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Algebraic subgroups of $\mathrm{GL}_2(\mathbb{C})$

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ABSTRACT

In this note we classify, up to conjugation, all algebraic subgroups of $\mathrm{GL}_2(\mathbb{C})$.

1. INTRODUCTION

Although the classification, up to conjugation, of the algebraic subgroups of $\mathrm{SL}_2(\mathbb{C})$ ([3, Theorem 4.12], [6, Theorem 4.29]), and the classification of subgroups of GL_2 over a finite field ([1], [8, Theorem 6.17]) are well known, it seems that the determination of all algebraic subgroups of $\mathrm{GL}_2(\mathbb{C})$ is not presented well in the literature. In this paper we give this classification, including full proofs. The final result is Theorem 4. We note that $\mathbb{C}$ can be replaced everywhere by any algebraically closed field of characteristic zero.

Notation. $\mu_n \subset \mathbb{C}^*$ denotes the nth roots of unity and ζ_n denotes a primitive nth root of unity. Let $\beta : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C}) = \mathrm{PSL}_2(\mathbb{C})$, $\gamma : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$ denote the canonical projections. For any algebraic subgroup $H \subset \mathrm{PSL}_2(\mathbb{C})$ we write $H^{\mathrm{SL}_2} = \gamma^{-1}(H) \subset \mathrm{SL}_2(\mathbb{C})$. Further

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$$

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and

$$D_\infty := \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \,\middle|\, c \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & -d \\ d^{-1} & 0 \end{pmatrix} \,\middle|\, d \in \mathbb{C}^* \right\}$$

are the Borel subgroup and the infinite dihedral subgroup of $\mathrm{SL}_2(\mathbb{C})$.

We first recall the classification of all algebraic subgroups of $\mathrm{PGL}_2(\mathbb{C})$.

Theorem 1. *Let H be an algebraic subgroup of* $\mathrm{PGL}_2(\mathbb{C})$. *Then, up to conjugation, one of the following cases occurs:*

1. $H = \mathrm{PGL}_2(\mathbb{C})$*;*
2. *H is a subgroup of the group $\gamma(B)$;*
3. $H = \gamma(D_\infty)$*;*
4. *$H = D_n$ (the dihedral group of order $2n$), A_4 (the tetrahedral group), S_4 (the octahedral group), or A_5 (the icosahedral group).*

The above theorem reduces the problem to describing the algebraic groups in $\mathrm{GL}_2(\mathbb{C})$ mapping to a given subgroup $G \subset \mathrm{PGL}_2(\mathbb{C})$. Each example is therefore a central extension of G and corresponds to an element in $H^2(G, \mu)$, where μ is either $\mathbb{C}^*$ or a finite cyclic subgroup of $\mathbb{C}^*$. The first case defines the Schur multiplier of G. In the interesting cases, μ is a finite group and the Schur multiplier does not provide information because the canonical map $H^2(G, \mu) \to H^2(G, \mathbb{C}^*)$ is not injective (see also Remark 3).

We note that Theorem 1 is a corollary of the following two well-known theorems.

Theorem 2 (Klein [4]). *A finite subgroup of* $\mathrm{PGL}_2(\mathbb{C})$ *is isomorphic to one of the following polyhedral groups:*

- *a cyclic group C_n;*
- *a dihedral group D_n of order $2n$, $n \geqslant 2$;*
- *the tetrahedral group A_4 of order 12;*
- *the octahedral group S_4 of order 24;*
- *the icosahedral group A_5 of order 60.*

Up to conjugation, all of these groups occur as subgroups of $\mathrm{PGL}_2(\mathbb{C})$ *exactly once.*

In Theorem 1, the cyclic groups C_n happen to be subgroups of $\gamma(B)$.

Theorem 3 ([3, Theorem 4.12]; [6, Theorem 4.29]). *Suppose that G is an algebraic subgroup of* $\mathrm{SL}_2(\mathbb{C})$. *Then, up to conjugation, one of the following cases occurs:*

1. $G = \mathrm{SL}_2(\mathbb{C})$*;*
2. *G is a subgroup of the Borel group B;*

(3) *G is not contained in the Borel group B and is a subgroup of the infinite dihedral group D_∞;*
(4) *G is one of the groups $A_4^{\mathrm{SL}_2},\ S_4^{\mathrm{SL}_2},\ A_5^{\mathrm{SL}_2}$.*

2. ALGEBRAIC SUBGROUPS OF $\mathrm{GL}_2(\mathbb{C})$

Given a group $H \subset \mathrm{PGL}_2(\mathbb{C})$ as in Theorem 1, we will determine *all* algebraic subgroups $G \subset \mathrm{GL}_2(\mathbb{C})$ such that $\beta(G) = H$. We first observe that there is only one maximal group with this property, namely $H_{\max} := \beta^{-1}(H)$. Any G with $\beta(G) = H$ satisfies $\mathbb{C}^* \cdot G = \mathbb{C}^* \cdot H^{\mathrm{SL}_2} = H_{\max}$.

By the Noetherian property, G contains a *minimal algebraic subgroup with image H*. We will denote any such minimal subgroup by $H_{\min}$. Any G with $\beta(G) = H$ has the form $\mu_k \cdot H_{\min}$ or $\mathbb{C}^* \cdot H_{\min} = H_{\max}$. Our problem now remains to determine *all minimal groups $H_{\min}$* (up to conjugation). We will proceed case by case based on Theorem 1.

2.1. $H = \mathrm{PGL}_2(\mathbb{C})$

Proposition 1. *For $H = \mathrm{PGL}_2(\mathbb{C})$ the only minimal group is* $\mathrm{SL}_2(\mathbb{C})$.

Proof. Clearly $H_{\max} = \mathrm{GL}_2(\mathbb{C})$. Let G be a minimal group with $\beta(G) = \mathrm{PGL}_2(\mathbb{C})$. The latter group is equal to its commutator subgroup and therefore $\beta([G,G]) = H$. Since G is minimal, one has $G = [G,G]$ and $G \subset \mathrm{SL}_2(\mathbb{C})$. By Theorem 3, G cannot be a proper subgroup of $\mathrm{SL}_2(\mathbb{C})$. □

2.2. H is a subgroup of the group $\gamma(B)$

Then $H = \gamma(F)$ for some algebraic subgroup F of $B \subset \mathrm{SL}_2(\mathbb{C})$. The algebraic subgroups of the Borel group $B \subset \mathrm{SL}_2(\mathbb{C})$ are listed below:

$$B;\quad \mathbf{G}_m = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \,\Big|\, a \in \mathbb{C}^*\right\};\qquad \mathbf{G}_a = \left\{\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} \,\Big|\, b \in \mathbb{C}\right\};$$
$$F_1^k = \left\{\begin{pmatrix} \xi & c \\ 0 & \xi^{-1}\end{pmatrix} \,\Big|\, \xi^k = 1, c \in \mathbb{C}\right\},\quad \text{with } k \in \mathbb{Z}_{\geqslant 1};$$
$$F_2^l = \left\{\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1}\end{pmatrix} \,\Big|\, \xi^l = 1\right\},\quad \text{with } l \in \mathbb{Z}_{\geqslant 1}.$$

We note that $\mu_l \cong F_2^l \subset \mathbf{G}_m \subset B$ and $F_1^1 = \mathbf{G}_a \subset F_1^k \subset B$.

2.2.1. $H = \gamma(B)$

Proposition 2. *For $H = \gamma(B)$ the minimal groups are*

$$H_{k,l} = \left\{\begin{pmatrix} a & b \\ 0 & c\end{pmatrix} \,\Big|\, a^k c^l = 1\right\}$$

with $k, l \in \mathbb{Z}$ satisfying $k + l \neq 0$ and $\gcd(k,l) = 1$.

Proof. Let $G \subset H_{\max} = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b, c \in \mathbb{C},\ ac \neq 0\}$ be minimal with $\beta(G) = H$. Then G contains an element of the form $A = \alpha \cdot \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ with $\alpha \in \mathbb{C}^*$. The unipotent component $A_u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of the multiplicative Jordan decomposition of A belongs to G. Then G contains the normal subgroup $N := \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \mid b \in \mathbb{C}\}$ and G/N is a proper subgroup of $H_{\max}/N \cong \mathbb{G}_m \times \mathbb{G}_m$. It follows that $G = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a^k c^l = 1\}$ for a certain pair $(k, l) \neq (0, 0)$. This group has projective image $\gamma(B)$ precisely when $k + l \neq 0$. By minimality $\gcd(k, l) = 1$. $\square$

2.2.2. $H = \gamma(\mathbf{G}_m)$

Proposition 3. *In this case, the minimal groups are*

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \, a^k b^l = 1 \right\}$$

with $k, l \in \mathbb{Z}$ *satisfying* $k + l \neq 0$ *and* $\gcd(k, l) = 1$.

Proof. A minimal subgroup G is a proper subgroup of $H_{\max} = \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid a, b \in \mathbb{C}^*\}$ with image $\mathbf{G}_m$ in $\mathrm{PGL}_2(\mathbb{C})$. Therefore it is of dimension one, hence it has the form $\{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid a^k b^l = 1\}$ for some pair of integers $(k, l) \neq (0, 0)$. This group has image $\mathbf{G}_m$ in $\mathrm{PGL}_2(\mathbb{C})$, if and only if $k + l \neq 0$. Since G is minimal one moreover has $\gcd(k, l) = 1$.

Remark 1. Two pairs (k, l) and (m, n) define conjugated minimal subgroups of $\mathrm{GL}_2(\mathbb{C})$ for Proposition 2 if and only if $(k, l) = \pm(m, n)$. For Proposition 3 the two pairs define conjugated groups if and only if $(k, l) \in \{\pm(m, n), \pm(n, m)\}$.

2.2.3. $H = \gamma(\mathbf{G}_a)$

In this case, we have $H^{\mathrm{SL}_2} = \{\pm 1\} \cdot \mathbf{G}_a$ and $H_{\max} = \mathbb{C}^* \cdot \mathbf{G}_a$.

Proposition 4. *In this case, the only minimal group is* $\mathbf{G}_a$.

Proof. Let G be minimal. Then G contains an element of the form $A = \alpha \cdot \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ with $\alpha \in \mathbb{C}^*$. The unipotent component $A_u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of the multiplicative Jordan decomposition of A also belongs to G and thus $G \supset \{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbb{C}\} = \mathbf{G}_a$. By minimality $G = \mathbf{G}_a$.

2.2.4. $H = \gamma(F_1^k)$

The group H is topologically (for the Zariski topology) generated by the images of the elements $\left(\begin{smallmatrix} \zeta_k^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{PGL}_2(\mathbb{C})$ (where ζ_k is a primitive kth root of the identity). Let G denote a minimal subgroup with $\beta(G) = H$. As before one concludes that $G \supset \{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbb{C}\} = \mathbf{G}_a$. Moreover, G is (topologically) generated by $\mathbf{G}_a$ and an element of the form $A := \alpha \cdot \left(\begin{smallmatrix} \zeta_k^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $\alpha \in \mathbb{C}^*$. If α is not a root of unity, then the group, topologically generated by A and $\mathbf{G}_a$, contains $\mathbb{C}^*$ and is equal to $H_{\max}$. By the minimality of G we have that α is some primitive nth root of unity. We define s by $s = k/2$ if k is divisible by 2 and $s = k$ otherwise. For every prime number p, not dividing s, we may consider the subgroup of G generated by A^p and $\mathbf{G}_a$. This group maps surjectively to H. Thus, by minimality, this group is equal to G and p does not divide the order n of α. We find that every prime divisor of n is also a prime divisor of s. Define, for any positive integer n with this property, and every primitive nth root of unity ζ_n, the group $H(\zeta_n)$ as generated by $\zeta_n \cdot \left(\begin{smallmatrix} \zeta_k^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathbf{G}_a$. This group $H(\zeta_n)$ depends on the choice of the primitive nth root of unity ζ_n. Further $\beta(H(\zeta_n)) = H$. The group $H(\zeta_n)$ is minimal since any proper subgroup of $H(\zeta_n)$, containing $\mathbf{G}_a$, is contained in the group generated by $(\zeta_n \cdot \left(\begin{smallmatrix} \zeta_k^2 & 0 \\ 0 & 1 \end{smallmatrix}\right))^p$ and $\mathbf{G}_a$, where the prime p divides s. The latter group does not map surjectively to H. Moreover we found $G \supset H(\zeta_n)$ for some n. Thus we found all minimal groups, namely the groups $H(\zeta_n)$.

Proposition 5. *For* $H = \gamma(F_1^k)$ *the minimal groups are the* $H(\zeta_n)$*, generated by* $\zeta_n \cdot \left(\begin{smallmatrix} \zeta_k^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *and* $\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbb{C}\} = \mathbf{G}_a$*, where every prime divisor of the positive integer* n *divides* k *if* k *is odd and divides* $k/2$ *if* k *is even.*

Remark 2. One has $H(\zeta_n)^o = \mathbf{G}_a$ and the order of the cyclic group $H(\zeta_n)/H(\zeta_n)^o$ is the smallest common multiple of n and k (for k odd) and that of n and $k/2$ (if k is even). Moreover, if $H(\zeta_n)$ is conjugated to H_m, then $n = m$. However the converse is not true in general.

2.2.5. $H = \gamma(F_2^l)$

Similarly to Section 2.2.4 one finds the following proposition:

Proposition 6. *For* $H = \gamma(F_2^l)$ *the minimal groups are the cyclic groups generated by* $\zeta_n \cdot \left(\begin{smallmatrix} \zeta_l^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *where* n *is a positive integer such that every prime divisor of* n *is a prime divisor of* l *if* l *is odd or of* $l/2$ *if* l *is even.*

2.3. $H = \gamma(D_\infty)$

Let G be minimal with $\beta(G) = H$. Then G is a proper subgroup of $H_{\max} = \mathbb{C}^* \cdot D_\infty$. The component of the identity $G^o \subset G$ has the form $\{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid a^k b^l = 1\}$ for some (k, l) with $\gcd(k, l) = 1$. Consider an element $B \in G$ with image (the class of) $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in H$. Thus $B = \beta \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for some $\beta \in \mathbb{C}^*$. From $BG^oB^{-1} = G^o$ it follows that $k = l$ and thus $G^o = \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid ab = 1\}$. By the minimality of G one has that $B^2 = \beta^2$ is a root of unity. The subgroup of G, generated by G^o and B^k, where k is any odd integer, is also mapped surjectively to H. The minimality of G implies that β^2 is a primitive 2^nth root of unity for some $n \geqslant 0$. Let H_n be the group generated by $\{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid ab = 1\}$ and $B_n := \zeta_{2^{n+1}} \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. This group does not depend on the choice of $\zeta_{2^{n+1}}$ since one may replace B_n by any odd power of B_n. Further $G \subset H_n$ for some n. The group G must contain $\{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid ab = 1\}$ and some element $\lambda \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The latter element has the form $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)(\zeta_{2^{n+1}} \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right))^p$ with $ab = 1$ and $p \in \mathbb{Z}$. One concludes that $a = b = \pm 1$ and p is odd. It follows that $G = H_n$ and we conclude: $\{H_n \mid n \geqslant 0\}$ *is the collection of the minimal groups*.

2.4. $H = D_n, A_4, S_4$ or A_5

We first note that if $H \subset \mathrm{PGL}_2(\mathbb{C})$ is a finite subgroup, then every $H_{\min} \subset \mathrm{GL}_2(\mathbb{C})$ is also finite. Indeed, it is clear that H^{SL_2} is finite. Because $H_{\min} \subsetneq \mathbb{C}^* \cdot H^{\mathrm{SL}_2}$, we see that $H_{\min}$ is finite.

2.4.1. $H = D_n$

We write $D_n = \langle a, b \mid a^n = b^2 = 1, ba = a^{-1}b \rangle \subset \mathrm{PGL}_2(\mathbb{C})$.

(i) n **odd and** $n \geqslant 3$**.** In this case, we may choose for a and b the images in $\mathrm{PGL}_2(\mathbb{C})$ of the matrices $\left(\begin{smallmatrix}\zeta_n & 0\\ 0 & \zeta_n^{-1}\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, with ζ_n a primitive nth root of unity.

Let G be a minimal group. As G is finite and generated by preimages of $a, b \in D_n$ one has that

$$G = \left\langle A = \lambda \begin{pmatrix}\zeta_n & 0\\ 0 & \zeta_n^{-1}\end{pmatrix},\ B = \mu \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \right\rangle$$

for certain roots of unity λ, μ. We have $A^n = \lambda^n$, $B^2 = \mu^2$, $BA = \lambda^2 A^{-1} B$. Every element of G has the form tA^k, or tA^kB, $k = 0, 1, \dots, n-1$, with $t \in \langle \lambda^2, \lambda^n, \mu^2 \rangle = \langle \lambda, \mu^2 \rangle$. Hence $G \cap \mathbb{C}^* = \langle \lambda, \mu^2 \rangle$. Since both $\lambda\left(\begin{smallmatrix}\zeta_n & 0\\ 0 & \zeta_n^{-1}\end{smallmatrix}\right) \in G$ and $\lambda \in G$, we can write

$$G = \left\langle A = \begin{pmatrix}\zeta_n & 0\\ 0 & \zeta_n^{-1}\end{pmatrix},\ B = \mu \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \right\rangle.$$

The subgroup of G generated by A and B^m, where $m \geqslant 1$ is odd, also maps surjectively to D_n. By the minimality of G, this implies that the order of μ is 2^k for some $k \geqslant 0$. Now define

$$H_k := \left\langle \begin{pmatrix}\zeta_n & 0\\ 0 & \zeta_n^{-1}\end{pmatrix},\ \zeta_{2^k} \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \right\rangle,$$

for $k \geqslant 0$. This group H_k does not depend on the choice of the primitive 2^kth root of unity because one can replace the second generator by any odd power of itself. The groups H_k are the only candidates for minimal groups.

We now show that H_k is indeed minimal. For $k = 0, 1$, the groups

$$H_0 = \left\langle \begin{pmatrix}\zeta_n & 0\\ 0 & \zeta_n^{-1}\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \right\rangle, \qquad H_1 = \left\langle \begin{pmatrix}\zeta_n & 0\\ 0 & \zeta_n^{-1}\end{pmatrix}, \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} \right\rangle$$

are minimal since they have order $2n$. The two groups are conjugated by the matrix $\left(\begin{smallmatrix}i & 0\\ 0 & -i\end{smallmatrix}\right)$. We note that $H_2 = D_n^{\mathrm{SL}_2}$. For $k \geqslant 2$, we see that $H_k \cap \mathbb{C}^* = \langle \zeta_{2^k}^2 \rangle$. Suppose that D is a subgroup of H_k which maps surjectively to D_n, then

$$D = \left\langle \begin{pmatrix}\zeta_n & 0\\ 0 & \zeta_n^{-1}\end{pmatrix},\ t\zeta_{2^k} \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \right\rangle$$

for some $t \in \langle \zeta_{2^k}^2 \rangle$. Since the order of $t\zeta_{2^k}$ is also 2^k, one has $D = H_k$ and thus H_k is minimal. For $k \geqslant 1$, the order of H_k is $2^k \cdot n$. Thus two minimal groups H_k and H_l with $k, l \geqslant 1$ are conjugated only if $k = l$.

(ii) n **even and** $n > 2$**.** A minimal G can be written as

$$G = \left\langle A = \lambda \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix},\ B = \mu \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle,$$

for certain roots of unity λ, μ. We have

$$A^n = -\lambda^n, \qquad B^2 = -\mu^2, \qquad BA = \lambda^2 A^{-1} B.$$

As before, this implies that $G \cap \mathbb{C}^* = \langle \lambda^2, -\lambda^n, -\mu^2 \rangle = \langle -1, \lambda^2, \mu^2 \rangle$. One can replace A and B by $c_1 A$ and $c_2 B$ with $c_1, c_2 \in \langle -1, \lambda^2, \mu^2 \rangle$. For a good choice of c_1, c_2, the group $\langle c_1 A, c_2 B \rangle$ will be a proper subgroup unless there exists an integer N with $\lambda, \mu \in \mu_{2^N}$. Thus the latter holds by the minimality of G. Then $\langle -1, \lambda, \mu \rangle = \mu_{2^{m+1}}$ for some $m \geqslant 0$.

For $m = 0$, we have $G \cap \mathbb{C}^* = \mu_2$ and this leads to only one group, namely

$$\left\langle \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle = D_n^{\mathrm{SL}_2}.$$

This group is clearly minimal. For $m \geqslant 1$, one has $G \cap \mathbb{C}^* = \mu_{2^m}$ and this leads to the three groups given by the table:

	λ	μ
$H_{1,m}$	$\zeta_{2^{m+1}}$	1
$H_{2,m}$	$\zeta_{2^{m+1}}$	$\zeta_{2^{m+1}}$
$H_{3,m}$	1	$\zeta_{2^{m+1}}$

They all are minimal and have order $2^m \cdot 2n$. However $H_{1,m}$ and $H_{2,m}$ are conjugated. Indeed, $\left(\begin{smallmatrix} \zeta_{2n} & 0 \\ 0 & 1 \end{smallmatrix}\right) H_{1,m} \left(\begin{smallmatrix} \zeta_{2n}^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) = H_{2,m}$ because

$$\begin{aligned} &\begin{pmatrix} \zeta_{2n} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \zeta_{2n}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad = \zeta_{2^{m+1}}^{-2} \cdot \left[\zeta_{2^{m+1}} \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix} \right] \cdot \left[\zeta_{2^{m+1}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right]. \end{aligned}$$

(iii) $n = 2$**.** As in (ii). In this case also $H_{1,m}$ and $H_{3,m}$ are also conjugated, namely by a matrix of the form $\left(\begin{smallmatrix} 0 & a \\ 1 & -1 \end{smallmatrix}\right)$.

2.4.2. $H = A_4$

Let $G \subset H_{\max} = \mathbb{C}^* \cdot A_4^{\mathrm{SL}_2}$ be a minimal group. Consider $G^+ \subset \mathbb{C}^* \times A_4^{\mathrm{SL}_2}$, the preimage of G under the obvious map $\alpha : \mathbb{C}^* \times A_4^{\mathrm{SL}_2} \to \mathbb{C}^* \cdot A_4^{\mathrm{SL}_2}$. We note that the kernel of α is $\{(1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)), (-1, \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right))\}$. Since $\beta(G) = A_4$, there exists for every $a \in A_4^{\mathrm{SL}_2}$ an element $(\lambda, a) \in G^+$. Let $\mu_k := \{\lambda \in \mathbb{C}^* \mid (\lambda, 1) \in G^+\}$. Then we obtain a homomorphism $h : A_4^{\mathrm{SL}_2} \to \mathbb{C}^*/\mu_k$ given by $h(a) = \lambda \bmod \mu_k$ if $(\lambda, a) \in G^+$.

This homomorphism factors as $A_4^{\mathrm{SL}_2} \to C_3 \xrightarrow{h_1} \mathbb{C}^*/\mu_k$, where $C_3 = \{1, \sigma, \sigma^2\}$ is the quotient of $A_4^{\mathrm{SL}_2}$ by its commutator subgroup. If h_1 is trivial, then G^+ contains $\{(1, a) \mid a \in A_4^{\mathrm{SL}_2}\}$ and by minimality $G = A_4^{\mathrm{SL}_2}$. By Theorem 3, the latter group of order 24 is minimal.

Now we suppose that h_1 is not trivial. Write $k = 3^r \ell$ with $\gcd(\ell, 3) = 1$. For any $a \in A_4^{\mathrm{SL}_2}$ there exists an element $(\lambda, a) \in G^+$ with $\lambda^3 \in \mu_{3^r \ell}$ and λ can be multiplied by any element in $\mu_{3^r \ell}$. Thus there exist a pair $(\lambda, a) \in G^+$ with $\lambda \in \mu_{3^{r+1}}$.

Now $G^+ \cap (\mu_{3^{r+1}} \times A_4^{\mathrm{SL}_2})$ is a subgroup of G^+ mapping surjectively to A_4. The minimality of G implies that $\ell = 1$ and $G^+ \subset \mu_{3^{r+1}} \times A_4^{\mathrm{SL}_2}$. Moreover, $\mu_{3^r} \subset G^+$ and the map $G^+ \to G$ is bijective. Then G has the form $\mu_{3^r} \cdot \{\delta(a)a \mid a \in A_4^{\mathrm{SL}_2}\}$, where $\delta =: A_4^{\mathrm{SL}_2} \to C_3 \xrightarrow{\delta_1} \{1, \zeta_{3^{r+1}}, \zeta_{3^{r+1}}^2\}$ for some map δ_1 which lifts the homomorphism $h_1 : C_3 \to \mu_{3^{r+1}}/\mu_{3^r} \subset \mathbb{C}^*/\mu_{3^r}$. There are two possibilities for nontrivial homomorphism h_1 (and thus for δ_1 and δ) and we find therefore two subgroups of $\mathrm{GL}_2(\mathbb{C})$, lying in $\mu_{3^{r+1}} \cdot A_4^{\mathrm{SL}_2}$. The last group is contained in $\mu_{3^{r+1}} \cdot S_4^{\mathrm{SL}_2}$. Conjugation by an element $\tau \in S_4 \setminus A_4$ induced on $C_3 = A_4/[A_4, A_4]$ the only non trivial automorphism and permutes the two possibilities for h_1. One lifts τ to an element $\tau' \in S_4^{\mathrm{SL}_2}$. Conjugation by τ' permutes the two possibilities for h_1 and therefore the above two groups are conjugated. It suffices to consider the group $H_r := \mu_{3^r} \cdot \{\delta(a)a \mid a \in A_4^{\mathrm{SL}_2}\}$ with δ_1 given by $\delta_1(1) = 1$, $\delta_1(\sigma) = \zeta_{3^{r+1}}$, $\delta_1(\sigma^2) = \zeta_{3^{r+1}}^2$. The order of H_r is $3^r \cdot 24$. The group H_0 is isomorphic to $A_4^{\mathrm{SL}_2}$, but not conjugated to $A_4^{\mathrm{SL}_2}$. The minimality of H_0 follows from the fact that A_4 does not have a faithful two-dimensional representation.

Finally, we will show that H_r is minimal for $r \geqslant 1$. Suppose that D is a subgroup of H_r with $\beta(D) = A_4$. Let $\tau \in A_4^{\mathrm{SL}_2}$ be an element of order 3. Then D contains an element $d = \lambda\delta(\tau)\tau$ for some $\lambda \in \{\pm 1\} \times \mu_{3^r}$. Now $\delta(\tau) \in \{\zeta_{3^{r+1}}, \zeta_{3^{r+1}}^2\}$ and $d^3 \in D \cap \mathbb{C}^*$ has order 3^r or $2 \cdot 3^r$. Thus D contains μ_{3^r} and it follows that $D = H_r$. Thus we found:

There are two minimal groups for A_4 *with order* 24 *and for every* $r \geqslant 1$ *there is one minimal group of order* $3^r \cdot 24$.

Remark 3. A minimal subgroup G for $H = A_4$ yields a central extension $1 \to \mu_k \to G \to A_4 \to 1$ for some k. The corresponding element ξ of $H^2(A_4, \mu_k)$ has, by the minimality of G, the property that ξ does not lie in the image of $H^2(A_4, \mu_d)$ for a proper divisor d of k. Since the order of A_4 in 12, we only have to consider the groups $H^2(A_4, \mu_{2^a 3^b})$. The minimal groups that we found above correspond to all the cases $(a, b) = (1, r)$. The central extensions with $a \neq 1$ produce, apparently, groups which do not have a faithful representation of degree two.

2.4.3. $H = S_4$

Let $G \subset H_{\max} = \mathbb{C}^* \cdot S_4^{\mathrm{SL}_2}$ be a minimal group. Consider $G^+ \subset \mathbb{C}^* \times S_4^{\mathrm{SL}_2}$, the preimage of G under the obvious map $\alpha : \mathbb{C}^* \times S_4^{\mathrm{SL}_2} \to \mathbb{C}^* \cdot S_4^{\mathrm{SL}_2}$. The kernel of α is $\{(1, \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)), (-1, \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right))\}$. Since $\beta(G) = S_4$, there exists for every $a \in S_4^{\mathrm{SL}_2}$

an element $(\lambda, a) \in G^+$. Let $\mu_k := \{\lambda \in \mathbb{C}^* \mid (\lambda, 1) \in G^+\}$. Then we obtain a homomorphism $h : S_4^{\mathrm{SL}_2} \to \mathbb{C}^*/\mu_k$ given by $h(a) = \lambda \bmod \mu_k$ if $(\lambda, a) \in G^+$. This homomorphism factors as $S_4^{\mathrm{SL}_2} \to C_2 \xrightarrow{h_1} \mathbb{C}^*/\mu_k$, where $C_2 = \{1, \sigma\}$ is the quotient of $S_4^{\mathrm{SL}_2}$ by its commutator subgroup. If h_1 is trivial, then G^+ contains $\{(1, a) \mid a \in S_4^{\mathrm{SL}_2}\}$ and by minimality $G = S_4^{\mathrm{SL}_2}$. According to Theorem 3, the latter group of order 48 is minimal.

Now we suppose that h_1 is not trivial. Write $k = 2^r \ell$ with ℓ odd. For any $a \in S_4^{\mathrm{SL}_2}$ there exists an element $(\lambda, a) \in G^+$ with $\lambda \in \mu_{2^{r+1}}$. Now $G^+ \cap (\mu_{2^{r+1}} \times S_4^{\mathrm{SL}_2})$ is a subgroup of G^+ mapping surjectively to S_4. The minimality of G implies that $\ell = 1$ and $G^+ \subset \mu_{2^{r+1}} \times S_4^{\mathrm{SL}_2}$. Define $\delta : S_4^{\mathrm{SL}_2} \to C_2 \xrightarrow{\delta_1} \{1, \zeta_{2^{r+1}}\}$ by $\delta_1(1) = 1$ and $\delta_1(\sigma) = \zeta_{2^{r+1}}$. All the elements of $\mu_{2^{r+1}}$ have the form $\zeta_{2^{r+1}}^{\varepsilon} \cdot \lambda$ with $\varepsilon \in \{0, 1\}$ and $\lambda \in \mu_{2^r}$. From this it follows that $G^+ = \{(\delta(a)\lambda, a) \mid a \in S_4^{\mathrm{SL}_2}, \ \lambda \in \mu_{2^r}\}$ and one concludes that $G = H_r := \mu_{2^r} \cdot \{\delta(a)a \mid a \in S_4^{\mathrm{SL}_2}\}$. The group H_r has order $2^r \cdot 48$. We note that H_0 is equal to $S_4^{\mathrm{SL}_2}$ and is minimal.

Let $r > 0$ and let $D \subset H_r$ be a subgroup satisfying $\beta(D) = S_4$. Let $\tau \in S_4^{\mathrm{SL}_2}$ be an element with image the permutation $(1, 2) \in S_4$. Then D contains an element of the form $d = \pm\lambda\delta(\tau)\tau$ with $\lambda \in \mu_{2^r}$. Then $d^2 = \lambda^2 \zeta_{2^r} \in D \cap \mathbb{C}^*$ has order 2^r. Thus D contains μ_{2^r} and it follows easily that $D = H_r$. Hence every H_r is minimal and we conclude that: *There is for every* $r \geqslant 0$ *a unique minimal group of order* $2^r \cdot 48$.

2.4.4. $H = A_5$

Let $G \subset \mathrm{GL}_2(\mathbb{C})$ be a minimal for H. Since $A_5 = [A_5, A_5]$, the group $[G, G]$ also satisfies $\beta([G, G]) = H$. By minimality $G = [G, G]$ and thus $G \subset \mathrm{SL}_2(\mathbb{C})$. This implies that $G \subset A_5^{\mathrm{SL}_2}$. Since, by Theorem 3, the latter group is minimal, we find that $A_5^{\mathrm{SL}_2}$ is the only minimal group.

In summary, we obtain the following result.

Theorem 4. *The list of all* minimal *groups, up to conjugation, for each algebraic subgroup* $H \subset \mathrm{PGL}_2(\mathbb{C})$ *(see Theorem* 1 *and Section* 2.2*) is:*

(1) $H = \mathrm{PGL}_2(\mathbb{C})$*: the only minimal group is* $\mathrm{SL}_2(\mathbb{C})$.

(2) H *is a subgroup of the group* B*:*
  (a) $H = \gamma(B)$*: for each pair of integers* (k, l) *with* $k + l \neq 0$ *and* $\gcd(k, l) = 1$ *there is a minimal one, namely* $\{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a^k c^l = 1\}$.
  (b) $H = \gamma(\mathbf{G}_m)$*: for each pair of integers* (k, l) *with* $k + l \neq 0$ *and* $\gcd(k, l) = 1$ *there is a minimal group, namely* $\{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid a^k b^l = 1\}$*. Further* (k, l) *and* (l, k) *define conjugated groups.*
  (c) $H = \gamma(\mathbf{G}_a)$*: there is only one minimal group, namely* $\mathbf{G}_a$.
  (d) $H = \gamma(F_1^k)$*: the minimal ones are the* $H(\zeta_n)$*, generated by* $\zeta_n \cdot \left(\begin{smallmatrix} \zeta_k^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *and* $\{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbb{C}\}$*, where every prime divisor of the positive integer* n *divides* k *if* k *is odd and divides* $k/2$ *if* k *is even.*

  - (e) $H = \gamma(F_2^l)$: *the minimal ones are the groups generated by* $\zeta_n \cdot \left(\begin{smallmatrix}\zeta_l^2 & 0\\ 0 & 1\end{smallmatrix}\right)$, *where every prime divisor of the positive integer* n *divides* l *if* l *is odd and divides* $l/2$ *if* l *is even.*

(3) $H = \gamma(D_\infty)$: *the minimal groups are* H_n *with* $n \geqslant 0$, *where* H_n *is generated by* $\{\left(\begin{smallmatrix}a & 0\\ 0 & b\end{smallmatrix}\right) \mid ab = 1\}$ *and* $\zeta_{2^{n+1}} \cdot \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$ *with* $\zeta_{2^{n+1}}$ *a primitive* 2^{n+1}*th root of unity.*

(4) *H finite:*

  - (a) $H = D_n$:
    - (i) $n \geqslant 3$ *odd: For every* $k \geqslant 1$, *there is one minimal group*

$$\left\langle \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}, \ \zeta_{2^k}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

with ζ_{2^k} *a primitive* 2^k*th root of unity;*

    - (ii) $n > 2$ *even: For* $k \geqslant 1$, *the minimal ones* $H_{1,k}, H_{2,k}, H_{3,k}$ *have the form*

$$\left\langle A = \lambda \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \ B = \mu \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle,$$

for certain roots of unity λ, μ *which are given in the table:*

	λ	μ
$H_{1,k}$	$\zeta_{2^{k+1}}$	1
$H_{2,k}$	$\zeta_{2^{k+1}}$	$\zeta_{2^{k+1}}$
$H_{3,k}$	1	$\zeta_{2^{k+1}}$

They all have order $2^k \cdot 2n$. *Further* $H_{1,k}$ *and* $H_{2,k}$ *are conjugated. For* $k = 0$, *there is only minimal group, namely the group*

$$\left\langle \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle = D_n^{\mathrm{SL}_2} \text{ of order } 4n;$$

    - (iii) $n = 2$: *As in* (ii), *but now* $H_{1,k}, H_{2,k}, H_{3,k}$ *are all conjugated.*
  - (b) $H = A_4$: *there are two minimal groups of order* 24. *For every* $n > 0$ *there is one minimal group of order* $3^n \cdot 24$.
  - (c) $H = S_4$: *For every* $n \geqslant 0$ *there is a minimal group of order* $2^n \cdot 48$.
  - (d) $H = A_5$: *There is only minimal group, namely* $A_5^{\mathrm{SL}_2}$.

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