

The first order convergence law fails for random perfect graphs

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Abstract

We consider first order expressible properties of random perfect graphs. That is, we pick a graph G_n uniformly at random from all (labelled) perfect graphs on n vertices and consider the probability that it satisfies some graph property that can be expressed in the first order language of graphs. We show that there exists such a first order expressible property for which the probability that G_n satisfies it does not converge as $n \rightarrow \infty$.

1 Introduction

A graph is perfect if the chromatic number equals the clique number in each of its induced subgraphs. Perfect graphs are a central topic in graph theory and play an important role in combinatorial optimization. In this paper we will study the random graph chosen uniformly at random from all (labelled) perfect graphs on n vertices. The first thing one might want in order to prove results about this object is a mechanism for generating random perfect graphs that is more descriptive than “put all n -vertex perfect graphs in a bag and pick one uniformly at random”. Such a mechanism has been introduced recently by McDiarmid and YOLOV [7]. Before presenting it, let us discuss as a preparation a simpler subclass of perfect graphs.

A graph is chordal if every cycle of length four or more has a chord, that is, an edge joining non-consecutive vertices in the cycle. A graph is split if its vertex set can be partitioned into a clique and an independent set (with arbitrary edges across the partition). It is easy to see that a split graph is chordal, but not conversely. On the other hand, it is known that almost all chordal graphs are split [1], in the sense that the proportion of chordal graphs that are split tends to 1 as the number of vertices n tends to infinity. Thus we arrive at a very simple process for generating random chordal graphs: (randomly) partition the vertex set into a clique A and an independent set B , and add an arbitrary set of edges between A and B (chosen uniformly at random from all possible sets of edges between A and B). The distribution we obtain in this way is not uniform as a split graph may arise from different partitions into a clique and an independent set, but it can be seen that when the size of A is suitably sampled then its total variational distance to the uniform distribution tends to zero as n tends to infinity. Now we turn to random perfect graphs. A graph G is *unipolar* if for

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some $k \geq 0$ its vertex set $V(G)$ can be partitioned into $k + 1$ cliques C_0, C_1, \dots, C_k , so that there are no edges between C_i and C_j for $1 \leq i < j \leq k$. Following [7] we call C_0 the *central clique*, and the C_i for $i \geq 1$ the *side cliques*. A graph G is *co-unipolar* if the complement \overline{G} is unipolar; and it is a *generalized split* graph if it is unipolar or co-unipolar. Notice that a graph can be both unipolar and co-unipolar, and that when the C_i for $i \geq 1$ are reduced to a single vertex, a generalized split graph is split. It can be shown that generalized split graphs are perfect, and it was proved in [8] that almost all perfect graphs are generalized split.

McDiarmid and Yolov [7] have devised the following process for generating random unipolar graphs. Choose an integer $m \in [n]$ according to a suitable distribution; choose a random m -subset $C_0 \subseteq [n]$; choose a (uniformly) random set partition $[n] \setminus C_0 = C_1 \cup \dots \cup C_k$ of the complement and make all the C_i into cliques; finally add edges between C_0 and $[n] \setminus C_0$ independently with probability $1/2$, and no further edges. Again this scheme is not uniform but it is shown in [7] that it approximates the uniform distribution on unipolar graphs on n vertices up to total variational distance $o_n(1)$. This gives a useful scheme for random perfect graphs: pick a random unipolar graph G on n vertices according to the previous scheme, and flip a fair coin: if the coin turns up heads then output G , otherwise output its complement \overline{G} . Several properties of random perfect graphs are proved in [7] using this scheme. One notable such result is that for every fixed graph H the probability that the random perfect graph on n vertices has an induced subgraph isomorphic to H tends to a limit that is either $0, 1/2$ or 1 .

In this paper we consider graph properties that can be expressed in the *first order language of graphs* (FO), on random perfect graphs. Formulas in this language are constructed using variables x, y, \dots ranging over the vertices of a graph, the usual quantifiers \forall, \exists , the usual logical connectives \neg, \vee, \wedge , etc., parentheses and the binary relations $=, \sim$, where $x \sim y$ denotes that x and y are adjacent. To aid readability we will also use commas and semicolons in the formulas in this paper. In FO one can for instance write “ G is triangle-free” as $\neg \exists x, y, z : (x \sim y) \wedge (x \sim z) \wedge (y \sim z)$. We say that a graph G is a model for the sentence $\varphi \in \text{FO}$ if G satisfies φ , and write $G \models \varphi$. (A sentence is a formula in which every variable is “bound” to a quantifier.)

Several restricted classes of graphs have been studied with respect to the limiting behaviour of FO properties, and usually one proves either a zero-one law (that is, every FO property has limiting probability $\in \{0, 1\}$) or a convergence law (that is, every FO property has a limiting probability). For instance, a zero-one law has been proved for trees [6] and for graphs not containing a clique of fixed size [4], while a convergence law has been proved for d -regular graphs for fixed d [5], and for forests and planar graphs [2].

In the light of the above mentioned result of McDiarmid and Yolov on the limiting probability of containing a fixed induced subgraph, one might expect the convergence law to hold for random perfect graphs, perhaps even with the limiting probabilities only taking the values $0, 1/2, 1$. The main result of this paper however states something rather different is the case.

Theorem 1 *There exists a sentence $\varphi \in \text{FO}$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}[P_n \models \varphi] \text{ does not exist,}$$

where P_n is chosen uniformly at random from all (labelled) perfect graphs on n vertices.

This is in stark contrast to random chordal graphs. The scheme we discussed above based on random split graphs is in fact very similar to the binomial bipartite random graph with

independent edge probabilities equal to $1/2$. A standard argument shows that in fact a zero-one law holds in this case, that is, the limiting probability that a FO property is satisfied tends either to 0 or 1 as $n \rightarrow \infty$ [10].

Our proof of Theorem 1 draws on the techniques introduced in the proof of the celebrated Shelah-Spencer result of non-convergence in the classical $G(n, p)$ model when $p = n^{-\alpha}$ and $\alpha \in (0, 1)$ is a rational number [9] (see also [10]). In fact, it is the richness of unipolar graphs together with the properties of random set partitions that allow us to produce a non-convergent first order sentence.

In addition we prove the following undecidability result.

Theorem 2 *There does not exist an algorithm that, given as input a $\varphi \in \text{FO}$ that is guaranteed to have either limiting probability zero or limiting probability one, decides whether the limiting probability equals one.*

For more discussion and open problems we refer the reader to Section 4

2 Preliminaries

Throughout this paper, we will say that a sequence of events E_1, E_2, \dots holds *with high probability* if $\lim_{n \rightarrow \infty} \mathbb{P}(E_n) = 1$.

Recall that the log-star function

$$\log^* n := \min\{k \in \mathbb{Z}_{\geq 0} : T(k) \geq n\},$$

is the least integer k for which $T(k)$ is at least n , where $T(\cdot)$ denotes the tower function – which can be defined recursively by $T(0) = 1$ and $T(n+1) = 2^{T(n)}$. Put differently, $T(n)$ is a “tower” of 2s of height n and $\log^* n$ is the number of iterations of the base two logarithm that are needed to reduce n to one or less.

The *spectrum* $\text{spec}(\varphi)$ of a sentence $\varphi \in \text{FO}$ is the set of all $n \in \mathbb{N}$ for which there exists a graph on n vertices that satisfies φ , that is

$$\text{spec}(\varphi) := \{v(G) : G \models \varphi\}.$$

The following lemma is a straightforward adaptation of a construction of Shelah and Spencer [9]

Lemma 3 *There exist $\varphi_0, \varphi_1 \in \text{FO}$ such that*

$$\log^* n \bmod 100 \in \{2, \dots, 49\} \Rightarrow n \in \text{spec}(\varphi_0) \setminus \text{spec}(\varphi_1),$$

$$\log^* n \bmod 100 \in \{52, \dots, 99\} \Rightarrow n \in \text{spec}(\varphi_1) \setminus \text{spec}(\varphi_0).$$

We remark that φ_0 is constructed explicitly in [10], pages 112–113, and that φ_1 is a straightforward adaptation of this construction.

We also need the following consequence of a more general theorem of Trakhtenbrot [11] (see also [3, page 303]) on undecidability in first order logic.

Theorem 4 (Trakhtenbrot) *There does not exist an algorithm that can decide, given an arbitrary sentence $\varphi \in \text{FO}$ as input, whether $\text{spec}(\varphi) = \emptyset$. (That is, whether or not φ is satisfied by at least one finite graph.)*

We next recall the scheme from [7] for generating random unipolar graphs with n vertices, together with some key properties of the construction.

- Choose the size m of the central clique C_0 according to a distribution proportional to $\binom{n}{m} 2^{m(n-m)} B(n-m)$, where $B(n)$ is the n -th Bell number (the exact distribution is not needed and is shown only for completeness), and choose C_0 as a random m -subset of $[n]$.
- Take a (uniformly) random set partition of the complement $[n] \setminus C_0 = C_1 \cup \dots \cup C_k$, and make each C_i a clique.
- Add edges arbitrarily between C_0 and $[n] \setminus C_0$ (i.e. add a set of edges chosen uniformly at random from all possible sets of edges between C_0 and $[n] \setminus C_0$), and no further edges.

A random perfect graph is obtained by taking a random unipolar graph G_n as generated above and flipping a fair coin: if the coin turns up heads then take G_n , otherwise take its complement. It is proved in [7] that the probability that a uniformly random perfect graph is both unipolar and co-unipolar is exponentially small, and that the distribution obtained by the above scheme has total variation distance $o(1)$ to the uniform random perfect graph, hence it can be used to prove properties of the uniform random perfect graph.

It is shown in [7] that the cliques C_i satisfy the following properties with probability tending to one as n tends to infinity:

- (i) $|C_0| = \frac{n}{2}(1 + o(1))$. This follows from [7, Theorem 2.5].
- (ii) Let r be the unique root of $re^r = n - |C_0|$. For $t = 1, \dots, (e - \varepsilon) \ln n$, with $\varepsilon > 0$ arbitrary but fixed, we have

$$|\{j : |C_j| = t\}| = \Omega(r^t/t!). \quad (1)$$

This follows from the results in [7, Section 2.2.3].

We note that, with high probability, we have

$$r = \ln n - (1 + o(1)) \ln \ln n. \quad (2)$$

3 Proofs

We start by noticing that it is easy to tell in FO whether the random perfect graph is unipolar.

Corollary 5 *There is a sentence $\text{UniP} \in \text{FO}$ such that if P_n denotes the random perfect graph then, with high probability, $P_n \models \text{UniP}$ if and only if P_n is unipolar.*

Proof: Let H be any graph that is unipolar but not co-unipolar, and let $\text{UniP} \in \text{FO}$ formalize that “ H is an induced subgraph”. The conclusion follows from [7, Theorem 2.3 and Lemma 4.1], implying that $\mathbb{P}(P_n \models \text{UniP} \mid P_n \text{ is unipolar}) = 1 - o(1)$, and $\mathbb{P}(P_n \models \text{UniP} \mid P_n \text{ is not unipolar}) = o(1)$ (the second statement is clear since the class of co-unipolar graphs is closed under taking induced subgraphs). ■

In what follows G_n will denote the alternative scheme of random unipolar graphs of McDiarmid and YOLOV. In the light of the above, it is enough for us to show the statement of Theorem 1 for G_n rather than P_n . This is because if φ is such that $\mathbb{P}(G_n \models \varphi)$ does not converge then

$$\mathbb{P}(P_n \models \text{UniP} \wedge \varphi) = (1/2 + o(1)) \cdot \mathbb{P}(G_n \models \varphi)$$

also does not converge. Similarly, it suffices to prove Theorem 2 for G_n rather than P_n . To see this, note that

$$\mathbb{P}(P_n \models (\text{UniP} \wedge \varphi) \vee (\neg \text{UniP} \wedge \bar{\varphi})) = \mathbb{P}(G_n \models \varphi) + o(1),$$

where $\bar{\varphi}$ is obtained from φ by swapping $a \sim b$ for $\neg(a \sim b)$ (and hence also $\neg(a \sim b)$ is replaced with $\neg\neg(a \sim b)$ which is equivalent to $a \sim b$), so that $G \models \bar{\varphi}$ if and only if $\bar{G} \models \varphi$.

In the remainder of this section we will therefore only work with G_n . In the proofs below, we usually think of revealing (conditioning on) the partition C_0, \dots, C_k of $[n]$ so that all computations of probabilities etc. will only be with respect to the random edges between C_0 and $\bigcup_{i>0} C_i$. This is justified because in the construction, we add the edges between C_0 and its complement last, after the partition C_0, \dots, C_k has been chosen.

The following observation provides us a useful way to distinguish whether a vertex is in C_0 or not.

Lemma 6 *With high probability it holds that, for each vertex v ,*

$$v \in C_0 \text{ if and only if } N(v) \text{ contains a stable set of size three.}$$

Proof: We first note that if $v \notin C_0$ then $v \in C_i$ for some $i > 0$ and then its neighbourhood $N(v) \subseteq C_0 \cup C_i$ is covered by two cliques. So in particular $N(v)$ does not contain a stable set of size three.

For the reverse, we first observe that by construction and estimates (1) and (2), there is a constant $c > 0$ such that with high probability there are at least $c \ln^2 n$ parts C_j of size $|C_j| = 2$. Moreover, if a vertex $v \in C_0$ does not have a stable set of size three in its neighbourhood, then v is adjacent to no more than four of the vertices in $\bigcup_{|C_j|=2} C_j$. Thus, if we let E denote the event that there is a $v \in C_0$ whose neighbourhood $N(v)$ does not contain a stable set of size three then

$$\begin{aligned} \mathbb{P}(E) &\leq \mathbb{P}(|\{j : |C_j| = 2\}| < c \ln^2 n) + n \cdot \mathbb{P}(\text{Bi}(2c \ln^2 n, 1/2) \leq 4) \\ &= o(1) + n e^{-\Omega(\ln^2 n)} = o(1), \end{aligned}$$

where we have used the Chernoff bound. ■

Corollary 7 *There exists an FO-formula $\text{In}C_0$ with one free variable such that, with high probability, $\text{In}C_0(x)$ holds for all $x \in C_0$ and $\neg \text{In}C_0(x)$ holds for all $x \notin C_0$.*

Proof: It is easily checked that the following formula states that the neighbourhood of x contains a stable set of size three:

$$\begin{aligned} \text{InC}_0(x) &:= \exists x_1, x_2, x_3 : (x \sim x_1) \wedge (x \sim x_2) \wedge (x \sim x_3) \\ &\quad \wedge \neg(x_1 = x_2) \wedge \neg(x_1 = x_3) \wedge \neg(x_2 = x_3) \\ &\quad \wedge \neg(x_1 \sim x_2) \wedge \neg(x_1 \sim x_3) \wedge \neg(x_2 \sim x_3). \end{aligned}$$

■

For $S \subseteq [n]$, we write $N(S) := \bigcap_{v \in S} N(v)$ for the set of common neighbours of S in our random graph G_n .

Corollary 8 *There exists an FO-formula CmNb with two free variables such that, with high probability, $\text{CmNb}(x, y)$ holds if and only if $x \in C_0, y \in C_i$ for some $i > 0$, and $x \in N(C_i)$.*

Proof: It is easily checked that the following definition works out (assuming that InC_0 expresses membership of C_0 as intended):

$$\begin{aligned} \text{CmNb}(x, y) &:= \text{InC}_0(x) \wedge \neg \text{InC}_0(y) \wedge (x \sim y) \\ &\quad \wedge (\forall z : (\neg \text{InC}_0(z) \wedge (z \sim y)) \Rightarrow (x \sim z)). \end{aligned}$$

■

For $S, T \subseteq [n]$ we let $H(S, T)$ denote the graph with vertex set S and an edge between $a, b \in S$ if and only if there is a $v \in T$ that is adjacent to both a and b .

Corollary 9 *There exists an FO-formula Edge with three free variables such that, with high probability, $\text{Edge}(x, y, z)$ holds if and only if $x, y \in C_0, x \neq y, z \in C_i$ for some $i > 0$, and xy is an edge of $H(C_0, C_i)$.*

Proof: It is easily checked that the following formula will do the trick (again assuming InC_0 expresses the right thing):

$$\begin{aligned} \text{Edge}(x, y, z) &:= \text{InC}_0(x) \wedge \text{InC}_0(y) \wedge \neg \text{InC}_0(z) \wedge \neg(x = y) \\ &\quad \wedge (\exists z_1 : ((z_1 = z) \vee (\neg \text{InC}_0(z_1) \wedge (z_1 \sim z)) \wedge (x \sim z_1) \wedge (y \sim z_1))). \end{aligned}$$

(To aid the reader, let us point out that the only purpose of the variable z is to represent C_i .)

■

Corollary 10 *For every $\varphi \in \text{FO}$ there exists an FO-formula $\Phi(x, y)$ with two free variables such that, with high probability, $\Phi(x, y)$ holds if and only if $x \in C_i, y \in C_j$ for some $i, j > 0$, and $H(N(C_i), C_j) \models \varphi$.*

Proof: The formula Φ can be read off from φ in a straightforward way as follows. In φ , we replace every occurrence of $a \sim b$ by $\text{Edge}(a, b, y)$ and we “relativize the quantifiers to $\text{CmNb}(\cdot, x)$ ”. That is:

- $\exists z : \psi$ is replaced by $\exists z : \text{CmNb}(z, x) \wedge \psi$, and;
- $\forall z : \psi$ is replaced by $\forall z : \text{CmNb}(z, x) \Rightarrow \psi$.

Finally we take the conjunction of the end result with $\neg \text{InC}_0(x) \wedge \neg \text{InC}_0(y)$. The reader can easily verify that the formula we obtain is as required (assuming InC_0 and CmNb take on their intended meanings). \blacksquare

Let us write

$$\ell := \lceil \ln \ln \ln n \rceil.$$

Lemma 11 *With high probability, for every $0 \leq \ell' \leq \ell$, there exist $n^{\Omega(1)}$ indices $i > 0$ with $|N(C_i)| = \ell'$.*

Proof: Let $t \in \mathbb{N}$ be such that $(n/2) \cdot (1/2)^{t-1} > \ell' \geq (n/2) \cdot (1/2)^t$. So we have $(n/2) \cdot (1/2)^t \in (\ell'/2, \ell']$ and

$$t = (1 + o(1)) \log_2 n = (1 + o(1)) \ln n / \ln 2. \quad (3)$$

Let us write $J := \{j : |C_j| = t\}$. In the McDiarmid-Yolov construction, with high probability, we have

$$\begin{aligned} |J| &= \Omega(r^t/t!) \\ &= \Omega(\exp[t \ln r - t \ln t + t + O(\ln t)]) \\ &= \Omega(\exp[t \ln(r/t) + t + O(\ln t)]) \\ &= \Omega(\exp[t \cdot (\ln \ln 2 + 1 + o(1)) + O(\ln t)]) \\ &= \exp[\Omega(\ln n)] \\ &= n^{\Omega(1)}, \end{aligned}$$

where we have used Stirling's approximation in the second line, and we have used that $r/t = \ln 2 + o(1)$ by (2) and (3) in the fourth line, and that $\ln \ln 2 + 1 \approx 0.633 > 0$ in the fifth line.

We have that (conditional on the partition C_0, \dots, C_k):

$$|N(C_j)| \stackrel{d}{=} \text{Bi}(|C_0|, (1/2)^t) \quad (\forall j \in J).$$

Hence

$$\mathbb{E}|N(C_j)| = |C_0|(1/2)^t = (1 + o(1))(n/2)(1/2)^t = \Theta(\ell').$$

(The expectation again being conditional on the partition C_0, \dots, C_k). Therefore, for each $j \in J$ (conditional on the partition C_0, \dots, C_k) we have

$$\begin{aligned} \mathbb{P}(|N(C_j)| = \ell') &= \binom{|C_0|}{\ell'} (1/2)^{t\ell'} (1 - (1/2)^t)^{|C_0| - \ell'} \\ &\geq (|C_0|/\ell')^{\ell'} (1/2)^{t\ell'} (1 - (1/2)^t)^{|C_0| - \ell'} \\ &= (|C_0|(1/2)^t/\ell')^{\ell'} (1 - (1/2)^t)^{|C_0| - \ell'} \\ &= (\Theta(\ell')/\ell')^{\ell'} (1 - (1/2)^t)^{|C_0| - \ell'} \\ &= \Theta(1)^{\ell'} (1 - (1/2)^t)^{|C_0| - \ell'} \\ &= \exp[\pm O(\ell') + (|C_0| - \ell') \ln(1 - (1/2)^t)] \\ &\geq \exp[-O(\ell') - O(|C_0|(1/2)^t)] \\ &\geq \exp[-O(\ell')]. \end{aligned}$$

Here we have used the standard bound $\binom{n}{k} \geq (n/k)^k$ in the second line, and the estimate $\ln(1-x) = -\Theta(x)$ as $x \downarrow 0$ in the seventh line. Let us denote by $I := \{j \in J : |N(C_j)| = \ell'\}$ the

number of $j \in J$ for which C_j has exactly ℓ' common neighbours. The previous considerations show that

$$\mathbb{E}|I| = |J| \cdot \exp[-O(\ell')] = \exp[\Omega(\ln n) - O(\ell')] = \exp[\Omega(\ln n)] = n^{\Omega(1)},$$

since $\ell' \leq \ell \ll \ln n$. Let us now point out that the random variables $\{|N(C_j)| : j \in J\}$ are in fact independent (since they depend on disjoint sets of edges – of course this is again all conditional on the partition C_0, \dots, C_k). So in particular $|I|$ is a binomial random variable, whose mean tends to infinity. Hence (for instance, by Chebyschev's inequality) $\mathbb{P}(|I| < \mathbb{E}|I|/2) = o(1)$. \blacksquare

Lemma 12 *With high probability, the following holds. For every $i, j > 0$ such that $|N(C_i) \cup N(C_j)| \leq 2\ell$, and for every (labelled) graph G with $V(G) = N(C_i) \cup N(C_j)$, there is a $k > 0$ such that $H(N(C_i) \cup N(C_j), C_k) = G$.*

Remark. We emphasize that when we say $G = H$ we do not just speak about isomorphism, but we really mean that $V(G) = V(H)$ and $E(G) = E(H)$.

Proof: We set $t := \lceil \sqrt{\ln n} \rceil$, and let $K := \{k : |C_k| = t\}$. As before, with high probability, we have

$$\begin{aligned} |K| &= \Omega(r^t/t!) \\ &= \Omega(\exp[t \ln r - t \ln t + t + O(\ln t)]) \\ &= \Omega(\exp[t \ln(r/t) + t + O(\ln t)]) \\ &= \exp[\Omega(t \ln \ln n)], \end{aligned}$$

where we have again used Stirling for the second line, and that $r/t = (1 + o(1))\sqrt{\ln n}$ by (2) for the last line.

For the moment, let us fix some set $S \subseteq C_0$ of cardinality $\leq 2\ell$, a $k \in K$ and a “target” graph G with $V(G) = S$. Let $E_{S,G,k}$ denote the event that $H(S, C_k) = G$. Since $|C_k| = t > \binom{|S|}{2}$ there is at least one way to choose the edges between S and C_k that would result in desired situation where $H(S, C_k) = G$. In other words,

$$\mathbb{P}(E_{S,G,k}) \geq (1/2)^{t|S|} \geq (1/2)^{2t\ell}.$$

Writing $E_{S,G} := \bigcup_{k \in K} E_{S,G,k}$ and denoting by A^c the complement of A , we find that

$$\begin{aligned} \mathbb{P}(E_{S,G}^c) &\leq (1 - (1/2)^{2\ell t})^{|K|} \\ &\leq \exp[-|K| \cdot (1/2)^{2\ell t}] \\ &= \exp[-\exp[\Omega(t \ln \ln n) - O(t\ell)]] \\ &= \exp[-\exp[\Omega(t \ln \ln n)]] . \end{aligned}$$

Let E denote the event that for every $S \subseteq C_0$ of the form $S = N(C_i) \cup N(C_j)$ with $|S| \leq 2\ell$ and for every target graph G with $V(G) = S$ there is some $k > 0$ such that $H(S, C_k) = G$.

We remark that there are at most n^2 choices of the set S (as it must be a union $N(C_i) \cup N(C_j)$) and at most $2^{\binom{2\ell}{2}}$ choices of the target graph G . Hence we have that

$$\begin{aligned} \mathbb{P}(E^c) &\leq n^2 \cdot 2^{\binom{2\ell}{2}} \cdot \exp[-\exp[\Omega(t \ln \ln n)]] \\ &= \exp[O(\ln n) + O(\ell^2) - \exp[\Omega(t \ln \ln n)]] \\ &= o(1), \end{aligned}$$

Since $\exp[\Omega(t \ln \ln n)] \gg \ln n (\gg \ell^2)$. ■

We now have all the tools to prove Theorem 2.

Proof of Theorem 2: Let $\varphi \in \text{FO}$ be an arbitrary sentence and let $\Phi(.,.)$ be as provided by Corollary 10. Consider the sentence

$$\psi := \exists x, y : \Phi(x, y).$$

Up to error probability $o(1)$, we have that ψ holds if and only if there exist $i, j > 0$ such that $H(N(C_i), C_j) \models \varphi$.

Thus, if $\text{spec}(\varphi) = \emptyset$, that is if there is no finite graph that satisfies φ , then clearly $\mathbb{P}(G_n \models \psi) = o(1)$.

On the other hand, if $\text{spec}(\varphi) \neq \emptyset$, that is, if there is some finite graph H such that $H \models \varphi$, then by Lemmas 11 and 12 for n sufficiently large (namely n such that $\ell \geq v(H)$) we will find indices $i, j > 0$ such that $H(N(C_i), C_j) = H$. This shows that, if $\text{spec}(\varphi) \neq \emptyset$, then $\mathbb{P}(G_n \models \psi) = 1 - o(1)$.

We have just shown that the constructed sentence ψ has limiting probability zero if $\text{spec}(\varphi) = \emptyset$ and limiting probability one otherwise. Thus any algorithm that can decide whether $\lim_{n \rightarrow \infty} \mathbb{P}(G_n \models \psi)$ equals zero or equals one will allow us to decide whether or not φ has a finite model. Therefore, there can be no such algorithm as this would contradict Trakhtenbrot's theorem. ■

Before we can prove Theorem 1 we need one more ingredient.

Corollary 13 *There exists an FO-formula Bgr with two free variables such that, with high probability:*

- *If $\text{Bgr}(x, y)$ holds then there exist $i, j > 0$ such that $x \in C_i, y \in C_j$ and $|N(C_i)| > |N(C_j)|$;*
- *If $x \in C_i, y \in C_j$ for some $i, j > 0$ with $|N(C_j)| < |N(C_i)| \leq \ell$ then $\text{Bgr}(x, y)$ holds.*

Proof: The main idea behind the FO-formula we'll give is that it expresses that there exists a $k > 0$ such that $H(C_j \Delta C_i, C_k)$ is a matching between $C_j \setminus C_i$ and $C_i \setminus C_j$ that saturates all of $C_j \setminus C_i$, but there is at least one unmatched vertex in $C_i \setminus C_j$. The reader can check that the following formula will do the trick (assuming that InC_0 and CmNb express the correct thing):

$$\begin{aligned} \text{Bgr}(x, y) \quad := & \neg \text{InC}_0(x) \wedge \neg \text{InC}_0(y) \wedge \neg(x = y) \wedge \neg(x \sim y) \\ & \wedge (\exists z : (\forall y_1 : (\text{CmNb}(y_1, y) \wedge \neg \text{CmNb}(y_1, x)) \Rightarrow \\ & (\exists! x_1 : \text{CmNb}(x_1, x) \wedge \neg \text{CmNb}(x_1, y) \wedge \text{Edge}(x_1, y_1, z)))) \\ & \wedge (\forall x_1, y_1, y_2 : (\text{CmNb}(x_1, x) \wedge \neg \text{CmNb}(x_1, y) \wedge \text{CmNb}(y_1, y) \\ & \wedge \neg \text{CmNb}(y_1, x) \wedge \text{CmNb}(y_2, y) \wedge \neg \text{CmNb}(y_2, x) \wedge \text{Edge}(x_1, y_1, z) \\ & \wedge \text{Edge}(x_1, y_2, z)) \Rightarrow (y_1 = y_2)) \\ & \wedge (\exists x_1 : \text{CmNb}(x_1, x) \wedge \neg \text{CmNb}(x_1, y) \\ & \wedge (\forall y_1 : \text{CmNb}(y_1, y) \wedge \neg \text{CmNb}(y_1, x) \Rightarrow \neg \text{Edge}(x_1, y_1, z))))). \end{aligned}$$

■

We are now ready to prove the main result.

Proof of Theorem 1: Let Φ_i denote the formula that Corollary 10 produces when applied to the sentence φ_i from Lemma 3. We define the following FO-sentence:

$$\varphi := \exists x, y : \Phi_1(x, y) \wedge \neg(\exists x', y' : \text{Bgr}(x', x) \wedge \Phi_0(x', y')).$$

Up to error probability $o(1)$, the sentence φ will hold if and only if $H(N(C_i), C_j) \models \varphi_1$ for some $i, j > 0$, and moreover if $H(N(C_{i'}), C_{j'}) \not\models \varphi_0$ for some $i', j' > 0$ then $\text{Bgr}(x, x')$ does not hold for any $x \in C_i, x' \in C_{i'}$. We briefly explain how this implies that φ does not have a limiting probability.

First we consider an increasing subsequence $(n_k)_k$ of the natural numbers for which $\log^* n \bmod 100 = 75$. Observe that

$$\log^* n - 10 \leq \log^* \ell \leq \log^* n. \quad (4)$$

With high probability there are lots of C_i for which $|N(C_i)| = \ell$ by Lemma 11, and by Lemma 12 for each of these there is a j such that $H(N(C_i), C_j) \models \varphi_1$ (since $\ell \in \text{spec}(\varphi_1)$) as $\log^* \ell \bmod 100 \in \{65, \dots, 75\}$ by (4) and the choice of n . So there are (lots of) pairs of vertices x, y such that $\Phi_1(x, y)$ holds and $x \in C_i$ for some $i > 0$ with $|N(C_i)| = \ell$. On the other hand, with high probability, for any x' such that $\text{Bgr}(x', x)$ it must hold that $x' \in C_{i'}$ for some $i' > 0$ with $\ell = |N(C_i)| < |N(C_{i'})| \leq n$. So in particular $\log^*(|N(C_{i'})|) \in \{65, \dots, 75\}$. Thus $|N(C_{i'})| \notin \text{spec}(\varphi_0)$, which shows that $H(N(C_{i'}), C_{j'}) \not\models \varphi_0$ for any $j' > 0$. In other words, if $\text{Bgr}(x', x)$ holds then there cannot be any y' such that $\Phi_0(x', y')$ holds. This shows that

$$\lim_{\substack{n \rightarrow \infty, \\ \log^* n \bmod 100 = 75}} \mathbb{P}(G_n \models \varphi) = 1.$$

Next, let us consider an increasing subsequence $(n_k)_k$ of the natural numbers for which $\log^* n \bmod 100 = 25$. In this case $\log^* \ell \bmod 100 \in \{15, \dots, 25\}$. In particular $\ell, \dots, n \notin \text{spec}(\varphi_1)$. So, with high probability, if there is pair x, y such that $\Phi_1(x, y)$ holds then we must have $x \in C_i$ for some $i > 0$ with $|N(C_i)|$ strictly smaller than ℓ . But then we can again apply Lemma's 11 and 12 to find that, with high probability, there exist x', y' with $x' \in C_{i'}, y' \in C_{j'}$ for some $i', j' > 0$ such that $|N(C_{i'})| = \ell$ and $H(N(C_{i'}), C_{j'}) \models \varphi_0$. Since $\ell = |N(C_{i'})| > |N(C_i)|$, with high probability, $\text{Bgr}(x', x)$ will hold by Corollary 13. This shows that

$$\lim_{\substack{n \rightarrow \infty, \\ \log^* n \bmod 100 = 25}} \mathbb{P}(G_n \models \varphi) = 0. \quad \blacksquare$$

4 Discussion and further work

We remark that with very minor variations on our proofs, it can be seen that Theorems 1 and 2 also hold for random unipolar and random co-unipolar graphs. Similarly, by a minor variation of the proof of Theorem 2, it can be shown that it is undecidable to determine, given a sentence $\varphi \in \text{FO}$ that is guaranteed to have limiting probability $\in \{0, 1/2\}$ (resp. $\{1/2, 1\}$), whether the limit is $1/2$.

Furthermore, by combining Corollary 10.37 from [3] with a minor variation of our proof of Theorem 2 it can be seen that there are formulas with a limiting probability but for which

the convergence is extremely slow, in the following precise sense. For every recursive function $f : \mathbb{N} \rightarrow \mathbb{N}$ and every k there exists a $\varphi \in \text{FO}$ of quantifier depth $\leq k$ (the definition of which can for instance be found in [3]) such that $\lim_{n \rightarrow \infty} \mathbb{P}(G_n \models \varphi) = 1$ yet $\max_{n \leq f(k)} \mathbb{P}(G_n \models \varphi) = o(1)$.

Recall that having a fixed graph H as an induced subgraph will have limiting probability $0, 1/2$ or 1 . During the exploratory stages of the research that led to the present paper, the last two authors spent some effort trying to construct a FO sentence with a limiting probability $\notin \{0, 1/2, 1\}$, without success. We thus pose this as an open problem to which we would love to know the answer.

Question. Does there exist a $\varphi \in \text{FO}$ such that $\lim_{n \rightarrow \infty} \mathbb{P}(G_n \models \varphi)$ exists and takes on a value other than $0, 1/2$ or 1 ?

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