Secular and oscillatory motions in dynamical systems

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1. Toroidal symmetry
2. Secular (slow) versus oscillatory (rapid)
3. Structures in slow dynamics
4. Examples & Conclusions


Action-angle variables given $H : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\dot{q} = \frac{\partial H}{\partial q}, \quad \dot{p} = -\frac{\partial H}{\partial p}$$

Define $A := A(H)$, then $T = \frac{dA}{dH}$,

$I := \frac{1}{2\pi} A, \ \varphi := \frac{T}{2\pi} t \leadsto$

$$\dot{I} = 0, \quad \dot{\varphi} = \omega(I),$$

$$\omega(I) = \frac{dH_0}{dI}(I)$$

General format of perturbation in $n$ dof:

$$\dot{\varphi} = \omega(I) + \varepsilon f(I, \varphi), \quad \dot{I} = \varepsilon g(I, \varphi)$$

where $H = H_0 + \varepsilon H_1, \ \omega(I) = \frac{dH_0}{dI}, \ f = \frac{\partial H_1}{\partial I}, \ \text{and} \ g = -\frac{\partial H_1}{\partial \varphi}$

Perturbation problems as usual
Integrability for $\varepsilon = 0$
Examples: 1 dof systems, central force field, Lagrange top, geodesics on ellipsoid, combinations of the above, ... 

Near integrability occurs persistently: think of Solar system as a perturbation of a number of uncoupled central force systems (Kepler, Newton)
Also in small neighbourhood of elliptic equilibrium points (see below ...)

Slow-fast dynamics: $I$–slow (secular), $\varphi$–fast (oscillatory)
Local theory at $0 \in \mathbb{R}^{2n} = \{p, q\}$

Birkhoff normal form at equilibrium

$$H(p, q) = \frac{1}{2}\langle \omega, (p^2 + q^2) \rangle + F(p^2 + q^2) + G(p, q)$$

with $\sigma = dp \wedge dq$, $F = O(4)$ and $G = O(N + 1)$ ($N$ depends on non-resonance properties of $\omega$)

Define $q = \sqrt{2I} \sin \varphi, p = \sqrt{2I} \cos \varphi$, then $p^2 + q^2 = 2I$, $\sigma = dI \wedge d\varphi$ and

$$H(I, \varphi) = \langle \omega, I \rangle + F(I) + G(I, \varphi)$$

$\Rightarrow$ system

$$\dot{\varphi} = \omega + \frac{\partial F}{\partial I} + \frac{\partial G}{\partial I}, \quad \dot{I} = -\frac{\partial G}{\partial \varphi}$$

where $G$ is small whenever $I$ is

Locally same perturbation problem
Proofs: by action of adjoint operator

\[ \omega \left( p \frac{\partial}{\partial q} - q \frac{\partial}{\partial p} \right) \]

on homogeneous parts of Taylor series

Simplification leaves only terms in kernel

Resonant examples later

Similar theory near periodic solutions,

action by, e.g.,

\[ \alpha \frac{\partial}{\partial t} + \omega \left( p \frac{\partial}{\partial q} - q \frac{\partial}{\partial p} \right) \]

and analogously near quasi-periodic tori
A simple Averaging Theorem

Given a (general) $2\pi$–periodic system

\[
\dot{\varphi} = \omega(I) + \varepsilon f(\varphi, I), \quad \dot{I} = \varepsilon g(\varphi, I), \quad (\varphi, I) \in \mathbb{T}^1 \times \mathbb{R}^n
\]

Suitable near-identity transformation $(\varphi, I) \mapsto (\varphi, J) \leadsto \text{truncating at order } O(\varepsilon^2)$

\[
\dot{J} = \varepsilon \bar{g}(J) \text{ where }
\]

\[
\bar{g}(J) = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi, J) d\varphi
\]

**Theorem:** If $\omega(J) > 0$ is bounded away from 0, then, for a constant $c > 0$

\[
|I(t) - J(t)| < c\varepsilon \text{ for } 0 \leq t \leq \frac{1}{\varepsilon}.
\]
Proof

Any transformation

\[ J = I + \varepsilon k(\varphi, I) \Leftrightarrow I = J + \varepsilon h(J, \varphi, \varepsilon) \]

\[ \Rightarrow \]

\[ \dot{J} = \dot{I} + \varepsilon \frac{\partial k}{\partial I} \dot{I} + \varepsilon \frac{\partial k}{\partial \varphi} \dot{\varphi} \]

\[ = \varepsilon \left( g(\varphi, I) + \frac{\partial k}{\partial \varphi} \omega(I) \right) + O(\varepsilon^2) \]

Define

\[ k(\varphi, I) = -\frac{1}{\omega(I)} \int_0^{2\pi} (g(\varphi, I) - \bar{g}(I)) d\varphi, \]

then

\[ \dot{J} = \varepsilon \bar{g}(I) + O(\varepsilon^2) = \varepsilon \bar{g}(J) + O(\varepsilon^2) \]
Extensions to many classes of systems, for instance to Hamiltonian systems

Generalization to the immediate vicinity of a quasi-periodic torus

Further normalization give estimates that are polynomial or exponential in $\varepsilon$ (in real analytic case)

Passage through resonance: then condition on $\omega$ not valid.


Application of Averaging Theorem

Hamiltonian time dependent, slowly varying system

\[ \dot{\varphi} = \omega(I, \lambda) + \varepsilon f(I, \varphi, \lambda), \quad \dot{I} = \varepsilon g(I, \varphi, \lambda), \quad \dot{\lambda} = \varepsilon \]

where \((\varphi, I, \lambda) \in T^1 \times \mathbb{R} \times \mathbb{R}\)

Compare with averaged system

\[ \dot{J} = \varepsilon \bar{g}(J), \quad \dot{\Lambda} = \varepsilon \]

Hamiltonian character gives \(\bar{g} \equiv 0\), hence

\[ |I(t) - I(0)| < c \varepsilon \text{ for } 0 \leq t \leq \frac{1}{\varepsilon} \]

\(\therefore I\) is adiabatic invariant
Planck’s: In the above take

\[ H(p, q, \ell) = \frac{p^2}{2\ell^2} + \ell g \frac{q^2}{2} \text{ with } \dot{\ell} = \varepsilon \]

Energy level \( H(p, q, \ell) = E \) is ellipse: axes \( a = \ell \sqrt{2E} \) and \( b = \sqrt{2E/\ell g} \)

Then action variable

\[ I = \frac{1}{2\pi} A = \frac{1}{2\pi} \pi ab = \frac{E}{\sqrt{g \ell}} = \frac{E}{\nu} \]

Laplace (more degrees of freedom) semi-major axes of Keplerian ellipses of the planets have no secular variations

KAM theory and adiabatic invariants: 2 dof versus more; Arnold diffusion . . .
Consider time periodic Hamiltonian (swing)

\[ H_{\alpha,\beta}(p, q, t) = \frac{1}{2}p^2 + (\alpha^2 + \beta \cos t)(1 - \cos q) \]

where \((p, q) \approx (0, 0)\) and \((\alpha, \beta) \approx (\frac{1}{2}, 0)\)
(near 1 : 2 resonance; \((p, q, t) \in \mathbb{R} \times T^1 \times T^1)\)

Two reversibilities: \(R(p, q, t) = (-p, q, -t)\) and \(S(p, q, t) = (p, -q, -t)\)

Complex notation \(z = p + \frac{1}{2}i q \leadsto\)

\[ \dot{z} = \frac{1}{2}iz - (\delta + \beta \cos t) \sin q - \frac{1}{4}(\sin q - q) \]
\[ \dot{t} = 1 \]

where \(\delta = \alpha^2 - \frac{1}{4}\) detunes the 1 : 2 resonance

Normalization/averaging (resonant) as described before

Double cover given by map

\[ \Pi : \mathbb{C} \times \mathbb{R} / (4\pi \mathbb{Z}) \rightarrow \mathbb{C} \times \mathbb{R} / (2\pi \mathbb{Z}) \]

\[ (\zeta, t) \mapsto (\zeta e^{\frac{1}{2}it}, t \mod 2\pi) \]

Deck transformation \((\zeta, t) \mapsto (\zeta e^{\pi i}, t - 2\pi)\)

Symmetry induced by reversions \(R\) and \(S \sim \mathbb{Z}_2 \oplus \mathbb{Z}_2\)-symmetry (all compatible)

Takens normal form Poincaré map

\[ P_{\alpha,\delta}(\zeta) = (-\text{Id}) \circ N_{\alpha,\delta}^{2\pi}(\zeta) + \text{h.o.t.}(\zeta, \alpha, \delta) \]

Singularity theory describes planar Hamilton functions of \(N_{\alpha,\delta}\) in ‘generic’ setting

Similarly for the other \(k : 2\) resonances
Bifurcation diagram of $N_{\alpha,\delta}$: showing the slow or secular dynamics near $1:2$ resonance
Poincaré map $P_{\alpha,\beta}$ for $\alpha \approx \frac{1}{2}$ and $\beta \approx .4$

toy model for botafumeiro (Santiago de Compostela)

coeexistence of periodicity, quasi-periodicity and ‘chaos’
Swing forced with fixed $\omega_1$ and $\omega_2$:

$$H_{\alpha,\beta,\omega_1,\omega_2}(p, q, t) = \frac{1}{2}p^2 + (\alpha^2 + \beta(\cos(\omega_1 t) + \cos(\omega_2 t)))(1 - \cos q)$$

gives on $T^2 \times \mathbb{R}^2$ system

$$\dot{\varphi}_1 = \omega_1, \quad \dot{\varphi}_2 = \omega_2$$
$$\dot{q} = p, \quad \dot{p} = -(\alpha^2 + \beta(\cos \varphi_1 + \cos \varphi_2)) \sin q$$

Resonances given by

$$k_1\omega_1 + k_2\omega_2 + \ell\alpha = 0,$$

for $k_1, k_2, \ell \in \mathbb{Z}$

If $\omega_1, \omega_2$ are rationally independent, then dense set of $\alpha$-values
A dense set of resonances

Arnold resonance tongues as these occur in many contexts
Diophantine conditions: For given $\tau > 1$ and $\gamma > 0$

$$|k_1 \omega_1 + k_2 \omega_2 + \ell \alpha| \geq \gamma(|k_1| + |k_2|)^{-\tau},$$

for all $(k_1, k_2) \in \mathbb{Z} \setminus \{(0, 0)\}$ and for all $\ell \in \mathbb{Z}$ with $|\ell| \leq N$

Nowhere dense, large positive measure (for small $\gamma$)

Normalized system (need Diophantine conditions) has same planar vector field truncation $N_{\alpha,\beta}$ as before; same secular motion as before


H. Hanßmann, *Local and Semi-Local Bifurcations in Hamiltonian Dynamical Systems* Results and Examples, LNM 1893 Springer 2007
Model for quasi-periodic center-saddle bifurcation

\[ H_\lambda(\varphi, I, p, q) = \langle \omega, I \rangle + \frac{1}{2}p^2 + (\lambda q - q^3) + \text{h.o.t.}(p, q, \lambda) \]

‘Cantorized’ fold: for \( q < 0 \) hyperbolic 2-tori occur and for \( q > 0 \) elliptic 2-tori
Universal model, in particular robust
Many further models in

Co-existing dynamics with periodicity and chaos in gaps;
often infinite regress inside gaps

Oxtoby ‘paradox’ nowhere dense versus positive measure
meagre versus full measure

Principle: mix of Singularity Theory and Kolmogorov Arnold Moser Theory
Dissipative analogues

Same idea of oscillatory versus secular dynamics, normalizing/averaging the former away

**Periodic** attractors and Cantorized families of quasi-periodic attractors of positive measure in parameter space

**Hopf-Neĭmark-Sacker** bifurcation between them

**Quasi-periodic** Hopf bifurcation from $n$- to $(n + 1)$-tori

**Onset** of turbulence: Hopf-Landau-Lifschitz-Ruelle-Takens in generic families co-existence of periodicity, quasi-periodicity (of positive measure in parameter space) and chaos (often idem dito)

**Cantorized** fold diagram also for bifurcations of diffeomorphisms from invariant circles to 2-tori

**Oxtoby ‘paradox’** as before
Analogies of Averaging Theorem valid in some cases (e.g., in cases with an attractor)

Singular perturbation theory is challenging

HWB, T.J. Kaper and M. Krupa, Geometric desingularization of a cusp singularity in slow fast systems with applications to Zeeman’s examples. *JDDE* 25(4) (2013) 925-958