

a. $\delta_0^{2-2p-q} \leq \frac{1}{14d_3} s_0^{4r+4}$, $\delta_0^{(2-2p-q)} \leq a^{4r+1}$ and $\delta_0^{\gamma(p(q-1))} \leq \frac{1}{14d_3}$;

b. $d_0 e^{-\frac{1}{2} d_j s_j} \leq \frac{1}{4C_3} \delta_j^{p+1}$, for all $j \geq 0$. 

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Then, for all \( j \geq 0 \),

i. \( \delta_j \leq \frac{1}{12C_1} s_j^{2r+2} \), \( \delta_j^2 \leq \frac{1}{48C_1} \varepsilon_j s_j^{2r+1} \);

ii. \( \delta_j^{2-2p-q} \leq \frac{1}{144C_3} s_j^{4r+4} \);

iii. \( |f^j|_{D_j^c} \leq \delta_j \), \( |g^j|_{D_j^c} \leq \delta_j^2 \), \( |h^j|_{D_j^c} \leq \delta_j^2 \).

**Remark B.18** Suppose that \( s_0, \delta_0 \) can be chosen such that they satisfy the assumptions of Lemma B.17, then conclusions (i) and (iii) imply that the assumptions of Proposition B.13, except for the condition that \( d_j s_j > 2n \), are satisfied for all \( j \). However, the inequality \( d_j s_j > 2n \) is implied by the assumption (94).

**Proof. (Lemma B.17)** We apply induction on the index \( j \).

i. Notice that conclusion (i) holds for \( j = 0 \). Now suppose that it also holds for the \( j \)th step. Then,

\[
\delta_{j+1} = \delta_j \delta_j^p \leq \frac{1}{12C_1} s_j^{2r+2} \delta_j^p \leq \frac{1}{12C_1} s_j^{2r+2} a^{-2r-2} \delta_j^p \leq \frac{1}{12C_1} s_j^{2r+2},
\]

and

\[
\delta_{j+1}^2 = \delta_j^2 \delta_j^{2p} \leq \frac{1}{48C_1} \varepsilon_j s_j^{2r+1} \delta_j^{2p} \leq \frac{1}{48C_1} \varepsilon_j s_j^{2r+1} \delta_j^{2p} a^{-2r-2},
\]

Since \( \delta_0^{2p-q} a^{-2r-2} \leq 1 \), the claim follows.

ii. Observe that (ii) holds for \( j = 0 \). Supposing now that conclusion (ii) is true for \( j \geq 0 \), we then have

\[
\delta_{j+1}^{2-2p-q} = \frac{1}{144C_3} s_j^{4r+4} \delta_j^{2-2p-q} \leq \frac{1}{144C_3} s_j^{4r+4} \delta_j^{2-2p-q} a^{4r+4}.
\]

Hence, we conclude that (ii) holds for all \( j \geq 0 \).

iii. Since all three the estimates of part (iii) are similar, we only show the first. Suppose that conclusion (iii) holds for the \( j \)th step. Then, by (i) and the assumption that \( s_0 \leq \frac{1}{2n+1} \), the assumptions in Proposition B.13 and hence the claims of Proposition B.13 hold. In particular, we have

\[
|f^{j+1}|_{D_j^c} \leq C_3 \left| f^j \right|_{D_j^c} \left( \mathcal{M} + \max \left\{ \frac{\varepsilon_j+1}{\varepsilon_j}, d_j e^{-\frac{1}{2}d_j s_j} \right\} \right).
\]
Observe that $\mathcal{M} \leq \delta_j^{2-q} s_j^{-4r-4}$ for all $j$. Hence,

$$\left| f_j \right|_{D_j^r} \mathcal{M} \leq \delta_j^{3-q} s_j^{-4r-4} \leq \delta_{j+1}^{2-2p} \delta_j^{2-4r-4} \leq \frac{1}{144C_3} \delta_{j+1}.$$  

Notice that $\frac{\varepsilon_{j+1}}{\varepsilon_j} = \delta_j^{pq}$. It follows that

$$\left| f_j \right|_{D_j^r} \frac{\varepsilon_{j+1}}{\varepsilon_j} \leq \delta_{j+1} \delta_0^{pq-p} \leq \frac{1}{144C_3} \delta_{j+1}.$$  

The above estimates together with the assumption that $d_n j \leq \frac{1}{2} d_j s_j \leq \frac{1}{4C_3} \delta_{j+1}$, yield the desired inequality $\left| f_{j+1} \right|_{D_j^r} \leq \delta_{j+1}$.

\[ \blacksquare \]

**Lemma B.19** For a sufficiently small value $\delta_0$, we find a positive value $s_0 \in (0, \min \{ \frac{c}{2}, \frac{1}{2q+1} \})$ such that assumptions (a) and (b) of Lemma B.17 are satisfied.

**Proof.** By [30] p. 153, condition (b) of Lemma B.17 is fulfilled if $s_0 \geq \frac{c_1 \log 1}{s_0} + c_2 \log \frac{1}{\delta_0} + c_3$ for suitable positive values $c_1$, $c_2$ and $c_3$. In order to prove Lemma B.19, we only need to find small $\delta_0$, $s_0$ such that the following inequalities hold simultaneously.

$$\begin{bmatrix}
\frac{1}{s_0} \\
\delta_0^{2-2p-q} \\
\delta_0^{p(2-2p-q)} \\
\delta_0^{p(q-1)}
\end{bmatrix} \geq \begin{bmatrix}
c_1 \log \frac{1}{s_0} + c_2 \log \frac{1}{s_0} + c_3 \\
c_4 s_0^{4r+4} \\
c_5 a^{4r+4} \\
c_5,
\end{bmatrix} \quad (95)$$

where $c_i$ ($i = 1, \ldots, 5$) are positive constants. The latter two inequalities of (95) are easily satisfied when $\delta_0$ is sufficiently small. Chose a constant $K \in (0, \frac{2-2p-q}{4r+4})$ and let $s_0 = \frac{1}{c_4 + 1} \delta_0^K$. Then, the second inequality of (95) also holds. Observe that $s_0 \to 0$, when $\delta_0 \to 0$. The first inequality of (95) becomes a special form of the following inequality

$$\frac{1}{s_0} - \tilde{c}_1 \log \frac{1}{s_0} - \tilde{c}_2 \geq 0, \quad (96)$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are constants. Since the function $f(x) = x - \tilde{c}_1 \log x - \tilde{c}_2$, defined on $\mathbb{R}$, is increasing for sufficiently large $x$, it follows that (96) holds for a sufficiently small $s_0$, which is ensured by choosing a sufficiently small value $\delta_0$.

\[ \blacksquare \]
Recall from Remark B.2 that for our proof of the main Theorem 1.6, we used the scaled families $\tilde{X}$ and $\tilde{X}_\varepsilon$ as the unperturbed and perturbed family, respectively. Our proof for Theorem 1.6 is now concluded by setting $\varepsilon = \delta_0$ and by letting the perturbation $\tilde{X}$ be in the real-analytic neighbourhood $A_{\gamma/\gamma_0, \sigma}$ of the unperturbed family $X$, where the set $A_{\gamma/\gamma_0, \sigma}$ is defined as in §B.1 (b).

Indeed, by applying the Inverse Approximation Lemma to the approximations $\Phi_j$ on the domains $D_j$, we obtain the limit $\Phi_\infty$ as a Whitney-smooth map on $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times \Gamma_{r,\gamma}(U')$, which is real analytic in the $x_\infty$-direction. By construction, this map $\Phi_\infty$ is normally affine and symplectic. Next we consider the limit $\tilde{X}_\infty = \Phi_\infty^* X$. By Proposition B.15, it follows that $f_\infty, g_\infty$ and $h_\infty$ vanish at the Diophantine tori $V = \{y_\infty = 0, z_\infty = 0\}$, where $h_\infty$ is quadratic in $z_\infty$. This implies that the image $\tilde{X}_\infty(V)$ is $\tilde{X}$-invariant and that

$$\tilde{X}_\infty(\tilde{x}, \tilde{y}, \tilde{z}) = \tilde{\omega} + O(1) \frac{\partial}{\partial x} + O(1) \frac{\partial}{\partial y} + \Omega(\tilde{\mu}) \frac{\partial}{\partial z},$$

where $\tilde{x} = x_\infty$ etc. and where the parameter $(\tilde{\omega}, \tilde{\mu})$ satisfies the Diophantine conditions. Finally, by the Whitney Extension Theorem we can extend the limit $\Phi_\infty$ to a smooth map $\Phi$ defined on $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times U$ such that $\Phi$ satisfies conclusions of Theorem 1.6.

### C Normal linearization on the symplectic normal bundle

In §3, we defined normal linearization of a given vector field with an invariant submanifold in the dissipative case by means of a scaling operator $D_e$, compare with (36). This procedure induces a vector field – the normal linear part – on the normal bundle of the invariant manifold. Normal linearization simplifies the perturbation problem: via the scaling $D_e$, normal linear stability of a given integrable family is implied by that of the corresponding normal linear part, see §3. However, the normal linearization in the dissipative setting and in the reversible setting, as explained in 3, is less satisfactory for the symplectic setting, since the normal linear part of a non-integrable Hamiltonian vector field contains a quadratic term, compare with Remark 3.3. To see this, we consider the symplectic phase space $(M, \sigma)$ endowed with an free $\mathbb{T}^n$-action, as introduced in §1.2, and a non-integrable (i.e., non-$\mathbb{T}^n$-symmetric) vector field $Y$ with the $\mathbb{T}^n$-orbit $V$ as an invariant submanifold. By the generalized Darboux Theorem [1], there exist local coordinates $(x, y, z) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_{2q}) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q}$, in which the submanifold $V$ is given by $V = \{(x, y, z) : (y, z) = (0, 0)\}$ and the 2-form $\sigma = dx \wedge dy + dz^2$. Suppose that $Y$ takes, in local coordinates $(x, y, z)$, the
shape
\[ Y(x,y,z) = f(x,y,z) \frac{\partial}{\partial x} + g(x,y,z) \frac{\partial}{\partial y} + h(x,y,z) \frac{\partial}{\partial z}. \]

Since \( V \) is \( Y \)-invariant, we have that \( g(x,0,0) \equiv 0 \) and \( h(x,0,0) \equiv 0 \). Using the scaling \( D_\varepsilon(x,y,z) = (x,\varepsilon^2 y, \varepsilon z) \), we obtain the normal linear part \( NY \) defined as
\[
NY = \lim_{\varepsilon \to 0} D_\varepsilon Y = f(x,0,0) \frac{\partial}{\partial x} + \left[ g_y(x,0,0)y + \frac{1}{2} G(x,0,z) \right] \frac{\partial}{\partial y} + h_z(x,0,0)z \frac{\partial}{\partial z},
\]
where \( g_y = \frac{\partial g}{\partial y}, h_z = \frac{\partial h}{\partial z} \) and \( G(x,0,z) = \left( \langle \frac{\partial^2 g_y}{\partial^2 y} z, z \rangle, \ldots, \langle \frac{\partial^2 g_n}{\partial^2 y} z, z \rangle \right) \). We see that the vector field \( NY \) contains, indeed, the nonlinear term \( G \) being quadratic in \( z \). To avoid such a nonlinear term, we use the (smaller) symplectic normal bundle [122] instead of the ordinary one, see [31] (§6b) to define (symplectic) normal linear part, compare with conclusion (iii) of Theorem 1.6.

We recall from [122, 31] the definition of the symplectic normal bundle and the notion of symplectic normal linearization. Consider the symplectic manifold \((M, \sigma)\) being our phase space and an isotropic submanifold \( V \subset M \). Let \( Y = Y(x,y,z) \) be a Hamiltonian vector field on \((M, \sigma)\) with \( V \) as an invariant submanifold. We define a normal linearization of such a vector field by considering the (smaller) symplectic normal bundle, see [122, 31]. Indeed, Let \((TV)^\perp\) be the \( \sigma \)-orthogonal complement of \( TV \), that is,
\[
(TV)^\perp = \{ u \in M : \sigma(u,v) = 0, \text{ for all } v \in V \}.
\]
By [122, 1], the quotient \( N_\sigma V = (TV)^\perp / TV \) is a symplectic vector bundle over \( V \), called the symplectic normal bundle of \( V \) in \( M \). Notice that \( \dim (TV)^\perp = n + 2q \) and \( \dim TV_x = n \), it follows that the fiber \((T_xV)^\perp / T_xV \) \( (x \in V) \) considered as a vector space is isomorphic to \( \mathbb{R}^{2q} \). The base of the bundle \( N_\sigma V \) is the submanifold \( V \cong \mathbb{T}^n \), we have a trivialization \( N_\sigma V \cong \mathbb{T}^n \times \mathbb{R}^{2q} \). Since the flow of \( Y \) leaves the submanifold \( V \) invariant, it induces a smooth flow on \( N_\sigma V \) and therefore a vector field \( N_\sigma Y \) on it, called a (symplectic) normal linearization of \( Y \). Following [31], we state that this vector field \( N_\sigma Y \) obtains the form
\[
N_\sigma Y(x,y,z) = f(x,0,0) \frac{\partial}{\partial x} + \left[ \frac{\partial h}{\partial z}(x,0,0)z \right] \frac{\partial}{\partial z}, \quad (98)
\]
where \( \frac{\partial h}{\partial z}(x,0,0) \in \mathfrak{sp}(2q, \mathbb{R}) \). Now preservation of the normal linear part in Theorem 1.6 is expressed in terms of the symplectic normal linearization.
A question one now may ask is what is the relation between the ordinary normal linear part \( NY \) and the symplectic normal linear part \( N^\sigma Y \). By a direct computation, we obtain the following

\[
g_\eta(x, 0, 0) = -\left( \frac{\partial f(x, 0, 0)}{\partial x} \right)^T \quad \text{and} \quad \frac{\partial^2 G(x, 0, z)}{\partial z^2} = \frac{\partial h_z(x, 0, 0)}{\partial x} J_{2q},
\]

where \( J_{2q} \) is the symplectic matrix given by (7). This shows that the vector field \( NY \) is completely determined by the \( N^\sigma Y \). In other words, the symplectic normal linear part \( N^\sigma Y \) contains all data of the vector field \( Y \) with respect to the ordinary normal bundle \( NV = T_Y M/TV \).