

# Multiperiodic dynamics – overview and some recent results

Henk Broer

Rijksuniversiteit Groningen

Instituut voor Wiskunde en Informatica

POBox 800

9700 AV Groningen

email: [broer@math.rug.nl](mailto:broer@math.rug.nl)

URL: <http://www.math.rug.nl/~broer>

Universiteit Utrecht

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# 1. Complex Linearization I

**Problem** Given  $F(z) = \lambda z + f(z)$   
holomorphic map (germ)  $F : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$   
 $f(0) = f'(0) = 0$ ;  
to find  $\Phi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$   
biholomorphic germ such that

$$\Phi \circ F = \lambda \cdot \Phi$$

Such  $\Phi$  is called linearization of  $F$

## Formal solution

Given  $f(z) = \sum_{j \geq 2} f_j z^j$   
want  $\Phi(z) = z + \sum_{j \geq 2} \phi_j z^j$

$\exists$  solution for  $\lambda \neq 0$  no root of 1  
Proof recursive / inductive:

**For**  $j = 2$  : get  $\lambda(1 - \lambda)\phi_2 = f_2$

**For**  $j = 3$  : get  $\lambda(1 - \lambda^2)\phi_3 = f_3 + 2\lambda f_2 \phi_2$

**For**  $j = n$  : get  $\lambda(1 - \lambda^{n-1})\phi_n = f_n + \text{known}$

# 1. Complex Linearization II: Convergent Solutions

**Hyperbolic case**  $0 < |\lambda| \neq 1$  :

Solved by H. Poincaré (proof by iteration)

**Elliptic case**  $\lambda \in \mathbb{T}^1$  (SMALL DIVISORS):

Solved by C.L. Siegel when  
for some  $\gamma > 0$  and  $\tau > 2$

$$|\lambda - e^{2\pi i \frac{p}{q}}| \geq \gamma |q|^{-\tau} \quad (1)$$

Diophantine (nonresonance) conditions:  
set of full measure in  $\mathbb{T}^1$

# 1. Complex Linearization III: H. Cremer's topological counter example (1928), version by M.R. Herman

$$F(z) = \lambda z + z^2, \lambda \in \mathbb{T}^1 \text{ not a root of unity}$$
$$F^q(z) = \lambda^q z + \dots + z^{2^q}$$

Point  $z$  is periodic of period  $q$  iff  $F^q(z) = z$

$$F^q(z) - z = z(\lambda^q - 1 + \dots + z^{2^q-1})$$

Call  $N = 2^q - 1$

$$z_1 \cdot z_2 \cdot \dots \cdot z_N = \lambda^q - 1$$

where  $z_j$  nontrivial solutions

$\Rightarrow \exists$  solution within radius

$$|\lambda^q - 1|^{1/N}$$

of  $z = 0$ . Consider  $\lambda \in \mathbb{T}^1$  with

$$\liminf_{q \rightarrow \infty} |\lambda^q - 1|^{1/N} = 0$$

form residual set (dense  $G_\delta$ , 2nd Baire)  
*Measure and category*, Oxtoby (1970)

For such  $\lambda$  linearization is impossible:

- Linearization  $z \mapsto \lambda z$  'irrational' rotation
- $\exists$  periodic points in any nbhd. of  $z = 0$

# 1. Circle Maps I

**Problem** Given

$$P_{\alpha, \varepsilon} : \mathbb{T}^1 \rightarrow \mathbb{T}^1 \quad x \mapsto x + 2\pi\alpha + \varepsilon a(x, \alpha, \varepsilon)$$

$C^\infty$ -family of circle maps

$$P_\varepsilon : \mathbb{T}^1 \times [0, 1] \rightarrow \mathbb{T}^1 \times [0, 1],$$

$$(x, \alpha) \mapsto (x + 2\pi\alpha + \varepsilon a(x, \alpha, \varepsilon), \alpha)$$

**Question** What is fate of the rigid rotations of  $P_0$  under perturbation 'as a family' ?

**Naively** look for conjugation  $\Phi_\varepsilon$

$$\begin{array}{ccc} \mathbb{T}^1 \times [0, 1] & \xrightarrow{P_0} & \mathbb{T}^1 \times [0, 1] \\ \downarrow \Phi_\varepsilon & & \downarrow \Phi_\varepsilon \\ \mathbb{T}^1 \times [0, 1] & \xrightarrow{P_\varepsilon} & \mathbb{T}^1 \times [0, 1] \end{array}$$

i.e., with

$$P_\varepsilon \circ \Phi_\varepsilon = \Phi_\varepsilon \circ P_0$$

# 1. Circle Maps II

## Small divisors again

(Skew) shape of  $\Phi_\varepsilon$  :

$$\Phi_\varepsilon(x, \alpha) = (x + \varepsilon U(x, \alpha, \varepsilon), \alpha + \varepsilon \sigma(\alpha, \varepsilon))$$

$\leadsto$  *nonlinear* equation:

$$\begin{aligned} U(x + 2\pi\alpha, \alpha, \varepsilon) - U(x, \alpha, \varepsilon) &= \\ &= 2\pi\sigma(\alpha, \varepsilon) + a(x + \varepsilon U(x, \alpha, \varepsilon), \alpha + \varepsilon\sigma(\alpha, \varepsilon), \varepsilon) \end{aligned}$$

Expand in powers of  $\varepsilon$

comparing lowest order  $\leadsto$  linear equation

$$U_0(x + 2\pi\alpha, \alpha) - U_0(x, \alpha) = 2\pi\sigma_0(\alpha) + a_0(x, \alpha)$$

Write in Fourier-series

$$a_0(x, \alpha) = \sum_{k \in \mathbf{Z}} a_{0k}(\alpha) e^{ikx}, U_0(x, \alpha) = \sum_{k \in \mathbf{Z}} U_{0k}(\alpha) e^{ikx}$$

Solutions  $\sigma_0 = -\frac{1}{2\pi} a_{00}$  (parameter shift) and

$$U_{0k}(\alpha) = \frac{a_{0k}(\alpha)}{e^{2\pi ik\alpha} - 1}$$

$\exists$  formal solution  $\Leftrightarrow \alpha$  irrational

Still small divisors ...

# 1. Circle Maps III

## Measure and Category

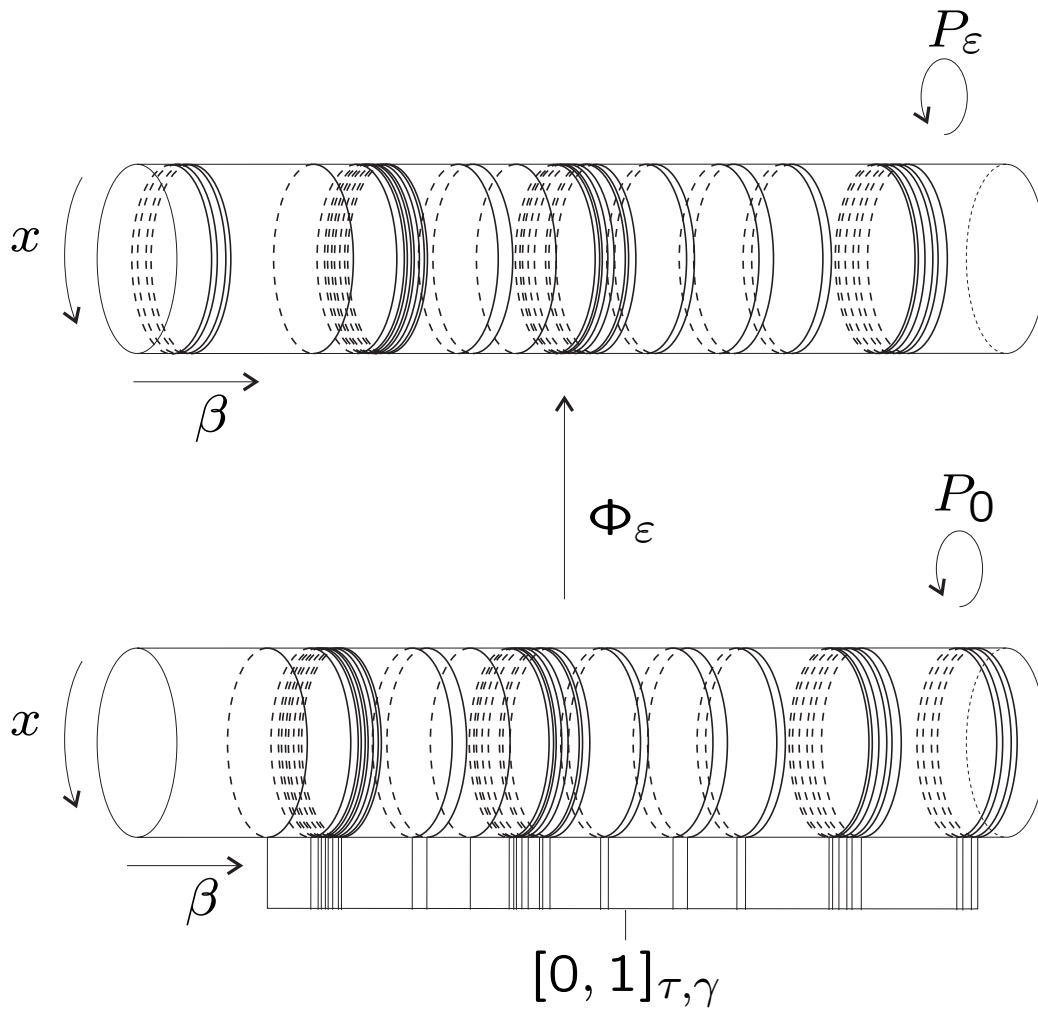
Diophantine nonresonance conditions  
for  $\tau > 2$  and  $\gamma > 0$  consider  $\alpha \in [0, 1]$  such  
that for all rationals  $p/q$

$$\left| \alpha - \frac{p}{q} \right| \geq \gamma q^{-\tau}, \quad (2)$$

Defines Cantor set  $[0, 1]_{\tau, \gamma}$  :  
nowhere dense, yet large measure,  
Oxtoby again ...

**Circle map theorem**  $\gamma$  small and  $\varepsilon$  a small  
in  $C^\infty$ -topology  $\Rightarrow \exists C^\infty$  transformation  $\Phi_\varepsilon :$   
 $\mathbb{T}^1 \times [0, 1] \rightarrow \mathbb{T}^1 \times [0, 1]$ , conjugating the re-  
striction  $P_0|_{[0, 1]_{\tau, \gamma}}$  to a subsystem of  $P_\varepsilon$

Formulation of Broer-Huitema-Takens (1990)  
in terms of 'quasi-periodic stability'



Smooth map of the cylinder, conjugating  
Diophantine invariant circles

# 1. Circle Maps IV

## Discussion

**Irrational rotation**  $\leftrightarrow$  quasi-periodicity

**Smoothness**  $\Phi_\varepsilon \Rightarrow$

Quasi-periodicity typically occurs with *positive* measure in parameter space

**Meaning** of  $\Phi_\varepsilon$  in gaps ?

**Perfectness** Cantor set  $\Rightarrow$

non-isolated occurrence of quasi-periodicity

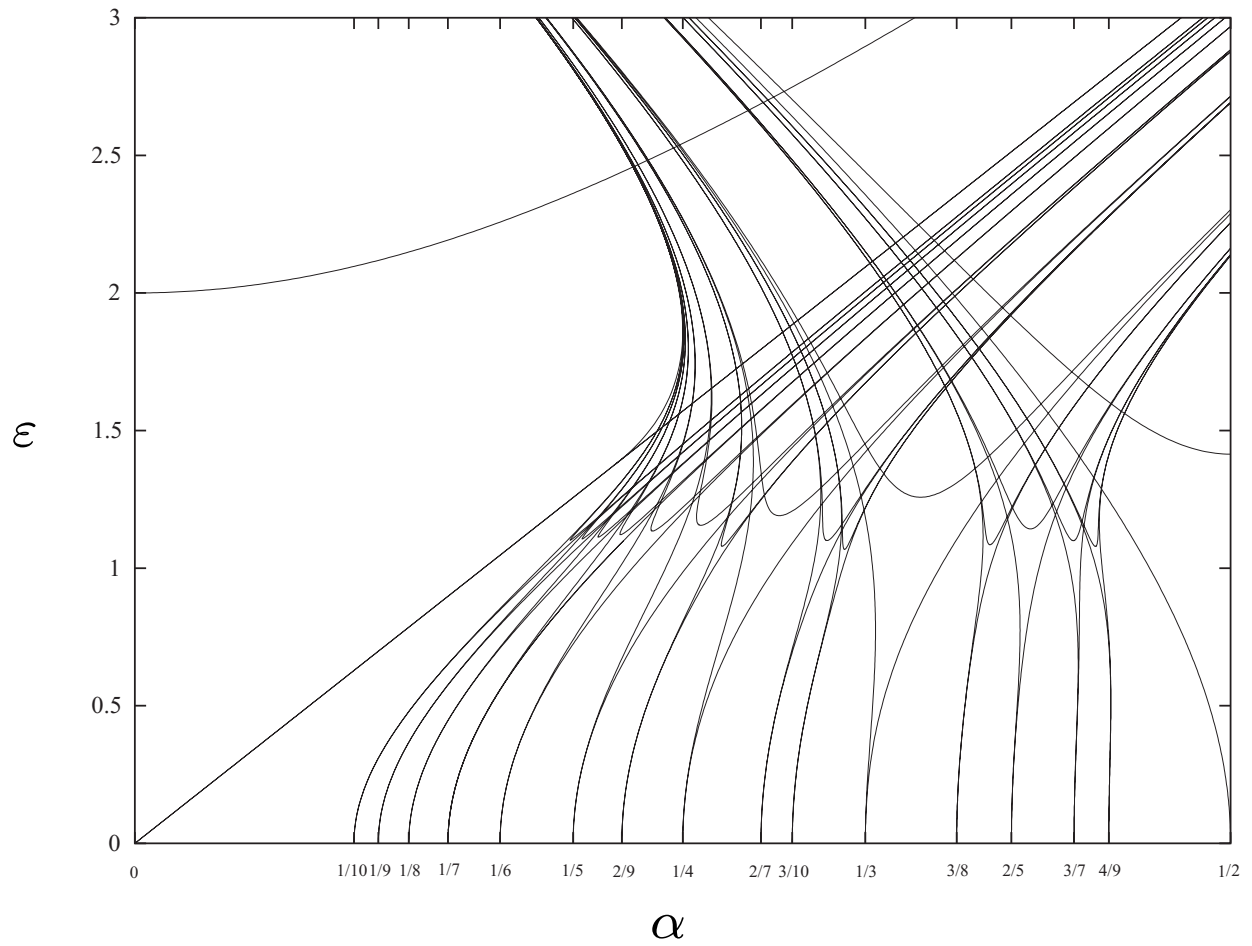
**Example: Arnold family** of circle maps

$$P_\varepsilon(x) = x + 2\pi\alpha + \varepsilon \sin x$$

periodic tongues (generic)

in between Diophantine quasi-periodic 'hairs'  
(positive measure)

**Herman-Yoccoz** non-perturbative version



Arnold resonance tongues; for  $\varepsilon \geq 1$  the maps are endomorphic

# 1. Circle Maps V

## Discussion

**Examples** Coupled Van der Pol type oscillators

$$\begin{aligned}\ddot{y}_1 + c_1\dot{y}_1 + a_1y_1 + f_1(y_1, \dot{y}_1) &= \varepsilon g_1(y_1, y_2, \dot{y}_1, \dot{y}_2) \\ \ddot{y}_2 + c_2\dot{y}_2 + a_2y_2 + f_2(y_2, \dot{y}_2) &= \varepsilon g_2(y_1, y_2, \dot{y}_1, \dot{y}_2)\end{aligned}$$

$\leadsto$  4-D vector field in  $\mathbb{R}^2 \times \mathbb{R}^2 = \{(y_1, z_1), (y_2, z_2)\}$   
with  $\dot{y}_j = z_j, j = 1, 2$

$\leadsto$  2-torus attractor in 4-D

**Poincaré map**  $P$  in 2-torus attractor  $\Rightarrow$   
similar results for vector fields / ODE's:  
phase-lock inside resonance tongues  
quasi-periodicity on Diophantine 'hairs'

**Huygens's** synchronised clocks  $\leftrightarrow$  1 : 1 tongue

**Families of** quasi-periodic attractors

## 2. Area preserving twist maps I J.K. Moser (1962)

Annulus  $\Delta \subseteq \mathbb{R}^2$  coordinates  $(\varphi, I) \in \mathbb{T}^1 \times \mathbf{K}$ ,  
 $\mathbf{K}$  interval; area form  $\sigma = d\varphi \wedge dI$   
 $\sigma$ -preserving smooth map  $P_\varepsilon : \Delta \rightarrow \Delta$

$$P_\varepsilon(\varphi, I) = (\varphi + 2\pi\alpha(I), I) + O(\varepsilon)$$

### **Assume:**

'frequency' map  $I \mapsto \alpha(I)$  diffeomorphism  
(twist condition  $\leftrightarrow$  Kolmogorov-nondegeneracy)

**Question** What is fate of the  $P_0$ -dynamics  
for  $|\varepsilon| \ll 1$  ?

Diophantine conditions as before:

for  $\tau > 2, \gamma > 0$  and for all rationals  $p/q$

$$\left| \alpha - \frac{p}{q} \right| \geq \gamma q^{-\tau},$$

$$\rightsquigarrow \Delta_{\tau, \gamma} \subseteq \Delta$$

**Twist theorem**  $\gamma$  small and perturbation small  
in  $C^\infty$ -topology  $\Rightarrow \exists$  smooth transformation  
 $\Phi_\varepsilon : \Delta \rightarrow \Delta$ , conjugating the restriction  $P_0|_{\Delta_{\tau, \gamma}}$   
to a subsystem of  $P_\varepsilon$

## 2. Area preserving twist maps II: Discussion and applications

**Compare** Theorems 1 and 2:  
parameter  $\alpha \longleftrightarrow$  coordinate  $I$

**Typically** quasi-periodicity occurs  
with positive measure in phase space

**In gaps** periodicity and ‘chaos’  
(generically  $\exists$  horseshoes  $\Rightarrow$   
positive topological entropy )

**Examples:** coupled pendula as before,  
but now *conservative*

**Map  $P$  (iso-energetic) Poincaré map** on  
families of 2-tori

## 2. Conservative KAM theory

**Kolmogorov (Amsterdam 1954)** -Arnold (1963)  
-Moser (1962)

**KAM theorem** *In Hamiltonian systems with  $n$  degrees of freedom typically  $n$  quasi-periodic Lagrangean tori occur with positive measure in phase space*

Format:

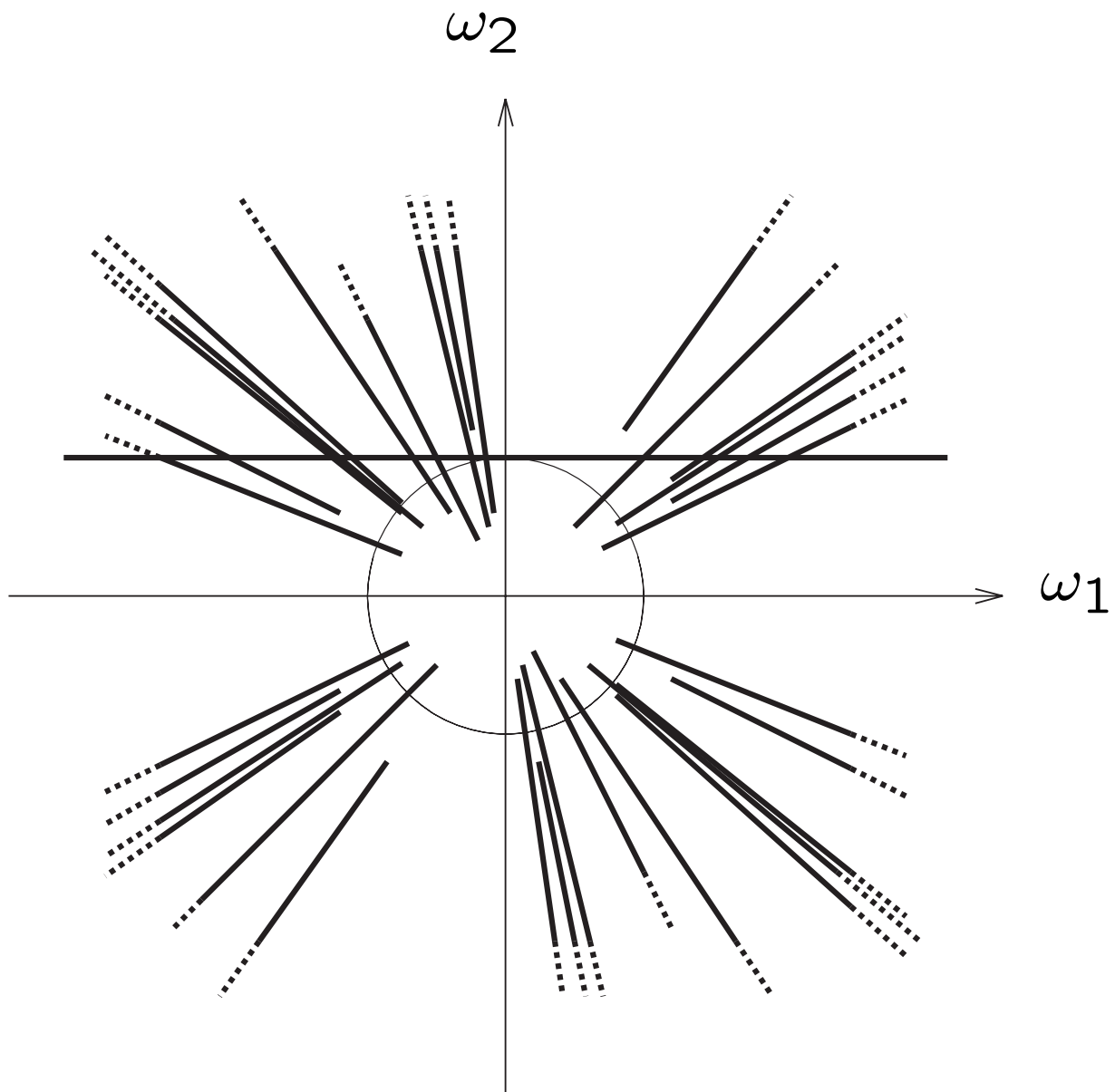
$$\begin{aligned}\dot{\varphi} &= \omega(I) + \varepsilon f(I, \varphi) \\ \dot{I} &= \varepsilon g(I, \varphi),\end{aligned}$$

Kolmogorov nondegeneracy: frequency map  $I \mapsto \omega(I)$  local diffeomorphism

Diophantine set of frequencies:  $\mathbb{R}_{\tau, \gamma}^n = \{\omega \in \mathbb{R}^n \mid |\langle \omega, k \rangle| \geq \gamma |k|^{-\tau}, k \in \mathbb{Z}^n \setminus \{0\}\}$ ,  
 $\tau > n - 1, \gamma > 0$

(More precise formulation below)

**Stability problems** Example: Solar system?



Diophantine set  $\mathbb{R}_{\tau, \gamma}^n$  has closed half line geometry

$\mathbb{S}^{n-1} \cap \mathbb{R}_{\tau, \gamma}^n$  Cantor set

measure  $\mathbb{S}^{n-1} \setminus \mathbb{R}_{\tau, \gamma}^n = O(\gamma)$  as  $\gamma \downarrow 0$

### 3. Quasi-periodic bifurcations I

Bifurcations of vector fields and diffeomorphisms:

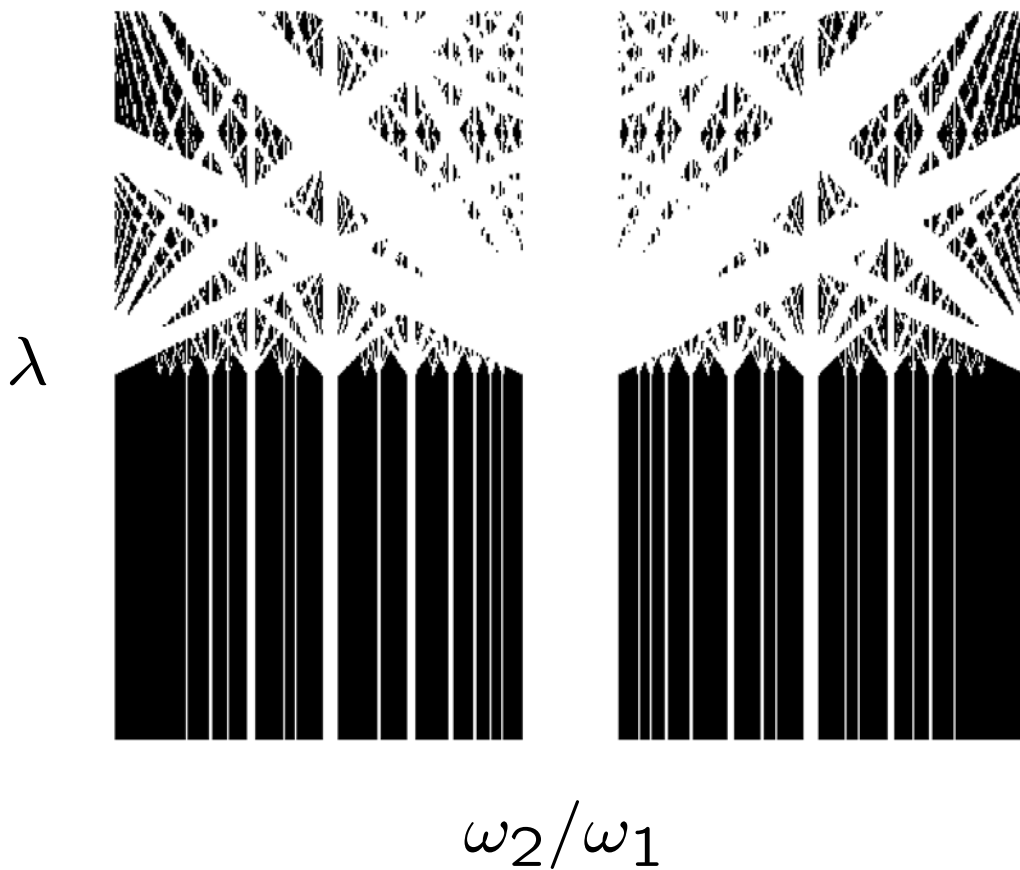
**Equilibria:** saddle-node and Hopf

**Periodic solutions / fixed points:** saddle-node, period-doubling, Hopf-Neĭmark-Sacker  
Compare with Arnold family:  
describe with **frequency parameter**

**Tori:**  $\mathbb{T}^n$ -symmetric (= 'integrable') systems:  
take product with above cases  
Perturbation by KAM  $\rightsquigarrow$  ( $C^\infty$ -) typicality  
restricted to Diophantine tori

**Example:** quasi-periodic bifurcation in diffeomorphism from circle to 2-torus  
(mind: Hopf-Landau-Lifschitz-Ruelle-Takens)

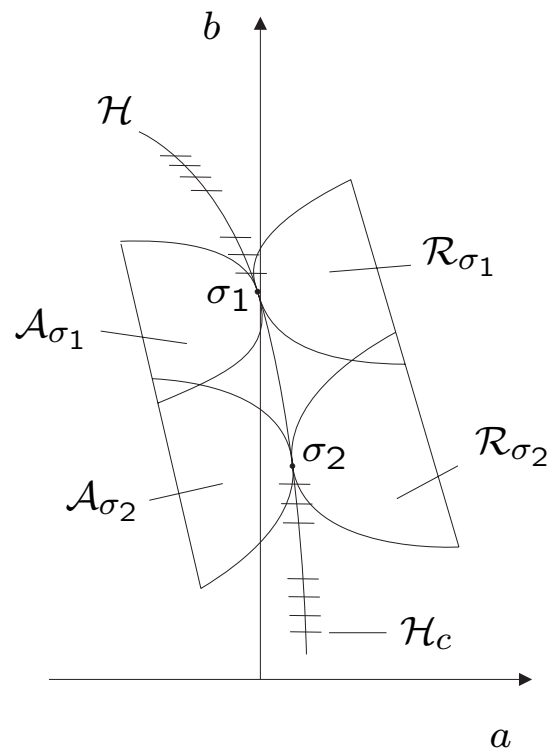
**'Cantorise'** above bifurcation geometry  
(Whitney, Thom, Arnold, *et al.*)



In the gaps ‘interesting’ dynamics may occur, like chaotic attractors. In generic family this coexists with positive measure of quasi-periodicity.

### 3. Quasi-periodic bifurcations II

Fattening torus domains by hyperbolicity  
~> Chenciner-bubbles around resonances



## 4. Theoretical background I

### Definition of Q-P by smooth conjugation

Fix  $n \in \mathbb{N}, n \geq 2$

Standard  $n$ -torus  $\mathbb{T}^n = \mathbb{R}^n / (2\pi\mathbb{Z})^n$

coordinates  $\varphi_1, \varphi_2, \dots, \varphi_n \bmod 2\pi$

For  $\omega \in \mathbb{R}^n$  consider  $\mathbb{X}_\omega = \sum_{j=1}^n \omega_j \frac{\partial}{\partial \varphi_j}$

$\omega_1, \omega_2, \dots, \omega_n$  called frequencies

Given smooth vector field  $X$  on manifold  $M$ ,  
with  $T \subseteq M$  an  $X$  invariant  $n$ -torus

Restriction  $X|_T$  called *parallel* iff

$\exists \omega \in \mathbb{R}^n$  and  $\Phi : T \rightarrow \mathbb{T}^n$ , smooth diffeo, such  
that  $\Phi_*(X|_T) = \mathbb{X}_\omega$

$X|_T$  is *quasi-periodic* if  $\omega_1, \omega_2, \dots, \omega_n$  are in-  
dependent over  $\mathbb{Q}$

**Remark.** Composing  $\Phi$  by translation on  $\mathbb{T}^n$   
does not change  $\omega$

However, composition with linear invertible  
 $S \in GL(n, \mathbb{Z})$  gives  $S_*\mathbb{X}_\omega = \mathbb{X}_{S\omega}$

## 4. Theoretical background II

### Integer affine structure

In Q-P case:

Self conjugations of  $\mathbb{X}_\omega \leftrightarrow$  translations of  $\mathbb{T}^n$   
 $\leftrightarrow$  affine structure on  $\mathbb{T}^n$

Thus given  $\Phi : T \rightarrow \mathbb{T}^n$  with  $\Phi_*(X|_T) = \mathbb{X}_\omega$ ,  
self conjugations of  $X|_T$  determine natural  
affine structure on  $T$

Conjugation  $\Phi$  unique modulo translations in  
 $T$  and  $\mathbb{T}^n$

### Remarks

Structure on  $T$  *integer* affine:

transition maps: translations and  $GL(n, \mathbb{Z})$

Fits with integrable Hamiltonian case where  
affine structure on all parallel or Liouville tori

## 5. Global KAM theory I

**Setting:** Example: Spherical pendulum  
consider Hamiltonian perturbations

**Question:** What is fate of nontrivial 2-torus bundle ?

**Background:** Liouville-Arnold and KAM  
Gives KAM 2-tori in charts

**Tools:** Unicity of KAM tori (restricted Diophantine)  
Partition of Unity: convex combinations of KAM conjugations in charts

**Result** (Rink, Broer-Cushman-Fassò-Takens):  
Extension of KAM theorem to bundles of Lagrangean tori (all near-identity Whitney interpolations isomorphic)

BCFT-approach preserves closed half line geometry

## 5. Global KAM theory II

### Example: spherical pendulum

Configuration space  $\mathbb{S}^2 = \{q \in \mathbb{R}^3 \mid \langle q, q \rangle = 1\}$

Phase space

$$T^*\mathbb{S}^2 \cong \{(q, p) \in \mathbb{R}^6 \mid \langle q, q \rangle = 1 \ \& \ \langle q, p \rangle = 0\}$$

Energy–momentum map  $\mathcal{EM} : T^*\mathbb{S}^2 \rightarrow \mathbb{R}^2$

$$(q, p) \mapsto (I, E) = \left( q_1 p_2 - q_2 p_1, \frac{1}{2} \langle p, p \rangle + q_3 \right)$$

Lagrangian fibration, fibers  $\mathbb{T}^2$

Equilibria (de and ude)

$$(q, p) = ((0, 0, \pm 1), (0, 0, 0)) \mapsto (I, E) = (0, \pm 1)$$

singularities

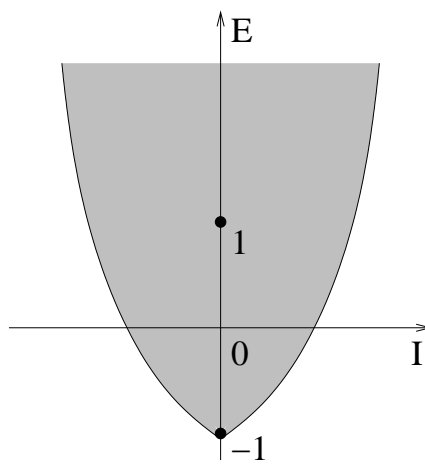
Duistermaat (1980): Circle around  $(I, E) = (0, 1)$  collects nontrivial *monodromy*

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{R})$$

(in suitable bases of period lattice)

$\Rightarrow$  *nontrivial*  $\mathbb{T}^2$ -bundle (Liouville-Arnold)

& nonexistence of *global* action angles



Range of the energy-momentum map of the spherical pendulum

## 5. Global KAM theory III

### Liouville-Arnold theorem

$(M, \sigma)$  symplectic manifold,  $\dim M = 2n$ ,  
 $B$   $n$ -dimensional manifold

**GIVEN**

integral  $\mathcal{F} : M \rightarrow B$ ,

$\mathcal{F} = (f_1, f_2, \dots, f_n)$  (locally)

with  $\{f_i, f_j\} = 0$ ,  $d\mathcal{F}$  maximal rank  
and compact fibres  $\mathcal{F}^{-1}(b)$

**THEN**

$\forall b \in B \exists$  open  $b \in V \subseteq B$  such that

$\mathcal{F}^{-1}(V) \subseteq M$  open and invariant under  $X_{f_i}$   
and

$\exists$  symplectic diffeo

$(I, \varphi) : \mathcal{F}^{-1}(V) \rightarrow U \times (\mathbb{R}/\mathbb{Z})^n$  with  $U \subseteq \mathbb{R}^n$   
open, i.e., such that:

$$(I, \varphi)^* \left( \sum_{j=1}^n d\varphi_j \wedge dI_j \right) = \sigma|_{\mathcal{F}^{-1}(V)}$$

# Global KAM virtual

Singularity ude *saddle* (double eigenvalues  $\pm 1$ )

Singular fiber: *pinched* 2-torus ( $= W^{u/s}(ude)$ )

**Theorem** (Matveev 1996, Nguyen 1997):

Given 4D symplectic manifold  $M$  fibered  
by level sets of map  $\mathcal{E}\mathcal{M}$

ASSUME

- $\mathcal{E}\mathcal{M}$  has only one critical value
- fibers of  $\mathcal{E}\mathcal{M}$  compact & connected
- singular fiber has  $k$  singular points,  
all real or complex saddle points

THEN

singular fiber is pinched torus

& Liouville-Arnold  $\mathbb{T}^2$ -bundle is non-trivial  
with monodromy similar to

$$\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \in Sl(2, \mathbb{R})$$

**Remark** Quantum monodromy *exists*

in semi-classical quantizations of classical in-  
tegrable system with nontrivial  $\mathbb{T}^n$ -bundles  
(San Vu Ngọc 1999)

# Global KAM theory V

## Back to Conservative KAM theorem:

Phase space  $\mathbb{T}^n \times A$

$A \subset \mathbb{R}^n$  is bounded, open & connected  
action angle coordinates  $(I, \varphi)$

symplectic form  $\mu = \sum_{j=1}^n dI_j \wedge d\varphi_j$

$h : \mathbb{T}^n \times A \rightarrow \mathbb{R}$   $C^\infty$  Hamilton function,  
*integrable* (not depending on  $\varphi$ )  $\rightarrow$

Hamiltonian vector field  $X_h$

$$X_h(\varphi, I) = \sum_{j=1}^n \omega_j(I) \frac{\partial}{\partial \varphi_j}, \text{ with } \omega(I) := \frac{\partial h}{\partial I}$$

**Kolmogorov nondegeneracy** frequency map

$\omega : A \rightarrow \mathbb{R}^n$  is diffeomorphism

## Diophantine frequencies

Given  $\tau > n - 1$  and  $\gamma > 0$ , recall definition

$$\mathbb{R}_{\tau, \gamma}^n := \{\omega \in \mathbb{R}^n \mid |\langle \omega, k \rangle| \geq \gamma |k|^{-\tau}, \forall k \in \mathbb{Z}^n \setminus \{0\}\}$$

# Global KAM theory VI

For  $\Gamma := \omega(A)$  let

$\Gamma' := \{\omega \in \Gamma \mid \text{dist}(\omega, \partial\Gamma) > \gamma\}$  and  $\Gamma'_\gamma := \Gamma' \cap \mathbb{R}_\gamma^n$

also define  $A'_\gamma \subset A$  by  $A'_\gamma := \omega^{-1}(\Gamma'_\gamma)$

**NB** measure  $(A \setminus A'_\gamma) = O(\gamma)$  as  $\gamma \downarrow 0$

Perturb to  $h + f : \mathbb{T}^n \times A \rightarrow \mathbb{R}$ , all of class  $C^\infty$   
with  $C^\infty$ -topology

**Local KAM Theorem** (Pöschel (82),  
Broer-Huitema-Takens (90))

*If  $\gamma > 0$  is sufficiently small and  $h + f$  is  
sufficiently near  $h$  in  $C^\infty$ -topology*

*then  $\exists$  map  $\Phi : \mathbb{T}^n \times A \rightarrow \mathbb{T}^n \times A$   
with following properties:*

- $\Phi$  is  $C^\infty$  diffeomorphism
- for  $\widehat{\Phi} := \Phi|_{\mathbb{T}^n \times D_\gamma(A'_\gamma)}$  we have

$$\widehat{\Phi}_* X_h = X_{h+f}$$

# Global KAM theory VII

## Discussion

Map  $\Phi$

- is generally *not* symplectic
- is near-identity in the  $C^\infty$ -topology

In global KAM theorem

- convex combinations on *unique* tori  
keeps closed half line geometry
- near-identity torus automorphisms are all translations
- Whitney extensions near Id  
 $\Rightarrow$  all extended  $\mathbb{T}^n$ -bundles isomorphic
- Quantum monodromy in semi-classical versions of integrable systems with monodromy
- How about the nearly-integrable case ?