Improper colouring of (random) unit disk graphs*

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Abstract

For any graph G, the *k-improper chromatic number* $\chi^k(G)$ is the smallest number of colours used in a colouring of G such that each colour class induces a subgraph of maximum degree k. We investigate χ^k for unit disk graphs and random unit disk graphs to generalise results of [24, 22] and [23].

1 Introduction

Given a set V of points in the plane and a distance threshold $r > 0$, we let $G(V, r)$ denote the following graph. The vertex set is V and distinct vertices are joined by an edge whenever the Euclidean distance between them is less than r. Any graph isomorphic to such a graph is called a *unit disk graph*. The study of the class of unit disk graphs stems partly from applications in communication networks. In particular, the problem of finding a proper vertex-colouring — in which the vertices of a graph are coloured so that adjacent vertices do not receive the same colour — of a given unit disk graph is closely associated with the so-called *frequency allocation problem* [11]. Consult Leese and Hurley [20] for a more general treatment of this important problem.

The authors McDiarmid, Reed, and Müller [24, 22, 23] investigated the chromatic number χ for unit disk graphs in two related cases. The first case is the asymptotic limit of χ where V is countably infinite and the

*This work was partially supported by Région Provence-Alpes-Côte D'Azur.

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distance threshold r approaches infinity: for countable sets V with finite upper density (to be defined below), the ratio of chromatic number over clique number approaches $\frac{2\sqrt{3}}{\pi}$ as $r \to \infty$ [24]. The second case is the asymptotic behaviour of χ for unit disk graphs based on randomly chosen points in the plane (where the distance threshold r approaches 0 as the number of points n approaches infinity). The papers [22, 23] establish almost sure (and in probability) convergence results for these random instances of unit disk graphs.

In this paper, we are also interested in vertex colourings of unit disk graphs; however, we partially relax the condition that any two vertices with the same colour may not be adjacent. Recall that, given an arbitrary colouring, a *colour class* is a set of vertices all assigned the same colour. Given $k \geq 0$, we say that a graph is *k-improperly colourable* if there is a colouring in which each colour class induces a subgraph with maximum degree at most k. We wish to find the *k-improper chromatic number* χ^k, i.e., the minimum number of colours used in such a colouring. Note that proper colouring is just 0-improper colouring and hence $\chi = \chi^0$. Our aim, in this paper, is to determine χ^k for unit disk graphs in the two cases mentioned above.

The k-improper chromatic number has been studied under various guises since the mid-1980's. The concept was introduced independently by Andrews and Jacobson [1], Harary and Fraughnaugh (née Jones) [12, 13], and Cowen *et al.* [6]. The first paper established several general lower bounds for the k-improper chromatic number; the second studied χ^k within the overall setting of generalised chromatic numbers (cf. Bollobás and West [4] and references cited therein); while the third provided the best upper bounds on k-improper chromatic numbers for planar graphs to generalise the Four Colour Theorem. Several related papers have appeared since then and these include (but are by no means limited to) [2, 9, 7, 8, 32, 3].

The problem of k-improperly colouring unit disk graphs arises in practice for instance when modelling certain satellite communications problems. More precisely, Alcatel Industries has proposed the following problem: a satellite sends information to receivers on earth. Because it is technically difficult to precisely focus the satellite's signal upon a receiver, part of the signal spills over into the surrounding area creating noise for nearby receivers listening on the same frequency. A receiver is able to distinguish its particular signal from the noise if the sum of total noise does not exceed a certain threshold. The problem is to optimally assign frequencies to the receivers in such a way that each receiver can obtain its intended signal properly. In the simplest model of this problem, we assume that each receiver's signal contributes noise to other receivers within a disk-shaped area surrounding it, and that the radii of the noise disks and the intensity of the noise created by the signals are independent of the frequency and the receiver. Hence, to distinguish its signal from the noise, a receiver must be in the noise disks of at most k receivers (where k is a fixed integer depending on the noise threshold and the intensity of the signals) listening on the same frequency. It is clear that, under this model, we are precisely asking to find χ^k for a given unit disk graph. The reader can refer to [15, 16] for further study of this problem.

Before going further, we must review and introduce some basic terminology and properties. We denote the maximum degree of G by $\Delta(G)$. A *clique* is a set of pairwise adjacent vertices; the *clique number* $\omega(G)$ is the maximum number of vertices in a clique of G. An *independent set* is a set of pairwise non-adjacent vertices; the *independence number* $\alpha(G)$ is the maximum number of vertices in an independent set of G. A *k-dependent set* is a set of vertices whose induced subgraph has maximum degree at most k; the *k-dependence number* $\alpha^k(G)$ is the maximum number of vertices in a k-dependent set of G. Recall that $\frac{|V(G)|}{\alpha(G)} \leq \chi(G)$. An analogous lower bound on $\chi^k(G)$ is as follows.

Proposition 1 *For any graph G and $k \geq 0$, $\chi^k(G) \geq \frac{|V(G)|}{\alpha^k(G)}$*

We leave the straightforward proof to the reader. Also recall that $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$ for any graph G. The following proposition shows that these bounds generalise to $\chi^k(G)$ in an appropriate sense—recalling that $\omega(G) \leq \chi(G)$.

Proposition 2 *For any graph G and $k \geq 0$, $\left\lceil \frac{\chi(G)}{k+1} \right\rceil \leq \chi^k(G) \leq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$.*

The second inequality is a corollary of a result due to Lovász [21]. To see that the first inequality holds, note that any colour class in a k-improper colouring induces a subgraph of maximum degree at most k and hence can be partitioned into $k+1$ independent sets.

Under the asymptotic models of unit disk graphs considered in this paper the lower bound in Proposition 2 more or less gives the right answer. We will see that in both models $(k+1)\chi^k$ approaches χ (in an appropriate sense, with some small exceptions). We mention here that under the standard asymptotic model for general graphs, i.e., the Erdős-Rényi $G(n,p)$ random graph model, there is qualitatively different behaviour. The perhaps rather counterintuitive result that, if $k = o(\ln(np))$ and $np \to \infty$ then $\chi^k(G(n,p))/\chi(G(n,p)) \to 1$ in probability, has been proved in [19].

We note that the clique number of a unit disk graph can be found in polynomial time by means of an $O(n^{4.5})$ algorithm [5] and even when an explicit representation in the plane is not available [30]. In contrast, the problem of finding the chromatic number of unit disk graphs is NP-complete [5]. Recent work [14] shows that the same holds for the k-improper chromatic number χ^k, for any fixed k. Furthermore, by the above two propositions and since

$\Delta(G) \leq 6\omega - 6$ for any unit disk graph G (to see this, consider that the vertices in each $\pi/3$-sector of a unit disk induce a clique), we have a heuristic for χ^k with approximation ratio 6. Except for the case $k = 0$ (where [27] gives a 3-approximation), 6 is the best known approximation ratio for χ^k.

The paper is divided as follows. In Sections 2 and 3, we consider the extensions of [24], stating definitions and results, then later giving proofs. Similarly, in Sections 4 and 5, we analyse improper colouring for random geometric graphs to extend results of [22, 23].

2 Asymptotically, improperly colouring unit disk graphs

This section discusses our extensions of McDiarmid and Reed [24]. Let V be any countable set of points in the plane. For $x > 0$, let $f(x)$ be the supremum of the ratio $\frac{|V \cap S|}{x^2}$ over all open $(x \times x)$ squares S with sides aligned with the axes. The *upper density* of V is $\sigma^+(V) = \inf_{x>0} f(x)$.

Theorem 1 ([24]) *Let V be a countable non-empty set of points in the plane with upper density $\sigma^+(V) = \sigma$.*

(i) $\frac{\omega(G(V,r))}{r^2} \geq \sigma\frac{\pi}{4}$ *and* $\frac{\chi(G(V,r))}{r^2} \geq \sigma\frac{\sqrt{3}}{2}$ *for any* $r > 0$*;*

(ii) $\frac{\Delta(G(V,r))}{r^2} \to \sigma\pi$, $\frac{\omega(G(V,r))}{r^2} \to \sigma\frac{\pi}{4}$ *and* $\frac{\chi(G(V,r))}{r^2} \to \sigma\frac{\sqrt{3}}{2}$ *as* $r \to \infty$*.*

We extend this theorem as follows.

Theorem 2 *Let V be a countable non-empty set of points in the plane with upper density $\sigma^+(V) = \sigma$, and let $\gamma = \sigma\frac{\sqrt{3}}{2}$. Then*

(i) $\frac{\chi^k(G(V,r))}{r^2} \geq \frac{\gamma}{k+1}$ *for any* $r > 0$*; and*

(ii) *as* $k \to \infty$, $(k+1)\frac{\chi^k(G(V,r))}{r^2} \to \gamma$ *if* $k = o(r)$*.*

In particular, the following holds.

Corollary 1 *Let $V \subseteq \mathbb{R}^2$ be a set with upper density $\sigma \in (0, \infty)$ and suppose that k satisfies $k = o(r)$. Then*

$$\frac{(k+1)\chi^k(G(V,r))}{\chi(G(V,r))} \to 1,$$

as $r \to \infty$*.*

It also holds that, for any countable set V of points in the plane with finite positive upper density, the ratio of $\chi^k(G(V,r))$ to $\frac{\omega(G(V,r))}{(k+1)}$ tends to $\frac{2\sqrt{3}}{\pi}$ as r approaches infinity. When k is zero, this result was proved in [24] and conjectured for the triangular lattice (defined below) in [10]. We have allowed k to vary as a function of r, but this does not detriment our results.

McDiarmid and Reed also tighten the upper bounds in Theorem 1 for the case where the points are approximately uniformly spread over the plane. Given a set V of points in the plane, a *cell structure* of V with density σ and radius ρ is a family $(C_v : v \in V)$ of sets that partition the plane and such that each C_v has area $\frac{1}{\sigma}$ and is contained in a ball of radius ρ about v.

Theorem 3 ([24]) *Let the set V of points in the plane have a cell structure with density σ and radius ρ, and let $\gamma = \sigma\frac{\sqrt{3}}{2}$. Then, for any $r > 0$,*

$$\omega(G(V,r)) \leq \frac{\sigma\pi}{4}(r+2\rho)^2 \text{ and}$$

$$\chi(G(V,r)) < \left(\gamma^{1/2}(r+2\rho) + \frac{2}{\sqrt{3}} + 1\right)^2.$$

Thus, combined with Theorem 1,

$$\omega(G(V,r)) = \sigma\frac{\pi}{4}r^2 + \mathrm{O}(r) \text{ and}$$

$$\chi(G(V,r)) = \gamma r^2 + \mathrm{O}(r) \text{ as } r \to \infty.$$

We extend this theorem as follows.

Theorem 4 *Let the set V of points in the plane have a cell structure with density σ and radius ρ, and let $\gamma = \sigma\frac{\sqrt{3}}{2}$. Then, if $r \geq \frac{3k}{2}$,*

$$\chi^k(G(V,r)) < \frac{\left(\gamma^{1/2}(r+2\rho) + \frac{2}{\sqrt{3}} + 2k + 1\right)\left(\gamma^{1/2}(r+2\rho) + \frac{2}{\sqrt{3}} + \frac{k}{2} + 1\right)}{(k+1)}.$$

Thus, if $r \geq \frac{3k}{2}$, then $(k+1)\chi^k(G(V,r)) = \gamma r^2 + \mathrm{O}(kr)$ as $r \to \infty$.

The key to all of the above theorems is the special case when V is the triangular lattice T, which is defined as the integer linear combinations of the vectors $a = (1,0)$ and $b = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. In what follows we will frequently make use of the observation that for $x, y \in \mathbb{R}$

$$\| ax + by \| = \sqrt{x^2 + xy + y^2}. \tag{1}$$

Let G_T denote the graph with vertex set T and an edge $uv \in E(G_T)$ whenever the vertices u, v are at distance 1 from each other. Note that the

Dirichlet-Voronoï cells of the set T constitute a cell structure with density $\frac{2}{\sqrt{3}}$ and radius $\frac{1}{\sqrt{3}}$, and hence Theorem 4 above gives good bounds on $\chi^k(G(V,r))$. However, we can obtain better results, and, indeed, for $k=0$, there is an exact result.

For any $r>0$, let $\hat{r}$ be the minimum distance between two points in T subject to that distance being at least r (from (1) it can be seen that $\hat{r}$ is the least value of $\sqrt{x^2+xy+y^2}$ greater than or equal to r so that x and y are integers). Note that $r \le \hat{r} \le \lceil r \rceil$, and the value of $\hat{r}^2$ can be computed in $\mathrm{O}(r)$ arithmetic operations.

Theorem 5 ([24]) *For any $r>0$, $\chi(G(T,r)) = \hat{r}^2$.*

Consult [24] for the origin of this result. Unfortunately, when we consider k-improper colouring, we do not obtain an exact result such as Theorem 5, but we give a good bound in the following theorem.

Theorem 6 *Suppose $k \ge 0$. If $r \ge \frac{3k}{2}$, then*

$$\chi^k(G(T,r)) \le \left\lceil \frac{r-1}{k+1} + 1 \right\rceil \left\lceil r + \frac{k}{2} \right\rceil < \frac{(r+2k+1)\left(r+\frac{k}{2}+1\right)}{k+1};$$

furthermore, if $r < \left\lceil \frac{k+1}{2} \right\rceil$, then $\chi^k(G(T,r)) \le \left\lceil \frac{2r}{\sqrt{3}} \right\rceil$.

3 Proofs for Section 2

As mentioned in Section 2, the main results rest on the special case when V is the set of points on the triangular lattice T so we will first focus our attention here. Theorem 6 follows from the following slightly more general result.

Theorem 7 *Suppose $k \ge 0$ and $1 \le \kappa \le \sqrt{k+1}$. Let $x_0 := \left\lfloor \frac{k+1}{\kappa} \right\rfloor$. If $r \ge \frac{3}{2}(x_0-1)$, then*

$$\chi^k(G(T,r)) \le \left\lceil \frac{1}{x_0}\left(r + \frac{\kappa-1}{2} + x_0 - 1\right) \right\rceil \left\lceil \frac{1}{\kappa}\left(r + \frac{x_0-1}{2} + \kappa - 1\right) \right\rceil;$$

furthermore, if $r < \left\lceil \frac{k+1}{2} \right\rceil$, then $\chi^k(G(T,r)) \le \left\lceil \frac{2r}{\sqrt{3}} \right\rceil$.

Proof. We just need to exhibit a k-improper colouring of T that satisfies the bound. It turns out that a *strict tiling* of T—a colouring such that each colour class is a translate $v+T'$ of some sublattice T' of T—suffices. We can describe such a colouring succinctly by using one of its "tiles", i.e., a finite subset $V' \subseteq T$ such that $V'+T'$ both covers and packs T.

Let us first define the tile V' and the sublattice T'. Set

$$x_1 := \alpha x_0 \text{ and } y_1 := \beta\kappa,$$

where $\alpha := \left\lceil \frac{1}{x_0}\left(r + \frac{\kappa-1}{2} + x_0 - 1\right)\right\rceil$ and $\beta := \left\lceil \frac{1}{\kappa}\left(r + \frac{x_0-1}{2} + \kappa - 1\right)\right\rceil$. We define V' to be all points $xa + yb$ such that $0 \le x < x_1$ and $0 \le y < y_1$. We let T' be all integer linear combinations of $x_1 a$ and $y_1 b$. Clearly, $V' + T'$ both covers and packs T.

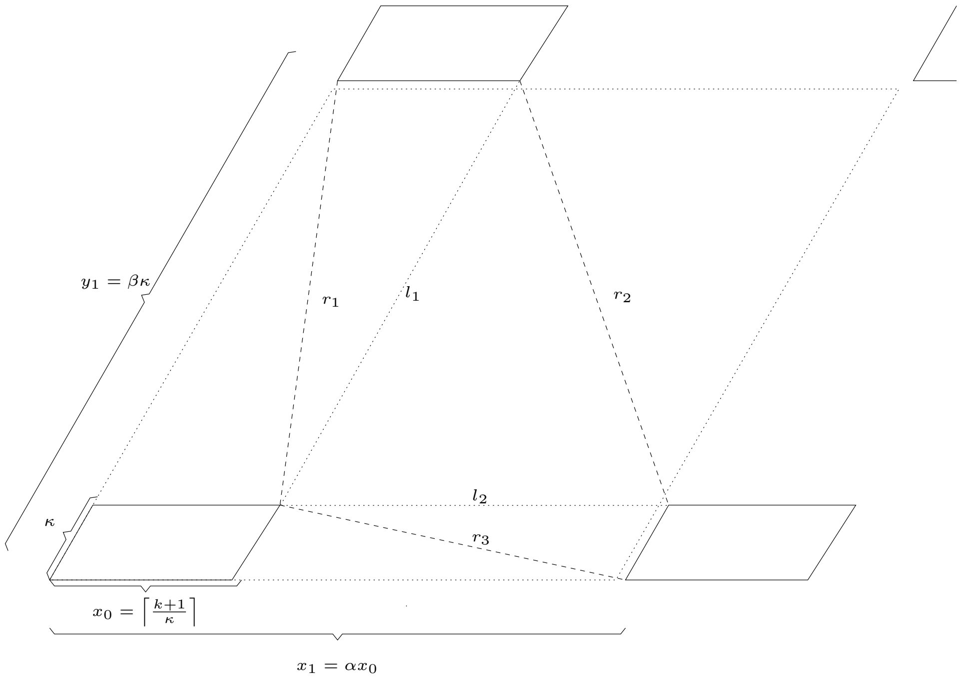

Figure 1: Illustration of the tiling of Theorem 7.

Define

$$V'_{i,j} := \{ai' + bj' : ix_0 \le i' \le (i+1)x_0 - 1 \text{ and } j\kappa \le j' \le (j+1)\kappa - 1\}$$

for $0 \le i < \alpha$ and $0 \le j < \beta$, and assign each set $V'_{i,j} + T'$ a distinct colour. Observe that this colouring uses $\alpha\beta$ colours, as required. To prove that it is a k-improper colouring, it suffices to show that the distance between any point in $V'_{0,0}$ and any point in $V'_{0,0} + ax_1, V'_{0,0} + by_1$ or $V'_{0,0} + ax_1 + by_1$ is at least r and that the distance between any point in $V'_{0,0} + ax_1$ and any point in $V'_{0,0} + by_1$ is at least r (see figure 1).

As can be easily seen from figure 1, the distance between any point in $V'_{0,0} + ax_1$ and any point in $V'_{0,0} + by_1$ is at least r_2 shown in figure 1.

The rightmost point of $V'_{0,0}$ has x-coordinate $x_0 - 1 + \frac{1}{2}(\kappa - 1)$ and the leftmost point of $V'_{0,0} + y_1 b$ has x-coordinate $\frac{1}{2}y_1 = \frac{1}{2}\beta\kappa \ge \frac{1}{2}(r + \frac{x_0-1}{2} +$

$\kappa - 1) \geq x_0 - 1 + \frac{1}{2}(\kappa - 1)$, using that $r \geq \frac{3}{2}(x_0 - 1)$ by assumption. Thus, $V'_{0,0} + by_1$ lies completely to the right of $V'_{0,0}$, and the distance between a point in $V'_{0,0}$ and any point in $V'_{0,0} + by_1$ is therefore at least r_1 shown in figure 1.

If we consider the projections onto $\mathcal{L}(\{b\})$ (or alernatively, if we rotate the picture clockwise by 60 degrees and consider the x-coordinates), then an analogous argument applies to $V'_{0,0}$ and $V'_{0,0} + ax_1$, showing that the distance between these sets is at least r_3 if $r \geq \frac{3}{2}(\kappa - 1)$ (and this condition is satisfied as $x_0 \geq \kappa$ and $r \geq \frac{3}{2}(x_0 - 1)$ by assumption).

Clearly $V'_{0,0} + ax_1 + by_1$ lies completely to the right of $V'_{0,0}$ so that the distance between these two sets is also at least r_1.

Thus, it suffices to show that $r_1, r_2, r_3 \geq r$. Using the formula (1) we see that

$$r_1 = \sqrt{l_1^2 - (x_0 - 1)l_1 + (x_0 - 1)^2},$$

where l_1 is as shown in figure 1. Thus $r_1 \geq r$ iff $l_1^2 - (x_0 - 1)l_1 + (x_0 - 1)^2 - r^2 \geq 0$ and by the quadratic formula this holds if

$$\begin{aligned} l_1 &\geq \frac{1}{2}\left((x_0 - 1) + \sqrt{(x_0 - 1)^2 + 4(r^2 - (x_0 - 1)^2)}\right) \\ &= \frac{x_0 - 1}{2} + \sqrt{r^2 - \frac{3}{4}(x_0 - 1)^2}. \end{aligned}$$

Now notice that $l_1 = \beta\kappa - (\kappa - 1) \geq r + \frac{x_0 - 1}{2} \geq \frac{x_0 - 1}{2} + \sqrt{r^2 - \frac{3}{4}(x_0 - 1)^2}$ by choice of β, so that indeed $r_1 \geq r$.

Analogous computations give that $r_3 \geq r$. Finally notice that $l_1 \geq r_1 \geq r$ (this can be seen by noting that l_1 is the distance between some point in $V'_{0,0}$ and some point in $V'_{0,0} + by_1$) and similarly $l_2 \geq r_3 \geq r$. Thus, $r_2 = \sqrt{l_1^2 - l_1 l_2 + l_2^2} \geq r$ and we see that the colouring defined by the tiling is indeed a k-improper colouring.

The "furthermore" condition implies that we may use one colour per row of T, and hence we need no more that $\lceil 2r/\sqrt{3} \rceil$ colours in total. □

Theorem 6 is just Theorem 7 for $\kappa = 1$. Note that other choices of κ will give better bounds when $k + 1$ is composite. Indeed, the best bound is when $k + 1$ is a square.

Corollary 2 *Suppose $k \geq 0$ and $k + 1 = \kappa^2$ is a square. If $r \geq \frac{3}{2}(\kappa - 1)$, then*

$$\chi^k(G(T, r)) \leq \left\lceil \frac{1}{\kappa}\left(r + \frac{3\kappa - 3}{2}\right)\right\rceil^2 < \frac{\left(r + (5\sqrt{k+1} - 3)/2\right)^2}{k + 1}.$$

Although Theorem 6 suffices for the remaining proofs of this section, it would also be interesting to know what is the value of $\chi^k(G(T,r))$ for all choices of k and r.

Let us continue with the proofs for Section 2. One way to prove the lower bound of Theorem 2 (and hence of Theorem 4), would be to mimic the approach given in [24], by establishing a lower bound on a k-improper analogue of the *stability quotient* (i.e., the maximum over all induced subgraphs $H \subseteq G$ of $\frac{|V(H)|}{\alpha(H)}$). However, it is sufficient to apply the lower bound of Proposition 2 to Theorem 1. Therefore, we just need to prove upper bounds, for which we shall generalise the arguments of [24].

Let us recall a definition from [24]. Given two sets A and B of points in the plane, and $w > 0$, we say that a function $\phi : A \to B$ is *w-wobbling* if the Euclidean distance $\| a - \phi(a) \|$ is at most w for each $a \in A$. Observe that, if there is a w-wobbling injection from A into B, then $\chi^k(G(A,r)) \leq \chi^k(G(B, r+2w))$ for any $r > 0$.

Proof of Theorem 2. Part **(i)** follows immediately from part **(i)** of Theorem 1 together with the lower bound in Proposition 2.

For the proof of part **(ii)**, we shall adapt the proof of Lemma 11 in [24]. Let $\varepsilon > 0$. We wish to show that $\frac{\chi^k(G)}{r^2} \leq (\sigma+\varepsilon)\frac{\sqrt{3}}{2(k+1)}$. First, we set T' to be T scaled so that its density is $(\sigma + \varepsilon/2)$, i.e., let $T' := \xi^{-1}T$ where ξ is $\left(\frac{(\sigma+\varepsilon/2)\sqrt{3}}{2}\right)^{1/2}$.

Let S denote the half-open unit square $S = [0,1)^2$. For any x sufficiently large, every translate of the square xS contains at least $(\sigma + \varepsilon/4)x^2$ points of T' and at most this number of points of V. If we partition the plane into a square grid with side length x, then for each grid square X there is a w-wobbling injection from $V \cap X$ into $T' \cap X$ where $w = \sqrt{2}x$. We may patch these injections together to obtain a w-wobbling injection $\phi : V \to T'$.

Now, using Theorem 6, we obtain

$$\begin{aligned}
\chi^k(G(V,r)) &\leq \chi^k(G(T', r+2w)) \\
&= \chi^k(G(T, \xi(r+2w))) \\
&< \frac{(\xi(r+2w) + 2k + 1)\left(\xi(r+2w) + \frac{k}{2} + 1\right)}{(k+1)} \\
&< r^2(\sigma+\varepsilon)\frac{\sqrt{3}}{2(k+1)}
\end{aligned}$$

if r is sufficiently large. □

The following lemma, proved by McDiarmid and Reed [24], will be used in the proof of Theorem 4. For all $w > 0$, two sets A and B are *w-close* if there exists a w-wobbling bijection between A and B.

Lemma 1 ([24]) *Let A (respectively B) be a set with a cell structure of density σ and radius r_A (respectively r_B). The sets A and B are $(r_A + r_B)$-close*

Proof of Theorem 4. We apply the proof of (3) in Theorem 2 in [24]. We first recall that the cells of the triangular lattice T constitute a cell structure with density $\frac{2}{\sqrt{3}}$ and radius $\frac{1}{\sqrt{3}}$. Let $\xi = \sqrt{\gamma} = \left(\sigma\frac{\sqrt{3}}{2}\right)^{1/2}$. Observe that ξV has the same density $\frac{2}{\sqrt{3}}$, but has radius $\xi\rho$. By Lemma 1, ξV and T are w-close where $w = \frac{1}{\sqrt{3}} + \xi\rho$ and hence, for any $r > 0$,

$$\chi^k(G(V,r)) = \chi^k\left(G\left(\xi V, \xi r\right)\right) \leq \chi^k(G(T,D))$$

where $D = \xi r + 2w = \xi(r + 2\rho) + \frac{2}{\sqrt{3}}$, whence,

$$\chi^k(G(V,r)) \leq \chi^k(G(T,D)) < \frac{(D + 2k + 1)\left(D + \frac{k}{2} + 1\right)}{k+1}$$

for r sufficiently large by Theorem 6. □

4 Improper colouring of random unit disk graphs

This section discusses our generalisations of [22] and [23]. We consider a sequence of graphs $(G_n)_n$ obtained as follows. We pick points $X_1, X_2, \ldots$ of $\mathbb{R}^2$ at random (i.i.d. according to some probability distribution ν on $\mathbb{R}^2$) and we set $G_n = G(\{X_1, \ldots, X_n\}, r(n))$, where we assume we are given a sequence of distances $r(n)$ that satisfies $r(n) \to 0$ as $n \to \infty$ and we will allow any choice of ν that has a bounded probability density function. We are interested in the behaviour of the clique number, the chromatic number, and the k-improper chromatic number of G_n as n grows large.

In this model, the distance $r(n)$ plays a role similar to that of $p(n)$ in the Erdős-Rényi $G(n,p)$ model. Depending on the choice of $r(n)$, qualitatively different types of behaviour can be observed. We prefer to describe the various cases in terms of the quantity nr^2, because nr^2 can be considered a measure of the average degree of the graph similar to np in the Erdős-Rényi $G(n,p)$ model. Intuitively, this should be obvious (consider for instance the case ν is uniform on $[0,1]^2$, so that the probability of an edge between X_1 and X_2 is $\approx \pi r^2$ when r is small and the expected degree of X_1 is therefore $\approx \pi(n-1)r^2$). For a somewhat more rigorous treatment of the relationship between nr^2 and the average degree, see [25].

In this section we will only consider the case when the parameter k is fixed. It is however possible to generalise Theorem 8 below to growing k as long as k does not grow too quickly. The results fully extend to arbitrary norm and dimension, i.e., the case when points are drawn from some distribution on $\mathbb{R}^d$ (replacing 2 by d appropriately in what follows) and an arbitrary norm is used to measure the distance between points. However, the scope of this paper is unit disk graphs on the plane.

We alluded to the following result in the introduction.

Theorem 8 *For $k \geq 0$ fixed and G_n as before the following holds.*

(i) *If $nr^2 \gg n^{-\delta}$ for all $\delta > 0$ then*

$$\frac{(k+1)\chi^k(G_n)}{\chi(G_n)} \to 1 \text{ almost surely;}$$

(ii) *If $nr^2 \ll n^{-\delta}$ for some $\delta > 0$ then*

$$\mathbb{P}\left(\chi^k(G_n) \in \left\{\left\lceil\frac{\chi(G_n)}{k+1}\right\rceil, \left\lceil\frac{\chi(G_n)}{k+1}\right\rceil + 1\right\} \text{ for all but finitely many } n\right) = 1.$$

This following proposition shows that the two point range for $\chi^k(G_n)$ in item **(ii)** cannot be reduced in general.

Proposition 3 *If $k \geq 1$ is fixed and r is chosen so that $nr^2 = \gamma n^{-\frac{1}{m(k+1)}}$ for some $\gamma > 0, m \in \mathbb{N}^*$, then there exists a $c = c(\gamma, m) \in (0,1)$ such that*

$$\mathbb{P}\left(\chi^k(G_n) = \left\lceil\frac{\chi(G_n)}{k+1}\right\rceil + 1\right) \to c.$$

When $nr^2 \ll n^{-\delta}$ for some $\delta > 0$ then it can be shown that $\chi^k(G_n)$ will remain bounded in the sense that $\mathbb{P}(\chi^k(G_n) \leq m$ for all but finitely many $n) = 1$ for some $m = m(\delta)$. Thus, Proposition 3 shows that when $nr^2 \ll n^{-\delta}$, almost sure convergence of the ratio $(k+1)\chi^k(G_n)/\chi(G_n)$ to 1 does not hold in general.

In contrast it was shown in [23] that for proper colouring it holds that $\mathbb{P}(\chi(G_n) = \omega(G_n)$ for all but finitely many $n) = 1$ whenever $nr^2 \ll n^{-\delta}$ for some $\delta > 0$.

It follows from the proof of Theorem 8 that when $nr^2 \ll n^{-\delta}$ for some $\delta > 0$ then there exists a sequence $m(n)$ such that

$$\mathbb{P}(\chi^k(G_n) \in \{m(n), m(n)+1\} \text{ for all but finitely many } n) = 1.$$

Thus, the probability distribution of χ^k becomes concentrated on two consecutive integers as n grows large in the sense that $\mathbb{P}(\chi^k(G_n) \in \{m(n), m(n)+1\}) \to 1$.

This phenomenon (of the probability measure becoming concentrated on two consecutive integers) is called focusing in [28, 29] and is well known to occur for various graph parameters in Erdős-Rényi random graphs. Recently, one of the authors proved a conjecture of Penrose stating that when

$nr^2 \ll \ln n$ then the clique number of G_n becomes focused and the same was shown to hold for the chromatic number [25]. It is a straightforward exercise adapt the proof in [25] to yield the analogous result for improper colouring as well: if $nr^2 \ll \ln n$ then there exists a sequence $m(n)$ such that $\mathbb{P}(\chi^k(G_n) \in \{m(n), m(n)+1\}) \to 1$.

5 Proofs for Section 4

We shall use the following result from [23]. Note that item **(i)** is indeed an observation made in the proof of item **(ii)**.

Theorem 9 ([23]) *If $nr^2 \ll n^{-\delta}$ for some $\delta > 0$ then the following holds.*

- **(i)** *There exists a sequence $m(n)$ such that $\mathbb{P}(\omega(G_n) \in \{m(n), m(n)+1\}$ and $\Delta(G_n) \in \{m(n), m(n)-1\}$ for all but finitely many $n) = 1$;*
- **(ii)** $\mathbb{P}(\chi(G_n) = \omega(G_n)$ *for all but finitely many* $n) = 1$.

Item **(ii)** of Theorem 8 is an easy consequence of this last theorem.
Proof of Theorem 8, item (ii). It follows from item **(i)** of Theorem 9 that, when $nr^2 \ll n^{-\delta}$,

$$\mathbb{P}(\Delta(G_n) \in \{\omega(G_n), \omega(G_n)-1\} \text{ for all but finitely many } n) = 1.$$

Combining this with part **(ii)** of Theorem 9, we see that also

$$\mathbb{P}(\chi(G_n) = \omega(G_n) \text{ and } \Delta(G_n) \leq \omega(G_n) \text{ for all but finitely many } n) = 1.$$

Now note that if $\chi(G_n) = \omega(G_n)$ and $\Delta(G_n) \leq \omega(G_n)$ then Proposition 2 gives

$$\left\lceil \frac{\chi(G_n)}{k+1} \right\rceil \leq \chi^k(G_n) \leq \left\lceil \frac{\chi(G_n)+1}{k+1} \right\rceil \leq \left\lceil \frac{\chi(G_n)}{k+1} \right\rceil + 1,$$

which concludes the proof. □

For the proof of Proposition 3 we will rely on some results from Chapter 3 of [29]. Recall that if Z, Z' are two integer valued random variables then their *total variational distance* is defined as

$$d_{TV}(Z, Z') := \sup_{A \subseteq \mathbb{Z}} |\mathbb{P}(Z \in A) - \mathbb{P}(Z' \in A)|.$$

Proposition 4 ([29]) *Let H be a connected unit disk graph with $l \geq 2$ vertices, let N denote the number of induced subgraphs of G_n isomorphic to H and let Z be a Poisson variable with mean $\mathbb{E}N$. The following hold.*

- **(i)** *There exists a constant $\mu = \mu(H) > 0$ such that $\mathbb{E}N \sim \mu n^l r^{2(l-1)}$;*

(ii) *There exists a constant $c = c(H)$ such that $d_{TV}(N, Z) \leq cnr^2$.*

Proposition 5 ([29]) *Let $H_1, \ldots, H_s$ be non-isomorphic connected unit disk graphs with $l \geq 2$ vertices. Let N_i denote the number of induced subgraphs of G_n isomorphic to H_i. Let $\mu_1 = \mu(H_1), \ldots, \mu_s = \mu(H_s)$ be as given by part* **(i)** *of Proposition 4. Suppose that $nr^2 \sim \gamma n^{-\frac{1}{l-1}}$ with $\gamma > 0$. Let $Z_1, \ldots, Z_s$ be independent Poisson variables with $\mathbb{E}Z_i = \gamma^{l-1}\mu_i$. Then*

$$(N_1, \ldots, N_s) \xrightarrow{d} (Z_1, \ldots, Z_s).$$

Proof of Proposition 3. According to part **(ii)** of Theorem 9 it suffices to consider $\mathbb{P}\left(\chi^k(G_n) = \left\lceil \frac{\omega(G_n)}{k+1} \right\rceil + 1\right)$, as $\mathbb{P}(\chi(G_n) \neq \omega(G_n)) \to 0$. Set $l := m(k+1)+1$. By the choice of $r(n)$, we have $n^l r^{2(l-1)} = \gamma^{l-1}$, and $n^{l+1}r^{2l} \to 0, n^{l-1}r^{2(l-2)} \to \infty$. If we denote the order of the largest component of G_n by $L(G_n)$ then Proposition 4 implies that

$$\mathbb{P}(\omega(G_n) \geq l-1, L(G_n) \leq l) \to 1.$$

To see this let N be the number of induced subgraphs of G_n isomorphic to K_{l-1} and let N' be the number of connected subgraphs of order $l+1$. Part **(i)** of Proposition 4 gives $\mathbb{E}N' = O(n^{l+1}r^{2l}) = o(1)$. Clearly, the largest component has size greater than l if and only if there exists a connected subgraph of order $l+1$. Therefore,

$$\mathbb{P}(L(G_n) > l) = \mathbb{P}(N' > 0) \leq \mathbb{E}N' = o(1).$$

On the other hand $\mathbb{E}N = \Omega(n^{l-1}r^{2(l-2)}) \to \infty$, and using part **(ii)** of Proposition 4 we get

$$\mathbb{P}(\omega(G_n) < l-1) = \mathbb{P}(N = 0) = \exp[-\mathbb{E}N] + o(1) = o(1).$$

Let $H_1, \ldots, H_s$ be all non-isomorphic connected unit disk graphs of order l that satisfy $\chi^k(H_i) = m+1$ yet H_i is not (isomorphic to) K_l. There exists at least one such graph, the unit disk graph $H := G(\{(\frac{i}{l-1}, 0) : i = 0, \ldots, l-1\}, 1)$, as depicted in Figure 2. This is simply the complete graph on l vertices with one edge removed. To see that there is no k-improper colouring of H with m colours, note that its vertices can be partitioned into a clique of size $l-1 = m(k+1)$ and a vertex v_0 which is adjacent to all but one of the other nodes. If there were a k-improper colouring with m colours then every colour would have to occur $k+1 \geq 2$ times amongst the vertices of the clique. Hence whichever of the m colours we assign to v_0 there will be a node in the clique adjacent to $k+1$ nodes of the same colour.

Let N_0 be the number of induced subgraphs of G_n isomorphic to K_l and let N_i be the number of induced subgraphs isomorphic to H_i. Observe that if both $\omega(G_n) \geq l-1$ and $L(G_n) \leq l$ hold then $\chi^k(G_n) = \left\lceil \frac{\chi(G_n)}{k+1} \right\rceil + 1$ holds

Figure 2: For Proposition 3, $H = K_{m(k+1)+1} - e$ satisfies $\omega(H) = m(k+1)$ and $\chi^k(H) = m+1$.

if and only if $N_0 = 0$ and $N_i > 0$ for some $1 \leq i \leq s$. Thus, we infer that the probability that $\chi^k(G_n)$ equals $\left\lceil \frac{\chi(G_n)}{k+1} \right\rceil + 1$ is

$$\mathbb{P}\left(N_0 = 0 \text{ and } N_i > 0 \text{ for some } 1 \leq i \leq s\right) + o(1).$$

Using Proposition 5, we may therefore conclude that

$$\mathbb{P}\left(\chi^k(G_n) = \left\lceil \frac{\chi(G_n)}{k+1} \right\rceil + 1\right) \to e^{-\mu_0 \gamma^{l-1}}(1 - e^{-(\mu_1 + \cdots + \mu_s)\gamma^{l-1}}),$$

for some $\mu_0, \ldots, \mu_s > 0$. $\square$

It should be noted that although the use of part **(ii)** of Proposition 4 is not crucial to show that $\mathbb{P}(\omega(G_n) < l-1) = o(1)$, since Proposition 5 will suffice instead, its use shortens the proof of Proposition 3.

The proof of item **(i)** of Theorem 8 relies on some results from [23] that were developed to study the behaviour of $\chi(G_n)$.

One important ingredient to the proof is the connection between graph colouring and integer linear programming. Recall that the chromatic number of a graph G is the optimum value of the following integer linear program (ILP for short).

$$\begin{aligned} \min \quad & 1^T x \\ \\ \text{subject to} \quad & Ax \geq 1, \\ & x \geq 0, x \text{ integers}, \end{aligned}$$

where A is the *vertex-independent set incidence matrix* of G. This is a $(0,1)$-matrix whose rows are indexed by the vertices of G and whose columns correspond to all possible independent sets in G. It has $a_{ij} = 1$ if vertex v_i is in the independent set corresponding to the j-th column and $a_{ij} = 0$ otherwise. Now, given a nonnegative integer vector $b = (b_1, \ldots, b_n)$, let the graph G' be obtained from G by replacing vertex $v_i \in G$ by a clique of size b_i and the vertices in the cliques corresponding to v_i and v_j are joined in G'

if and only if v_i and v_j are joined in G. Then $\chi(G')$ is the objective value of the following ILP.

$$\begin{aligned} \min \quad & 1^T x \\ \text{subject to} \quad & Ax \geq b, \\ & x \geq 0, x \text{ integers.} \end{aligned}$$

Furthermore, $\chi^k(G')$ does not exceed the objective value of the following ILP.

$$\begin{aligned} \min \quad & 1^T x \\ \text{subject to} \quad & (k+1)Ax \geq b, \\ & x \geq 0, x \text{ integers.} \end{aligned}$$

This is because taking $k+1$ copies of each node in a stable set in G gives a k-dependent set in G' (but not every k-dependent set can be constructed in this way of course).

As mentioned in Section 4 we assume that the probability measure ν on the plane used to generate the X_i has a bounded density function f. Let us denote the *essential supremum* of f by $f_{\max}$, i.e.,

$$f_{\max} := \sup\{t : \text{vol}(\{x : f(x) > t\}) > 0\},$$

where vol denotes the Lebesgue measure. We say that a measurable set $A \subseteq \mathbb{R}^2$ has a *small neighbourhood* if $\lim_{\varepsilon \to 0} \text{vol}(A_\varepsilon) = \text{vol}(A)$ where A_ε denotes $A + B(0, \varepsilon)$.

We shall denote by $\mathcal{F}$ the collection of all bounded, nonnegative functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$ with bounded support that satisfy the regularity conditon that $\{x : \varphi(x) > t\}$ has a small neighbourhood for all t, and that are not almost everywhere zero—i.e., their support has nonzero area. For $\varphi \in \mathcal{F}$, let us define the random variable M_φ as

$$M_\varphi := \sup_{x \in \mathbb{R}^2} \sum_{i=1}^{n} \varphi\left(\frac{X_i - x}{r}\right).$$

It turns out that the random variables M_φ play an important role when studying the (k-improper) chromatic number of G_n, see [23].

Proposition 6 ([23]) *Let $\varphi = 1_W$ for some bounded set $W \subset \mathbb{R}^2$ with a small neighbourhood and nonempty interior. For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $n^{-\delta} \leq nr^2 \leq \delta \ln n$ then*

$$\mathbb{P}\left((1-\varepsilon)m \leq M_\varphi \leq (1+\varepsilon)m \text{ for all but finitely many } n\right) = 1,$$

where $m = m(n) := (\ln n)/\ln\left(\frac{\ln n}{nr^2}\right)$.

Proposition 7 ([23]) *Pick $\varphi \in \mathcal{F}$. For every $\varepsilon > 0$ there exists a $T = T(\varepsilon)$ such that if $nr^2 \geq T \ln n$ then*

$$\mathbb{P}\left((1-\varepsilon)m \leq M_\varphi \leq (1+\varepsilon)m \text{ for all but finitely many } n\right) = 1,$$

where $m = m(n) = f_{\max} nr^2 \int_{\mathbb{R}^2} \varphi(x)dx$.

Proposition 8 ([23]) *Pick $\varphi \in \mathcal{F}$. If $nr^2 \sim t \ln n$ for some $t \in (0, \infty)$ then*

$$\frac{M_\varphi}{nr^2} \to f_{\max} \int_{\mathbb{R}^2} \varphi(x) e^{\varphi(x)s} dx \text{ almost surely,}$$

where $s = s(\varphi, t) \geq 0$ solves

$$\int_{\mathbb{R}^2} (s\varphi(x)e^{\varphi(x)s} - e^{\varphi(x)s} + 1)dx = \frac{1}{tf_{\max}}. \tag{2}$$

For $\varphi \in \mathcal{F}, 0 < t < \infty$, we shall set

$$\xi(\varphi, t) := \int_{\mathbb{R}^2} \varphi(x) e^{\varphi(x)s(\varphi,t)} dx,$$

where $s(\varphi, t)$ is the unique nonnegative solution of (2) above. To see that (2) indeed has a unique nonnegative solution (unless φ is almost everywhere 0, in which case there is no solution to (2)), notice that the left-hand side of (2) is finite for all $s > 0$ (as $\varphi \in \mathcal{F}$ is bounded and has bounded support), and is increasing with s for $s \geq 0$ (as the integrand $s\varphi(x)e^{\varphi(x)s} - e^{\varphi(x)s} + 1 = H(e^{\varphi(x)s})$ with $H(x) := x \ln x - x + 1$ is strictly increasing in s for any fixed x with $\varphi(x) > 0$). We will need the following observation from [23].

Lemma 2 ([23]) *For $\varphi \in \mathcal{F}$ and $\lambda \in (0, 1)$, let φ_λ be given by $\varphi_\lambda(x) := \varphi(\lambda x)$. Then*

$$\xi(\varphi_\lambda, t) \leq \lambda^{-2} \xi(\varphi, t).$$

Proof of Theorem 8, item (i). We adapt a proof from [23]. We will first derive an upper bound on $\chi^k(G_n)$. To this end, let us fix $\varepsilon > 0$ and consider the graphs G'_n constucted as follows. For $x \in \mathbb{R}^2$ let S_x denote the square $x + [0, \varepsilon r)^2$ of side εr and lower left hand corner x. So the squares $S_p, p \in \varepsilon r \mathbb{Z}^2$ form a dissection of $\mathbb{R}^2$. Let Γ be the graph with vertex set $\varepsilon r \mathbb{Z}^2$ and an edge $pq \in E(\Gamma)$ if $\| p - q \| < r(1 + \varepsilon\sqrt{2})$. For $W \subseteq \mathbb{R}^2$, we denote by $N(W)$ the number of points in W, i.e.,

$$N(W) := |\{X_1, \ldots, X_n\} \cap W|.$$

We will now consider the graph G'_n, constructed by replacing each point p of Γ by a clique of size $N(S_p)$. Note that G_n is a subgraph of G'_n, so in particular $\chi^k(G_n) \leq \chi^k(G'_n)$. Let us fix $K > 0$ (large) such that ε divides

K. For $p \in \varepsilon r\mathbb{Z}^2$, denote by H_p the subgraph of G_n induced by the points of Γ inside $p + [0, Kr)^2$, and by H'_p the corresponding subgraph of G'_n. By remarks made before the start of the proof we know that $\chi(H'_p)$ is no more than the objective value of the following ILP.

$$\begin{array}{ll} \min & 1^T x \\ \\ \text{subject to} & Ax \geq \frac{1}{k+1} b(p), \\ & x \geq 0, x \text{ integers}, \end{array}$$

where A is the vertex-independent set incidence matrix of the subgraph Γ_K of Γ induced by the points of Γ inside $p + [0, Kr)^2$, and $b(p)$ is the (random) vector $(N(S_{p+p_1}), \ldots, N(S_{p+p_l}))$ whose entries are the number of points in each of the squares S_{p+p_i} for $p_i \in [0, Kr)^2 \cap \varepsilon r\mathbb{Z}^2$. We now consider the LP-relaxation of this program (we drop the condition that the variables need to be integers), and denote by $M(p)$ the optimum value of this LP-relaxation. As A depends only on ε and K, there is a constant $c = c(K, \varepsilon)$ such that

$$\chi^k(H'_p) \leq M(p) + c(K, \varepsilon), \tag{3}$$

because rounding up all the variables of a feasible point of the LP-relaxation gives a feasible point of the ILP. So, in particular, we may take $c(K, \varepsilon)$ equal to the number of columns of A, which equals the number of stable sets in Γ_K. This upper bound on the difference $\chi(H'_p) - M(p)$ can be further improved, but it does not concern us here as for the proof it is only relevant that $c(K, \varepsilon)$ is a constant that does not depend on n. By *LP*-duality, $M(p)$ is also equal to the value of the program

$$\begin{array}{ll} \max & \frac{1}{k+1} b^T y \\ \\ \text{subject to} & A^T y \leq 1, \\ & y \geq 0. \end{array}$$

This formulation has the advantage that the polytope defined by $A^T y \leq 1$ depends only on ε, K. Given ε, K we can therefore list the vertices of this polytope, which we will denote by $y_1, \ldots, y_m$. Because the optimum of the LP will be attained in one of the vertices, we can write

$$M(p) = \frac{1}{k+1} \max_i y_i^T b(p). \tag{4}$$

We now remark that the y_i correspond to functions in a natural way. Let $\psi_i : \mathbb{R}^2 \to [0, 1]$ be the function which is given by

$$\psi_i(x) := \begin{cases} (y_i)_j & \text{if } x \in S_{p_j} \text{ for } 1 \leq j \leq l, \\ 0 & \text{if } x \notin [0, Kr)^2. \end{cases}$$

Here we used the same enumeration $p_1, \ldots, p_l$ of $[0, Kr)^2 \cap \varepsilon\mathbb{Z}^2$ used earlier in the construction of the ILP. It is not hard to see that

$$b^T(p)y_i = \sum_{j=1}^{l} (y_i)_j N(S_{p+p_j}) = \sum_{k=1}^{n} \psi_i(X_k - p).$$

Let us now set $\varphi_i(x) := \psi_i(rx)$. The functions $\varphi_1, \ldots, \varphi_m$ in fact do not depend on r (or n) anymore, but merely on ε, K. Furthermore $\varphi_i \in \mathcal{F}$ for all i and

$$b^T(p)y_i = \sum_{j=1}^{n} \varphi_i\left(\frac{X_j - p}{r}\right).$$

Together with (3) and (4) this shows that for all p we have

$$\chi^k(H'_p) \leq \frac{1}{k+1} \max_{i=1,\ldots,m} M_{\varphi_i} + c(K, \varepsilon).$$

We now remark that not only H'_p can be coloured with this many colours for any $p \in \varepsilon r\mathbb{Z}^2$, but also the subgraph of G_n induced by the points in the set $W_p := p + [0, Kr)^2 + (K+1)r\mathbb{Z}^2$, as depicted in Figure 3.

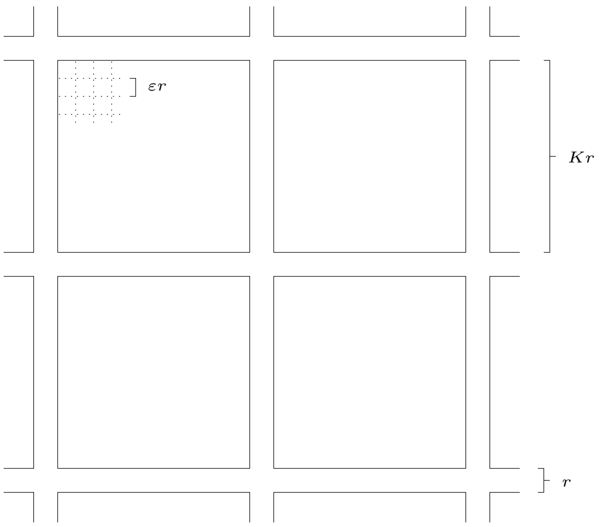

Figure 3: Artists impression of the sets W_p.

To see this note that if $x \in p+[0, Kr)^2+(K+1)rz, y \in p+[0, Kr)^2+(K+1)rz'$ for $z \neq z' \in \mathbb{Z}^2$, then $\| x - y \| \geq r$.

We may assume without loss of generality that $K \in \mathbb{N}$. Now consider the collection of all $\mathcal{W} := \{W_p : p \in r\{0, \ldots, K\}^2\}$ and note that each small square S_q, $(q \in \varepsilon r\mathbb{Z}^2)$, is covered by exactly K^2 of the $(K+1)^2$ sets $W_p \in \mathcal{W}$ considered.

Let us now consider the graphs $G'_{n,p}$ which are obtained by replacing each point q of $\Gamma \cap W_p$ by a clique of size $\left\lceil \frac{N(S_q)}{K^2} \right\rceil$ rather than $N(S_q)$. The subgraph $G'_{n,p}$ can be k-improperly coloured with no more than

$$\max_p \tfrac{1}{k+1} \max_i \left(\left\lceil \tfrac{N(S_{p+p_1})}{K^2} \right\rceil, \ldots, \left\lceil \tfrac{N(S_{p+p_l})}{K^2} \right\rceil \right) \cdot y_i + c(K,\varepsilon) \\ \leq \\ \tfrac{1}{(k+1)K^2} \max_i M_{\varphi_i} + c(K,\varepsilon) + l$$

colours (as $(y_i)_j \leq 1$ for all i, j, so that the difference due to rounding is at most l). The colourings of the $G'_{n,p}$ can be combined to give a colouring of G_n with a total of at most

$$\frac{1}{k+1} \left(\frac{K+1}{K} \right)^2 \max_{i=1,\ldots,m} M_{\varphi_i} + (K+1)^2 (c(K,\varepsilon) + l)$$

colours.

Next we wish to lower bound $\chi(G_n)$. For convenience let us set $r' := r\frac{1-\varepsilon\sqrt{2}}{1+\varepsilon\sqrt{2}}$ and $S'_y := y + [0, \varepsilon r')^2$ for $y \in \mathbb{R}^2$. For $x \in \mathbb{R}^2$ consider the subgraph H_x of G_n induced by the points in the square $x + [0, r'K)^2$.

Let Γ'_K be the graph with vertices $x + p'_i$, with p'_i running through $[0, Kr')^2 \cap \varepsilon r' \mathbb{Z}^2$ and an edge $yz \in E(\Gamma'_K)$ if $\| y - z \| < r'(1 + \varepsilon\sqrt{2}) = r(1 - \varepsilon\sqrt{2})$. Then Γ'_K is in fact isomorphic to Γ_K and in particular has the same vertex-independent set incidence matrix A. Let H'_x be the graph we get by replacing a vertex $x + p'_i$ of Γ'_K by a clique of size $N(S'_{x+p'_i})$. We remark that H'_x is a subgraph of G_n and $\chi(H'_x)$ is at least the objective value of the linear program

$$\begin{aligned} \max \quad & b'(x)^T y \\ \text{subject to} \quad & A^T y \leq 1, \\ & y \geq 0, \end{aligned}$$

where $b'(x) := (N(S'_{x+p'_1}), \ldots, N(S'_{x+p'_l}))$. The vertices $y_1, \ldots, y_m$ of the polytope are still the same. However, they now correspond to the sums

$$b'(x)^T y_i = \sum_{j=1}^{n} \varphi'_i \left(\frac{X_j - x}{r} \right),$$

where we $\varphi'_i(x) := \varphi_i \left(\frac{1+\varepsilon\sqrt{2}}{1-\varepsilon\sqrt{2}} x \right)$. Maximising over all choices of $x \in \mathbb{R}^2$ we get

$$\chi(G_n) \geq \max_{i=1,\ldots,m} M_{\varphi'_i}. \tag{5}$$

It follows that

$$1 \leq \frac{(k+1)\chi^k(G_n)}{\chi(G_n)} \leq \left(\frac{K+1}{K} \right)^2 \frac{\max_i M_{\varphi_i}}{\max_i M_{\varphi'_i}} + \frac{\alpha}{\max_i M_{\varphi'_i}}, \tag{6}$$

where $\alpha = \alpha(K,\varepsilon) := (K+1)^2(c(K,\varepsilon)+l)$.

Let us pick $t_0 < t_1 < \cdots < t_a$ with $t_{i+1} \leq (1+\varepsilon)t_i$, t_0 small and t_a large (to be made precise later) and let us "split" the sequence r into subsequences $r_0, \ldots, r_{a+1}$:

$$r_0(n) := \begin{cases} r(n) & \text{if } nr^2 \leq t_0 \ln n, \\ \sqrt{\frac{t_0 \ln n}{n}} & \text{otherwise.} \end{cases}, \quad r_{a+1}(n) := \begin{cases} r(n) & \text{if } nr^2 \geq t_a \ln n, \\ \sqrt{\frac{t_a \ln n}{n}} & \text{otherwise.} \end{cases}$$

$$r_i(n) := \begin{cases} r(n) & \text{if } t_{i-1} \ln n \leq nr^2 \leq t_i \ln n, \\ \sqrt{\frac{t_i \ln n}{n}} & \text{otherwise.} \end{cases}, \quad \text{for } 1 \leq i \leq a,$$

and let us set $G_n^i := G(\{X_1, \ldots, X_n\}, r_i(n))$. We claim that for all i

$$\mathbb{P}\left(1 \leq \frac{(k+1)\chi^k(G_n^i)}{\chi(G_n^i)} \leq \gamma(K,\varepsilon) \text{ for all but finitely many } n\right) = 1, \quad (7)$$

where $\gamma(K,\varepsilon) := \frac{(1+\varepsilon)^2}{1-\varepsilon}\left(\frac{1+\varepsilon\sqrt{2}}{1-\varepsilon\sqrt{2}}\right)^2\left(\frac{K+1}{K}\right)^2 + \varepsilon$. From this it will immediately follow that also

$$\mathbb{P}\left(1 \leq \frac{(k+1)\chi^k(G_n)}{\chi(G_n)} \leq \gamma(K,\varepsilon) \text{ for all but finitely many } n\right) = 1,$$

as G_n always coincides with one of the G_n^i and the intersection of finitely many events of probability one has probability one. Taking $K \to \infty, \varepsilon \to 0$ will then give that

$$\frac{(k+1)\chi^k(G_n)}{\chi(G_n)} \to 1 \text{ almost surely,}$$

as required. Thus it remains to establish (7) for all i (for a suitable choice of $t_0, \ldots, t_a$).

Let us first consider G_n^0. Let us set $\psi_1 := 1_{B(0,\frac{1}{2})}, \psi_2 := 1_{B(0,1)}$ and notice that M_{ψ_1}, M_{ψ_2} are respectively the maximum number of points of G_n in a disk of radius $\frac{r}{2}$ and r. Notice that we must have

$$M_{\psi_1} \leq \omega(G_n) \leq \chi(G_n) \leq \Delta(G_n) + 1 \leq M_{\psi_2}.$$

Applying Proposition 6 and the upper bound from Proposition 2, we get

$$\mathbb{P}\left(\chi(G_n^0) \geq (1-\varepsilon)b \text{ for all but finitely many } n\right) = 1,$$
$$\mathbb{P}\left(\chi^k(G_n^0) \leq \frac{1}{k+1}(1+\varepsilon)b + 1 \text{ for all but finitely many } n\right) = 1,$$

where $b = b(n) := (\ln n)/\ln\left(\frac{\ln n}{nr_0^2}\right)$. Now notice that $b \to \infty$, since $nr_0^2 \geq n^{-\delta}$ implies that $b \geq \frac{1}{\delta} + o(1)$. We see that

$$\mathbb{P}\left(\frac{(k+1)\chi^k(G_n^0)}{\chi(G_n^0)} \leq \frac{1+\varepsilon}{1-\varepsilon} + \varepsilon \text{ for all but finitely many } n\right) = 1.$$

Let us now consider G_n^{a+1}. Provided t_a was chosen large enough we have by Proposition 7

$$\begin{array}{l}\mathbb{P}(M_{\varphi_i} \leq (1+\varepsilon)b_i \text{ for all but finitely many } n) = 1, \\ \mathbb{P}(M_{\varphi_i'} \geq (1-\varepsilon)b_i' \text{ for all but finitely many } n) = 1,\end{array}$$

for $i \in \{1, \ldots, m\}$, where $b_i := f_{\max} n r_i^2 \int_{\mathbb{R}^d} \varphi_i(x)dx, b_i' := f_{\max} n r_i^2 \int_{\mathbb{R}^2} \varphi_i'(x)dx = \left(\frac{1-\varepsilon\sqrt{2}}{1+\varepsilon\sqrt{2}}\right)^2 b_i$ (we have used the substitution $y = \left(\frac{1+\varepsilon\sqrt{2}}{1-\varepsilon\sqrt{2}}\right) x$ for the last identity). By (6) and the fact that $b_i' \to \infty$,

$$\mathbb{P}\left(\frac{(k+1)\chi^k(G_n^{a+1})}{\chi(G_n^{a+1})} \leq \frac{1+\varepsilon}{1-\varepsilon}\left(\frac{1+\varepsilon\sqrt{2}}{1-\varepsilon\sqrt{2}}\right)^2 \left(\frac{K+1}{K}\right)^2 + \varepsilon \text{ for all but finitely many } n\right) = 1.$$

Now let us consider G_n^i for $1 \leq i \leq k$. We may assume that the t_i have been chosen in such a way that $t_i \leq (1+\varepsilon)t_{i-1}$. Let us set $r_i^- := \sqrt{\frac{t_{i-1}\ln n}{n}}, r_i^+ := \sqrt{\frac{t_i \ln n}{n}}$. Because $\chi^k(G(V,\rho)), \chi(G(V,\rho))$ are both increasing in ρ,

$$\chi(G_n^i) \geq \chi(G(\{X_1, \ldots, X_n\}, r_i^-)), \quad \chi^k(G_n^i) \leq \chi^k(G(\{X_1, \ldots, X_n\}, r_i^+)).$$

Applying Proposition 8 and (5) we see that (with probability one, for all but finitely many n)

$$\begin{array}{l}\chi(G_n^i) \geq (1-\varepsilon) f_{\max} n (r_i^-)^2 \max_j \xi(\varphi_j', t_{i-1}), \\ \chi^k(G_n^i) \leq \frac{1}{k+1}(\frac{K+1}{K})^2 (1+\varepsilon) f_{\max} n (r_i^+)^2 \max_j \xi(\varphi_j, t_{i-1}) + \alpha.\end{array}$$

In the language of Lemma 2 we have $\varphi_i = (\varphi_i')_\lambda$ with $\lambda := \left(\frac{1-\varepsilon\sqrt{2}}{1+\varepsilon\sqrt{2}}\right)$. So we see that (with probability one, for all but finitely many n)

$$\begin{array}{rcl}\frac{(k+1)\chi^k(G_n^i)}{\chi(G_n^i)} & \leq & \left(\frac{1+\varepsilon}{1-\varepsilon}\right) \frac{t_i}{t_{i-1}} \left(\frac{1+\varepsilon\sqrt{2}}{1-\varepsilon\sqrt{2}}\right)^2 \left(\frac{K+1}{K}\right)^2 + \varepsilon \\ & \leq & \frac{(1+\varepsilon)^2}{1-\varepsilon} \left(\frac{1+\varepsilon\sqrt{2}}{1-\varepsilon\sqrt{2}}\right)^2 \left(\frac{K+1}{K}\right)^2 + \varepsilon,\end{array}$$

which concludes the proof. □

As pointed out by one of the referees, the method used in our proof of colouring subgraphs induced by the points inside the sets W_p, which are unions of squares of side Kr with a space of r between the different squares (see Figure 3), and then shifting this pattern $(K+1)^2$ times and overlaying the colourings obtained, is similar to the shifting strategy developed in [17], see also [18].

6 Conclusion

In Sections 2 and 3, we studied the asymptotic behaviour of χ^k when $r \to \infty$ and V is countably infinite to extend results of [24]. For these results, the bound for the k-improper chromatic number of the generalised triangular lattice given in Theorem 6 suffices; however, we would be interested to know an exact expression for $\chi^k(G(T, r))$ for any k and r. In Sections 4 and 5, we studied the k-improper chromatic number of random unit disk graphs to extend results in [22, 23]. An essential element of one of the proofs was an LP-formulation of the problem and an appropriate partition of the space. An issue that we have not studied for random unit disk graphs, but that might be of practical importance, is the rates of convergence of our results.

In both cases, with minor exceptions, we have seen that χ^k is well-approximated by the lower bound of Proposition 2, in other words $(k+1)\chi^k$ approaches χ in some appropriately defined manner. This behaviour differs notably from that of Erdős-Rényi random graphs, where $\chi^k/\chi \to 1$ in probability if $k(n) = o(\ln(np))$ and $np \to \infty$ [19].

Due to the motivating application in satellite communications, we have focused upon the case with Euclidean norm and dimension two (i.e., unit disk graphs); however, we note here that our results naturally generalise to arbitrary norm and higher dimensions (see [26, 31]).

A major purpose of this study was to gain insight into the problem of finding the k-improper chromatic number of unit disk graphs. As mentioned in the introduction, the best polynomial approximation for χ^k is 6 (and 3 for $k = 0$). The results of this paper suggest that, for fixed k, given randomly generated instances G_n, the polynomially computable value $\omega(G_n)/(k+1)$ multiplied by the factor $2\sqrt{3}/\pi \approx 1.103$ (which is much smaller than 6) is a reasonable approximation for the k-improper chromatic number when n is large enough.

Acknowledgements

The authors would like to acknowledge the two anonymous referees for pointers to relevant literature and helpful comments, which have greatly improved the presentation of our results. We would also like to thank Colin McDiarmid for helpful discussions and sound advice.

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