Multiperiodic dynamics –
overview and some recent results

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Contents

1. Complex Linearization I

Problem Given $F(z) = \lambda z + f(z)$
holomorphic map (germ) $F : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$
$f(0) = f'(0) = 0$;
to find $\Phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$
biholomorphic germ such that

$$\Phi \circ F = \lambda \cdot \Phi$$

Such Φ is called linearization of F

Formal solution

Given $f(z) = \sum_{j \geq 2} f_j z^j$
want $\Phi(z) = z + \sum_{j \geq 2} \phi_j z^j$

$\exists$ solution for $\lambda \neq 0$ no root of 1
Proof recursive / inductive:

For $j = 2:$ get $\lambda(1 - \lambda)\phi_2 = f_2$

For $j = 3:$ get $\lambda(1 - \lambda^2)\phi_3 = f_3 + 2\lambda f_2 \phi_2$

For $j = n:$ get $\lambda(1 - \lambda^{n-1})\phi_n = f_n +$ known

1. Complex Linearization II: Convergent Solutions

Hyperbolic case $0 < |\lambda| \neq 1$:

Solved by H. Poincaré (proof by iteration)

Elliptic case $\lambda \in \mathbb{T}^1$ (SMALL DIVISORS):

Solved by C.L. Siegel when for some $\gamma > 0$ and $\tau > 2$

$$|\lambda - \mathrm{e}^{2\pi \mathrm{i}\frac{p}{q}}| \geq \gamma |q|^{-\tau} \tag{1}$$

Diophantine (nonresonance) conditions: set of full measure in $\mathbb{T}^1$

1. Complex Linearization III: H. Cremer's topological counter example (1928), version by M.R. Herman

$F(z) = \lambda z + z^2$, $\lambda \in \mathbb{T}^1$ not a root of unity
$F^q(z) = \lambda^q z + \cdots + z^{2^q}$

Point z is periodic of period q iff $F^q(z) = z$

$$F^q(z) - z = z(\lambda^q - 1 + \cdots + z^{2^q - 1})$$

Call $N = 2^q - 1$

$$z_1 \cdot z_2 \cdot \cdots \cdot z_N = \lambda^q - 1$$

where z_j nontrivial solutions

$\Rightarrow \exists$ solution within radius

$$|\lambda^q - 1|^{1/N}$$

of $z = 0$. Consider $\lambda \in \mathbb{T}^1$ with

$$\mathrm{liminf}_{q \to \infty} |\lambda^q - 1|^{1/N} = 0$$

form residual set (dense G_δ, 2nd Baire)
Measure and category, Oxtoby (1970)

For such λ linearization is impossible:
- Linearization $z \mapsto \lambda z$ 'irrational' rotation
- $\exists$ periodic points in any nbhd. of $z = 0$

1. Circle Maps I

Problem Given
$P_{\alpha,\varepsilon} : \mathbb{T}^1 \to \mathbb{T}^1 \; x \mapsto x + 2\pi\alpha + \varepsilon a(x, \alpha, \varepsilon)$
C^∞-family of circle maps

$P_\varepsilon : \mathbb{T}^1 \times [0, 1] \to \mathbb{T}^1 \times [0, 1]$,
$(x, \alpha) \mapsto (x + 2\pi\alpha + \varepsilon a(x, \alpha, \varepsilon), \alpha)$

Question What is fate of the rigid rotations of P_0 under perturbation 'as a family' ?

Naively look for conjugation Φ_ε

$$\begin{array}{ccc} \mathbb{T}^1 \times [0,1] & \xrightarrow{P_0} & \mathbb{T}^1 \times [0,1] \\ \downarrow \Phi_\varepsilon & & \downarrow \Phi_\varepsilon \\ \mathbb{T}^1 \times [0,1] & \xrightarrow{P_\varepsilon} & \mathbb{T}^1 \times [0,1] \end{array}$$

i.e., with

$$P_\varepsilon \circ \Phi_\varepsilon = \Phi_\varepsilon \circ P_0$$

1. Circle Maps II
Small divisors again

(Skew) shape of Φ_ε :
$\Phi_\varepsilon(x,\alpha) = (x + \varepsilon U(x,\alpha,\varepsilon), \alpha + \varepsilon\sigma(\alpha,\varepsilon))$
$\rightsquigarrow$ *nonlinear* equation:

$$U(x + 2\pi\alpha, \alpha, \varepsilon) - U(x,\alpha,\varepsilon) =$$
$$= 2\pi\sigma(\alpha,\varepsilon) + a\,(x + \varepsilon U(x,\alpha,\varepsilon), \alpha + \varepsilon\sigma(\alpha,\varepsilon), \varepsilon)$$

Expand in powers of ε
comparing lowest order $\rightsquigarrow$ linear equation

$$U_0(x+2\pi\alpha,\alpha) - U_0(x,\alpha) = 2\pi\sigma_0(\alpha) + a_0(x,\alpha)$$

Write in Fourier-series

$$a_0(x,\alpha) = \sum_{k\in\mathbf{Z}} a_{0k}(\alpha)\mathrm{e}^{\mathrm{i}kx}, U_0(x,\alpha) = \sum_{k\in\mathbf{Z}} U_{0k}(\alpha)\mathrm{e}^{\mathrm{i}kx}$$

Solutions $\sigma_0 = -\frac{1}{2\pi}a_{00}$ (parameter shift) and

$$U_{0k}(\alpha) = \frac{a_{0k}(\alpha)}{\mathrm{e}^{2\pi\mathrm{i}k\alpha} - 1}$$

$\exists$ formal solution $\Leftrightarrow$ α irrational
Still small divisors . . .

1. Circle Maps III
Measure and Category

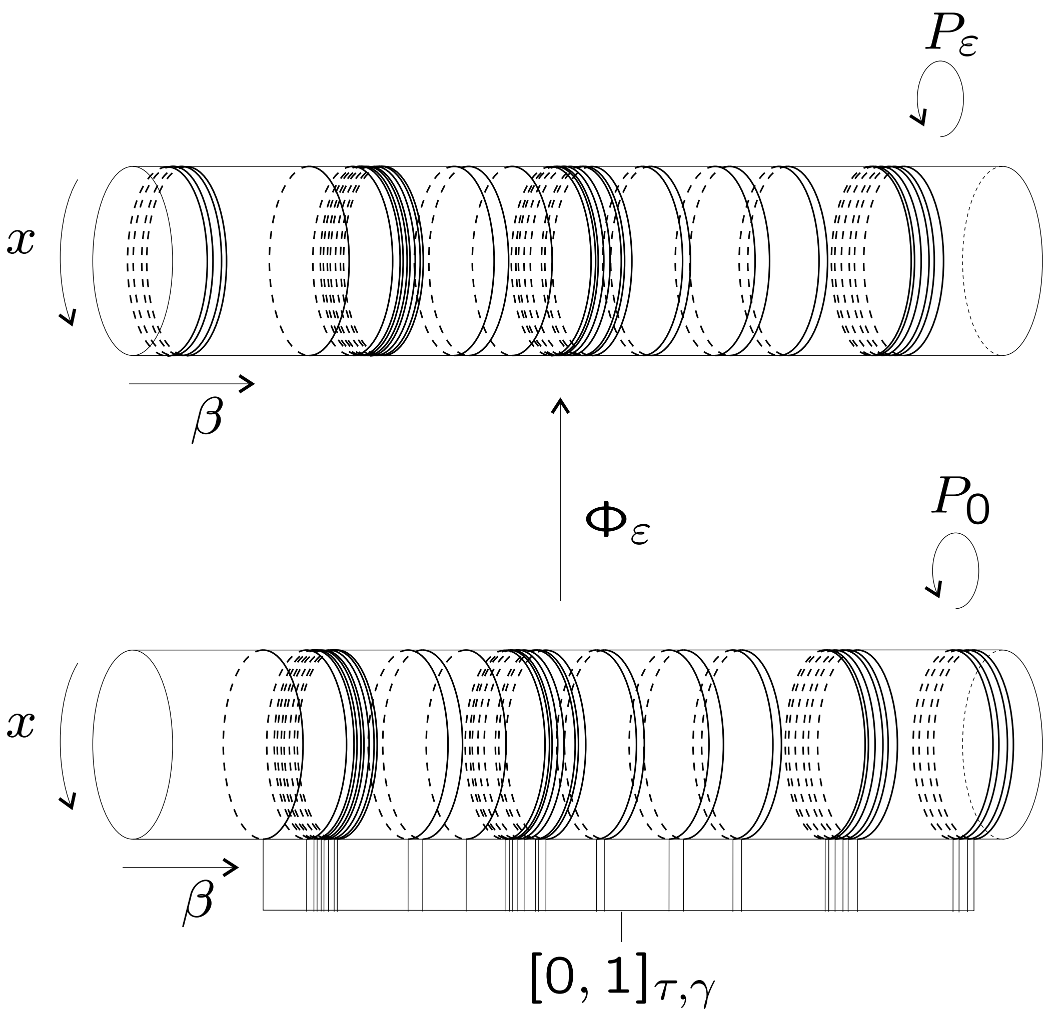
Diophantine nonresonance conditions
for $\tau > 2$ and $\gamma > 0$ consider $\alpha \in [0,1]$ such that for all rationals p/q

$$\left|\alpha - \frac{p}{q}\right| \geq \gamma q^{-\tau}, \qquad (2)$$

Defines Cantor set $[0,1]_{\tau,\gamma}$:
nowhere dense, yet large measure,
Oxtoby again ...

Circle map theorem γ *small and* $\varepsilon\, a$ *small in* C^∞*-topology* $\Rightarrow \exists\, C^\infty$ *transformation* Φ_ε : $\mathbb{T}^1 \times [0,1] \to \mathbb{T}^1 \times [0,1]$, *conjugating the restriction* $P_0|_{[0,1]_{\tau,\gamma}}$ *to a subsystem of* P_ε

Formulation of Broer-Huitema-Takens (1990) in terms of 'quasi-periodic stability'



Smooth map of the cylinder, conjugating Diophantine invariant circles

1. Circle Maps IV Discussion

Irrational rotation $\leftrightarrow$ quasi-periodicity

Smoothness $\Phi_\varepsilon \Rightarrow$
Quasi-periodicity typically occurs with *positive* measure in parameter space

Meaning of Φ_ε in gaps ?

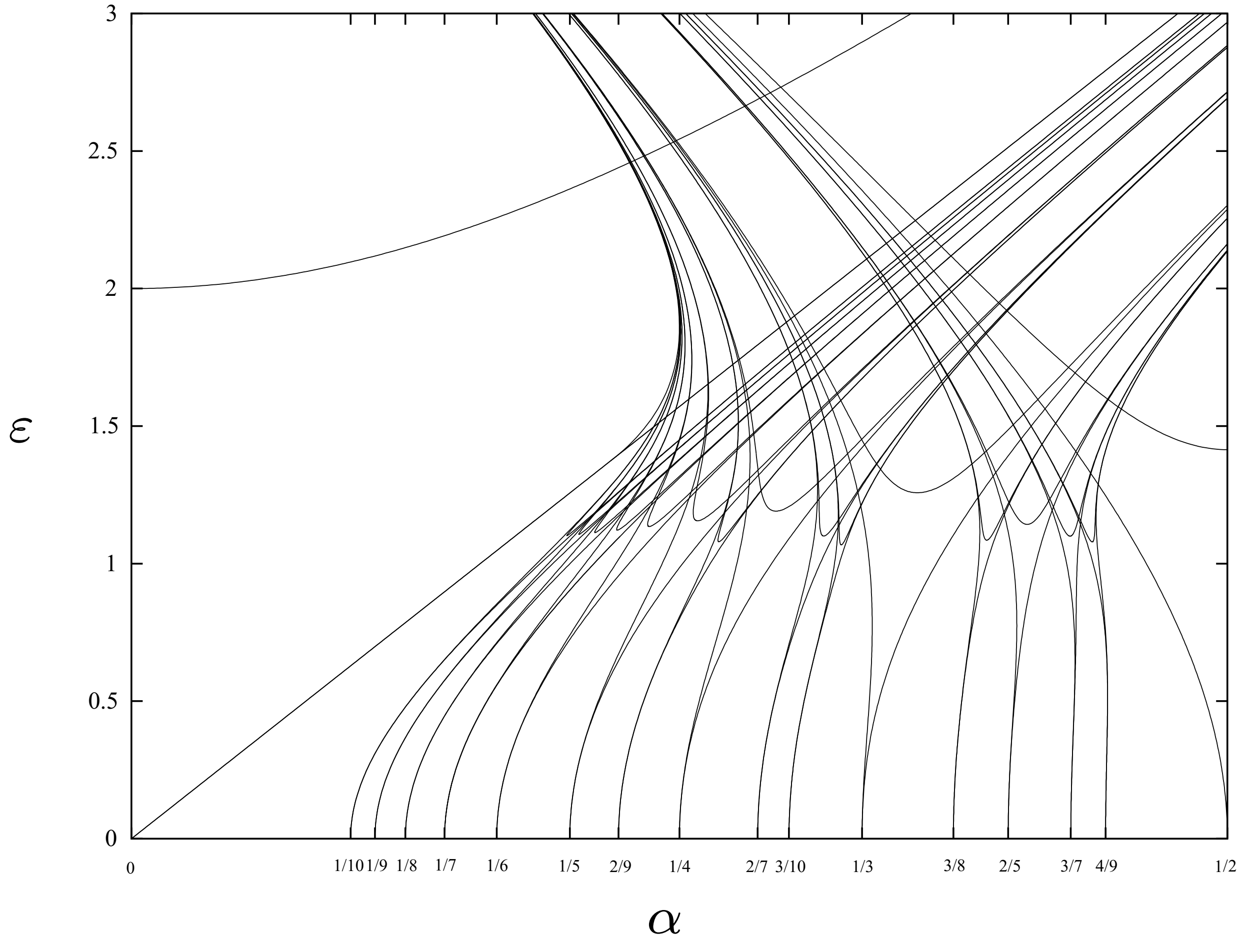
Perfectness Cantor set $\Rightarrow$
non-isolated occurrence of quasi-periodicity

Example: Arnold family of circle maps

$$P_\varepsilon(x) = x + 2\pi\alpha + \varepsilon \sin x$$

periodic tongues (generic)
in between Diophantine quasi-periodic 'hairs' (positive measure)

Herman-Yoccoz non-perturbative version



Arnold resonance tongues; for $\varepsilon \geq 1$ the maps are endomorphic

1. Circle Maps V
Discussion

Examples Coupled Van der Pol type oscillators

$$\begin{aligned} \ddot{y}_1 + c_1\dot{y}_1 + a_1 y_1 + f_1(y_1,\dot{y}_1) &= \varepsilon g_1(y_1,y_2,\dot{y}_1,\dot{y}_2) \\ \ddot{y}_2 + c_2\dot{y}_2 + a_2 y_2 + f_2(y_2,\dot{y}_2) &= \varepsilon g_2(y_1,y_2,\dot{y}_1,\dot{y}_2) \end{aligned}$$

$\rightsquigarrow$ 4-D vector field in $\mathbb{R}^2 \times \mathbb{R}^2 = \{(y_1, z_1), (y_2, z_2)\}$
with $\dot{y}_j = z_j, j = 1, 2$
$\rightsquigarrow$ 2-torus attractor in 4-D

Poincaré map P in 2-torus attractor $\Rightarrow$
similar results for vector fields / ODE's:

phase-lock inside resonance tongues
quasi-periodicity on Diophantine 'hairs'

Huygens's synchronised clocks $\leftrightarrow$ 1 : 1 tongue

Families of quasi-periodic attractors

2. Area preserving twist maps I J.K. Moser (1962)

Annulus $\Delta \subseteq \mathbb{R}^2$ coordinates $(\varphi, I) \in \mathbb{T}^1 \times \mathbf{K}$, $\mathbf{K}$ interval; area form $\sigma = d\varphi \wedge dI$
σ-preserving smooth map $P_\varepsilon : \Delta \to \Delta$

$$P_\varepsilon(\varphi, I) = (\varphi + 2\pi\alpha(I), I) + O(\varepsilon)$$

Assume:
'frequency' map $I \mapsto \alpha(I)$ diffeomorphism
(twist condition $\leftrightarrow$ Kolmogorov-nondegeracy)

Question What is fate of the P_0-dynamics for $|\varepsilon| \ll 1$?

Diophantine conditions as before:
for $\tau > 2, \gamma > 0$ and for all rationals p/q

$$\left| \alpha - \frac{p}{q} \right| \geq \gamma q^{-\tau},$$

$\rightsquigarrow \Delta_{\tau,\gamma} \subseteq \Delta$

Twist theorem γ *small and perturbation small in* C^∞*-topology* $\Rightarrow \exists$ *smooth transformation* $\Phi_\varepsilon : \Delta \to \Delta$, *conjugating the restriction* $P_0|_{\Delta_{\tau,\gamma}}$ *to a subsystem of* P_ε

2. Area preserving twist maps II: Discussion and applications

Compare Theorems 1 and 2:
parameter $\alpha \longleftrightarrow$ coordinate I

Typically quasi-periodicity occurs
with positive measure in phase space

In gaps periodicity and 'chaos'
(generically $\exists$ horseshoes $\Rightarrow$
positive topological entropy)

Examples: coupled pendula as before,
but now *conservative*

Map P **(iso-energetic) Poincaré map** on
families of 2-tori

2. Conservative KAM theory

Kolmogorov (Amsterdam 1954) -Arnold (1963) -Moser (1962)

KAM theorem *In Hamiltonian systems with* n *degrees of freedom typically* n *quasi-periodic Lagrangean tori occur with positive measure in phase space*

Format:

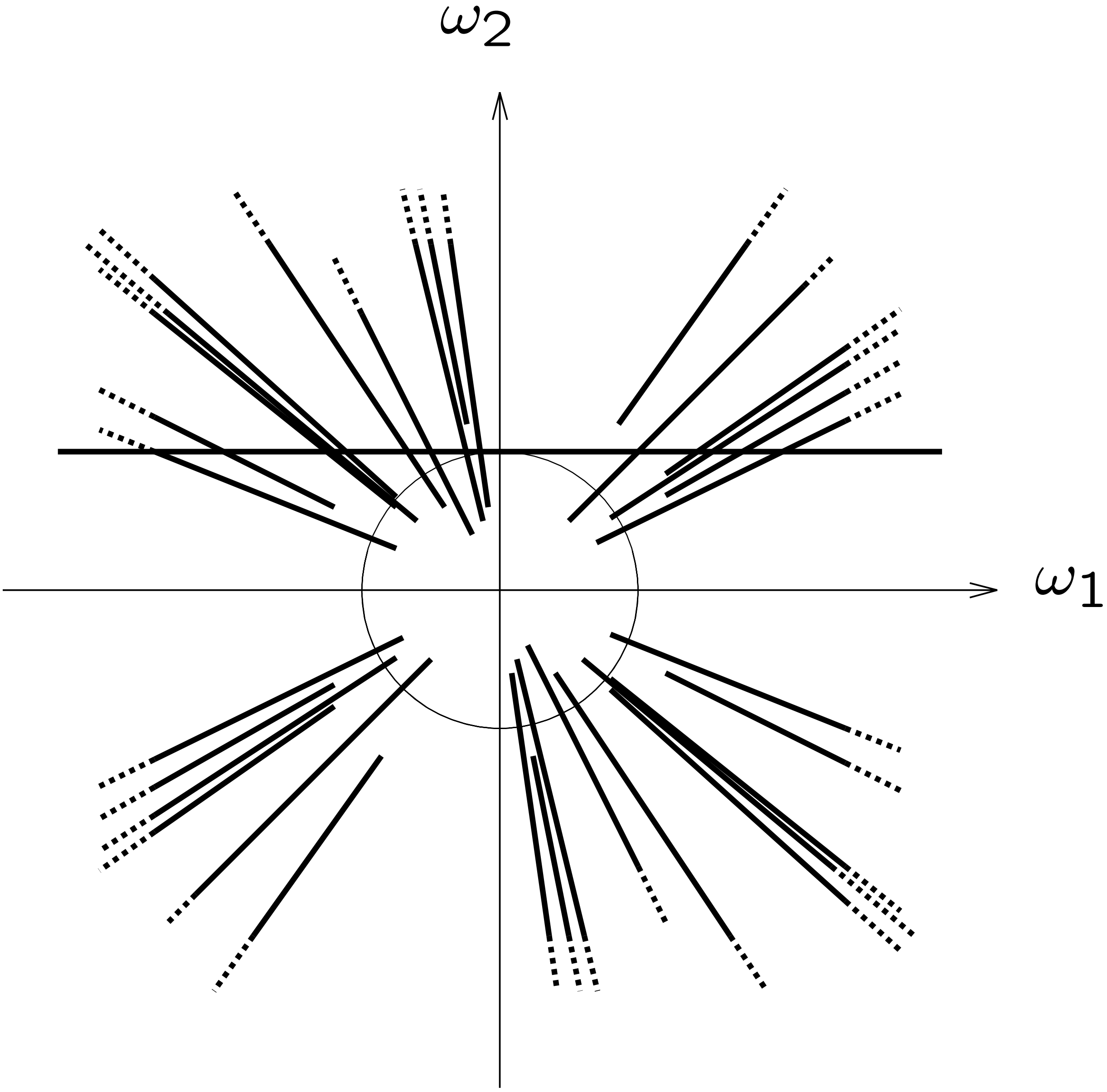
$$\begin{aligned} \dot{\varphi} &= \omega(I) + \varepsilon f(I, \varphi) \\ \dot{I} &= \varepsilon g(I, \varphi), \end{aligned}$$

Kolmogorov nondegeneracy: frequency map $I \mapsto \omega(I)$ local diffeomorphism

Diophantine set of frequencies: $\mathbb{R}^n_{\tau,\gamma} = \{\omega \in \mathbb{R}^n \mid |\langle \omega, k \rangle| \geq \gamma |k|^{-\tau}, k \in \mathbb{Z}^n \setminus \{0\}\}$, $\tau > n - 1, \gamma > 0$

(More precise formulation below)

Stability problems Example: Solar system?



Diophantine set $\mathbb{R}^n_{\tau,\gamma}$ has closed half line geometry

$\mathbb{S}^{n-1} \cap \mathbb{R}^n_{\tau,\gamma}$ Cantor set

measure $\mathbb{S}^{n-1} \setminus \mathbb{R}^n_{\tau,\gamma} = O(\gamma)$ as $\gamma \downarrow 0$

3. Quasi-periodic bifurcations I

Bifurcations of vector fields and diffeomorphisms:

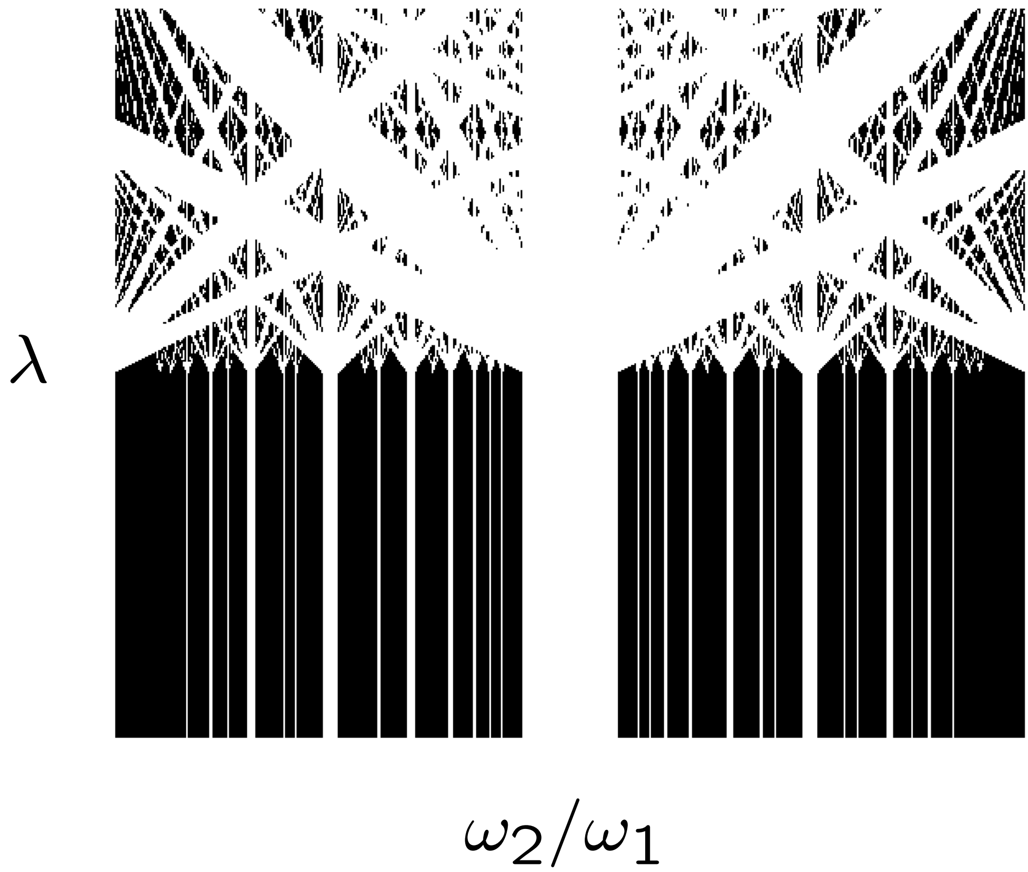
Equilibria: saddle-node and Hopf

Periodic solutions / fixed points: saddle-node, period-doubling, Hopf-Neĭmark-Sacker
Compare with Arnold family:
describe with **frequency parameter**

Tori: $\mathbb{T}^n$-symmetric (= 'integrable') systems:
take product with above cases
Perturbation by KAM $\rightsquigarrow$ (C^∞-) typicality restricted to Diophantine tori

Example: quasi-periodic bifurcation in diffeomorphism from circle to 2-torus
(mind: Hopf-Landau-Lifschitz-Ruelle-Takens)

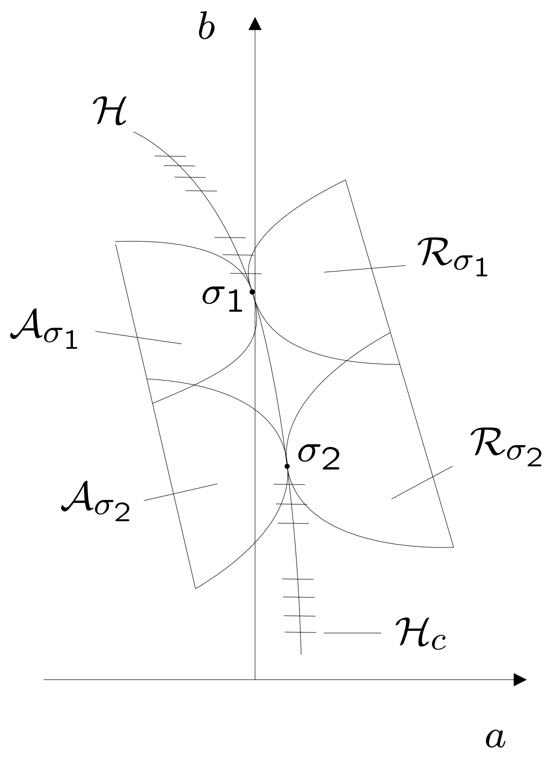
'Cantorise' above bifurcation geometry
(Whitney, Thom, Arnold, *et al.*)



In the gaps 'interesting' dynamics may occur, like chaotic attractors. In generic family this coexists with positive measure of quasi-periodicity.

3. Quasi-periodic bifurcations II

Fattening torus domains by hyperbolicity
$\rightsquigarrow$ Chenciner-bubbles around resonances



4. Theoretical background I

Definition of Q-P by smooth conjugation

Fix $n \in \mathbb{N}, n \geq 2$
Standard n-torus $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$
coordinates $\varphi_1, \varphi_2, \ldots, \varphi_n$ mod 2π
For $\omega \in \mathbb{R}^n$ consider $\mathbb{X}_\omega = \sum_{j=1}^n \omega_j \frac{\partial}{\partial \varphi_j}$
$\omega_1, \omega_2, \ldots \omega_n$ called frequencies

Given smooth vector field X on manifold M, with $T \subseteq M$ an X invariant n-torus
Restriction $X|_T$ called *parallel* iff
$\exists \omega \in \mathbb{R}^n$ and $\Phi : T \to \mathbb{T}^n$, smooth diffeo, such that $\Phi_*(X|_T) = \mathbb{X}_\omega$

$X|_T$ is *quasi-periodic* if $\omega_1, \omega_2, \ldots, \omega_n$ are independent over $\mathbb{Q}$

Remark. Composing Φ by translation on $\mathbb{T}^n$ does not change ω
However, composition with linear invertible $S \in GL(n, \mathbb{Z})$ gives $S_* \mathbb{X}_\omega = \mathbb{X}_{S\omega}$

4. Theoretical background II

Integer affine structure

In Q-P case:
Self conjugations of $\mathbb{X}_\omega \leftrightarrow$ translations of $\mathbb{T}^n$
$\leftrightarrow$ affine structure on $\mathbb{T}^n$

Thus given $\Phi : T \to \mathbb{T}^n$ with $\Phi_*(X|_T) = \mathbb{X}_\omega$, self conjuguations of $X|_T$ determine natural affine structure on T
Conjugation Φ unique modulo translations in T and $\mathbb{T}^n$

Remarks

Structure on T *integer* affine:
transition maps: translations and $GL(n, \mathbb{Z})$

Fits with integrable Hamiltonian case where affine structure on all parallel or Liouville tori

5. Global KAM theory I

Setting: Example: Spherical pendulum
consider Hamiltonian perturbations

Question: What is fate of nontrivial 2-torus bundle ?

Background: Liouville-Arnold and KAM
Gives KAM 2-tori in charts

Tools: Unicity of KAM tori (restricted Diophantine)
Partition of Unity: convex combinations of KAM conjugations in charts

Result (Rink, Broer-Cushman-Fassò-Takens):
Extension of KAM theorem to bundles of Lagrangean tori (all near-identity Whitney interpolations isomorphic)

BCFT-approach preserves closed half line geometry

5. Global KAM theory II

Example: spherical pendulum
Configuration space $\mathbb{S}^2 = \{q \in \mathbb{R}^3 \mid \langle q, q \rangle = 1\}$
Phase space
$T^*\mathbb{S}^2 \cong \{(q,p) \in \mathbb{R}^6 \mid \langle q, q \rangle = 1 \,\&\, \langle q, p \rangle = 0\}$

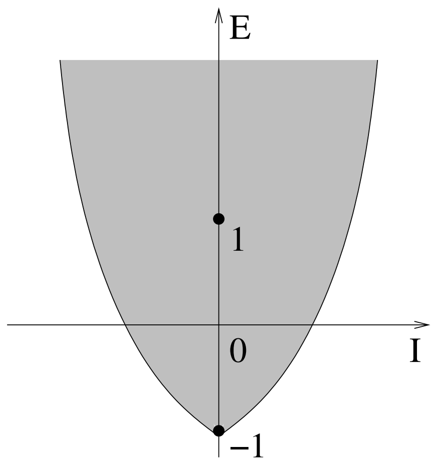
Energy–momentum map $\mathcal{EM} : T^*\mathbb{S}^2 \to \mathbb{R}^2$
$(q,p) \mapsto (I,E) = \left(q_1 p_2 - q_2 p_1, \frac{1}{2}\langle p, p \rangle + q_3\right)$
Lagrangean fibration, fibers $\mathbb{T}^2$

Equilibria (de and ude)
$(q,p) = ((0,0,\pm 1),(0,0,0)) \mapsto (I,E) = (0, \pm 1)$
singularities

Duistermaat (1980): Circle around $(I,E) = (0,1)$ collects nontrivial *monodromy*

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in GL(2,\mathbb{R})$$

(in suitable bases of period lattice)
$\Rightarrow$ *nontrivial* $\mathbb{T}^2$-bundle (Liouville-Arnold)
& nonexistence of *global* action angles



Range of the energy-momentum map of the spherical pendulum

5. Global KAM theory III

Liouville-Arnold theorem

(M, σ) *symplectic manifold,* $\dim M = 2n$,
B n*-dimensional manifold*

GIVEN
integral $\mathcal{F} : M \to B$,
$\mathcal{F} = (f_1, f_2, \ldots, f_n)$ *(locally)*
with $\{f_i, f_j\} = 0$, $d\mathcal{F}$ *maximal rank*
and compact fibres $\mathcal{F}^{-1}(b)$

THEN
$\forall b \in B$ $\exists$ *open* $b \in V \subseteq B$ *such that*
$\mathcal{F}^{-1}(V) \subseteq M$ *open and invariant under* X_{f_i}
and
$\exists$ *symplectic diffeo*
$(I, \varphi) : \mathcal{F}^{-1}(V) \to U \times (\mathbb{R}/\mathbb{Z})^n$ *with* $U \subseteq \mathbb{R}^n$
open, i.e., such that:

$$(I, \varphi)^*\left(\sum_{j=1}^{n} d\varphi_j \wedge dI_j\right) = \sigma\Big|_{\mathcal{F}^{-1}(V)}$$

Global KAM virtual

Singularity ude *saddle* (double eigenvalues ± 1)
Singular fiber: *pinched* 2-torus ($= W^{u/s}$(ude))

Theorem (Matveev 1996, Nguyen 1997):
Given 4D symplectic manifold M fibered by level sets of map $\mathcal{EM}$
ASSUME
- $\mathcal{EM}$ has only one critical value
- fibers of $\mathcal{EM}$ compact & connected
- singular fiber has k singular points, all real or complex saddle points

THEN
singular fiber is pinched torus
& Liouville-Arnold $\mathbb{T}^2$-bundle is non-trivial with monodromy similar to

$$\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \in Sl(2, \mathbb{R})$$

Remark Quantum monodromy *exists* in semi-classical quantizations of classical integrable system with nontrivial $\mathbb{T}^n$-bundles (San Vu Ngọc 1999)

Global KAM theory V

Back to Conservative KAM theorem:

Phase space $\mathbb{T}^n \times A$
$A \subset \mathbb{R}^n$ is bounded, open & connected
action angle coordinates (I, φ)
symplectic form $\mu = \sum_{j=1}^n dI_j \wedge d\varphi_j$

$h : \mathbb{T}^n \times A \to \mathbb{R}$ C^∞ Hamilton function,
integrable (not depending on φ) $\to$
Hamiltonian vector field X_h

$$X_h(\varphi, I) = \sum_{j=1}^{n} \omega_j(I) \frac{\partial}{\partial \varphi_j}, \text{ with } \omega(I) := \frac{\partial h}{\partial I}$$

Kolmogorov nondegeneracy frequency map
$\omega : A \to \mathbb{R}^n$ is diffeomorphism

Diophantine frequencies
Given $\tau > n - 1$ and $\gamma > 0$, recall definition

$$\mathbb{R}^n_{\tau,\gamma} := \{\omega \in \mathbb{R}^n \mid |\langle \omega, k \rangle| \geq \gamma |k|^{-\tau}, \forall k \in \mathbb{Z}^n \backslash \{0\}\}$$

Global KAM theory VI

For $\Gamma := \omega(A)$ let

$\Gamma' := \{\omega \in \Gamma \mid \text{dist}(\omega, \partial\Gamma) > \gamma\}$ and $\Gamma'_\gamma := \Gamma' \cap \mathbb{R}^n_\gamma$
also define $A'_\gamma \subset A$ by $A'_\gamma := \omega^{-1}(\Gamma'_\gamma)$

NB measure $(A \setminus A'_\gamma) = O(\gamma)$ as $\gamma \downarrow 0$

Perturb to $h + f : \mathbb{T}^n \times A \to \mathbb{R}$, all of class C^∞ with C^∞-topology

Local KAM Theorem (Pöschel (82), Broer-Huitema-Takens (90))
If $\gamma > 0$ is suffiently small and $h + f$ is sufficienly near h in C^∞-topology then $\exists$ map $\Phi : \mathbb{T}^n \times A \to \mathbb{T}^n \times A$ with following properties:

- *Φ is C^∞ diffeomorphism*
- *for $\widehat{\Phi} := \Phi|_{\mathbb{T}^n \times \mathrm{D}_\gamma(A'_\gamma)}$ we have*

$$\widehat{\Phi}_* X_h = X_{h+f}$$

Global KAM theory VII

Discussion

Map Φ

- is generally *not* symplectic
- is near-identity in the C^∞-topology

In global KAM theorem

- convex combinations on *unique* tori
  keeps closed half line geometry
- near-identity torus automorphisms are all
  translations
- Whitney extensions near Id
  $\Rightarrow$ all extended $\mathbb{T}^n$-bundles isomorphic

- Quantum monodromy in semi-classical
  versions of integrable systems
  with monodromy
- How about the nearly-integrable case ?