

Integer realizations of disk and segment graphs

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Abstract

A disk graph is the intersection graph of disks in the plane, a unit disk graph is the intersection graph of same radius disks in the plane, and a segment graph is an intersection graph of line segments in the plane. Every disk graph can be realized by disks with centers on the integer grid and with integer radii; and similarly every unit disk graph can be realized by disks with centers on the integer grid and equal (integer) radius; and every segment graph can be realized by segments whose endpoints lie on the integer grid. Here we show that there exist disk graphs on n vertices such that in every realization by integer disks at least one coordinate or radius is $2^{2^{\Omega(n)}}$ and on the other hand every disk graph can be realized by disks with integer coordinates and radii that are at most $2^{2^{O(n)}}$; and we show the analogous results for unit disk graphs and segment graphs. For (unit) disk graphs this answers a question of Spinrad, and for segment graphs this improves over a previous result by Kratochvíl and Matoušek.

Keywords: disk graphs, segment graphs.

1 Introduction and statement of results

In this paper we will consider intersection graphs of disks and segments in the plane. Over the past 20 years or so, intersection graphs of geometric objects in the plane, especially unit disk graphs (intersection graphs of equal radius disks), have been considered by many different authors. Partly because of their relevance for practical applications (for example in cellular communication systems) disk graphs have been the subject of a sustained research effort by many different authors. Aspects of these graphs that have been studied include the algorithmic decision problem of recognizing (unit) disk graphs [4, 19], computing or approximating the chromatic number [7, 16, 33], computing the clique number [7, 38], computing or approximating the independence number [7, 31, 33, 36] and finding or approximating a smallest dominating set [7, 10].

It is well known and easy to see that every disk graph can be realized by disks with centers on the integer grid \mathbb{Z}^2 and with integer radii; and similarly every unit disk graph can be realized by disks with centers on the integer grid and equal (integer) radius. (To prove this folklore fact we may start with a representation by open disks, shrink radii slightly so that disjoint disks have disjoint closures, perturb so that each centre and radius is rational, then rescale all coordinates and radii by the product of the denominators so that we have integers.) Spinrad [44] (Section 3.3.3, see also Chapter 4) asked whether every (unit) disk graph has such a representation where all the integers involved are at most $2^{O(n^K)}$ for some fixed K . This question was also studied by Van Leeuwen and Van Leeuwen [29] who called it the *Polynomial Representation Hypothesis* (PRH). They showed that a positive answer is equivalent to a statement about spacings between the points of a realization and they discussed some of its implications. (See also Chapter 4 of E.J. van Leeuwen's doctoral thesis [28].) Theorems 1.1 and 1.2 below show that the PRH in fact

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fails as there are (unit) disk graphs that require integers that are doubly exponentially large in the number of vertices n . Theorem 1.3 establishes the analogous results for segment graphs if we place all the endpoints of the segments on \mathbb{Z}^2 . This improves over earlier work of Kratochvíl and Matoušek [27] who had showed that integers that are doubly exponentially large in the *square root* of the number of vertices may be needed for segment graphs.

Breu and Kirkpatrick [4] proved that the algorithmic decision problem of recognizing unit disk graphs (i.e. given a graph in adjacency matrix form as input, decide whether it is a unit disk graph) is NP-hard; and Hliněný and Kratochvíl [19] proved that recognizing disk graphs is NP-hard. Had it been true, the PRH would have proved that these problems are also members of the complexity class NP. For a decision problem to be in NP we need a “polynomial certificate”; that is, for each graph that is a (unit) disk graph there should be a proof of this fact in the form of a polynomial size bit string that can be checked by an algorithm in polynomial time. An obvious candidate for such a certificate is a list of the radii and the coordinates of the centres of the disks representing the graph. For this to be a good certificate, we would however need to guarantee that these coordinates and radii can be stored using polynomially many bits (which would be the case if they were all integers bounded by $2^{O(n^K)}$).

Another reason to be interested in the number of bits needed to store the coordinates and radii is that many algorithms proposed in the literature (such as those in [7, 10, 33]) assume the input contains the radii and the coordinates of the centres of the disks (rather than just a list of edges or an adjacency matrix). This allows the algorithms to directly exploit geometric information. A bound on the number of bits needed to store these radii and coordinates is thus relevant to the practical feasibility of these algorithms.

Yet another reason that one might be interested in the number of bits needed to store coordinates and radii, is that one might hope to prove a special case of the *implicit graph conjecture* of Kannan, Naor and Rudich [22]. This conjecture states that, for every hereditary graph class \mathcal{C} (i.e. \mathcal{C} is closed under taking induced subgraphs) such that the number of graphs on n vertices in \mathcal{C} is $2^{O(n \log n)}$, there exists a “labelling scheme” that encodes graphs in the class by assigning bit strings of length $O(\log n)$ to the vertices in such a way that the adjacency of two vertices u, v can be tested by examining only the labels of u and v . Using Warren’s theorem [49] it is easily shown that the class of (unit) disk graphs has no more than $2^{O(n \log n)}$ members with n vertices (see Chapter 4 of [44]). A natural candidate for a labelling scheme would of course be to use as labels the coordinates and radii of a set of disks representing the graph. For this to give us a proof of the implicit graph conjecture for the special case of (unit) disk graphs, we would however need to guarantee that each coordinate and radius can be stored using only logarithmically many bits.

Our results show that exponentially many bits may be needed to store the coordinates and radii of a geometric representation when we require it to be an integer valued representation. We will also see that if we use rational coordinates and radii rather than integers then we still need an exponential number of bits.

Our results fit into a line of research on the smallest piece of the integer grid needed for various kinds of drawings of graphs. This line of research goes back to a seminal work by De Fraysseix et al. [9] and independently Schnyder [41], who considered (crossing-free) straight-line drawings of planar graphs on the integer grid. Not much later Bienstock [1] considered the smallest piece of the grid needed for straight-line drawings of a general graph where the drawing is restricted to have as few crossings as possible. There are also results of a similar nature on intersection graphs of same-size squares [8], intersection graphs of (general) convex polygons [35, 46], intersection graphs of continuous curves [26] and intersection graphs of general convex sets [37, 40]. We mention further related work in Section 8.

We will now proceed to give the necessary definitions to state our results more formally.

1.1 Statement of results

If $\mathcal{A} = (A(v) : v \in V)$ is a tuple of sets, then the *intersection graph* of \mathcal{A} is the graph $G = (V, E)$ with vertex set V , and an edge $uv \in E$ if and only if $A(v) \cap A(u) \neq \emptyset$. We say that \mathcal{A} *realizes* G . If the sets $A(v)$ are closed line segments in the plane then we speak of a *segment graph*. A

disk graph is an intersection graph of open disks in the plane, and if all the disks can be taken to have the same radius, then we speak of a *unit disk graph*. We have taken closed segments, resp. open disks merely out of convention: observe that if we had defined a segment graph as the intersection graph of open segments in the plane we would have defined exactly the same class of graphs – and similarly for (unit) disk graphs. Let us denote by DG the set of all graphs that are (isomorphic to) disk graphs; and similarly let UDG resp. SEG denote the set of all graphs that are (isomorphic to) unit disk graphs resp. segment graphs. Let us also set $\text{DG}(n) := \{G \in \text{DG} : |V(G)| = n\}$, $\text{UDG}(n) := \{G \in \text{UDG} : |V(G)| = n\}$, $\text{SEG}(n) := \{G \in \text{SEG} : |V(G)| = n\}$.

If $\mathcal{A} = (A(v) : v \in V(G))$ is a realization of $G \in \text{DG}$ and $A(v)$ is a disk for every $v \in V(G)$ then we also say that \mathcal{A} is a *DG-realization* of G or a realization of G as a disk graph. Similarly if the $A(v)$ are disks, all of the same radius, then we say \mathcal{A} is a *UDG-realization* of G ; and if all the $A(v)$ are segments then we say that \mathcal{A} is a *SEG-realization* of G .

As we noted earlier, every $G \in \text{DG}$ has a DG-realization in which the centers of the disks $A(v)$ lie on \mathbb{Z}^2 and the radii are integers. We shall call such a realization an *integer DG-realization* or just an integer realization if no confusion can arise. Similarly in an *integer UDG-realization* of $G \in \text{UDG}$ all the disks $A(v)$ have centers $\in \mathbb{Z}^2$ and a common radius $\in \mathbb{N}$; and in an *integer SEG-realization* of $G \in \text{SEG}$ both endpoints of each segment $A(v)$ lie on \mathbb{Z}^2 .

If \mathcal{A} is a collection of bounded sets in the plane, then we will denote:

$$k(\mathcal{A}) := \min\{k \in \mathbb{N} : A \subseteq [-k, k]^2 \text{ for all } A \in \mathcal{A}\}.$$

For $G \in \text{DG}$ we will denote

$$f_{\text{DG}}(G) = \min_{\substack{\mathcal{A} \text{ integer DG-realization} \\ \text{of } G}} k(\mathcal{A}),$$

and let us define $f_{\text{UDG}}(G)$ for $G \in \text{UDG}$ and $f_{\text{SEG}}(G)$ for $G \in \text{SEG}$ analogously. We now define

$$f_{\text{DG}}(n) := \max_{G \in \text{DG}(n)} f_{\text{DG}}(G),$$

and we define $f_{\text{UDG}}(n)$ and $f_{\text{SEG}}(n)$ analogously. Phrased differently, $f_{\text{DG}}(n)$ tells us the smallest square piece of the integer grid on which we can realize any disk graph on n vertices, and similarly for f_{UDG} and f_{SEG} . Let us observe that, by definition, every $G \in \text{SEG}(n)$ has an integer realization where all coordinates of all endpoints are at most $f_{\text{SEG}}(n)$ in absolute value, and there is at least one $G \in \text{SEG}(n)$ such that in every integer realization there is some coordinate of some endpoint that is at least $f_{\text{SEG}}(n)$ in absolute value. Similarly every $G \in \text{DG}(n)$ (resp. $G \in \text{UDG}(n)$) has an integer realization where all radii and all coordinates of the centers of the disks are at most $f_{\text{DG}}(n)$ (resp. $f_{\text{UDG}}(n)$) in absolute value, and there is at least one $G \in \text{DG}(n)$ (resp. $G \in \text{UDG}(n)$) such that in every integer realization there is some disk such that either one of the coordinates of its center or its radius is at least $f_{\text{DG}}(n)/2$ (resp. $f_{\text{UDG}}(n)/2$) in absolute value.

Both Spinrad [44] (Section 3.3.3, see also Chapter 4) and Van Leeuwen and Van Leeuwen [29] (see also chapter 4 of [28]) asked whether $f_{\text{DG}}(n)$ and $f_{\text{UDG}}(n)$ are bounded by $2^{O(n^K)}$ for some constant K . This was called the *polynomial representation hypothesis* (for disk/unit disk graphs) by Van Leeuwen and Van Leeuwen. Here we will disprove the polynomial representation hypothesis:

Theorem 1.1 $f_{\text{DG}}(n) = 2^{2^{\Theta(n)}}$.

Theorem 1.2 $f_{\text{UDG}}(n) = 2^{2^{\Theta(n)}}$.

Theorem 1.2 also improves over the conference version [34] of this work, where we proved a lower bound of $f_{\text{UDG}}(n) = 2^{2^{\Omega(\sqrt{n})}}$.

As it happens, $f_{\text{SEG}}(n)$ has been previously studied by Kratochvíl and Matoušek [27], who gave a lower bound of $f_{\text{SEG}}(n) \geq 2^{2^{\Omega(\sqrt{n})}}$. Here we will improve over their lower bound, and give a matching upper bound:

Theorem 1.3 $f_{\text{SEG}}(n) = 2^{2^{\Theta(n)}}$.

A standard convention (see [42]) is to store rational numbers in the memory of a computer as a pair of integers (the denominator and numerator) that are relatively prime and those integers are stored in the binary number format. The bit size of an integer $a \in \mathbb{Z}$ is

$$\text{bitsize}(a) := 1 + \lceil \log_2(|a| + 1) \rceil,$$

(the extra one is for the sign) and the bit size of a rational number $q \in \mathbb{Q}$ is $\text{bitsize}(q) = \text{bitsize}(a) + \text{bitsize}(b)$ if $q = \frac{a}{b}$ and $a, b \in \mathbb{Z}$ are relatively prime. For $G \in \text{DG}$, let $g_{\text{DG}}(G)$ denote the minimum, over all realizations by disks with centers in \mathbb{Q}^2 and rational radii, of the sum of the bit sizes of the coordinates of the centers and the radii; and let $g_{\text{DG}}(n)$ denote the maximum of $g_{\text{DG}}(G)$ over all $G \in \text{DG}(n)$. Let us define $g_{\text{UDG}}(n)$ and $g_{\text{SEG}}(n)$ analogously. By definition of $f_{\text{DG}}(n)$, every $G \in \text{DG}(n)$ has a rational realization where each of the $3n$ coordinates and radii has numerator at most $f_{\text{DG}}(n)$ in absolute value and denominator equal to 1. Hence

$$g_{\text{DG}}(n) \leq 3n \cdot \text{bitsize}(f_{\text{DG}}(n)) + 6n. \quad (1)$$

(Here the term $6n$ comes from storing the numerator for each of the $3n$ rational numbers – i.e. storing the number one $3n$ times.) If we multiply all the coordinates and radii of a rational realization by the product of their denominators we get an integer realization. In the resulting integer realization each coordinate or radius will have a bit size that does not exceed the sum of the bit sizes of the coordinates and radii of the original rational realization. (Observe that for two integers n, m we have $\text{bitsize}(nm) \leq \text{bitsize}(n) + \text{bitsize}(m)$.) Thus:

$$\text{bitsize}(\lceil f_{\text{DG}}(n)/2 \rceil) \leq g_{\text{DG}}(n), \quad (2)$$

From the definition of $\text{bitsize}(\cdot)$ and Theorem 1.1 we have $\text{bitsize}(f_{\text{DG}}(n)) = 2^{\Theta(n)}$. Observe also that $\text{bitsize}(\lceil a/2 \rceil) \geq \text{bitsize}(a) - 1$ for all $a \in \mathbb{Z}$. So in particular $\text{bitsize}(\lceil f_{\text{DG}}(n)/2 \rceil) = 2^{\Theta(n)}$ as well. Combining this with (1) and (2) we find:

Corollary 1.4 $g_{\text{DG}}(n) = 2^{\Theta(n)}$. ■

Similarly we have

Corollary 1.5 $g_{\text{UDG}}(n) = 2^{\Theta(n)}$. ■

Corollary 1.6 $g_{\text{SEG}}(n) = 2^{\Theta(n)}$. ■

Before beginning our proofs in earnest we will give a brief overview of the structure of the paper and the main ideas in the proofs.

1.2 Overview of the paper and sketch of the main ideas in the proofs

The proofs of the doubly exponential upper bounds in Theorems 1.1, 1.2 and 1.3 can be found in Section 7. The upper bound of Theorem 1.3 is a direct consequence of a result of Goodman, Pollack and Sturmfels [15], and the upper bounds in Theorems 1.1 and 1.2 are both relatively straightforward consequences of a result of Grigor'ev and Vorobjov [17].

The proofs of the lower bounds are substantially more involved. The main ingredient is Theorem 3.2. This theorem gives a construction of an oriented line arrangement \mathcal{L} and a set of points \mathcal{P} with the following property. Whenever some set of points $\tilde{\mathcal{P}}$ have the same sign vectors with respect to an oriented line arrangement $\tilde{\mathcal{L}}$ as the points \mathcal{P} have with respect to \mathcal{L} then there are two pairs of intersection points of the lines $\tilde{\mathcal{L}}$ such that the distance between one pair is a factor $2^{2^{\Omega(|\tilde{\mathcal{L}}|)}}$ larger than the distance between the other pair.

An important property of the construction is that the number of points $|\mathcal{P}|$ is linear in the number of lines $|\mathcal{L}|$. This ultimately allows us to give lower bounds that are doubly exponential in the number of vertices n rather than the *square root* \sqrt{n} . This allows us to get sharp lower

bounds, and improve over our own result for unit disk graphs in the conference version [34] of this paper, and the lower bound of Kratochvíl and Matoušek [27] for segment graphs. The reason why in these previous works \sqrt{n} comes up (naturally) is because the combinatorial description of an oriented line arrangement of n lines has $\Theta(n^2)$ distinct sign vectors (see for instance chapter 6 of Matoušek’s monograph [32]). Kratochvíl and Matoušek [27] used unoriented line arrangements in their proof and a notion of the combinatorial description of a line arrangement that does not use sign vectors, but the reason why they ended up with the \sqrt{n} is essentially the same. Theorem 3.2 makes it possible to ensure that a line arrangement has the desired (for our proofs) property by specifying only $O(n)$ sign vectors rather than all $\Theta(n^2)$.

A brief sketch of the construction of \mathcal{L} and \mathcal{P} is as follows. We start with a constructible point configuration \mathcal{Q} such that, in every point configuration $\tilde{\mathcal{Q}}$ that is projectively equivalent to it, there are four points with a large cross ratio. (We construct such a constructible point configuration using Von Staudt sequences, a classical way to encode arithmetic operations into point configurations.) We now construct \mathcal{L}, \mathcal{P} step-by-step by adding four lines and eleven points for each $q \in \mathcal{Q}$. These four lines and eleven points per point of \mathcal{Q} are placed in such a way (inspired by constructions of Jaggi et al. [20] and Shor [43]) that in any $\tilde{\mathcal{L}}$ for which some point set $\tilde{\mathcal{P}}$ has the same sign vectors with respect to $\tilde{\mathcal{L}}$ as \mathcal{P} has with respect to \mathcal{L} we can construct a point configuration $\tilde{\mathcal{Q}}$ that is projectively equivalent to \mathcal{Q} and whose points lie in prescribed cells of $\tilde{\mathcal{L}}$.

With our main tool at hand we can construct unit disk graphs, disk graphs, and segment graphs that need large parts of the integer grid to be realized. For unit disk graphs the idea is to take the oriented line arrangement from Theorem 3.2 and construct a unit disk graph with a pair of vertices for each line of \mathcal{L} , one corresponding to each halfplane defined by the line, and one vertex for each point of \mathcal{P} . The constructed unit disk graph is such that, in every realization of it, we can construct a line arrangement $\tilde{\mathcal{L}}$ and a set of points $\tilde{\mathcal{P}}$ out of the coordinates of the centers of the disks such that the sign vectors of $\tilde{\mathcal{P}}$ with respect to $\tilde{\mathcal{L}}$ are the same as the sign vectors of \mathcal{P} with respect to \mathcal{L} . It then follows after some calculations that at least one coordinate or radius is doubly exponentially large.

The construction for disk graphs is very similar, except that now we are forced to place an induced copy of a certain graph H for each point in \mathcal{P} rather than a single vertex. The graph H has the property that for every realization \mathcal{A} of it there is a point $p(\mathcal{A})$ such that any disk that intersects all disks of \mathcal{A} contains $p(\mathcal{A})$. The construction of H is pretty involved. It relies heavily on the fact (Lemma 5.2 below) that in every realization of a triangle-free disk graph G with minimum degree at least two, the centers of the disks define a crossing free straight-line drawing of G .

For segment graphs the construction of such a graph that needs a large portion of the integer grid makes use of a convenient result of Kratochvíl and Matoušek [27], the “order forcing lemma”. This time we use one vertex per line of \mathcal{L} , and two per point of \mathcal{P} , and a constant number of vertices that are needed to set the construction up in such a way that we can apply the order forcing lemma.

In Section 2 we do some preliminary work needed for the proof of Theorem 3.2 in Section 3. The material in Section 2 is classical and may be skimmed by readers familiar with constructible point configurations and Von Staudt sequences. In Section 3 we prove Theorem 3.2, our main tool. Section 4 contains the lower bound for unit disk graphs, Section 5 contains the lower bound for disk graphs and Section 6 has the lower bound for segment graphs. As mentioned earlier, upper bounds are proved in Section 7.

2 Constructible point configurations

Although we are mainly interested in intersection graphs of objects in the (ordinary) euclidean plane it is convenient to do some preliminary work in the projective setting. Recall that the *real projective plane* $\mathbb{R}P^2$ has as its points the one-dimensional linear subspaces of \mathbb{R}^3 , and as its lines the two-dimensional linear subspaces of \mathbb{R}^3 . We denote a point of $\mathbb{R}P^2$ in homogeneous coordinates

as $(x : y : z) := \{(\lambda x, \lambda y, \lambda z)^T : \lambda \in \mathbb{R}\} -$ where $(x, y, z) \neq (0, 0, 0)$. We say that $v \in \mathbb{R}^3 \setminus \{0\}$ is a *representative* of $p = (x : y : z) \in \mathbb{R}P^2$ if $v \in p$. The euclidean plane \mathbb{R}^2 is contained in the projective plane via the *canonical embedding* $(x, y)^T \mapsto (x : y : 1)$. The points of $\mathbb{R}P^2$ that do not lie on \mathbb{R}^2 are all points of the form $(x : y : 0)$, and they form a line of $\mathbb{R}P^2$ (they correspond to the plane $\{z = 0\}$ in \mathbb{R}^3), called the *line at infinity*. A convenient property of the projective plane is that *every two lines meet in a point*. If two lines are parallel in the euclidean plane, then they have an intersection point on the line at infinity in the projective plane. A *projective transformation* is the action that a non-singular linear map $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ induces on $\mathbb{R}P^2$. (Observe that it sends the points of $\mathbb{R}P^2$ to points of $\mathbb{R}P^2$ and the lines of $\mathbb{R}P^2$ to lines of $\mathbb{R}P^2$.) Recall that an *isometry* of the euclidean plane is a map that preserves distance, and that an isometry can always be written as a translation followed by the action of an orthogonal linear map. We omit the straightforward proof of the following observation.

Lemma 2.1 *If $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an isometry then there exists a projective transformation $T : \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ such that the restriction of T to \mathbb{R}^2 coincides with f .* ■

For vectors $u, v, w \in \mathbb{R}^3$ we will write

$$[u, v, w] := \det \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}.$$

If $a, b, c, d \in \mathbb{R}P^2$ are four collinear points, and $p \in \mathbb{R}P^2$ is a point not on the line spanned by them, then the *cross ratio* can be defined as:

$$\text{cross}(a, b, c, d) := \frac{[p, a, c][p, b, d]}{[p, a, d][p, b, c]}. \quad (3)$$

Here we take arbitrary representatives of a, b, c, d, p in the right-hand side. (That is, if $a = (a_x : a_y : a_z)$ then we may take $(\lambda a_x, \lambda a_y, \lambda a_z)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$, etc.) To see that (3) is a valid definition, recall that the determinant is linear in each of its rows (meaning that $[u_1 + u_2, v, w] = [u_1, v, w] + [u_2, v, w]$, $[\lambda u, v, w] = \lambda[u, v, w]$ etc.). Thus, if instead of (a_x, a_y, a_z) we take $(\lambda a_x, \lambda a_y, \lambda a_z)$ in (3) then we just get a factor of λ in both the denominator and the numerator. Similarly for b, c, d . Let ℓ denote the line that a, b, c, d are on, and let $H \subseteq \mathbb{R}^3$ denote the corresponding two-dimensional linear subspace. To see that the choice of p does not matter, let us fix a $u \in \mathbb{R}^3 \setminus \{0\}$ that is orthogonal to H .

Pick $p \in \mathbb{R}P^2 \setminus \ell$, and let z be an arbitrary representative of p . So in other words $z \in \mathbb{R}^3 \setminus H$. We can write $z = u_1 + \lambda u$ with $u_1 \in H$. Then we get $[z, a, c] = [u_1, a, c] + \lambda[u, a, c] = \lambda[u, a, c]$, and similarly $[z, a, d] = \lambda[u, a, d]$, $[z, b, c] = \lambda[u, b, c]$, $[z, b, d] = \lambda[u, b, d]$. We see that

$$\frac{[z, a, c][z, b, d]}{[z, a, d][z, b, c]} = \frac{[u, a, c][u, b, d]}{[u, a, d][u, b, c]},$$

no matter which $p \in \mathbb{R}P^2 \setminus \ell$ resp. $z \in \mathbb{R}^3 \setminus H$ we start from. So the definition of $\text{cross}(a, b, c, d)$ in (3) does indeed not depend on the choice of representatives for a, b, c, d or the choice of $p \in \mathbb{R}P^2 \setminus \ell$.

Next, let us also remark that, since $\det(AB) = \det(A)\det(B)$ for square matrices A, B , we have $[Tu, Tv, Tw] = \det(T)[u, v, w]$ for any projective transformation T . This implies that the cross ratio is preserved under projective transformations:

Lemma 2.2 *If T is a projective transformation and $a, b, c, d \in \mathbb{R}P^2$ are collinear, then $\text{cross}(a, b, c, d) = \text{cross}(Ta, Tb, Tc, Td)$.* ■

In the projective plane there is no obvious way to define a notion of distance, but for collinear points in the euclidean plane the cross ratio can be related to euclidean distances:

Lemma 2.3 *If $a, b, c, d \in \mathbb{R}^2$ are collinear, then*

$$|\text{cross}(a, b, c, d)| = \frac{\|a - c\| \cdot \|b - d\|}{\|a - d\| \cdot \|b - c\|}.$$

Proof: If we apply an isometry that maps the line that contains a, b, c, d to the x -axis, then the cross-ratio does not change by Lemma 2.1 and Lemma 2.2. Hence we can assume $a = (\alpha, 0)^T, b = (\beta, 0)^T, c = (\gamma, 0)^T, d = (\delta, 0)^T$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Or, in projective terms we have $a = (\alpha : 0 : 1), b = (\beta : 0 : 1), c = (\gamma : 0 : 1), d = (\delta : 0 : 1)$. Taking $p = (1 : 1 : 1)$ we see that

$$[p, a, c] = \det \begin{pmatrix} 1 & 1 & 1 \\ \alpha & 0 & 1 \\ \gamma & 0 & 1 \end{pmatrix} = \gamma - \alpha = \pm \|a - c\|, \quad (4)$$

and similarly $[p, a, d] = \pm \|a - d\|, [p, b, c] = \pm \|b - c\|, [p, b, d] = \pm \|b - d\|$. This proves the lemma. \blacksquare

A *point configuration* is a tuple $\mathcal{P} = (p_1, \dots, p_n)$ of (labelled) points in the projective plane. If all the points lie in the euclidean plane \mathbb{R}^2 then we speak of a *euclidean point configuration*. For two distinct points p, q we shall denote by $\ell(p, q)$ the unique line through p and q . Recall that we say a configuration of points in the (euclidean or projective) plane is *in general position* if no three of them are collinear. We call a point configuration *constructible* if

(CPC-1) p_1, p_2, p_3, p_4 are in general position;

(CPC-2) For each $i \geq 5$ there are $j_1, j_2, j_3, j_4 < i$ such that $\{p_i\} = \ell(p_{j_1}, p_{j_2}) \cap \ell(p_{j_3}, p_{j_4})$.

We will say that two point configurations $\mathcal{P} = (p_1, \dots, p_n), \tilde{\mathcal{P}} = (\tilde{p}_1, \dots, \tilde{p}_n)$ are *projectively equivalent* if there exists a projective transformation T such that $\tilde{p}_i = Tp_i$ for all $i = 1, \dots, n$. Observe that if T is a projective transformation, then $T[\ell(p, q)] = \ell(Tp, Tq)$. Thus:

Lemma 2.4 *If \mathcal{P} is a constructible point configuration and $\tilde{\mathcal{P}}$ is projectively equivalent to \mathcal{P} then $\tilde{\mathcal{P}}$ is also constructible.* \blacksquare

The following observation will be needed in the next section.

Lemma 2.5 *If $\mathcal{P} = (p_1, p_2, p_3, p_4)$ and $\tilde{\mathcal{P}} = (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4)$ are two point configurations on 4 points in general position, then \mathcal{P} and $\tilde{\mathcal{P}}$ are projectively equivalent.*

Proof: Since the inverse of a projective transformation is again a projective transformation and the composition of two projective transformations is a projective transformation, it suffices to prove the result for $p_1 = (1 : 0 : 0), p_2 = (0 : 1 : 0), p_3 = (0 : 0 : 1), p_4 = (1 : 1 : 1)$ and $\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4$ arbitrary (but in general position). Let us pick representatives $v_i \in \mathbb{R}^3 \setminus \{0\}$ of \tilde{p}_i for $i = 1, \dots, 4$. Then no v_i is a linear combination of only two of the other v s. Hence there are nonzero λ_i s such that:

$$v_4 = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3.$$

Let us now define a linear map $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by setting $e_i \mapsto \lambda_i v_i$ for $i = 1, \dots, 3$. Then it is easy to see that, when viewed as a projective transformation, T in fact maps p_i to \tilde{p}_i for $i = 1, \dots, 4$. \blacksquare

The so-called *Von Staudt sequences*, originally invented by Von Staudt [47] in 1847, allow us to encode arithmetic operations in terms of constructible point configurations. Let us write

$$\begin{aligned} P_0 &:= (0 : 0 : 1), & P_\infty &:= (1 : 0 : 0), \\ Q &:= (0 : 1 : 0), & R &:= (1 : 1 : 1). \end{aligned} \quad (5)$$

And, for $a \in \mathbb{R}$ let us set

$$P_a := (a : 0 : 1). \quad (6)$$

(Observe that this definition is consistent with the earlier definition of P_0 .) The idea of the Von Staudt sequences is the following. Given a point configuration that contains P_0, P_∞, R, Q we can “construct” the point P_1 as an intersection point (in the sense of **(CPC-2)** above). If the point configuration also contains P_a, P_b for some $a, b \in \mathbb{R}$ then we can “construct” the point P_{a+b} resp. $P_{a \cdot b}$ by defining a number of intersection points (in the sense of **(CPC-2)**) the last of which will be P_{a+b} resp. $P_{a \cdot b}$. There are also Von Staudt sequences for subtraction and division, but we shall not need them here.

The Von Staudt sequence for one is:

$$\{O_1\} := \ell(R, Q) \cap \ell(P_0, P_\infty). \quad (7)$$

See Figure 1, left. The Von Staudt sequence for addition is to set (in this order):

$$\begin{aligned} \{A_1\} &= \ell(R, P_\infty) \cap \ell(P_a, Q), \\ \{A_2\} &= \ell(P_0, A_1) \cap \ell(P_\infty, Q), \\ \{A_3\} &= \ell(P_b, A_2) \cap \ell(R, P_\infty), \\ \{A_4\} &= \ell(A_3, Q) \cap \ell(P_0, P_\infty). \end{aligned} \quad (8)$$

See Figure 1, middle. The Von Staudt sequence for multiplication is to set (in this order):

$$\begin{aligned} \{M_1\} &= \ell(P_0, R) \cap \ell(P_b, Q), \\ \{M_2\} &= \ell(P_a, R) \cap \ell(P_\infty, Q), \\ \{M_3\} &= \ell(M_1, M_2) \cap \ell(P_0, P_\infty). \end{aligned} \quad (9)$$

See Figure 1, right.

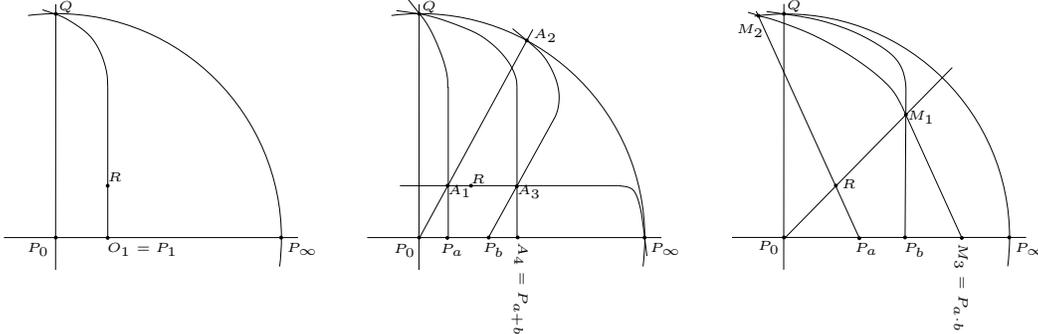


Figure 1: The Von Staudt constructions for one (left), addition (middle) and multiplication (right). The line at infinity (the line through P_∞ and Q) is drawn as a circular arc.

Lemma 2.6 *Let P_0, P_∞, Q, R be as defined in (5) and for arbitrary $a, b \in \mathbb{R}$ let P_a, P_b be as defined by (6). Then the following hold.*

- (i) *Let O_1 be as defined by (7): then $O_1 = P_1$;*
- (ii) *Let A_1, \dots, A_4 be as defined by (8): then $A_4 = P_{a+b}$;*
- (iii) *Let M_1, M_2, M_3 be as defined by (9): then $M_3 = P_{a \cdot b}$.*

Proof: It is convenient to consider what the Von Staudt sequences look like in the euclidean plane if we embed it in the projective plane via the canonical embedding $(x, y)^T \mapsto (x : y : 1)$. The point P_a then corresponds to $(a, 0)^T$ for all $a \in \mathbb{R}$. Also observe that the x -axis corresponds to $\ell(P_0, P_\infty)$, and the y -axis to $\ell(P_0, Q)$. Two lines are parallel in the euclidean plane precisely if

in the projective plane they intersect in a point on the line at infinity $\ell(P_\infty, Q)$. Thus horizontal lines in the euclidean plane are precisely the lines which intersect $\ell(P_\infty, Q)$ in the point P_∞ , and vertical lines are precisely the lines that intersect $\ell(P_\infty, Q)$ in Q .

Proof of (i): In euclidean terms R is the point $(1, 1)^T$ and $\ell(R, Q)$ is the vertical line through R , and $\ell(P_0, P_\infty)$ is the x -axis. (See Figure 1, left.) Hence O_1 , the intersection point of $\ell(R, Q)$ and $\ell(P_0, P_\infty)$, must correspond to $(1, 0)^T$.

Proof of (ii): The points A_1, A_3 lie on the horizontal line $\ell(R, P_\infty)$, the lines $\ell(P_a, A_1), \ell(A_4, A_3)$ are vertical and the lines $\ell(P_0, A_1)$ and $\ell(P_b, A_3)$ are parallel. Hence the triangle $P_b A_3 A_4$ is a translate of the triangle $P_0 A_1 P_a$. (See Figure 1, middle.) In particular the segments $[P_0, P_a]$ and $[P_b, A_4]$ have the same length, and so we must indeed have that A_4 coincides with the point $(a + b, 0)^T$.

Proof of (iii): The line $\ell(P_0, R)$ coincides with the line $y = x$ in the euclidean plane. Since the line $\ell(M_1, P_b)$ is vertical, the point M_1 corresponds to $(b, b)^T$. Since the lines $\ell(P_a, R)$ and $\ell(M_3, M_1)$ are parallel, the triangles $P_0 R P_a$ and $P_0 M_1 M_3$ are similar. The height of $P_0 R P_a$ is 1, and its base is a . Since the height of $P_0 M_1 M_3$ is b , its base must be ab . Hence M_3 coincides with $(ab, 0)^T$ as required. ■

Lemma 2.7 *Let $P_0, P_\infty, Q, R \in \mathbb{R}P^2$ be as defined by (5), and let $P_1, P_x \in \mathbb{R}P^2$ be as defined by (6). Then $\text{cross}(P_1, P_x, P_\infty, P_0) = x$.*

Proof: Observe that for all $a, b \in \mathbb{R}$:

$$\det \begin{pmatrix} 1 & 1 & 1 \\ a & 0 & 1 \\ b & 0 & 1 \end{pmatrix} = b - a, \quad \det \begin{pmatrix} 1 & 1 & 1 \\ a & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = 1.$$

It is easily seen that R does not lie on the line $\ell(P_0, P_\infty)$. Hence, by definition (3) of the cross ratio:

$$\text{cross}(P_1, P_x, P_\infty, P_0) = \frac{[R, P_1, P_\infty] \cdot [R, P_x, P_0]}{[R, P_1, P_0] \cdot [R, P_x, P_\infty]} = x,$$

for all $x \in \mathbb{R}$, as required. ■

We are now in a position to give a quick proof of the following lemma, which will play an important role in the next section. The lemma is already proved implicitly in the beginning of the proof of Theorem 1 in the seminal work of Goodman, Pollack and Sturmfels [15] (see also [14]). A similar construction was also invented independently by Kratochvíl and Matoušek in [25], the technical report version of [27].

Lemma 2.8 *For every $r \in \mathbb{N}$, there exists a constructible euclidean point configuration $\mathcal{P} = (p_1, \dots, p_n)$ on $n = 3r + 6$ points such that p_1, p_2, p_5, p_n are collinear and $\text{cross}(p_5, p_n, p_2, p_1) = 2^{2^r}$.*

Proof: For any finite set $S \subseteq \mathbb{R}P^2$ we can find a projective transformation such that $T[S] \subseteq \mathbb{R}^2$ (if T is the action of a matrix with i.i.d. standard normal entries then T will do the trick with probability one, for instance), so by Lemmas 2.4 and 2.2 it suffices to define a suitable constructible point configuration in the projective plane. Our four initial points will be $(p_1, p_2, p_3, p_4) = (P_0, P_\infty, Q, R)$, and we set $p_5 = P_1$. Then we append the Von Staudt sequence for P_{1+1} , followed by the Von Staudt sequences for P_{2-2}, P_{4-4} and so on until $P_{2^{2^r}}$. This gives a constructible point configuration on $n = 4 + 1 + 4 + 3(r - 1) = 3r + 6$ points, and by Lemma 2.7 we have $\text{cross}(p_5, p_n, p_2, p_1) = \text{cross}(P_1, P_{2^{2^r}}, P_\infty, P_0) = 2^{2^r}$, as required. ■

3 Oriented line arrangements

In this section all the action takes place exclusively in the euclidean plane again. A line ℓ divides $\mathbb{R}^2 \setminus \ell$ into two pieces. In an *orientation* of ℓ we distinguish between these two pieces by (arbitrarily) calling one of them ℓ^- the “negative side” and the other ℓ^+ the “positive side”. An *oriented line arrangement* is a tuple $\mathcal{L} := (\ell_1, \dots, \ell_n)$ of distinct lines in the plane, each with an orientation.

The *sign vector* of a point $p \in \mathbb{R}^2$ wrt. an oriented line arrangement $\mathcal{L} = (\ell_1, \dots, \ell_n)$ is the vector $\sigma(p; \mathcal{L}) \in \{-, 0, +\}^n$ defined as follows:

$$(\sigma(p; \mathcal{L}))_i = \begin{cases} - & \text{if } p \in \ell_i^-, \\ 0 & \text{if } p \in \ell_i, \\ + & \text{if } p \in \ell_i^+. \end{cases}$$

If $\mathcal{P} \subseteq \mathbb{R}^2$ is a set of points then we write

$$\sigma(\mathcal{P}; \mathcal{L}) := \{\sigma(p; \mathcal{L}) : p \in \mathcal{P}\}.$$

The *combinatorial description* $\mathcal{D}(\mathcal{L})$ of \mathcal{L} is the set of all sign vectors $\mathcal{D}(\mathcal{L}) := \sigma(\mathbb{R}^2; \mathcal{L})$. The combinatorial description $\mathcal{D}(\mathcal{L})$ is almost the same thing as the covectors of an oriented matroid and in fact it determines the oriented matroid associated with \mathcal{L} (see [2] for more details). It should be mentioned that various other notions of a combinatorial description of a line arrangement are in use such as a local sequences, allowable sequences and wiring diagrams (see for instance [12]). We have chosen to work with oriented line arrangements and the notion of combinatorial description in terms of sign vectors, because this is the most convenient for the proofs that will follow.

Each connected component of $\mathbb{R}^2 \setminus (\ell_1 \cup \dots \cup \ell_n)$ is called a *cell*. All points in the same cell have the same sign vector, in which zero does not occur. A sign vector with two or more zeroes corresponds to an intersection point of two or more lines, and a sign vector with exactly one zero corresponds to a line segment. (See Figure 2.)

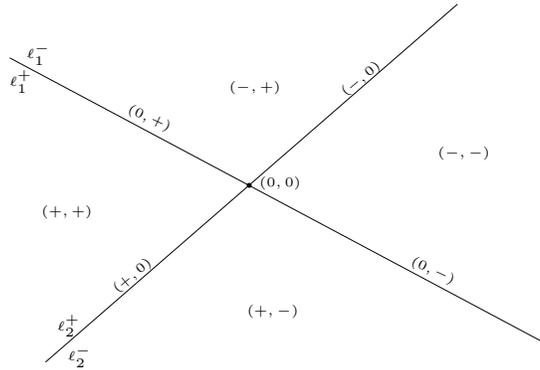


Figure 2: An oriented line arrangement and its sign vectors.

Moreover, let us observe that from the set of sign vectors $\mathcal{D}(\mathcal{L})$ alone we can determine all relevant combinatorial/topological information, such as whether a given cell is a k -gon, which cells/segments/points are incident with a given cell/segment/point, etc. If $\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{L}')$ then we say that \mathcal{L} and \mathcal{L}' are *isomorphic*. Informally speaking, isomorphic oriented line arrangements have the same “combinatorial structure”.

If every two lines of \mathcal{L} intersect, and no point is on more than two lines then we say that \mathcal{L} is *simple*. It can be seen that a simple oriented line arrangement has exactly $\binom{n+1}{2} + 1$ cells (for a proof of a generalization see for instance Proposition 6.1.1 of [32]). We will need the following standard elementary observation:

Lemma 3.1 *If \mathcal{L} is simple and \mathcal{L}' has the same number of lines, then $\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{L}')$ if and only if $\{-, +\}^n \cap \mathcal{D}(\mathcal{L}) = \{-, +\}^n \cap \mathcal{D}(\mathcal{L}')$. ■*

(This is just the observation that in a simple oriented line arrangement we can reconstruct all other sign vectors from the nonzero ones.)

For \mathcal{L} an oriented line arrangement, let $\mathcal{I}(\mathcal{L})$ denote the set of *intersection points*, that is all points $p \in \mathbb{R}^2$ that lie on more than one line. (These points are sometimes also referred to as the *vertices* of \mathcal{L} .) The *span* of an oriented line arrangement can be defined as

$$\text{span}(\mathcal{L}) := \frac{\max_{p,q \in \mathcal{I}(\mathcal{L})} \|p - q\|}{\min_{\substack{p,q \in \mathcal{I}(\mathcal{L}), \\ p \neq q}} \|p - q\|}. \quad (10)$$

(Thus $\text{span}(\mathcal{L})$ is the ratio of the furthest distance between two intersection points to the smallest distance between two intersection points.)

The main tool in the proofs of the lower bounds in Theorems 1.1, 1.2 and 1.3 will be the following result.

Theorem 3.2 *For every $k \in \mathbb{N}$, there exists a set $S \subseteq \{-, +\}^m$ with $m \leq 12k + 37$ and $|S| \leq 33k + 103$ such that*

- (i) *There exists a line arrangement \mathcal{L} with $S \subseteq \mathcal{D}(\mathcal{L})$;*
- (ii) *For every line arrangement $\tilde{\mathcal{L}}$ with $S \subseteq \mathcal{D}(\tilde{\mathcal{L}})$ we have $\text{span}(\tilde{\mathcal{L}}) \geq 2^{2^k}$.*

Proof: We start by constructing the set S and proving (i). Let $\mathcal{P} = (p_1, \dots, p_n)$ be a constructible euclidean point configuration such that $\text{cross}(p_1, p_n, p_2, p_5) = 2^{2^{k+1}}$ and $n \leq 3k + 9$. Such a point configuration exists by Lemma 2.8. We can assume without loss of generality that all the points of \mathcal{P} are distinct (if a point occurs more than once then we can drop all occurrences after the first from the sequence (p_1, \dots, p_n) and we will still have a constructible point configuration). We shall first construct an oriented line arrangement \mathcal{L} on $4n + 1$ lines, and a point configuration \mathcal{Q} of $11n + 4$ points, and set $S := \sigma(\mathcal{Q}; \mathcal{L})$ to establish (i).

Our construction is inspired by the proof of Lemma 4 in [43]. For each $i \geq 5$ let us fix a 4-tuple $f(i) = (j_1, j_2; j_3, j_4)$ such that $\{p_i\} = \ell(p_{j_1}, p_{j_2}) \cap \ell(p_{j_3}, p_{j_4})$ and $i > j_1, j_2, j_3, j_4$. (In principle there can be many different 4-tuples that define the same point p_i , but it is useful to fix a definite choice for the construction.) We will also need $0 < \varepsilon_1 < \varepsilon_2 < \dots < \varepsilon_n$, chosen sufficiently small for the construction that we are about to follow to work. (How small exactly depends on the point configuration \mathcal{P} .)

For every point $p_i \in \mathcal{P}$ there will be four oriented lines $\ell_{4i-3}, \ell_{4i-2}, \ell_{4i-1}, \ell_{4i}$ such that the oriented line arrangement $\mathcal{L}_i := (\ell_{4i-3}, \ell_{4i-2}, \ell_{4i-1}, \ell_{4i})$ is isomorphic to the one shown in Figure 3, and

$$p_i \in E_i := \ell_{4i-3}^- \cap \ell_{4i-2}^- \cap \ell_{4i-1}^- \cap \ell_{4i}^-. \quad (11)$$

So in particular E_i is a quadrilateral with opposite sides on ℓ_{4i-3} and ℓ_{4i-2} ; and the other pair of opposite sides on ℓ_{4i-1} and ℓ_{4i} . There will also be eleven points $q_{11i-10}, \dots, q_{11i}$ of \mathcal{Q} associated with p_i , one in each cell of \mathcal{L}_i . For notational convenience, let us write $\mathcal{Q}_i := (q_{11i-10}, \dots, q_{11i})$ and let \mathcal{Q}_i^E denote the set of those points of \mathcal{Q}_i that lie in cells of \mathcal{L}_i sharing at least one corner point with E_i (so \mathcal{Q}_i^E has nine elements). In the construction we shall make sure that the following demands are met:

(\mathcal{L} -1) $\mathcal{Q}_i^E \subseteq B(p_i, \varepsilon_i)$ for all $i = 1, \dots, n$;

(\mathcal{L} -2) For each $1 \leq i \neq j \leq n$ there is some $4i - 3 \leq k \leq 4i$ such that $B(p_j, \varepsilon_j) \subseteq \ell_k^+$.

(Here and in the rest of the paper $B(p, r)$ denotes the open disk of radius r with center p .) Let us now begin the construction of \mathcal{L} and \mathcal{Q} in earnest. To avoid treating definitions of $\mathcal{Q}_i, \mathcal{L}_i$ with $i = 1, \dots, 4$ as special cases, it is convenient to define points $p_0, p_{-1}, \dots, p_{-15}$ (no three collinear) such that $\{p_i\} = \ell(p_{-4i+1}, p_{-4i+2}) \cap \ell(p_{-4i+3}, p_{-4i+4})$ for each $i = 1, \dots, 4$. We then set

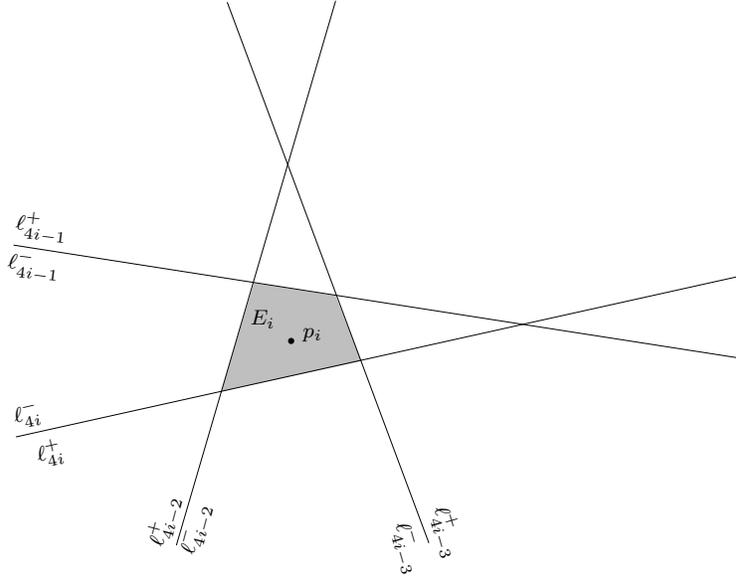


Figure 3: The oriented line arrangement \mathcal{L}_i .

$f(i) = (-4i + 1, -4i + 2; -4i + 3, -4i + 4)$ for $i = 1, \dots, 4$; and $\mathcal{Q}_j := (p_j, \dots, p_j)$ and $E_j = \{p_j\}$ for $j \leq 0$.

Suppose that, for some $i \geq 1$, we have already defined \mathcal{Q}_j and \mathcal{L}_j for all $j < i$ and that, thus far, the demands **(L-1)**-**(L-2)** are met. Let $f(i) = (j_1, j_2; j_3, j_4)$. We shall place ℓ_{4i-3}, ℓ_{4i-2} both at a very small angle to $\ell(p_{j_1}, p_{j_2})$ in such a way that:

- a) if p_i lies on the segment $[p_{j_1}, p_{j_2}]$ then ℓ_{4i-3}, ℓ_{4i-2} intersect in a point $\in \ell(p_{j_1}, p_{j_2}) \setminus [p_{j_1}, p_{j_2}]$ and $p_i, \mathcal{Q}_{j_1}^E, \mathcal{Q}_{j_2}^E$ will lie in the same cell of the line arrangement $(\ell_{4i-3}, \ell_{4i-2})$;
- b) if p_i lies outside the segment $[p_{j_1}, p_{j_2}]$ then ℓ_{4i-3}, ℓ_{4i-2} intersect in a point $\in [p_{j_1}, p_{j_2}]$ and $\mathcal{Q}_{j_1}^E, \mathcal{Q}_{j_2}^E$ will lie in opposite (i.e. not sharing a segment) cells of the line arrangement $(\ell_{4i-3}, \ell_{4i-2})$ and p_i will lie either in the same cell as $\mathcal{Q}_{j_1}^E$ or in the same cell as $\mathcal{Q}_{j_2}^E$.

(See Figure 4.)

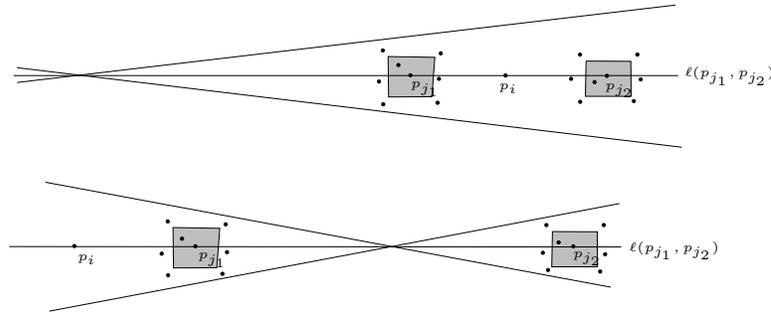


Figure 4: Placing ℓ_{4i-3}, ℓ_{4i-2} depending on whether p_i lies on the segment $[p_{j_1}, p_{j_2}]$ or not. The grey quadrangles represent E_{j_1} and E_{j_2} .

We place ℓ_{4i-1}, ℓ_{4i} in relation to $\mathcal{Q}_{j_3}^E, \mathcal{Q}_{j_4}^E$ analogously according to whether $p_i \in [p_{j_3}, p_{j_4}]$ or not (again both at a small angle to $\ell(p_{j_3}, p_{j_4})$).

Observe that, provided $\varepsilon_1, \dots, \varepsilon_{i-1}$ were small enough, we can place the lines $\ell_{4i-3}, \dots, \ell_{4i}$ such that a) and b) above hold and in addition the angles between $\ell(p_{j_1}, p_{j_2})$ and ℓ_{4i-3}, ℓ_{4i-2} and the angles between $\ell(p_{j_3}, p_{j_4})$ and ℓ_{4i-1}, ℓ_{4i} are small enough to make sure that:

- 1) $\mathcal{L}_i = (\ell_{4i-3}, \ell_{4i-2}, \ell_{4i-1}, \ell_{4i})$ is isomorphic to the oriented line arrangement in Figure 3;
- 2) p_i lies in the quadrangular cell E_i of \mathcal{L}_i ;
- 3) $\text{cl}(E_i) \subseteq B(p_i, \varepsilon_i)$.

(Here $\text{cl}(\cdot)$ denotes topological closure.) We can then orient the lines $\ell_{4i-3}, \ell_{4i-2}, \ell_{4i-1}, \ell_{4i}$ in such a way that $E_i = \ell_{4i-3}^- \cap \ell_{4i-2}^- \cap \ell_{4i-1}^- \cap \ell_{4i}^-$. We now place \mathcal{Q}_i in such a way that $\mathcal{Q}_i^E \subseteq B(p_i, \varepsilon_i)$ (recall that \mathcal{Q}_i has one point in each cell of \mathcal{L}_i and \mathcal{Q}_i^E consists of those points of \mathcal{Q}_i in cells sharing at least one corner with E_i). Because of 3) this is possible. Thus, **(L-1)** holds up to i . To see that we can also satisfy **(L-2)**, notice that for each $j \neq i$, either $p_j \notin \ell(p_{j_1}, p_{j_2})$ or $p_j \notin \ell(p_{j_3}, p_{j_4})$, because otherwise we would have $p_j = p_i$. Without loss of generality $p_j \notin \ell(p_{j_1}, p_{j_2})$. We can then also assume that ε_j was chosen such that $B(p_j, \varepsilon_j)$ misses $\ell(p_{j_1}, p_{j_2})$, and hence if we place ℓ_{4i-3}, ℓ_{4i-2} close enough to $\ell(p_{j_1}, p_{j_2})$ then either $B(p_j, \varepsilon_j) \subseteq \ell_{4i-3}^+$ or $B(p_j, \varepsilon_j) \subseteq \ell_{4i-2}^+$.

Let A_i denote the cell of \mathcal{L}_i that contains $\mathcal{Q}_{j_1}^E$; let B_i denote the cell of \mathcal{L}_i that contains $\mathcal{Q}_{j_2}^E$; let C_i denote the cell of \mathcal{L}_i that contains $\mathcal{Q}_{j_3}^E$; and let D_i denote the cell of \mathcal{L}_i that contains $\mathcal{Q}_{j_4}^E$. Observe that the situation must be one of the situations as in Figure 5, up to swapping of the labels A and B and/or swapping of the labels C and D .

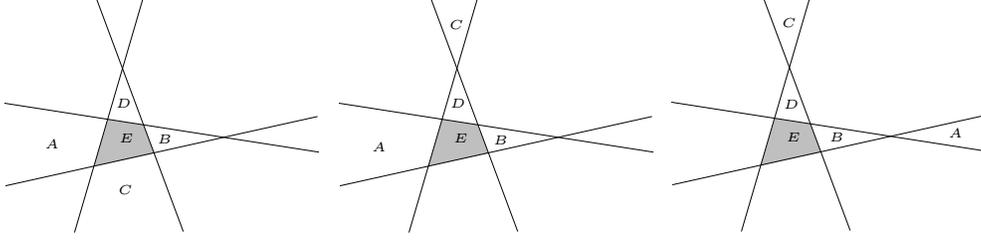


Figure 5: The different positions of $\mathcal{Q}_{j_1}^E, \dots, \mathcal{Q}_{j_4}^E$ in the cells of \mathcal{L}_i . The left figure corresponds to case a) twice, the middle figure to case a) once and case b) once, and the right figure to case b) twice.

We now set $\ell_{4n+1} := \ell(p_1, p_2)$ (oriented in an arbitrary way) and we pick

$$\begin{aligned} q_{11n+1} &\in E_1 \cap \ell_{4n+1}^-, & q_{11n+2} &\in E_1 \cap \ell_{4n+1}^+, \\ q_{11n+3} &\in E_2 \cap \ell_{4n+1}^-, & q_{11n+4} &\in E_2 \cap \ell_{4n+1}^+. \end{aligned} \quad (12)$$

To finalize the construction, let us set $\mathcal{Q} := (q_1, \dots, q_{11n+4})$ and $S := \sigma(\mathcal{Q}; \mathcal{L})$ (so trivially $S \subseteq \mathcal{D}(\mathcal{L})$).

It remains to show that S satisfies **(ii)**, which we will now show. To this end, let $\tilde{\mathcal{L}} = (\tilde{\ell}_1, \dots, \tilde{\ell}_{4n+1})$ be an oriented line arrangement with $S \subseteq \mathcal{D}(\tilde{\mathcal{L}})$. Let us fix points $\tilde{\mathcal{Q}} = (\tilde{q}_1, \dots, \tilde{q}_{11n+4})$ with $\sigma(\tilde{q}_i; \tilde{\mathcal{L}}) = \sigma(q_i; \mathcal{L})$ for all $i = 1, \dots, 11n+4$ and let $\tilde{\mathcal{Q}}_i, \tilde{\mathcal{Q}}_i^E, \tilde{\mathcal{L}}_i$ be defined in the obvious way. Observe that for each $i = 1, \dots, n$

$$\{\sigma(q; \mathcal{L}_i) : q \in \mathcal{Q}_i\} = \{\sigma(\tilde{q}; \tilde{\mathcal{L}}_i) : \tilde{q} \in \tilde{\mathcal{Q}}_i\},$$

so that, using Lemma 3.1, \mathcal{L}_i and $\tilde{\mathcal{L}}_i$ are isomorphic. In particular $\tilde{\mathcal{L}}_i$ is again isomorphic to the oriented line arrangement shown in Figure 3. Let us thus define $\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i, \tilde{E}_i$ as the cells of $\tilde{\mathcal{L}}_i$ corresponding to A_i, B_i, C_i, D_i, E_i . Observe that (see Figure 5):

$$\begin{aligned} \text{For all } a \in \tilde{A}_i, b \in \tilde{B}_i, c \in \tilde{C}_i, d \in \tilde{D}_i \text{ the lines } \ell(a, b), \ell(c, d) \\ \text{intersect in a point } e \in \tilde{E}_i. \end{aligned} \quad (13)$$

This observation shall play a key role below.

It follows from **(L-1)**-**(L-2)** that for every $i \neq j$ there is a $4i - 3 \leq k \leq 4i$ such that $\mathcal{Q}_j^E \subseteq \ell_k^+$. We must then also have $\tilde{\mathcal{Q}}_j^E \subseteq \tilde{\ell}_k^+$. By convexity, this also gives $\text{conv}(\tilde{\mathcal{Q}}_j^E) \subseteq \tilde{\ell}_k^+$. Because $\tilde{\mathcal{Q}}_j^E$ contains a point in each cell of $\tilde{\mathcal{L}}_j$ sharing at least one corner with \tilde{E}_j , we have that

$$\text{cl}(\tilde{E}_i) \subseteq \text{conv}(\tilde{\mathcal{Q}}_i^E) \text{ for all } i = 1, \dots, n, \quad (14)$$

which implies

$$\text{cl}(\tilde{E}_i) \cap \text{cl}(\tilde{E}_j) = \emptyset \text{ for all } 1 \leq i \neq j \leq n. \quad (15)$$

Also observe that, from (12) it follows that $\tilde{\ell}_{4n+1}$ intersects both \tilde{E}_1 and \tilde{E}_2 .

We will now construct a point set $\tilde{\mathcal{P}} = (\tilde{p}_1, \dots, \tilde{p}_n)$ that will turn out to be projectively equivalent to \mathcal{P} . We first pick $\tilde{p}_1 \in \tilde{\ell}_{4n+1} \cap \tilde{E}_1$ and $\tilde{p}_2 \in \tilde{\ell}_{4n+1} \cap \tilde{E}_2$, and then we pick $\tilde{p}_3 \in \tilde{E}_3, \tilde{p}_4 \in \tilde{E}_4$ in such a way that $\tilde{p}_1, \dots, \tilde{p}_4$ are in general position (this can clearly be done because \tilde{E}_3, \tilde{E}_4 are nonempty and open). Once $\tilde{p}_1, \dots, \tilde{p}_{i-1}$ have been constructed for some $i \geq 5$, we set

$$\{\tilde{p}_i\} := \ell(\tilde{p}_{j_1}, \tilde{p}_{j_2}) \cap \ell(\tilde{p}_{j_3}, \tilde{p}_{j_4}),$$

where $f(i) = (j_1, j_2; j_3, j_4)$. Since $\mathcal{Q}_{j_1}^E \subseteq A_i, \mathcal{Q}_{j_2}^E \subseteq B_i, \mathcal{Q}_{j_3}^E \subseteq C_i, \mathcal{Q}_{j_4}^E \subseteq D_i$ it follows from (14) that $\tilde{E}_{j_1} \subseteq \tilde{A}_i, \tilde{E}_{j_2} \subseteq \tilde{B}_i, \tilde{E}_{j_3} \subseteq \tilde{C}_i, \tilde{E}_{j_4} \subseteq \tilde{D}_i$. Applying the observation (13) gives that $\tilde{p}_i \in \tilde{E}_i$.

By Lemma 2.5 there is a projective transformation T that maps p_i to \tilde{p}_i for $i = 1, \dots, 4$. We now claim that in fact we must have $T(p_i) = \tilde{p}_i$ for all $i = 1, \dots, n$. To see this suppose that, for some $i \geq 5$, we have $T(p_j) = \tilde{p}_j$ for all $j < i$. Let us again write $f(i) = (j_1, j_2; j_3, j_4)$. Since projective transformations map lines to lines, we have that $T[\ell(p_{j_1}, p_{j_2})] = \ell(\tilde{p}_{j_1}, \tilde{p}_{j_2})$ and $T[\ell(p_{j_3}, p_{j_4})] = \ell(\tilde{p}_{j_3}, \tilde{p}_{j_4})$. This implies that indeed $T(p_i) = \tilde{p}_i$. The claim follows.

Using Lemma 2.2 we find

$$\text{cross}(\tilde{p}_5, \tilde{p}_n, \tilde{p}_2, \tilde{p}_1) = \text{cross}(p_5, p_n, p_2, p_1) = 2^{2^{k+1}}.$$

Thus, by Lemma 2.3, we have either $\|\tilde{p}_5 - \tilde{p}_2\| / \|\tilde{p}_5 - \tilde{p}_1\| \geq \sqrt{2^{2^{k+1}}} = 2^{2^k}$ or $\|\tilde{p}_n - \tilde{p}_2\| / \|\tilde{p}_n - \tilde{p}_1\| \geq 2^{2^k}$. Without loss of generality $\|\tilde{p}_n - \tilde{p}_2\| / \|\tilde{p}_n - \tilde{p}_1\| \geq 2^{2^k}$. Observe that $\tilde{p}_1, \tilde{p}_2, \tilde{p}_n \in \tilde{\ell}_{4n+1} = \ell(\tilde{p}_1, \tilde{p}_2)$ as $p_1, p_2, p_n \in \ell_{4n+1} = \ell(p_1, p_2)$. Let the segments I_1, I_2, I_n be defined by $I_j := \tilde{\ell}_{4n+1} \cap \tilde{E}_j$ ($j = 1, 2, n$). The distance $\|\tilde{p}_2 - \tilde{p}_n\|$ is at most the furthest distance between an endpoint of I_2 and an endpoint of I_n . Similarly, $\|\tilde{p}_1 - \tilde{p}_n\|$ is at least the shortest distance between an endpoint of I_1 and an endpoint of I_n , and this distance is positive by (15). Since the endpoints of the I_j s are intersection points of $\tilde{\ell}_{4n+1}$ with some other lines of $\tilde{\mathcal{L}}$, we see that $\text{span}(\tilde{\mathcal{L}}) \geq 2^{2^k}$, as required. \blacksquare

Another ingredient we need for the proofs of the lower bounds is the following lemma relating the span of oriented line arrangements to the numbers used to express the oriented line arrangement as linear inequalities.

Lemma 3.3 *Let $\mathcal{L} = (\ell_1, \dots, \ell_n)$ be an oriented line arrangement. Suppose that, for some $k \in \mathbb{N}$ there are nonzero $w_1, \dots, w_n \in \{-k, \dots, k\}^2$ and $c_1, \dots, c_n \in \{-k, \dots, k\}$ such that we can express the lines as:*

$$\ell_i = \{z : w_i^T z = c_i\}.$$

Then $\text{span}(\mathcal{L}) \leq 2^{9/2} \cdot k^6$.

Proof: Any point $p \in \mathcal{I}(\mathcal{L})$ is the solution to a 2×2 linear system $Az = b$. More precisely, if $\{p\} = \ell_i \cap \ell_j$ then

$$A = \begin{pmatrix} (w_i)_x & (w_i)_y \\ (w_j)_x & (w_j)_y \end{pmatrix}, \quad b = \begin{pmatrix} c_i \\ c_j \end{pmatrix},$$

(Observe that, since p must be the unique solution, A is non-singular.) From the familiar formula

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{-a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ \frac{a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \end{pmatrix},$$

we see that $p = A^{-1}b$ has coordinates $|p_x|, |p_y| \leq 2k^2$; and both coordinates are ratios $\frac{s}{t}$ of two integers with denominator $1 \leq t \leq 2k^2$. Hence we have

$$\max_{p, q \in \mathcal{I}(\mathcal{L})} \|p - q\| \leq 4k^2\sqrt{2}.$$

Similarly, because if $p \neq q$ then either $p_x \neq q_x$ or $p_y \neq q_y$, we have (also recall $\frac{s_1}{t_1} - \frac{s_2}{t_2} = \frac{s_1t_2 - s_2t_1}{t_1t_2}$):

$$\min_{\substack{p, q \in \mathcal{I}(\mathcal{L}), \\ p \neq q}} \|p - q\| \geq \frac{1}{4k^4}.$$

The lemma follows. ■

4 The lower bound for unit disk graphs

For convenient reference later on, we have separated out the following observation as a lemma.

Lemma 4.1 *Let $\mathcal{L} = (\ell_1, \dots, \ell_n)$ be an oriented line arrangement and let $\mathcal{P} \subseteq \mathbb{R}^2$ be a finite set of points. Then there exist equal radius disks $D_1^-, D_1^+, \dots, D_n^-, D_n^+$ such that $\mathcal{P} \cap \ell_i^- \subseteq D_i^- \subseteq \ell_i^-$ and $\mathcal{P} \cap \ell_i^+ \subseteq D_i^+ \subseteq \ell_i^+$ for all $i = 1, \dots, n$.*

Proof: We first claim that for each $1 \leq i \leq n$ and each $s \in \{-, +\}$ there is a number $R_0(i, s)$ such that for all $R \in \mathbb{R}$ bigger than $R_0(i, s)$, there exists a point $q_i^s(R)$ such that $\mathcal{P} \cap \ell_i^s \subseteq B(q_i^s(R); R) \subseteq \ell_i^s$.

To see this, fix an $1 \leq i \leq n$ and an $s \in \{-, +\}$. Let us write $\mathcal{P} \cap \ell_i^s =: \{a_1, \dots, a_m\}$. Observe that by applying a suitable isometry if needed we can assume without loss of generality that $\ell_i^s = \{(x, y)^T : y > 0\}$ and, thus $a_j = (x_j, y_j)^T$ with $y_j > 0$ for all $1 \leq j \leq m$. Let us set $q(R) := (0, R)^T$. We have $\|q(R) - a_j\|^2 = x_j^2 + (R - y_j)^2 = R^2 - 2y_jR + (x_j^2 + y_j^2)$, and therefore $\|q(R) - a_j\| < R$ for $R > \|a_j\|^2/2y_j$. In other words, for $R > R_0 := \max_j \frac{\|a_j\|^2}{2y_j}$ we have $a_j \in B(q(R); R) \subseteq \ell_+$ for all j , proving the claim.

If we pick $R > \max_{i,s} R_0(i, s)$ and we set $D_i^s = B(q_i^s(R); R)$ then the lemma follows. ■

The following proposition allows us to encode a combinatorial description of an oriented line arrangement into a unit disk graph.

Lemma 4.2 *Let $\mathcal{L} = (\ell_1, \dots, \ell_n)$ be an oriented line arrangement and $\mathcal{S} \subseteq \mathcal{D}(\mathcal{L}) \cap \{-, +\}^n$. There exists a unit disk graph G on vertex set*

$$V(G) = \{v_i^-, v_i^+ : i = 1, \dots, n\} \cup \{u_j : j = 1, \dots, |\mathcal{S}|\},$$

such that in any UDG-realization $(B(\tilde{p}(v), r) : v \in V(G))$ of G we have $\mathcal{S} \subseteq \mathcal{D}(\tilde{\mathcal{L}})$, where $\tilde{\mathcal{L}} = (\tilde{\ell}_1, \dots, \tilde{\ell}_n)$ is the oriented line arrangement defined by setting, for each $i = 1, \dots, n$

$$\tilde{\ell}_i^- := \{z : \|z - \tilde{p}(v_i^-)\| < \|z - \tilde{p}(v_i^+)\|\}$$

(so $\tilde{\ell}_i^-$ is the perpendicular bisector of the line segment $[\tilde{p}(v_i^-), \tilde{p}(v_i^+)]$).

Proof: Let $\mathcal{P} = \{p_1, \dots, p_{|\mathcal{S}|}\}$ be a set of points such that $\mathcal{S} = \{\sigma(p; \mathcal{L}) : p \in \mathcal{P}\}$. Let $D_1^-, D_1^+, \dots, D_n^-, D_n^+$ be as provided by Lemma 4.1, let R denote their common radius and let $p(v_i^s)$ denote the center of D_i^s for $1 \leq i \leq n$ and $s \in \{-, +\}$. Define

$$\begin{aligned} D(v_i^s) &:= B(p(v_i^s); R/2) \quad \text{for } s \in \{-, +\}, i = 1, \dots, 2n, \\ D(u_j) &:= B(p_j; R/2) \quad \text{for } j = 1, \dots, |\mathcal{S}|; \end{aligned} \tag{16}$$

and let G be the corresponding intersection graph. Observe that there is an edge between v_i^s and u_j if and only if $p_j \in B(p(v_i^s); R) = D_i^s$; and this happens if and only if $p_j \in \ell_i^s$ by choice of the D_i^s s.

Now let $(B(\tilde{p}(v), r) : v \in V(G))$ be an arbitrary realization of G as the intersection graph of disks of equal radius r , and let $\tilde{\mathcal{L}}$ be as in the statement of the lemma. Pick a $1 \leq j \leq |\mathcal{S}|$ and a $1 \leq i \leq n$, and suppose that $p_j \in \ell_i^-$. Then, by definition (16) of G , we must have that $B(\tilde{p}(v_i^-), r) \cap B(\tilde{p}(u_j), r) \neq \emptyset$ and $B(\tilde{p}(v_i^+), r) \cap B(\tilde{p}(u_j), r) = \emptyset$. This gives $\|\tilde{p}(u_j) - \tilde{p}(v_i^-)\| < \|\tilde{p}(u_j) - \tilde{p}(v_i^+)\|$. Or, in other words, $\tilde{p}(u_j) \in \tilde{\ell}_i^-$. Similarly we have $\tilde{p}(u_j) \in \tilde{\ell}_i^+$ if $p_j \in \ell_i^+$.

Since this holds for all $1 \leq i \leq n$, we see that

$$\sigma(\tilde{p}(u_j); \tilde{\mathcal{L}}) = \sigma(p_j; \mathcal{L}),$$

for all $j = 1, \dots, |\mathcal{S}|$. Hence $\mathcal{S} \subseteq \mathcal{D}(\tilde{\mathcal{L}})$ as required. \blacksquare

We are now in a position to prove:

Lemma 4.3 $f_{\text{UDG}}(n) = 2^{2^{\Omega(n)}}$.

Proof: It suffices to show that for every $k \in \mathbb{N}$ there exists a unit disk graph G on $O(k)$ vertices with $f_{\text{UDG}}(G) = 2^{2^{\Omega(k)}}$. Let us thus pick an arbitrary $k \in \mathbb{N}$, let \mathcal{L}, \mathcal{S} be as provided by Theorem 3.2 and let G be as provided by Lemma 4.2. Then $|V(G)| = 2|\mathcal{L}| + |\mathcal{S}| = O(k)$. Let $(B(p(v); r) : v \in V(G))$ be an arbitrary UDG-realization of G with $p(v) \in \{-m, \dots, m\}^2$ for all v and $r \in \{1, \dots, m\}$ for some $m \in \mathbb{N}$.

Consider the line $\tilde{\ell}_i$ from Lemma 4.2. It satisfies

$$\begin{aligned} \tilde{\ell}_i &= \{z : \|z - p(v_i^-)\| = \|z - p(v_i^+)\|\} \\ &= \{z : (p(v_i^+) - p(v_i^-))^T z = (p(v_i^+) - p(v_i^-))^T \left(\frac{p(v_i^+) + p(v_i^-)}{2} \right)\} \\ &= \{z : 2(p(v_i^+) - p(v_i^-))^T z = (p(v_i^+) - p(v_i^-))^T (p(v_i^+) + p(v_i^-))\} \\ &=: \{z : w_i z = c_i\}. \end{aligned}$$

Observe that the w_i s have integer coordinates and the c_i s are integers, whose absolute values are all upper bounded by $8m^2$. We can thus apply Lemma 3.3 to get that

$$2^{9/2}(8m^2)^6 = 2^{45/2}m^{12} \geq \text{span}(\tilde{\mathcal{L}}) \geq 2^{2^k}.$$

Hence

$$m \geq \sqrt[12]{2^{2^k}/2^{45/2}} = 2^{2^k/12 - 45/24} = 2^{2^k - \log(12) - o(1)} = 2^{2^{\Omega(k)}},$$

which proves the lemma. \blacksquare

5 The lower bound for disk graphs

To prove the lower bound in Theorem 1.1 we develop a construction for “embedding” a line arrangement into a disk graph that is analogous to that for unit disk graphs in Lemma 4.2; and then we will again apply Lemma 3.3. In the proof of Lemma 4.2 we used two vertices for every oriented line of \mathcal{L} and one vertex for each sign vector of \mathcal{S} . For disk graphs we can still use two vertices for each line; but for each sign vector, rather than a single vertex, we will need to place an induced copy of a special disk graph H with a certain desirable property. The construction of H is pretty involved and takes up most of this section. Rather than giving a list of vertices and

edges, we will give a (geometric) procedure for constructing a realization of H as a disk graph. But before we can begin the construction of H , we will need to do some preliminary work.

By a result of Koebe [24] every planar graph is an intersection graph of touching disks (i.e. closed disks with disjoint interiors). This also shows that every planar graph is the intersection graph of open disks.

Theorem 5.1 *Every planar graph is a disk graph.* ■

Recall that a planar *embedding* of a planar graph assigns each vertex $v \in V(G)$ to a point $p(v) \in \mathbb{R}^2$ in the plane, and each edge $uv \in E(G)$ to a simple closed curve $\gamma(uv)$ with endpoints $p(u), p(v)$ such that for any distinct $e, f \in E(G)$, the curves $\gamma(e), \gamma(f)$ do not intersect except possibly in a common endpoint.

In a *Fáry embedding* the curves $\gamma(e)$ are straight-line segments, i.e. $\gamma(uv) = [p(u), p(v)]$. A fundamental result that was proved at least three separate times by Wagner [48], Fáry [11] and Stein [45] states that every planar graph has a Fáry embedding. The following observation gives a partial converse to Theorem 5.1. It is essentially the same as Theorem 3.4 in Breu's PhD thesis [3] and Lemma 4.1 in Malesińska's PhD thesis [30]. We give a proof for completeness.

Lemma 5.2 *Let G be a triangle-free disk graph of minimum degree at least two, and let $(B(p(v), r(v)) : v \in V(G))$ be a realization of G as a disk graph. Then G is planar and the points $p(v)$ define a Fáry embedding of G .*

Proof: Observe first that if u, v are distinct vertices then $\|p(u) - p(v)\| > r(u) - r(v)$. For if not then $B(p(v), r(v)) \subseteq B(p(u), r(u))$ and then G would have a triangle since v has degree at least two.

Let $uv, uw \in E(G)$ be two distinct edges that share an endpoint. We claim that

$$[p(u), p(v)] \cap [p(u), p(w)] = \{p(u)\}. \quad (17)$$

This is easily seen to hold unless either $p(v) \in [p(u), p(w)]$ or $p(w) \in [p(u), p(v)]$. If $p(u) \in [p(v), p(w)]$ then (17) is again easily seen to hold. Let us thus suppose that $p(v) \in [p(u), p(w)]$. We have

$$\begin{aligned} \|p(v) - p(w)\| &= \|p(u) - p(w)\| - \|p(u) - p(v)\| \\ &< r(u) + r(w) - (r(u) - r(v)) \\ &= r(w) + r(v). \end{aligned}$$

But then we must have $vw \in E(G)$, contradicting that G is triangle-free. Similarly we cannot have $p(w) \in [p(u), p(v)]$. Thus (17) holds, as claimed.

Now consider two edges $uv, st \in E(G)$ with u, v, s, t distinct. We claim that

$$[p(u), p(v)] \cap [p(s), p(t)] = \emptyset. \quad (18)$$

Aiming for a contradiction, let us suppose that the point q lies in both segments $[p(u), p(v)]$ and $[p(s), p(t)]$. Observe that

$$\begin{aligned} \|p(u) - p(s)\| + \|p(v) - p(t)\| &\leq \|p(u) - q\| + \|q - p(s)\| \\ &\quad + \|p(v) - q\| + \|q - p(t)\| \\ &= \|p(u) - p(v)\| + \|p(s) - p(t)\| \\ &< r(u) + r(v) + r(s) + r(t). \end{aligned}$$

Hence either $\|p(u) - p(s)\| < r(u) + r(s)$ or $\|p(v) - p(t)\| < r(v) + r(t)$. In other words, either $us \in E(G)$ or $vt \in E(G)$. Similarly either $ut \in E(G)$ or $vs \in E(G)$. It is easily checked that in each of the four cases there is a triangle, contradicting that G is triangle-free.

It follows that (18) holds as claimed, and the proof is complete. ■

Lemma 5.3 *For every $\varepsilon > 0$ there is a $k = k(\varepsilon)$ such that the following holds. Let $G = K_{1,k}$ be the star on $k + 1$ vertices and let $u \in V(G)$ denote the vertex of degree k . For any DG-realization $(B(p(v), r(v)) : v \in V(G))$ of G , there is a $w \in V(G)$ such that $r(w) < \varepsilon \cdot r(u)$.*

Proof: Let $\varepsilon > 0$ be arbitrary and let $k = \lceil (1 + \varepsilon)^2 / \varepsilon^2 \rceil + 1$. Let $(B(p(v), r(v)) : v \in V(K_{1,k}))$ be an arbitrary realization of $K_{1,k}$ as a disk graph, and suppose that $r(u) = 1$ and $r(v) \geq \varepsilon$ for all $v \neq u$. We can assume that $r(v) = \varepsilon$ for all $v \neq u$, by replacing $D(v)$ with a smaller ball $D'(v) := B(p'(v), \varepsilon) \subseteq D(v)$ that still intersects $D(u)$. Since $D(v)$ intersects $D(u)$ and has radius ε , we have $D(v) \subseteq B(p(u), 1 + \varepsilon)$ for all $v \neq u$. Since $D(v) \cap D(w) = \emptyset$ for all $v \neq w \in V(G) \setminus \{u\}$, by area considerations we must have

$$k \leq \frac{\pi(1 + \varepsilon)^2}{\pi\varepsilon^2} = (1 + \varepsilon)^2 / \varepsilon^2 < k,$$

a contradiction. ■

For an odd $k \in \mathbb{N}$, let O_k denote the graph obtained as follows. We start with a path u_0, \dots, u_{2k} of length $2k$. Now we add vertices a, b each joined to u_j for all even j . Let c denote u_k , the middle vertex of the path, and let us also denote the endpoints of the path by $s = u_0, t = u_{2k}$. Observe that O_k is triangle-free. See Figure 6 for a depiction of O_k .

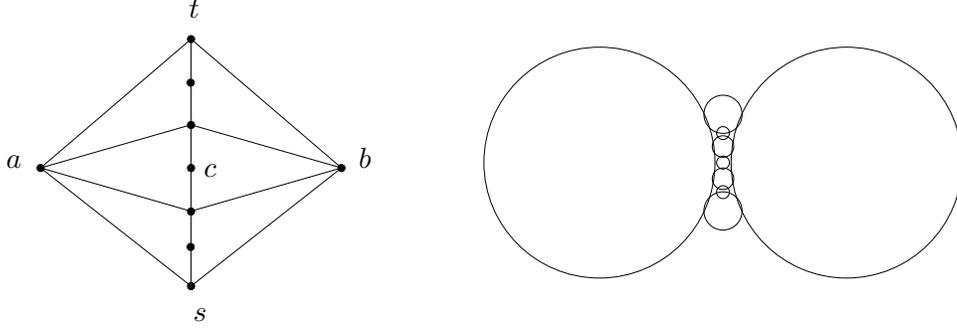


Figure 6: The graph O_3 and a realization of it as a disk graph.

Lemma 5.4 *For every $\varepsilon > 0$ there is a $k = k(\varepsilon)$ such that if $(B(p(v), r(v)) : v \in V(O_k))$ is a DG-realization of O_k , and $p(s), p(a), p(t), p(b)$ lie on the outer face of the corresponding Fáry embedding, then $r(c) < \varepsilon \cdot r(a)$.*

Proof: Let k be large, and let $(B(p(v), r(v)) : v \in V(O_k))$ be an embedding of O_k as a disk graph such that $p(s), p(a), p(t), p(b)$ lie on the outer face of the corresponding Fáry embedding.

By Lemma 5.3 above, if k was chosen sufficiently large, there is an even $0 \leq i < k - 1$ such that $r(u_i) < \varepsilon \cdot r(a)$ and an even $k + 1 < j \leq 2k$ such that $r(u_j) < \varepsilon \cdot r(a)$.

Observe that, since the outer face of the Fáry embedding is the quadrilateral with corners $p(s), p(a), p(t), p(b)$, we must have that $p(c)$ lies inside the quadrilateral with vertices $p(u_i), p(a), p(u_j), p(b)$.

Let us also observe that $\mathbb{R}^2 \setminus (D(u_i) \cup D(a) \cup D(u_j) \cup D(b))$ consists of two connected regions, a bounded and an unbounded one. Let R denote the bounded one, and let Q denote the (inside of) the quadrangle with corners $p(u_i), p(a), p(u_j), p(b)$. Then R is clearly contained in Q . By the previous we have $D(c) \subseteq R$ (as its center $p(c)$ lies in Q and $D(c)$ is disjoint from $D(u_i) \cup D(a) \cup D(u_j) \cup D(b)$).

Observe that R is also contained in the quadrilateral Q' whose corner points are: an intersection point q_1 of the bounding circles $\partial D(u_i)$ and $\partial D(a)$; an intersection point q_2 of $\partial D(a)$ and $\partial D(u_j)$; an intersection point q_3 of $\partial D(u_j)$ and $\partial D(b)$; and an intersection point q_4 of $\partial D(b)$ and $\partial D(u_i)$ (see Figure 7).

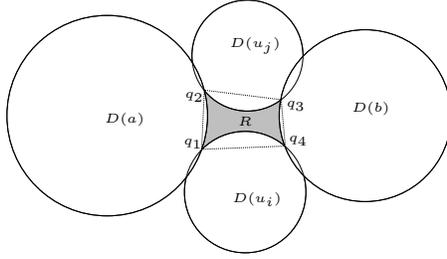


Figure 7: R is contained in the quadrilateral with corners q_1, q_2, q_3, q_4 .

Clearly $\|q_3 - q_2\| \leq 2r(u_j)$ and $\|q_4 - q_1\| \leq 2r(u_i)$. Thus, two opposite sides of Q' have length $< 2\varepsilon \cdot r(a)$. Since also $D(c) \subseteq Q'$ we then must have $r(c) < \varepsilon \cdot r(a)$, as required. \blacksquare

Consider a realization of O_k with $k = k(1/1000)$ as in Lemma 5.4 with $p(a), p(s), p(t), p(b)$ on the outer face of the corresponding Fáry embedding (such a realization is depicted in Figure 6 for $k = 3$). For notational convenience let us denote $X := O_k$.

We now define a disk graph Y on vertex set

$$V(Y) = \{a\} \cup \{v_i : v \in V(X) \setminus \{a\}, i = 0, \dots, N\},$$

as follows. We let $D(a)$ be as in the chosen realization of X . For each $v \in V(X) \setminus \{a\}$ and $i = 0, \dots, N$ we place a disk $D(v_i) = B(p(v_i), r(v_i))$ where $r(v_i) := r(v)$ and $p(v_i)$ is obtained by rotating $p(v)$ counterclockwise about $p(a)$ by an angle of $i \cdot \alpha$. Here α, N are chosen so that $C = c_0 \dots c_N$ will constitute an induced cycle in the resulting intersection graph of disks Y . (See Figure 8.) For notational convenience, let us also write $a_i := a$, and let X_i denote the i -th rotated copy of X (i.e. it has vertex set $V(X_i) = \{v_i : v \in V(X)\}$). Observe that $D(a)$ does not intersect any $D(c_i)$ and it is contained in the bounded region of $\mathbb{R}^2 \setminus \bigcup_i D(c_i)$.

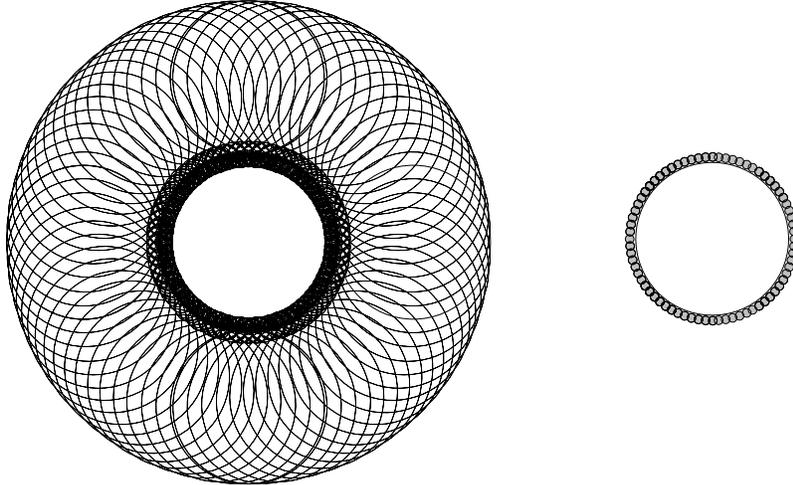


Figure 8: The graph Y is obtained by rotating copies of a realization of X about $p(a)$ in such a way that the c_i s form an induced cycle. On the right only $D(a)$ and the $D(c_i)$ s are shown.

We now construct the disk graph H as follows. We start with a four cycle F and consider a realization of it by equal size disks. Inside each of the two regions of $\mathbb{R}^2 \setminus \bigcup_{v \in V(F)} D(v)$ we place a suitably shrunken copy of the realization of Y we have just constructed. (See Figure 9.)

Let $Y^{(1)}, Y^{(2)}$ denote these copies of Y ; for each vertex $v \in V(Y)$ let $v^{(j)}$ denote the corresponding vertex in $Y^{(j)}$; and let $C^{(j)}$ and $X_i^{(j)}$ be defined in the obvious way.

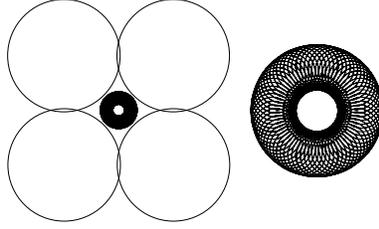


Figure 9: Placing copies of Y inside each region of $\mathbb{R}^2 \setminus \bigcup_{v \in F} D(v)$.

We now add four internally vertex disjoint (meaning they do not share vertices other than their endpoints) paths P_1, \dots, P_4 to our construction such that each of them joins $a^{(1)}$ to $a^{(2)}$ and passes through a vertex of $C^{(1)}$, a vertex of F , and a vertex of $C^{(2)}$; and the subgraph Z of H induced by the vertices

$$V(Z) := \{a^{(1)}, a^{(2)}\} \cup V(F) \cup V(C^{(1)}) \cup V(C^{(2)}) \cup \bigcup_{m=1}^4 V(P_m),$$

is triangle free. We can do this if we represent the vertices we are adding by small enough disks – see Figure 10. Only the stated properties of the paths P_1, \dots, P_4 will play a role in the proof of Lemma 5.5 below; the lengths and any additional edges that may have been created inadvertently are irrelevant as long as the induced subgraph Z is triangle free and the paths P_1, \dots, P_4 have the properties stated.

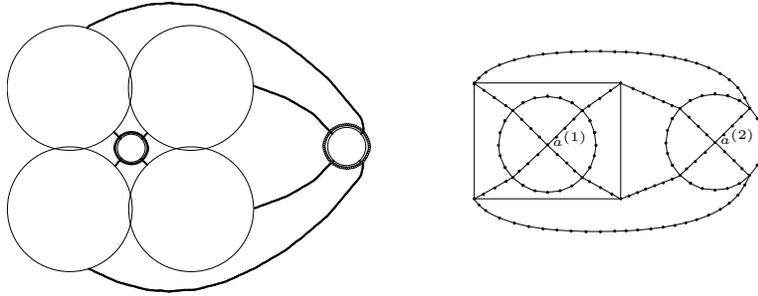


Figure 10: Adding paths P_1, \dots, P_4 between $a^{(1)}$ and $a^{(2)}$.

For $0 \leq i \leq N$ and $j = 1, 2$, let us add vertex disjoint paths $Q_1^{(i,j)}, \dots, Q_4^{(i,j)}$ to our construction, each running from one of the vertices $a^{(j)}, b_i^{(j)}, s_i^{(j)}, t_i^{(j)}$ to a vertex on F , in such a way that the subgraph $H^{(i,j)}$ of H induced by the vertices

$$V(H^{(i,j)}) := V(F) \cup V(X_i^{(j)}) \cup \bigcup_{m=1}^4 V(Q_m^{(i,j)}),$$

is triangle free. Again this is possible if we choose the radii of the disks making up the internal vertices of the paths small enough – see Figure 11.

This concludes the construction of H . The following lemma gives the key property of H that will be crucial in the proof of the lower bound in Theorem 1.1.

Lemma 5.5 *Let $\mathcal{D} = (D(v) : v \in V(H))$ be an arbitrary realization of H as a disk graph. Then there exists a point $p = p(\mathcal{D})$ such that the following holds for all convex $W \subseteq \mathbb{R}^2$:*

- (i) *If $W \cap D(v) \neq \emptyset$ for all $v \in V(H)$ then $p \in W$;*

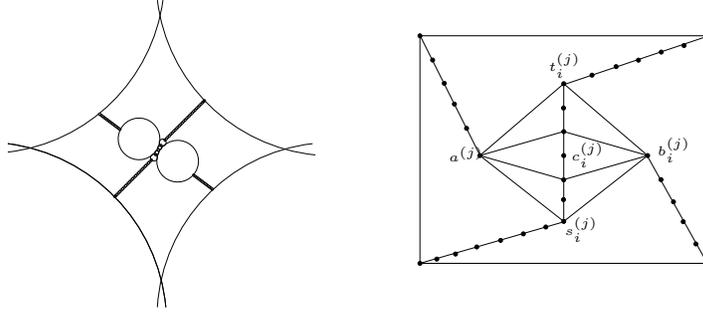


Figure 11: Adding paths $Q_1^{(i,j)}, \dots, Q_4^{(i,j)}$ between F and $a^{(j)}, b_i^{(j)}, s_i^{(j)}, t_i^{(j)}$.

(ii) If $W \cap D(v) = \emptyset$ for all $v \in V(H)$ then $p \notin W$.

Proof: For $uv \in E(H)$ let us write $\gamma(uv) = [p(u), p(v)]$ and for $H' \subseteq H$ a subgraph let us write $\gamma(H') := \bigcup_{e \in H'} \gamma(e)$. If H' is an induced cycle of H , then $\gamma(H')$ is a simple closed curve (by Lemma 5.2) and hence $\mathbb{R}^2 \setminus \gamma(H')$ consists of two regions, a bounded and an unbounded one. We say that a point x lies inside $\gamma(H')$ if it lies in the bounded component of $\mathbb{R}^2 \setminus \gamma(H')$.

Recall that Z denotes the subgraph of H induced by the vertices $\{a^{(1)}, a^{(2)}\} \cup V(F) \cup V(C^{(1)}) \cup V(C^{(2)}) \cup \bigcup_{m=1}^4 V(P_m)$. By construction, Z is triangle free and has minimum degree at least two. Since it is an induced subgraph of H , by Lemma 5.2, the points $p(v) : v \in V(Z)$ define a Fáry embedding of Z .

Let us observe that in any planar embedding of Z either $p(a^{(1)})$ lies inside $\gamma(F)$ or $p(a^{(2)})$ lies inside $\gamma(F)$ – otherwise we could not embed the paths P_1, \dots, P_4 without crossings (see Figure 10, right). Without loss of generality it is $p(a^{(1)})$ in our Fáry embedding. We must then also have (see again Figure 10, right) that $p(a^{(1)})$ lies inside $\gamma(C^{(1)})$. This also gives that $D(a^{(1)})$ is contained in the bounded region of $\mathbb{R}^2 \setminus \bigcup_{v \in C^{(1)}} D(v)$. (In other words $D(a^{(1)})$ is “surrounded” by the $D(c_i^{(1)})$ s.)

Recall that $H^{(i,1)}$ denotes the subgraph of H induced by the vertices $V(F) \cup V(X_i^{(1)}) \cup \bigcup_{m=1}^4 V(Q_m^{(i,1)})$. By construction $H^{(i,1)}$ is triangle free and of minimum degree at least two. Hence Lemma 5.2 again gives that $p(v) : v \in V(H^{(i,1)})$ defines a Fáry embedding of H . We already know that $p(a^{(1)})$ lies inside $\gamma(F)$. It now follows that $p(a^{(1)}), p(s_i^{(1)}), p(t_i^{(1)}), p(b_i^{(1)})$ must lie on the outer face in the Fáry embedding of $X_i^{(1)}$, because otherwise we could not embed $Q_1^{(i,1)}, \dots, Q_4^{(i,1)}$ without crossings (see Figure 11, right). Thus, by Lemma 5.4 and the choice of $X = O_k$, we have that $r(c_i^{(1)}) < r(a^{(1)})/1000$ for all $0 \leq i \leq N$.

Now set $p := p(a^{(1)})$, let ℓ_x be the horizontal line through p and ℓ_y the vertical line through p . Observe that, because $D(a)$ is surrounded by the $D(c_i^{(1)})$ s and $r(c_i^{(1)}) < r(a)/1000$ for all i , each of the four quadrants of $\mathbb{R}^2 \setminus (\ell_x \cup \ell_y)$ contains one of the disks $D(c_i^{(1)})$. (See Figure 12.)

Hence, any W that intersects $D(v)$ for all $v \in V(H)$, has a point in each of the four regions of $\mathbb{R}^2 \setminus (\ell_x \cup \ell_y)$; and hence if such a W is convex then it must contain p . This proves part (i) of the Lemma.

That part (ii) holds is immediate from the choice of $p = p(a^{(1)})$ as the center of $D(a^{(1)})$. ■

Lemma 5.6 Let $\mathcal{L} = (\ell_1, \dots, \ell_n)$ be an oriented line arrangement and $\mathcal{S} \subseteq \mathcal{D}(\mathcal{L}) \cap \{-, +\}^n$. There exists a disk graph G on vertex set $V(G) = \{v_i^-, v_i^+ : i = 1, \dots, n\} \cup \{v^{(j)} : v \in V(H), j = 1, \dots, |\mathcal{S}|\}$, such that in any DG-realization $(B(p(v), r(v)) : v \in V(G))$ of it, the oriented line arrangement $\tilde{\mathcal{L}} = (\tilde{\ell}_1, \dots, \tilde{\ell}_n)$ defined by

$$\tilde{\ell}_i^- := \{z : w_i^T z < c_i\},$$

where

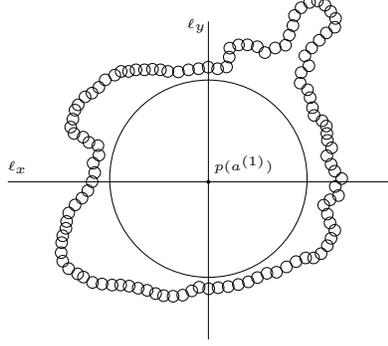


Figure 12: The $D(c_i^{(1)})$ s surround $D(a^{(1)})$, so that each of the four quadrants defined by ℓ_x, ℓ_y contains one of the $D(c_i^{(1)})$ s.

$$w_i := p(v_i^+) - p(v_i^-), \quad c_i := w_i^T \left(\left(\frac{r(v_i^+)}{r(v_i^+) + r(v_i^-)} \right) p(v_i^-) + \left(\frac{r(v_i^-)}{r(v_i^+) + r(v_i^-)} \right) p(v_i^+) \right),$$

has $\mathcal{S} \subseteq \mathcal{D}(\tilde{\mathcal{L}})$.

Proof: Let $\mathcal{P} = \{p_1, \dots, p_{|\mathcal{S}|}\}$ be a set of points such that $\mathcal{S} = \{\sigma(p; \mathcal{L}) : p \in \mathcal{P}\}$.

Let $D_1^-, D_1^+, \dots, D_n^-, D_n^+$ be as provided by Lemma 4.1, and let us set

$$D(v_i^s) := D_i^s \quad \text{for } s \in \{-, +\}, j = 1, \dots, n, \quad (19)$$

Now consider a $1 \leq j \leq |\mathcal{S}|$. Then $O_j := \bigcap \{D_i^s : p_j \in D_i^s\}$ is open. Hence we can place a suitably shrunken copy of a realization of H inside O_j . Let $H^{(j)}$ denote the copy of H placed in O_j and let $u^{(j)} : u \in V(H)$ denote the vertices of $H^{(j)}$.

Let G be the corresponding intersection graph of disks, and let $(\tilde{D}(v) : v \in V(G))$ be an arbitrary realization of G as a disk graph. Let us write $\tilde{D}(v) = B(\tilde{p}(v), \tilde{r}(v))$ for all $v \in V(G)$. For each $1 \leq j \leq |\mathcal{S}|$, let $\tilde{p}_j = p(\tilde{D}(u^{(j)} : u \in V(H)))$ be the point provided by Lemma 5.5, applied to $H^{(j)}$.

Suppose that, for some $1 \leq j \leq |\mathcal{S}|$ and $1 \leq i \leq n$ and $s \in \{-, +\}$ we have $p_j \in D(v_i^s)$. Then we have that $D(u^{(j)}) \cap D(v_i^s) \neq \emptyset$ for all $u \in V(H)$ by construction. We must then also have $\tilde{D}(u^{(j)}) \cap \tilde{D}(v_i^s) \neq \emptyset$ for all $u \in V(H)$, because the \tilde{D} s and the D s define the same intersection graph. Since $\tilde{D}(v_i^s)$ is convex, by the property of \tilde{p}_j certified by Lemma 5.5, we also have $\tilde{p}_j \in \tilde{D}(v_i^s)$.

Now suppose that $p_j \notin D(v_i^s)$. Then $p_j \in D(v_i^{-s})$ by choice of the $D(v_i^s)$ s (Lemma 4.1). And thus, by the argument we just gave, $\tilde{p}_j \in \tilde{D}(v_i^{-s})$. Since $\tilde{D}(v_i^s) \cap \tilde{D}(v_i^{-s}) = \emptyset$ we thus have $\tilde{p}_j \notin \tilde{D}(v_i^s)$. We have just proved that

$$p_j \in D(v_i^s) \text{ if and only if } \tilde{p}_j \in \tilde{D}(v_i^s). \quad (20)$$

Let the oriented line arrangement $\tilde{\mathcal{L}} = (\tilde{\ell}_1, \dots, \tilde{\ell}_n)$ be as in the statement of the lemma. Pick an arbitrary $z \in \tilde{D}(v_i^-)$. Then we can write $z = \tilde{p}(v_i^-) + \tilde{r}(v_i^-)u$ with $\|u\| < 1$. Hence, with w_i and c_i as in the statement of the lemma, we have

$$\begin{aligned} w_i^T z &= w_i^T \left(\tilde{p}(v_i^-) + \tilde{r}(v_i^-)u \right) \\ &< w_i^T \left(\tilde{p}(v_i^-) + \tilde{r}(v_i^-) \frac{w_i}{\|w_i\|} \right) \\ &\leq w_i^T \left(\tilde{p}(v_i^-) + \left(\frac{\tilde{r}(v_i^-)}{\tilde{r}(v_i^-) + \tilde{r}(v_i^+)} \right) (\tilde{p}(v_i^+) - \tilde{p}(v_i^-)) \right) \\ &= w_i^T \left(\left(\frac{\tilde{r}(v_i^+)}{\tilde{r}(v_i^+) + \tilde{r}(v_i^-)} \right) \tilde{p}(v_i^-) + \left(\frac{\tilde{r}(v_i^-)}{\tilde{r}(v_i^+) + \tilde{r}(v_i^-)} \right) \tilde{p}(v_i^+) \right) \\ &= c_i, \end{aligned}$$

where we have used that $\|w\| = \|\tilde{p}(v_i^+) - \tilde{p}(v_i^-)\| \geq \tilde{r}(v_i^+) + \tilde{r}(v_i^-)$ (since $\tilde{D}(v_i^+)$ and $\tilde{D}(v_i^-)$ are disjoint) in the third line. We have just proved that $\tilde{D}(v_i^-) \subseteq \tilde{\ell}_i^-$ for all i . Completely analogously $\tilde{D}(v_i^+) \subseteq \tilde{\ell}_i^+$ for all i . By (20) and the choice of $D(v_i^s) := D_i^s$ (recall the D_i^s s are chosen such that $D_i^s \subseteq \ell_i^s$, and that either $p_j \in D_i^-$ or $p_j \in D_i^+$ for all j) we see that

$$\sigma(p_j; \mathcal{L}) = \sigma(\tilde{p}_j; \tilde{\mathcal{L}}),$$

for all j . This proves the lemma. \blacksquare

Now we are finally in a position to prove the lower bound of Theorem 1.1.

Lemma 5.7 $f_{\text{DG}}(n) = 2^{2^{\Omega(n)}}$.

Proof: It again suffices to prove that for every $k \in \mathbb{N}$ there exists a disk graph G on $O(k)$ vertices with $f_{\text{DG}}(G) = 2^{2^{\Omega(k)}}$. Let us thus pick an arbitrary $k \in \mathbb{N}$, let \mathcal{L}, \mathcal{S} be as provided by Theorem 3.2, and let G be as provided by Lemma 5.6. Then $|V(G)| = 2|\mathcal{L}| + |V(H)| \cdot |\mathcal{S}| = O(k)$. Let $(B(p(v), r(v)) : v \in V(G))$ be an arbitrary DG-realization of G with $p(v) \in \{-m, \dots, m\}^2$ and $r(v) \in \{1, \dots, m\}$ for all v for some integer $m \in \mathbb{N}$. Let $\tilde{\mathcal{L}}$ be as defined in the statement of Lemma 5.6. Then we can write, with w_i, c_i as in Lemma 5.6:

$$\begin{aligned} \tilde{\ell}_i &= \{z : w_i^T z = c_i\} \\ &= \{z : (r(v_i^+) + r(v_i^-))w_i^T z = w_i^T (r(v_i^-)p(v_i^+) + r(v_i^+)p(v_i^-))\} \\ &=: \{z : (w_i')^T z = c_i'\}. \end{aligned}$$

Observe that the w_i' s have integer coordinates and the c_i' s are integers, whose absolute values are all upper bounded by $8m^3$. We can thus apply Lemma 3.3 to get that $2^{9/2}(8m^3)^6 \geq \text{span}(\mathcal{L}) \geq 2^{2^k}$, and hence $m = 2^{2^{\Omega(k)}}$. \blacksquare

6 The lower bound for segment graphs

An important tool in this section will be part (b) of the “order forcing lemma” of Kratochvíl and Matoušek [27]:

Lemma 6.1 ([27]) *Let G be a segment graph and $(S(v) : v \in V(G))$ a SEG-realization such that all parallel segments are disjoint and no three segments share a point. Then there exists a segment graph G' with $G \subseteq G'$ such that for every SEG-realization $(\tilde{S}(v) : v \in V(G'))$ there exists an open set $O \subseteq \mathbb{R}^2$ and a homeomorphism $\varphi : \mathbb{R}^2 \rightarrow O$ such that $\varphi[S(v)] \subseteq \tilde{S}(v)$ for all $v \in V(G)$.*

Lemma 6.2 *Let $\mathcal{L} = (\ell_1, \dots, \ell_n)$ be an oriented line arrangement and $\mathcal{S} \subseteq \mathcal{D}(\mathcal{L})$. There exists a segment graph G on $|\mathcal{L}| + 2|\mathcal{S}|$ vertices such that for every SEG-realization $(S(v) : v \in V(G))$ there is an oriented line arrangement $\tilde{\mathcal{L}} = (\tilde{\ell}_1, \dots, \tilde{\ell}_n)$ such that each line $\tilde{\ell}_i$ contains some segment $S(v)$ and $\mathcal{S} \subseteq \mathcal{D}(\tilde{\mathcal{L}})$.*

Proof: We start with the segment graph G_0 on the vertices $x_1, \dots, x_4, y_1, y_2, t, m, b$ and the realization S of it given in figure 13. Let O_t denote the inside of the quadrilateral bounded by $S(y_1), S(x_1), S(y_2), S(x_2)$; let O_m denote the inside of the quadrilateral bounded by $S(y_1), S(x_2), S(y_2), S(x_3)$; and let O_b denote the inside of the quadrilateral bounded by $S(y_1), S(x_3), S(y_2), S(x_4)$. Let us list three key properties of the embedding S for convenient future reference.

(S-1) $S(t) \subseteq O_t, S(m) \subseteq O_m, S(b) \subseteq O_b$;

(S-2) $S(y_1)$ and $S(y_2)$ each intersect $S(x_1), \dots, S(x_4)$ in the order of the indices;

(S-3) If ℓ is a line that intersects $S(y_1)$ between its intersection points with $S(x_2)$ and $S(x_3)$, and ℓ intersects $S(y_2)$ between its intersection points with $S(x_2)$ and $S(x_3)$, then ℓ separates $S(t)$ from $S(b)$.

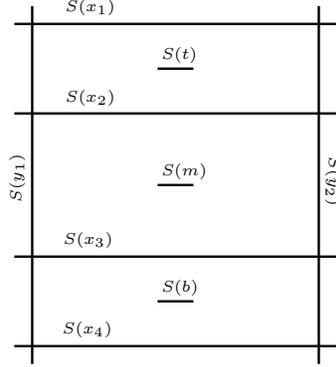


Figure 13: The SEG-embedding of G_0 we are starting from.

Let G_1 be the graph that Lemma 6.1 constructs out of G_0 , and consider an arbitrary SEG-realization S' of G_1 . Let us define O'_t, O'_m, O'_b in the obvious way. By construction of G_1 , the properties **(S-1)**-**(S-3)** hold also for S' and O'_t, O'_m, O'_b . (It can be seen that **(S-1)**-**(S-3)** are “preserved under homeomorphism” in the sense of Lemma 6.1). Now let an oriented line arrangement $\mathcal{L} = (\ell_1, \dots, \ell_n)$ and a set of sign vectors $\mathcal{S} \subseteq \mathcal{D}(\mathcal{L})$ be given, and let $\mathcal{P} = \{p_1, \dots, p_{|\mathcal{S}|}\}$ be such that $\sigma(\mathcal{P}; \mathcal{L}) = \mathcal{S}$. By applying a suitable affine transformation if needed (i.e. we define a new oriented line arrangement and point set by setting $\tilde{\ell}_i^- := T[\ell_i^-]$ and $\tilde{p}_j := T(p_j)$ – clearly this does not affect sign vectors), we can assume without loss of generality that $\mathcal{P} \subseteq O'_m$; that each line ℓ_i intersects $S'(m)$ and it intersects $S'(y_1)$ between the intersection points with $S'(x_2)$ and $S'(x_3)$, and it intersects $S'(y_2)$ between the intersection points with $S'(x_2)$ and $S'(x_3)$. (So in particular ℓ_i separates $S'(t)$ from $S'(b)$ for all $1 \leq i \leq n$.) For each $1 \leq i \leq n$ let $S'(v_i) \subseteq \ell_i$ denote the segment between the intersection point of ℓ_i with $S'(y_1)$ and the intersection point of ℓ_i with $S'(y_2)$.

For each $1 \leq j \leq |\mathcal{S}|$ let $S'(t_j)$ be a line segment between p_j and a point on $S'(t)$; and let $S'(b_j)$ be a line segment between p_j and a point on $S'(b)$. We let G be the intersection graph of the segments $(S'(v) : v \in V(G_1) \cup \{v_1, \dots, v_n\} \cup \{b_j, t_j : 1 \leq j \leq |\mathcal{S}|\})$ we just constructed. (See Figure 14 for a depiction of this construction.)

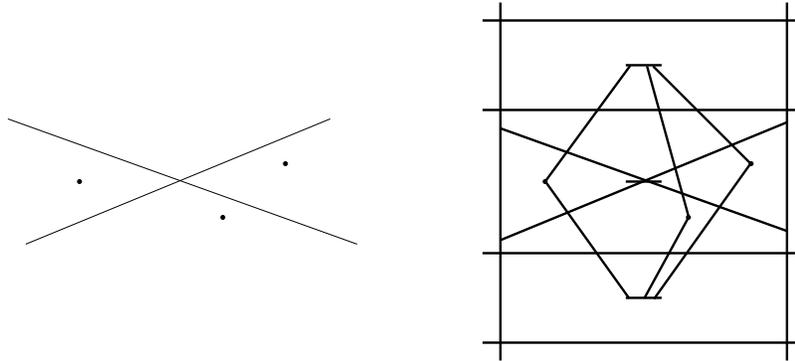


Figure 14: \mathcal{L} and \mathcal{P} (left) and the a segment graph G we construct from them (right).

Now let $(\tilde{S}(v) : v \in V(G))$ be an arbitrary realization of G as a segment graph, and let $\tilde{O}_t, \tilde{O}_m, \tilde{O}_b$ be defined in the obvious way. Again **(S-1)**-**(S-3)** hold. Let $\tilde{\ell}_i$ denote the line that contains $\tilde{S}(v_i)$ for $i = 1, \dots, n$. Since $\tilde{S}(v_i)$ intersects all of $\tilde{S}(m), \tilde{S}(y_1), \tilde{S}(y_2)$ and none of $\tilde{S}(x_1), \dots, \tilde{S}(x_4)$ (by construction of G_1), it must hold that $\tilde{S}(v_i)$ intersects $\tilde{S}(y_1)$ between the

intersection points with $\tilde{S}(x_3)$ and $\tilde{S}(x_4)$; and $\tilde{S}(v_i)$ intersects $\tilde{S}(y_2)$ between the intersection points with $\tilde{S}(x_3)$ and $\tilde{S}(x_4)$. Hence $\tilde{\ell}_i$ separates $\tilde{S}(t)$ from $\tilde{S}(b)$. Let us orient the lines $\tilde{\ell}_i$ such that $\tilde{\ell}_i^+ \supseteq \tilde{S}(t)$ if and only if $\ell_i^+ \supseteq S'(t)$. For $1 \leq j \leq |\mathcal{S}|$ let \tilde{p}_j be a point of $\tilde{S}(b_j) \cap \tilde{S}(t_j)$ (such a point exists as $S'(b_j), S'(t_j)$ intersect), and let us write $\tilde{\mathcal{P}} = \{\tilde{p}_1, \dots, \tilde{p}_{|\mathcal{S}|}\}$. We claim that

$$\sigma(\mathcal{P}; \mathcal{L}) = \sigma(\tilde{\mathcal{P}}; \tilde{\mathcal{L}}). \quad (21)$$

To see that (21) holds, pick an arbitrary $1 \leq i \leq n$ and an arbitrary $1 \leq j \leq |\mathcal{S}|$. Suppose that p_j and $S'(t)$ lie on the same side of ℓ_i . Since $\tilde{S}(t_j)$ hits both \tilde{p}_j and $\tilde{S}(t)$ but it does not hit $\tilde{\ell}_i$, we see that \tilde{p}_j and $\tilde{S}(t)$ are also on the same side of $\tilde{\ell}_i$. Similarly, if p_j and $S'(b)$ are on the same side of ℓ_i then \tilde{p}_j and $\tilde{S}(b)$ are on the same side of $\tilde{\ell}_i$.

By choice of (the orientation of) $\tilde{\mathcal{L}}$, this proves (21) and hence the lemma. \blacksquare

Lemma 6.3 $f_{\text{SEG}}(n) = 2^{2^{\Omega(n)}}$.

Proof: It again suffices to prove that for every $k \in \mathbb{N}$ there exists a segment graph G on $O(k)$ vertices with $f_{\text{SEG}}(G) = 2^{2^{\Omega(k)}}$. Let us thus pick an arbitrary $k \in \mathbb{N}$, let \mathcal{L}, \mathcal{S} be as provided by Theorem 3.2, and let G be as provided by Lemma 6.2. Then $|V(G)| = |V(G_1)| + |\mathcal{L}| + 2|\mathcal{S}| = O(k)$. Let $(S(v) : v \in V(G))$ be an arbitrary SEG-realization of G such that, for some $m \in \mathbb{N}$, we can write $S(v) = [a(v), b(v)]$ with $a(v), b(v) \in \{-m, \dots, m\}^2$ for all $v \in V(G)$. Let us order $V(G)$ arbitrarily as v_1, \dots, v_n and let the oriented line arrangement $\mathcal{L} = (\ell_1, \dots, \ell_n)$ be such that ℓ_i contains $S(v_i)$ for $i = 1, \dots, n$ (the orientation does not matter in the sequel). Then we can write

$$\ell_i = \{z : w^T z = c\},$$

with

$$w = \begin{pmatrix} (b(v_i))_y - (a(v_i))_y \\ (b(v_i))_x - (a(v_i))_x \end{pmatrix}, \quad c = w^T a(v_i).$$

Thus the coordinates of w and c are integers whose absolute values are upper bounded by $4m^2$. We can again apply Lemma 3.3 to see that $2^{7/2}(4m^2)^6 \geq \text{span}(\mathcal{L}) \geq 2^{2^k}$, and hence $m = 2^{2^{\Omega(k)}}$. \blacksquare

7 Proofs of the upper bounds

The *order type* of a point configuration $\mathcal{P} = (p_1, \dots, p_n)$ stores for each triple of indices $1 \leq i_1 < i_2 < i_3 \leq n$ whether the points $p_{i_1}, p_{i_2}, p_{i_3}$ are in clockwise position, in counter clockwise position or collinear. Recall that a point configuration is in *general position* if no three points are collinear. We need the following result of Goodman, Pollack and Sturmfels [15], stated here only for two dimensions.

Theorem 7.1 ([15]) *Let $f(n)$ denote the least k such that every order type of n points in general position in the plane can be realized by points on $\{1, \dots, k\}^2$. Then $f(n) = 2^{2^{\Theta(n)}}$.*

Observe that every segment graph has a realization in which the endpoints of the segments are in general position. (To see this, start with an arbitrary realization. By making the segments slightly longer if needed we can ensure that every two intersecting segments intersect in a point that is interior to both. Now we can perturb the endpoints very slightly so that they are in general position and the intersection graph of segments remains the same.) Let us also observe that we can tell whether two segments $[a, b], [c, d]$ intersect or not from the order type of (a, b, c, d) , unless a, b, c, d are all collinear. Hence if $(S(v) : v \in V(G))$ is a SEG-realization of G whose endpoints are in general position, then any point configuration with the same order type as the endpoints of the segments also gives a SEG-realization of G . Therefore Theorem 7.1 immediately implies the upper bound in Theorem 1.3.

Corollary 7.2 $f_{\text{SEG}}(n) = 2^{2^{O(n)}}$.

The upper bounds in Theorems 1.1, 1.2 are relatively straightforward consequences of a result of Grigor'ev and Vorobjov that was also the main ingredient in the proof of the upper bound in Theorem 7.1. The following is a reformulation of Lemma 10 in [17]:

Lemma 7.3 ([17]) *For each $d \in \mathbb{N}$ there exists a constant $C = C(d)$ such that the following hold. Suppose that h_1, \dots, h_k are polynomials in n variables with integer coefficients, and degrees $\deg(h_i) < d$. Suppose further that the bit sizes of the all coefficients are less than B . If there exists a solution $(x_1, \dots, x_n) \in \mathbb{R}^n$ of the system $\{h_1 \geq 0, \dots, h_k \geq 0\}$, then there also exists one with $|x_1|, \dots, |x_n| \leq \exp[(B + \ln k)C^n]$.*

Lemma 7.4 $f_{\text{UDG}}(n) = 2^{2^{O(n)}}$.

Proof: Let G be a unit disk graph on n vertices. Any UDG-realization is a nothing more than a solution $(x_1, y_1, \dots, x_n, y_n, r) \in \mathbb{R}^{2n+1}$ of the system of polynomial of inequalities

$$\begin{aligned} (x_i - x_j)^2 + (y_i - y_j)^2 &< r^2, & \text{for all } ij \in E(G), \\ (x_i - x_j)^2 + (y_i - y_j)^2 &\geq r^2, & \text{for all } ij \notin E(G), \\ r &> 0. \end{aligned} \tag{22}$$

Observe that any solution of (22) can be perturbed to a solution in which all inequalities are strict (if we fix $r' = r$ and set $x'_i = \lambda x_i, y'_i = \lambda y_i$ for all $1 \leq i \leq n$ then, if we chose $\lambda > 1$ but very very close to 1, then we have a new solution in which all inequalities are strict). Similarly, if we now multiply all variables by the same (very large) scalar μ we get a solution of:

$$\begin{aligned} (x_i - x_j)^2 + (y_i - y_j)^2 &\leq (r - 10)^2, & \text{for all } ij \in E(G), \\ (x_i - x_j)^2 + (y_i - y_j)^2 &\geq (r + 10)^2, & \text{for all } ij \notin E(G), \\ r &\geq 100. \end{aligned} \tag{23}$$

This is a system of $1 + \binom{n}{2}$ polynomial inequalities of degree less than 3 in $2n + 1$ variables, with all coefficients small integers. Since the system has a solution, by lemma 7.3, there exists a solution to this system with all numbers less than $\lfloor \exp[\gamma^n] \rfloor$ in absolute value for some γ (we absorb the factor $\ln(1 + \binom{n}{2}) + O(1)$ by taking $\gamma > C$). Let us now round down all numbers to the next integer, i.e. we set $\tilde{x}_i := \lfloor x_i \rfloor, \tilde{y}_i := \lfloor y_i \rfloor, \tilde{r} := \lfloor r \rfloor$. If $ij \in E(G)$ then we have

$$\begin{aligned} (\tilde{x}_i - \tilde{x}_j)^2 + (\tilde{y}_i - \tilde{y}_j)^2 &\leq (|x_i - x_j| + 1)^2 + (|y_i - y_j| + 1)^2 \\ &= (x_i - x_j)^2 + (y_i - y_j)^2 + 2(|x_i - x_j| + |y_i - y_j|) + 2 \\ &\leq (r - 10)^2 + 4(r - 10) + 2 \\ &= r^2 - 16r + 62 \\ &\leq (r - 1)^2 \\ &< \tilde{r}^2 \end{aligned}$$

(Here we have used $|x_i - x_j| + |y_i - y_j| \leq 2\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \leq 2(r - 10)$ in the third line; and in the fifth line we used that $r \geq 100$.) Similarly, if $ij \notin E(G)$ then

$$\begin{aligned} (\tilde{x}_i - \tilde{x}_j)^2 + (\tilde{y}_i - \tilde{y}_j)^2 &\geq (|x_i - x_j| - 1)^2 + (|y_i - y_j| - 1)^2 \\ &= (x_i - x_j)^2 + (y_i - y_j)^2 - 2(|x_i - x_j| + |y_i - y_j|) + 2 \\ &\geq (x_i - x_j)^2 + (y_i - y_j)^2 - 4\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} + 2 \\ &\geq (r + 10)^2 - 4(r + 10) + 2 \\ &= r^2 + 16r + 62 \\ &\geq r^2 \\ &\geq \tilde{r}. \end{aligned}$$

(Here we have used that $d^2 - 4d + 2$ is increasing for $d \geq 2$ in the fourth line.) Thus $(\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_n, \tilde{y}_n, \tilde{r})$ is a solution of (22). Since all variables of this solution are integers of absolute value at most

$$\exp[\gamma^n] = 2^{\log(e) \cdot \gamma^n} = 2^{2^n \log \gamma + \log \log e} = 2^{2^{O(n)}},$$

this proves the lemma. ■

The proof of the upper bound in Theorem 1.1 is very similar to the proof of Lemma 7.4 and we therefore omit it.

Lemma 7.5 $f_{\text{DG}}(n) = 2^{2^{O(n)}}$. ■

8 Concluding remarks.

In this paper, we have shown that every (unit) disk graph on n vertices has a realization by disks where all radii and all coordinates of the centers are integers no larger than $2^{2^{O(n)}}$, and that there exist (unit) disk graphs for which in every such integer realization, at least one radius or coordinate of a center is as large as $2^{2^{\Omega(n)}}$. We have also proved the analogous result for segment graphs if we wish the endpoints of the segments to lie on the integer grid. The most interesting parts of our proofs were the constructions of the graphs that need large portions of the integer grid. In all cases the idea was to “encode” some set of sign vectors S belonging to an oriented line arrangement into a unit disk/disk/segment graph G in such a way that every realization of G defines an oriented line arrangement that contains S in its combinatorial description. We then applied these constructions to a set of sign vectors S that forces the *span* (as defined by (10)) to be large; and finally some relatively straightforward calculations then gave that in every integer realization of G some coordinate or radius must be large. Most of this argument does not depend on the fact that we are dealing with integer realizations. For example, a straightforward adaptation of the proof of Theorem 1.2 will show that there exist unit disk graphs such that in *every* (not necessarily integer) realization the ratio of the largest distance between centers (of disks) to the shortest distance between centers is doubly exponentially large.

A natural direction for future work would be to consider the analogous problem in higher dimensions. The natural generalization of a disk graph is an intersection graph of d -dimensional balls, and the natural generalization of a unit disk graph is an intersection graph of d -dimensional balls of the same radius. The paper [21] by Kang and the second author builds on the work in the current paper, and amongst other things generalizes Theorem 1.2 to intersection graphs of d -dimensional balls of the same radius, for every dimension $d \geq 3$. (It is a follow-up paper for which the publication process was unusually quick.)

Whether Theorem 1.1 can be generalized to arbitrary dimension is an open problem. While this is probably true, proving it appears to be more difficult than the same-radius case. An important role in the proof of the lower bound in Theorem 1.1 is played by the graph H which has the property that for every realization \mathcal{A} of it by disks, there is a point $p(\mathcal{A})$ such that every disk that intersects every $A \in \mathcal{A}$ must contain $p(\mathcal{A})$. The construction of the graph H in Section 5 is pretty involved and relies on reasoning that is essentially two-dimensional. Constructing an analogue of the graph H with similar properties in general dimension seems to require new ideas. But, perhaps a very different proof structure is possible – one that does not rely on Theorem 3.2 or a variant of it.

There is also a body of work on representations of graphs as intersection graphs of d -dimensional axis-parallel boxes or hypercubes (see for instance [6, 39]). A natural question is whether in that setting the methods from the current paper might also be used to construct such intersection graphs such that if we want to realize them by boxes/hypercubes with all corners on the d -dimensional integer grid, then some coordinate of some corner will be doubly exponentially large. As it turns out this is not possible: arguments in a recent paper by E.J. van Leeuwen, J. van Leeuwen and the second author [35] can be adapted to show that in this case all coordinates can be kept no larger than $O(n^2)$.

Recall from early in Section 5 that every planar graph is a *coin graph*. That is, an intersection graph of closed disks with disjoint interiors. One might wonder whether it is always possible to find such a representation of a planar graph with all radii integers; but this is not the case, as was shown by Brightwell and Scheinerman [5] (Section 3). Observe that any coin graph representation with all radii integers will define a crossing-free straight-line drawing with all edges of integer length, but not every such straight-line drawing corresponds to a coin graph representation. So it might still be possible that every planar graph has a crossing-free straight-line drawing with all edges of integer length. This is an old problem posed by Harborth et al. [18], who proved that the one-skeleton of a Platonic solid has such a drawing. The question was upgraded to a conjecture by Kemnitz and Harborth in [23], who described a possible approach towards a solution. Some further progress towards a positive solution of the conjecture was recently made by Geelen et al. in [13], who showed that such a drawing can be found for every planar graph of maximum degree three.

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