The number of bits needed to represent a unit disk graph

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Abstract. We prove that for sufficiently large n, there exist unit disk graphs on n vertices such that for every representation with disks in the plane at least $c^{\sqrt{n}}$ bits are needed to write down the coordinates of the centers of the disks, for some $c > 1$. We also show that d^n bits always suffice, for some $d > 1$.

1 Introduction and statement of results

A *unit disk graph* is the intersection graph of equal sized disks in the plane. That is, we can represent the vertices by disks $D_1, \dots, D_n \subseteq \mathbb{R}^2$ of equal radius in such a way that $ij \in E$ if and only if $D_i \cap D_j \neq \emptyset$. Equivalently, we can represent G by a sequence of points $\mathcal{V} = (z_1, \dots, z_n)$ in the plane such that $ij \in E(G)$ if and only if $\|z_i - z_j\| \leq 1$. We say that such a $\mathcal{V}$ *realizes* G.

Over the past 20 years or so, unit disk graphs have been the subject of a sustained research effort by many different authors. Partly because of their relevance for practical applications one of the main foci is the design of (efficient) algorithms for them.

One can of course store the unit disk graph G in a computer as an adjacency matrix or a list of edges, but for many purposes (algorithms) it is useful to actually store a representation as points in the plane. In this article we will study the number of bits that are needed to store such a representation. There are of course infinitely many realizations, but we will focus on a realization whose coordinates have the smallest possible bit size. Here we shall use the convention that a rational number is stored as a pair of integers (the denominator and numerator) that are relatively prime and those integers are stored in the binary number format (see for instance [7]).

We will denote the bit size of a rational number $q \in \mathbb{Q}$ by $\text{size}(q)$. The bit size of a point $z \in \mathbb{Q}^2$ will be the sum of the bit sizes of its coordinates $\text{size}(z) := \text{size}(z_x) + \text{size}(z_y)$, and the bit size of a realization $\mathcal{V} = (z_1, \dots, z_n)$ of a unit disk graph G will be $\text{size}(\mathcal{V}) := \sum_{i=1}^{n} \text{size}(z_i)$. We are thus interested in the following quantity for G a unit disk graph:

$$\text{size}(G) := \min_{\mathcal{V} \text{ realizes } G} \text{size}(\mathcal{V}).$$

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Our main result in this paper is the following:

Theorem 1. *There exists a* $\gamma > 1$ *such that for each* n*, there exists a unit disk graph on* n *vertices with* $\text{size}(G) > \gamma^{\sqrt{n}}$.

Theorem 1 answers a question of Spinrad [8]. This question was also studied by Van Leeuwen and Van Leeuwen [6], who dubbed it the *Polynomial Representation Hypothesis* (PRH) for unit disk graphs. The PRH for unit disk graphs states that a unit disk graph can always be realized by points whose bit sizes are bounded by some polynomial in the number of vertices n. Theorem 1 above thus shows that this hypothesis is false. It is known that unit disk graph recognition is NP-hard [1], but membership in NP is still an open problem. Had the PRH been true, then this would have proved membership in NP, but as it is this remains an open problem.

Theorem 1 could be seen as bad news for those wishing to design algorithms for unit disk graphs. On the slightly positive side we offer the following upper bound:

Theorem 2. *There exists a constant* γ *such that for each* n*, each unit disk graph* G *on* n *vertices has* $\text{size}(G) \leq \gamma^n$.

Our results also hold for disk graphs (intersection graphs of disks not all of the same radius), but the proofs are more involved. We therefore postpone these proofs to the journal version of this paper.

2 Proofs

A *line arrangement* is a family $\mathcal{L} = (\ell_1, \ldots, \ell_m)$ of lines in the plane. A line arrangement is *simple* if every two lines intersect (there are no parallel lines), and there is no point on three lines.

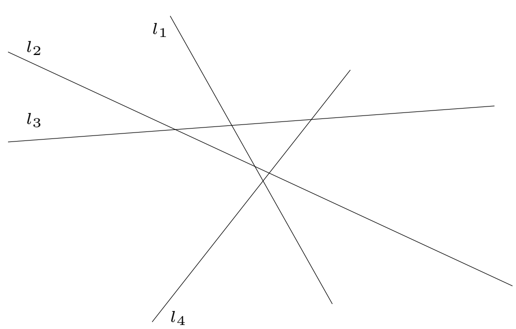

Fig. 1. A simple line arrangement

A *segment* is the portion of a line between two intersections with other lines. The *size* of a line arrangement $\mathcal{L}$ is the quotient of the length of a longest segment and a shortest segment.

The *combinatorial description* $\mathcal{D}$ of a line arrangement $\mathcal{L}$ is obtained as follows. We assume without loss of generality that no line other than l_0 is vertical. We add an auxiliary vertical line ℓ_0 to the left of all intersection points of the lines and we store the order in which each line $\ell_0, \ldots, \ell_k$ intersects the others from left to right (top to bottom for ℓ_0). For instance, the combinatorial description of the line arrangement in figure 1 is given by the sequences $(1, 2, 3, 4), (0, 3, 2, 4), (0, 3, 1, 4), (0, 2, 1, 4), (0, 1, 2, 3)$. If the line arrangement $\mathcal{L}$ has combinatorial description $\mathcal{D}$ then we say that $\mathcal{L}$ *realizes* $\mathcal{D}$.

We shall make use of the following impressive result of Kratochvil and Matousek [5] and independently Goodman, Pollack and Sturmfels [2].

Theorem 3 ([5],[2]). *For every k, there exists a combinatorial description of a simple line arrangement on $O(k)$ lines such that every realization of it has size at least 2^{2^k}.*

Here it should be mentioned that Kratochvil and Matousek's proof of Theorem 3 can only be found in the technical report version [4]. The following proposition allows us to encode a combinatorial description of a simple line arrangement into a unit disk graph.

Lemma 1. *Let $\mathcal{D}$ be a combinatorial description of a simple line arrangement on k lines. There exists a unit disk graph G on $O(k^2)$ vertices such that for every realization $\mathcal{V} = (z_0, \ldots, z_m)$ of it, up to isometry, the line arrangement $\mathcal{L} = \{\ell_1, \ldots, \ell_k\}$ where*

$$\ell_i := \{z : \|z - z_{2i}\| = \|z - z_{2i+1}\|\}, \quad i = 1, \ldots, k,$$

is a realization of $\mathcal{D}$. Moreover, all the segments defined by $\mathcal{L}$ will have length at most one.

Proof. Let $\mathcal{L}$ be a realization of $\mathcal{D}$, and let ℓ_0 be a vertical line to the left of all intersection points. We will call a connected component of $\mathbb{R}^2 \setminus (\bigcup_{i=0}^k \ell_i)$ a *cell*. Let c denote the number of cells and put $m = 2k + 1 + c$. It is easily seen that $c = 1 + 1 + 2 + 3 + \cdots + k = 1 + \binom{k+1}{2}$. So in particular $m = O(k^2)$. Let us arbitrarily label the cells as $C_1, \ldots, C_c$. For $i = 1, \ldots, c$, we place a point p_{2k+1+i} in the interior of C_i (we shall define points $p_0, \ldots, p_{2k+1}$ shortly).

Since none of the p_js that have been defined until now lie on the line l_i, for any sufficiently large radius R, we can place disks $D_i^0(R)$ and $D_i^1(R)$ of radius R on either side of ℓ_i such that all the p_js are contained in one of $D_i^0(R)$ and $D_i^1(R)$ (see figure 2). We now choose R big enough so that $D_i^0(R), D_i^1(R)$s can be constructed with this property for all $i = 0, \ldots, k$ and moreover we make sure that R is bigger than the distance between any of the p_js that have been defined until now. We let p_{2i} be the center of $D_i^0(R)$ and p_{2i+1} the center of $D_i^1(R)$ for $i = 0, \ldots, k$. To finish our construction, we set $z_i := \frac{1}{R} p_i$ for $i = 0, \ldots, m$

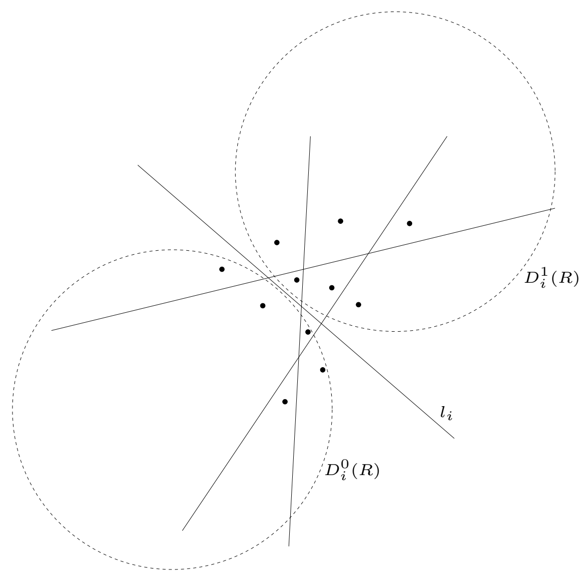

Fig. 2. The construction of $D_i^1(R), D_i^2(R)$.

and $\mathcal{V} := (z_0, \dots, z_m)$, and we let G be the corresponding unit disk graph on vertex set $\{0, \dots, m\}$. To avoid confusion, let us stress that we use the second definition of a unit disk graph given in the introduction, i.e. $ij \in E(G)$ if and only if $\|z_i - z_j\| \leq 1$. This corresponds to the intersection graph of disks of radius $\frac{1}{2}$ centered on the z_is (or disks of radius $\frac{R}{2}$ centered on the p_is). Let us observe that the set of vertices $C = \{2k+2, \dots, m\}$ forms a clique in G, and the neighbourhoods $N(2i), N(2i+1)$ partition C into two non-empty parts for all $i = 0, \dots, k$.

We claim that G is as required. To this end, let $\mathcal{V}' := (z_0', \dots, z_m')$ be an arbitrary realization of G. For $i = 0, \dots, k$ we set

$$\ell_i' := \{z : \|z - z_{2i}'\| = \|z - z_{2i+1}'\|\}.$$

One of p_0, p_1 was to the left of ℓ_0 in our original construction, without loss of generality assume it was p_0. By applying a suitable isometry if needed, we can assume that ℓ_0' is vertical, and z_0' lies to the left of l_0' (and z_1' to its right).

Now consider an arbitrary line ℓ_i for some $i \in \{0, \dots, k\}$. In the original arrangement $\mathcal{L}$, the line ℓ_i intersects the other lines in some order $(i_1, i_2, \dots, i_k)$ from left to right (top to bottom if $i = 0$ – in the next few paragraphs "left" should be replaced by "top" and "right" by "bottom" in case $i = 0$). We wish to show that ℓ_i' intersects the other ℓ_j's in the same order.

Let us relabel the points in the cells that are neighbouring ℓ_i as $t_0, \dots, t_k$ and $b_0, \dots, b_k$ where the t_js lie in the cells above ℓ_i and the b_j lie in the cells below ℓ_i, and t_0, b_0 lie in the leftmost cells, t_1, b_1 in the second leftmost cells and so on (see figure 3). For $r = 0, \ldots, k$, let t'_r denote the point of $\mathcal{V}'$ corresponding

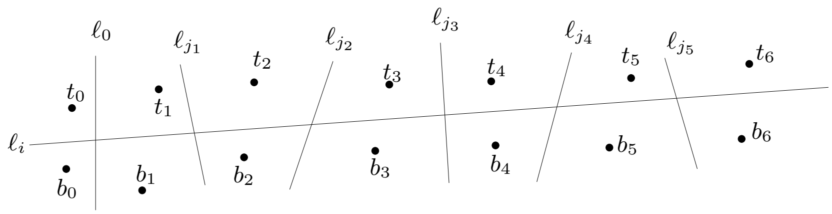

Fig. 3. The intersections of ℓ_i with the other lines.

to t_r and let b'_r denote the point corresponding to b_r (by "corresponding to" we mean that both represent the same vertex of G).

We say that a line ℓ *separates* two sets $A, B \subseteq \mathbb{R}^2$ if A and B lie on different sides of ℓ. For $j = 1, \ldots, k$ let us set $A_j := \{t_r, b_r : r \le j\}, B_j := \{t_r, b_r : r > j\}$ and $A'_j := \{t'_r, b'_r : r \le j\}, B'_j := \{t'_r, b'_r : r > j\}$. By our choice of $p_0, \ldots, p_m$ above, all points in A_j have distance $< R$ from p_{2i_j} and distance $> R$ from p_{2i_j+1}, and these inequalities are reversed for the points in B_j (swapping the labels of p_{2i_j}, p_{2i_j+1} if necessary). By construction of G, we must then also have that all the points in A'_j have distance ≤ 1 to z'_{2i_j} and distance > 1 to z'_{2i_j+1} and these inequalities are reversed for the points in B'_j. Thus, $\ell'_{i_j} := \{z : \|z - z'_{2i_j}\| = \|z - z'_{2i_j+1}\|\}$ separates A'_j from B'_j for all $j = 1, \ldots, k$. For $r = 0, \ldots, k$, let u'_r be the intersection point of the segment $[t'_r, b'_r]$ with ℓ'_i and let v'_r be the intersection point of ℓ'_{j_r} with ℓ'_i (see figure 4). We must have that on ℓ'_i the points $\{u'_0, \ldots, u'_j\}$

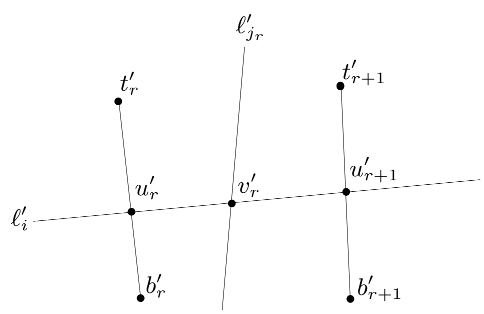

Fig. 4. The definition of u'_r and v'_r.

and $\{u'_{j+1}, \ldots, u'_k\}$ lie on different sides of v'_j (here we use that the line segment $[t'_r, b'_r]$ stays on the same side of ℓ'_{i_j} that t'_r, b'_r are on). From this last observation it now follows that the order of the u_j's and v_j's on ℓ_i' from left to right is either $(u_0', v_1', u_1', v_2', \dots, v_k', u_k')$ or the reverse order $(u_k', v_k', u_{k-1}', v_{k-1}, \dots, v_1', u_0')$.

Suppose that $i > 0$. In this case we must have that $i_1 = 0$, because the line ℓ_0 lies to the left of all intersection points between other lines in the original line arrangement $\mathcal{L}$. Note that t_0', b_0' and hence also u_0' lie to the left of l_0', since t_0', b_0' have distance ≤ 1 to z_0' and distance > 1 to z_1' and we have made sure that z_0' lies to the left of l_0' and z_1' to the right. From this it follows that u_0' is the leftmost point among the u_j's, which in turn shows that ℓ_i intersects the other lines in the desired order.

Suppose that $i = 0$. We have seen that ℓ_0' intersects the other lines from top to bottom either in the correct order, or in the reverse of the correct order. In this last case we can reflect our point set $(z_0', \dots, z_m')$ through the x-axis (notice this does not change the left-to-right orders on other lines) to fix it.

This proves that indeed, after applying a suitable isometry, $\mathcal{L}'$ has combinatorial description $\mathcal{D}$.

It remains to check that all the segments of the line arrangement $\mathcal{L}'$ have length at most 1. To this end, let p' be the intersection point of two lines ℓ_i' and ℓ_j'. Each of the four connected components of $\mathbb{R}^2 \setminus (\ell_i' \cup \ell_j')$ must contain at least one element of $C' := \{z_{2k+2}', \dots, z_m'\}$. This is because the points of C' inside each of these regions corresponds with one of the non-empty sets of vertices $C \cap N(2i) \cap N(2j), C \cap N(2i+1) \cap N(2j), C \cap N(2i) \cap N(2j+1), C \cap N(2i+1) \cap N(2j+1)$ in G, where $C = \{2k+2, \dots, m\}$ and $N(j)$ denotes the neighbourhood of j in G as before. It follows that p' lies in the convex hull $\mathrm{conv}(C')$ of C' (see figure 5). Finally notice that, as C is a clique in G, the distance between any

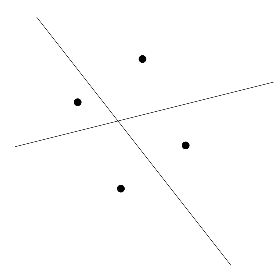

Fig. 5. An intersection point lies in the convex hull of the cell-points.

two points of C' is at most 1. Thus, if p', q' are two intersection points of the lines, then

$$\|p' - q'\| \leq \mathrm{diam}(\mathrm{conv}(C')) = \mathrm{diam}(C') \leq 1.$$

This concludes the proof of Lemma 1 $\square$

The last ingredient for the proof of Theorem 1 is the following elementary observation. For completeness we provide a proof in the appendix.

Lemma 2. *Let $a, b \in \mathbb{Q}$ be two rational numbers with bit sizes* $\text{size}(a), \text{size}(b) \leq B$. *Then* $\text{size}(a+b), \text{size}(a-b), \text{size}(ab), \text{size}(a/b) \leq 4B$. □

Proof of Theorem 1. For an arbitrary $k \in \mathbb{N}$, let $\mathcal{D}$ be the combinatorial description from theorem 3, and let G be the unit disk graph that Lemma 1 constructs from it. It suffices to show that any realization of G has bit size at least $\Omega(2^k)$.

To this end, let $\mathcal{V} = (z_0, \ldots, z_m)$ be an arbitrary realization of G, and let B be such that each coordinate of each z_i is stored using at most B bits. Note that we can also write l_i as

$$l_i = \{z : (z_{2i+1} - z_{2i})^T z = (z_{2i+1} - z_{2i})^T (z_{2i+1} + z_{2i})/2\}.$$

This shows that the intersection point between two lines is the solution to a 2×2 linear system $Az = b$ where each entry of A and b can be obtained by applying a bounded number of arithmetic operations (i.e. addition, substraction, multiplication, division) to the coordinates of a bounded number of z_js. By Lemma 2, the entries of A and b thus all have bit size $O(B)$. From the familiar formula

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{-a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ \frac{-a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \end{pmatrix},$$

we see that the intersection point $z = A^{-1}b$ can be obtained by a bounded number of arithmetic operations from the entries of A and b. By another application of Lemma 2 we thus have that all intersection points have bit size $O(B)$.

By construction of G, there are two intersection points v, w with $0 < \|v - w\| \leq 1/2^{2^k}$ (the longest segment has length at most 1 and the ratio of the longest to the smallest segment is at least 2^{2^k}). Let $s_x := |v_x - w_x|, s_y := |v_y - w_y|$. At least one of these numbers must be positive, assume without loss of generality it is s_x. Recall that rational numbers are stored as a pair (denominator, numerator) of integers. We see from $0 < s_x \leq 1/2^{2^k}$ that its numerator must be at least 2^{2^k}, which gives $\text{size}(s_x) \geq 2^k$. On the other hand, we have already seen that $\text{size}(s_x) = O(B)$. Hence $B = \Omega(2^k)$, which concludes the proof. □

Theorem 2 is a straightforward consequence of a result of Grigor'ev and Vorobjov. The following is a reformulation of Lemma 10 in [3]:

Lemma 3 ([3]). *For each $d \in \mathbb{N}$ there exists a constant $C = C(d)$ such that the following hold. Suppose that $h_1, \ldots, h_k$ are polynomials in n variables with integer coefficients, and degrees* $\deg(h_i) < d$. *Suppose further that the bit sizes of the all coefficients are less than B. If there exists a solution $(x_1, \ldots, x_n) \in \mathbb{R}^n$ of the system $\{h_1 \geq 0, \ldots, h_k \geq 0\}$, then there also exists one with $|x_1|, \ldots, |x_n| \leq$* $\exp[(B + \ln k)C^n]$.

Proof of Theorem 2. Let G be a unit disk graph on n vertices. Consider the set of all $(x_1, y_1, \ldots, x_n, y_n, R) \in \mathbb{R}^{2n+1}$ that satisfy:

$$(x_i - x_j)^2 + (y_i - y_j)^2 \leq (R - 10)^2, \text{ for all } ij \in E(G),$$
$$(x_i - x_j)^2 + (y_i - y_j)^2 \geq (R + 10)^2, \text{ for all } ij \notin E(G),$$
$$R \geq 100.$$

This is a system of $1 + \binom{n}{2}$ polynomial inequalities of degree less than 3 in $2n+1$ variables, with all coefficients small integers. It follows from the fact that any unit disk graph has a realization with all distances different from 1 (see for instance Proposition 1 of [6]) that the system has a solution, by "inflating" such a realization. Hence, by lemma 3 there exists a solution to this system with all numbers less than $\exp[\gamma^n]$ in absolute value for some γ (we absorb the factor $\ln(1 + \binom{n}{2}) + O(1)$ by taking $\gamma > C$). Let us now round down all numbers to the next integer. It is easily checked that we get a $(x'_1, y'_1, \ldots, x'_n, y'_n, R') \in \mathbb{Z}^{2n+1}$ that satisfies

$$(x'_i - x'_j)^2 + (y'_i - y'_j)^2 \leq (R')^2, \text{ for } ij \in E(G),$$
$$(x'_i - x'_j)^2 + (y'_i - y'_j)^2 > (R')^2, \text{ for } ij \notin E(G).$$

Hence, if we divide the x'_i, y'_i by R' we get a realization of G that uses $O(\gamma^n)$ bits per coordinate. □

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A The proof of Lemma 2

Proof of Lemma 2. Let $a, b \in \mathbb{Q}$ be arbitrary with $\text{size}(a), \text{size}(b) \leq B$. Let us write $a = p_1/q_1, b = p_2/q_2$ with p_i, q_i relatively prime integers for $i = 1, 2$. Note that an integer $n \in \mathbb{Z}$ has bit size

$$\text{size}(n) = 1 + \lceil \log_2(|n| + 1) \rceil, \tag{1}$$

(the extra one is for the sign). From (1) it follows that for two integers $n, m \in \mathbb{Z}$:

$$\begin{aligned} &\text{size}(nm) \leq \text{size}(n) + \text{size}(m), \\ &\text{size}(n + m) \leq 1 + \max(\text{size}(n), \text{size}(m)). \end{aligned} \tag{2}$$

From $ab = p_1 p_2 / q_1 q_2$ and (2), we see that

$$\begin{aligned} \text{size}(ab) &\leq \text{size}(p_1 p_2) + \text{size}(q_1 q_2) \\ &\leq \text{size}(p_1) + \text{size}(p_2) + \text{size}(q_1) + \text{size}(q_2) \\ &= \text{size}(a) + \text{size}(b) \\ &\leq 2B. \end{aligned}$$

Completely analogously, $\text{size}(a/b) \leq 2B$.

From $a + b = (p_1 q_2 + p_2 q_1)/q_1 q_2$ and (2) we see that

$$\begin{aligned} \text{size}(a + b) &\leq \text{size}(p_1 q_2 + p_2 q_1) + \text{size}(q_1 q_2) \\ &\leq 1 + \max(\text{size}(p_1) + \text{size}(q_2), \text{size}(p_2) + \text{size}(q_1)) \\ &\quad + \text{size}(q_1) + \text{size}(q_2) \\ &\leq 1 + 3B \\ &\leq 4B. \end{aligned}$$

Completely analogously, $\text{size}(a - b) \leq 4B$. □