A counterexample to a conjecture of Grünbaum on piercing convex sets in the plane

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July 22, 2013

Abstract
A collection of sets \( F \) has the \((p,q)\)-property if out of every \( p \) elements of \( F \) there are \( q \) that have a point in common. A transversal of a collection of sets \( F \) is a set \( A \) that intersects every member of \( F \). Grünbaum conjectured that every family \( F \) of closed, convex sets in the plane with the \((4,3)\)-property and at least two elements that are compact has a transversal of bounded cardinality. Here we construct a counterexample to his conjecture. On the positive side, we also show that if such a collection \( F \) contains two disjoint compacta then there is a transversal of cardinality at most 13.

1 Introduction and statement of results

Let \( F \) be a collection of sets. A transversal of \( F \) is a set \( A \) that intersects every member of \( F \) (that is, \( A \cap F \neq \emptyset \) for all \( F \in F \)). The transversal number of piercing number \( \tau(F) \) of \( F \) is the smallest size of a transversal, i.e.

\[
\tau(F) := \min_{A \text{ transversal of } F} |A|.
\]

(Note that \( \tau(F) = \infty \) if no finite transversal exists.)

A collection of sets \( F \) has the \((p,q)\)-property if out of every \( p \) sets of \( F \) there are \( q \) that have a point in common. In 1957, Hadwiger and Debrunner [2] conjectured that for every \( d \) and every \( p \geq q \geq d + 1 \) there is a universal constant \( c = c(d;p,q) \) such that every finite collection \( F \) of convex sets in \( \mathbb{R}^d \) with the \((p,q)\)-property satisfies \( \tau(F) \leq c \). (By considering hyperplanes in general position it is easily seen that for \( q \leq d \) no such universal constant \( c \) can exist.) Many years later, in 1992, Alon and Kleitman [1] finally proved the conjecture of Hadwiger and Debrunner by cleverly combining various pre-existing tools from the literature.

In the special case when \( p = q = d + 1 \) the Hadwiger-Debrunner conjecture reduces to the classical theorem of Helly [3] which states that if \( F \) is a finite family of convex sets in \( \mathbb{R}^d \) such that every \( d + 1 \) members of \( F \) have a point in common then \( \tau(F) = 1 \). A variant of Helly’s theorem states that if \( F \) is an infinite collection of closed, convex sets in \( \mathbb{R}^d \) and at least one member of \( F \) is compact then \( \tau(F) = 1 \).

Erdős conjectured that in the first nontrivial case of the Hadwiger-Debrunner problem, a similar variant would be true. That is, he conjectured that if \( F \) is a collection of closed, convex sets in the plane with the \((4,3)\)-property and one of the members of \( F \) is compact, then \( \tau(F) \leq c \) for some universal constant \( c \). Boltyanski and Soifer included this conjecture in the first edition of their book “Geometric Etudes in Combinatorial Mathematics” and they offered a prize of $25 for its solution. Eighteen years later, Grünbaum found a counterexample while proofreading the second edition, earning the reward. Grünbaum also made a conjecture of his own, stating that if \( F \) is a collection of closed, convex sets in the plane, and two members of \( F \) are compact then \( \tau(F) \) is finite. (See [5], pages 198-199.) Here we show that Grünbaum’s conjecture fails as well:

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Theorem 1 There exists a collection $\mathcal{F}$ of closed, convex subsets of the plane such that

(i) $\mathcal{F}$ has the $(4,3)$-property, and;

(ii) Two of the elements of $\mathcal{F}$ are compact, and;

(iii) $\tau(\mathcal{F}) = \infty$.

On the positive side, we show that any collection $\mathcal{F}$ of closed, convex sets in the plane that contains two disjoint compacta and satisfies the $(4,3)$-property does have universally bounded transversal number:

Theorem 2 If $\mathcal{F}$ is a collection of closed, convex sets in the plane such that

(i) $\mathcal{F}$ has the $(4,3)$-property, and;

(ii) $\mathcal{F}$ contains two disjoint compacta,

then $\tau(\mathcal{F}) \leq 13$.

2 The counterexample

Let us set

$$F_1 := [-1,1] \times \{0\}, \quad F_2 := [0,2] \times \{0\}.$$ 

Let $t_1 < t_2 < t_3 < \ldots$ be a strictly increasing sequence of numbers between 0 and 1, and let $s_1 > s_2 > \ldots$ be a strictly decreasing sequence of negative numbers that tends to $-\infty$. (For instance $t_n := 1 - \frac{1}{n}, s_n := -n$ would be a valid choice.) Set $p_n := (t_n,0)$; let $\ell_n$ denote the vertical line through $p_n$, and let $\ell_n'$ denote the line through $p_n$ of slope $s_n$.

For $n \geq 3$ we now let $F_n$ be the set of all points either on or to the left of $\ell_n$ and either on or above $\ell_n'$. See figure 1.

![Figure 1: The construction of $F_n$ for $n \geq 3$ (left) and part of the collection $\mathcal{F}$ (right).](image)

Observe that by construction $F_n$ contains all sufficiently high points on the $y$-axis for all $n \geq 3$:

For each $n \geq 3$ there exists a $y_n > 0$ such that $\{(0,y) : y \geq y_n\} \subseteq F_n$. \hfill (1)

Let $\mathcal{F} := \{F_1, F_2, \ldots\}$ be the resulting infinite collection of closed convex sets. We first establish that $\mathcal{F}$ has the $(4,3)$-property.
Lemma 3 \( \mathcal{F} \) has the \((4,3)\)-property.

**Proof:** Let us pick four arbitrary distinct indices \( i_1 < i_2 < i_3 < i_4 \) and consider the quadruple \( F_{i_1}, F_{i_2}, F_{i_3}, F_{i_4} \in \mathcal{F} \).

If \( i_1 = 1 \) and \( i_2 = 2 \) then clearly \( F_{i_1} \cap F_{i_2} \cap F_{i_3} = \{p_{i_3}\} \), so that \( F_{i_1}, F_{i_2}, F_{i_3} \) is an intersecting triple. We can thus assume that \( i_2, i_3, i_4 > 2 \). In this case \( F_{i_2} \cap F_{i_3} \cap F_{i_4} \neq \emptyset \) by the observation (1).

It remains to show that \( \mathcal{F} \) does not have a finite transversal.

**Lemma 4** \( \tau(\mathcal{F}) = \infty \).

**Proof:** It suffices to show that every point of the plane is in finitely many elements of \( \mathcal{F} \). Let \( a = (a_x, a_y) \in \mathbb{R}^2 \) be arbitrary. If \( a_y \leq 0 \) then \( a \) is in at most three elements of \( \mathcal{F} \). Let us therefore assume \( a_y > 0 \). In this case, if \( a_x \geq t_n \) for all \( n \in \mathbb{N} \) then \( a \) is in no element of \( \mathcal{F} \). Let us therefore assume that there is at least one \( n \in \mathbb{N} \) such that \( a_x < t_n \). Let us fix an \( n_0 \) such that \( a_x < t_{n_0} \), and set

\[
    s := -\frac{a_y}{t_{n_0} - a_x}.
\]

(Note that \( s \) is exactly the slope of the line through \( a \) and \( p_{n_0} \).) Since \( s_n \to -\infty \), there is an \( m_0 \) such that \( s_n < s \) for all \( n \geq m_0 \).

Observe that for all \( n \geq \max(n_0, m_0) \) the point \( a \) is below the line \( t'_n \) (as the point \( p_n \) is to the right of \( p_{n_0} \) and \( t'_n \) has a steeper slope than \( s \)). This shows that \( a \notin F_n \) for all \( n \geq \max(n_0, m_0) \). Hence \( a \) is in infinitely many elements of \( \mathcal{F} \) as required.

**Remark:** By adding additional compact sets to \( \mathcal{F} \) that each contain \([0, 1] \times \{0\}\) we can obtain a collection \( \mathcal{F}' \) that contains an arbitrary number of compacta, and still has the \((4,3)\)-property and \( \tau(\mathcal{F}') = \infty \).

3 The proof of Theorem 2

The proof of the Hadwiger-Debrunner conjecture by Alon and Kleitman [1] does not give a good bound on the universal constant \( c \). A better bound on this constant for the special case when \( p = 4, q = 3 \) was later given by Kleitman, Gyárfás and Tóth [4].

**Theorem 5** (Kleitman et al. [4]) If \( \mathcal{F} \) is a finite collection of convex sets in the plane with the \((4,3)\)-property then \( \tau(\mathcal{F}) \leq 13 \).

A standard compactness argument (which we do not repeat here) shows that the same also holds if \( \mathcal{F} \) is an infinite collection of convex compacta with the \((4,3)\)-property.

**Corollary 6** If \( \mathcal{F} \) is an infinite collection of convex, compact sets in the plane and \( \mathcal{F} \) has the \((4,3)\)-property then \( \tau(\mathcal{F}) \leq 13 \).

**Proof of Theorem 2:** Let \( \mathcal{F} \) be an arbitrary infinite collection of closed, convex sets with the \((4,3)\)-property with two sets \( A, B \in \mathcal{F} \) that are disjoint and compact. Let us set

\[
    F_0 := \text{conv}(A \cup B).
\]

Let us first observe that

\[
    F \cap F_0 \neq \emptyset \quad \text{for all} \quad F \in \mathcal{F}. \quad (2)
\]

To see this, suppose that some \( F \in \mathcal{F} \) is disjoint from \( F_0 \), and let \( F' \in \mathcal{F} \) be an arbitrary element distinct from \( F, F_1, F_2 \) and \( F_0 \). Then the quadruple \( F_1, F_2, F, F' \) does not have an intersecting
triple as every triple contains a pair of disjoint sets. But this contradicts the (4,3)-property! Hence (2) holds as claimed.

Next, we claim that

$$\text{If } F_1, F_2, F_3 \in \mathcal{F} \text{ are such that } F_1 \cap F_2 \cap F_3 \neq \emptyset \text{ then also } F_0 \cap F_1 \cap F_2 \cap F_3 \neq \emptyset. \quad (3)$$

To see that the claim (3) holds, consider an arbitrary triple $F_1, F_2, F_3$ such that $F_1 \cap F_2 \cap F_3 \neq \emptyset$. Let us assume $F_1 \cap F_2 \cap F_3 \subseteq F_0$ (otherwise we are done), and fix a $q \in (F_1 \cap F_2 \cap F_3) \setminus F_0$. By considering the quadruple $A, B, F_1, F_2$ we see that we either have $A \cap F_1 \cap F_2 \neq \emptyset$ or $B \cap F_1 \cap F_2 \neq \emptyset$.

In either case, there is a point $p_{12} \in F_0 \cap F_1 \cap F_2$. Similarly there are points $p_{13} \in F_0 \cap F_1 \cap F_2, p_{23} \in F_0 \cap F_2 \cap F_3$.

By Radon’s lemma the set $\{q, p_{12}, p_{13}, p_{23}\}$ can be partitioned into two sets whose convex hulls intersect. Note that we cannot have that $q \in \text{conv}(\{p_{12}, p_{13}, p_{23}\})$ since $q \notin F_0$ and $p_{12}, p_{13}, p_{23} \in F_0$ and $F_0$ is convex. Hence, up to relabelling of the indices we have either $p_{23} \in \text{conv}(\{q, p_{12}, p_{13}\})$ or $[q, p_{23}] \cap [p_{12}, p_{13}] \neq \emptyset$.

In the first case we have that $p_{23} \in F_0 \cap F_1 \cap F_2 \cap F_3$ since we have chosen $p_{23} \in F_0 \cap F_2 \cap F_3$ and $\text{conv}(\{q, p_{12}, p_{13}\}) \subseteq F_1$ as all three of $q, p_{12}, p_{13}$ are in $F_1$ and $F_1$ is convex.

In the second case we have that the intersection point of $[q, p_{23}]$ and $[p_{12}, p_{13}]$ is in $F_0 \cap F_1 \cap F_2 \cap F_3$. This is because $[q, p_{23}] \subseteq F_2 \cap F_3$ and $[p_{12}, p_{13}] \subseteq F_0 \cap F_1$.

Thus, (3) holds as claimed.

We now define a new collection of sets by setting:

$$\mathcal{F}^\prime := \{F \cap F_0 : F \in \mathcal{F}\}.$$  

Since the members of $\mathcal{F}$ are closed and convex and $F_0$ is compact and convex, each element of $\mathcal{F}^\prime$ is compact and convex. By (2) each set of $\mathcal{F}^\prime$ is nonempty (this is needed since otherwise there cannot be any transversal of $\mathcal{F}^\prime$), and by (3) together with the fact that $\mathcal{F}$ satisfies the $(4,3)$-property, the collection $\mathcal{F}^\prime$ also satisfies the $(4,3)$-property. The theorem now follows from Corollary 6 as every transversal of $\mathcal{F}^\prime$ is also a transversal of $\mathcal{F}$.

\section*{Acknowledgement}

I thank Bart de Keijzer and Branko Grünbaum for helpful discussions.

\section*{References}


