A Simple Answer to Gelfand's Question

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Abstract. Using elementary techniques, a question named after the famous Russian mathematician I. M. Gelfand is answered. This concerns the leading (i.e., most significant) digit in the decimal expansion of integers $2^n, 3^n, \ldots, 9^n$. The history of this question, some of which is very recent, is reviewed.

1. INTRODUCTION AND RESULTS. It is well known that any positive integer n has a decimal expansion

$$n = a_m 10^m + \cdots + a_1 10 + a_0,$$

for some integer $m \geq 0$ and integers $a_j \in \{0, 1, 2, \ldots, 8, 9\}$ with $a_m \neq 0$. Moreover, this expansion is unique. In what follows, the integer a_m appearing at the beginning of the decimal expansion of n is denoted by

$$\langle\!\langle n \rangle\!\rangle := a_m,$$

and it is called the *leading digit* of n. So by definition,

$$\langle\!\langle n \rangle\!\rangle \in \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}.$$

The leading digit of positive numbers was already studied in 1881 by S. Newcomb [**9**]; in fact he considered real numbers appearing in logarithm tables and he noted that numbers with leading digit 1 appeared much more frequently in such tables than, e.g., numbers with leading digit 9. This eventually led to the famous "Benford's law" for the distribution of first digits, predicted to hold in many real life sets of numbers. This law is named after the physicist F. Benford, who published his paper [**3**] stating it in 1938.

According to p. 37 of a book [**2**] by A. Avez published in 1966, the famous Russian mathematician Israel M. Gelfand (1913–2009) posed the following question concerning leading digits.

Question 1. Does $n > 1$ exist such that $\langle\!\langle 2^n \rangle\!\rangle = 9$?

The question was also included (Example 3.2 on p. 10) in the volume [**1**] by V. I. Arnold and A. Avez which was published two years later, however, with '9' replaced by '7.' On pp. 135–136 (Application A12.5) of [**1**], a detailed and complete answer is provided. In fact, this answer shows that the distribution of the first digits of the numbers 2^n for positive n satisfies "Benford's law." It may be noted that in [**1**] no remark appears relating the question to Gelfand.

The web pages of *Wolfram MathWorld* recall the question attributed to Gelfand, and add three more questions to it [**13**].

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Question 2. Does $n > 1$ exist such that

$$(\langle\!\langle 2^n \rangle\!\rangle,\ \langle\!\langle 3^n \rangle\!\rangle, \ldots, \langle\!\langle 8^n \rangle\!\rangle,\ \langle\!\langle 9^n \rangle\!\rangle) = (2, 3, 4, 5, 6, 7, 8, 9)?$$

More particularly, compute

$$\lim_{N\to\infty} \frac{\#\{n \in \mathbb{Z}_{>0}\ ;\ n < N \text{ and } \langle\!\langle \ell^n \rangle\!\rangle = \ell \text{ for } 2 \le \ell \le 9\}}{N}$$

if this limit exists.

Question 3. Do any $n > 0$ and $a \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ exist such that

$$(\langle\!\langle 2^n \rangle\!\rangle,\ \langle\!\langle 3^n \rangle\!\rangle, \ldots, \langle\!\langle 8^n \rangle\!\rangle,\ \langle\!\langle 9^n \rangle\!\rangle) = (a, a, a, a, a, a, a, a)?$$

Question 4. Does $n \ge 1$ exist such that

$$\langle\!\langle 2^n \rangle\!\rangle 10^7 + \langle\!\langle 3^n \rangle\!\rangle 10^6 + \langle\!\langle 4^n \rangle\!\rangle 10^5 + \cdots + \langle\!\langle 8^n \rangle\!\rangle 10 + \langle\!\langle 9^n \rangle\!\rangle$$

is a prime number?

The *Wolfram MathWorld* page discussing these questions is titled 'Gelfand's Question,' although it clearly states that only the simple Question 1 was attributed to Gelfand. No reference is given to the book by Arnold and Avez which, as indicated above, contains a detailed and complete answer to Question 1. A reference discussing all four questions stated above is Jonathan L. King's prize winning paper [**8**], which was honored in 1995 with the Lester R. Ford Award of the Mathematical Association of America. Page 610 of King's paper states the four questions, and pp. 619–624 contain a detailed answer to Question 1 and even several arguments which lead in the direction of an answer to Questions 2 and 3. However, no answer is given. Since the *MathWorld* page uses [**8**] as its main reference, it seems plausible that this is how they learned about the questions. It is somewhat remarkable that the *MathWorld* pages do not mention the fact that King provides a complete answer to Question 1. In the same vein, although King cites the textbook [**2**], he does not mention [**1**] and therefore misses the fact that this book (also) answers Question 1.

Concerning Questions 2 and 3, the *MathWorld* page mentions that this was tested for $n \le 10^5$ and no examples were found. In a post (19 June 2013) on his blog *The Endeavour* [**4**], the American mathematician John D. Cook informs the readers that he has extended the search to $n < 10^{10}$. This did not result in any examples. Cook's post [**5**] written on the same day discusses the problem of actually calculating $\langle\!\langle a^n \rangle\!\rangle$ for small a and exponent n in the search range mentioned above.

Question 4 led to a list in the Online Encyclopedia of Integer Sequences [**12**]. This contains the 53 smallest positive integers n such that $\langle\!\langle 2^n \rangle\!\rangle 10^7 + \langle\!\langle 3^n \rangle\!\rangle 10^6 + \langle\!\langle 4^n \rangle\!\rangle 10^5 + \cdots + \langle\!\langle 8^n \rangle\!\rangle 10 + \langle\!\langle 9^n \rangle\!\rangle$ is prime.

It turns out that not only do Questions 1 and 4 have a simple answer that can be explained using classical and quite well-known arguments; the same holds for Questions 2 and 3. Indeed, we have the following result.

Theorem 1.

1. *(see Appendix 12 in [**1**] and [**8**, p. 621]) Suppose k, ℓ are integers satisfying $2 \le k, \ell \le 9$. Then*

$$\lim_{N\to\infty} \frac{\#\{n \in \mathbb{Z}_{>0}\ ;\ n < N \text{ and } \langle\langle \ell^n \rangle\rangle = k\}}{N} = \log_{10}\left(\frac{k+1}{k}\right).$$

This means precisely that the integers $\{\ell^n\ ;\ n \in \mathbb{Z}_{>0}\}$ *satisfy Benford's law.*

2. *The only integer* $n > 0$ *such that* $\langle\langle \ell^n \rangle\rangle = \ell$ *holds for every* $\ell \in \{2, \ldots, 9\}$ *is* $n = 1$.
3. *For a fixed integer* $n > 0$*, the* 8 *numbers in the sequence*

$$\langle\langle 2^n \rangle\rangle,\ \langle\langle 3^n \rangle\rangle, \ldots, \langle\langle 8^n \rangle\rangle,\ \langle\langle 9^n \rangle\rangle$$

   are not all equal.
4. *The set*

$$\left\{ \sum_{\ell=2}^{9} \langle\langle \ell^n \rangle\rangle 10^{9-\ell}\ :\ n > 0 \right\}$$

   contains precisely 17 596 *integers, of which* 1127 *are prime numbers.*

The essential parts of this result were posted on his blog MATHBLAG [**11**] by the second author of our paper on the same day that David Cook discussed Gelfand's question (19 June 2013). Independent of this, the theorem was proven in the bachelor's thesis [**6**] of the first author of our paper. However, due to a small mistake in an algorithm, the numbers appearing in Part (4) of our result were higher in this thesis. A comparison with the data provided on [**11**] revealed this mistake and moreover showed that also the numbers originally presented on the website were too high. The remainder of this paper explains the proof of Theorem 1.

2. TOOLS. The operation $m \mapsto \langle\langle m \rangle\rangle$, which assigns to a positive integer m its leading digit, can easily be extended to arbitrary positive real numbers. To make this precise, put

$$M := \{x \in \mathbb{R}\ :\ 1 \le x < 10\}.$$

Any $x \in M$ has its integer part $\lfloor x \rfloor \in \{1, 2, \ldots, 8, 9\}$, which is by definition the largest integer $\le x$.

By multiplying all numbers in M by 10, one obtains $10M$ which consists of all y such that $10 \le y < 100$; similarly dividing by 1000 yields $10^{-3}M$, the numbers between $1/1000$ and $1/100$ with the left boundary included. If one uses all powers of 10 in this way, a subdivision

$$\mathbb{R}_{>0} = \bigcup_{k\in\mathbb{Z}} 10^k M$$

of the positive real numbers as a union of pairwise disjoint half open intervals $10^k M$ is obtained. This means that every positive real number x can be written in a unique way as

$$x = 10^k m$$

for some integer k and some $m \in M$. Now define

$$\langle\langle x \rangle\rangle := \lfloor m \rfloor.$$

Note that for positive integers $x = n$ this definition coincides with the one given in Section 1.

Next, define for $x, y \in M$ the product $xy \in M$ as

$$xy := \begin{cases} x \cdot y & \text{if} \quad x \cdot y < 10; \\ x \cdot y/10 & \text{if} \quad x \cdot y \geq 10. \end{cases}$$

Observe that this product provides M with the structure of a commutative group. The inverse of $x \in M$ is $10/x$ in case $x \neq 1$; the unit element $1 \in M$ is obviously its own inverse.

Quite analogously, a group A can be defined as follows. As a set, take

$$A := \{x \in \mathbb{R} \ : \ 0 \leq x < 1\}$$

and as a group structure on A define for $a, b \in A$ their sum

$$a + b \in A$$

to be the usual sum of the real numbers a, b when this sum is < 1, and one less than this usual sum otherwise.

The groups M and A are isomorphic: $m \mapsto \log_{10}(m)$ is an isomorphism with inverse $a \mapsto 10^a$. In fact, both groups A and M are well known. A is the group $\mathbb{R}/\mathbb{Z}$, the additive group of real numbers modulo the subgroup of all integers, and M is the group $\mathbb{R}^\times_{>0}/10^{\mathbb{Z}}$, the multiplicative group of positive real numbers modulo the subgroup $\{10^k \ : \ k \in \mathbb{Z}\}$. This last observation provides one with a group homomorphism

$$\mathbb{R}^\times_{>0} \longrightarrow \mathbb{R}^\times_{>0}/10^{\mathbb{Z}} \cong M$$

which will be denoted $x \mapsto \tilde{x}$. Writing $x = 10^k m$ as before, it maps $x \in \mathbb{R}_{>0}$ to $\tilde{x} = m \in M$. By definition,

$$\langle\langle x \rangle\rangle = \lfloor \tilde{x} \rfloor$$

and we obtain a diagram

$$\begin{tikzcd} \mathbb{R}_{>0} \arrow[rr, "x \mapsto \tilde{x}"] \arrow[dr, "\langle\langle\cdot\rangle\rangle"'] & & M \arrow[dl, "\lfloor\cdot\rfloor"] \\ & \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}. & \end{tikzcd}$$

The group A is the well known group of points on the circle C. Indeed, the map $A \to C$, given by $a \mapsto e^{2\pi i a}$ defines an isomorphism from A to the subgroup $C = \{z \in \mathbb{C} \ : \ |z| = 1\}$ of the multiplicative group consisting of all nonzero complex numbers. The main mathematical tool that we use in this paper is the Kronecker–Weyl equidistribution theorem. Kronecker proved in 1884 the following result.

Theorem 2. *Let* $(\alpha_1, \dots, \alpha_m) \in A \times \cdots \times A$ *be such that, as real numbers,* $\lambda_0 + \lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_m\alpha_m \neq 0$ *for all nonzero* $(\lambda_0, \dots, \lambda_m) \in \mathbb{Q}^{m+1}$*. Then* $\{n(\alpha_1, \dots, \alpha_m) : n \in \mathbb{Z}_{>0}\}$ *is dense in* $A \times \cdots \times A$.

In fact, the name "equidistribution theorem" refers to a much stronger result which was not yet claimed by Kronecker in 1884, but was proved by several people including Hermann Weyl around 1909–1910. A nice two page exposition of the result and its proof is given in [**10**]. We only mention a special and simple consequence (compare [**7**, § 23.10]).

Theorem 3. *Let* $\alpha \in A$ *be an element of infinite order (i.e.,* $\alpha \notin \mathbb{Q}$*) and let* $I \subseteq A$ *be an interval of length* $|I|$*. Then*

$$\lim_{N\to\infty} \frac{\#\{n \in \mathbb{Z}_{>0} \ : \ n \leq N \quad and \quad n\alpha \in I\}}{N} = |I|.$$

Obviously, using the isomorphism $A \cong M$ one obtains analogous results in the multiplicative setup. This will be used in proving our answers to "Gelfand's questions."

3. PROOFS. We will now prove the assertions presented in Theorem 1. For completeness we include a proof of the first part, although this may also be found in [**1**] and in [**8**].

Part 1. Let k, ℓ be integers such that $2 \leq k, \ell \leq 9$. For any positive integer n we have

$$\begin{aligned} \langle\!\langle \ell^n \rangle\!\rangle = k \quad &\Leftrightarrow \quad k \leq \widetilde{\ell^n} < k+1 \\ &\Leftrightarrow \quad \log_{10}(k) \leq n\log_{10}(\ell) < \log_{10}(k+1), \end{aligned}$$

where $\alpha := \log_{10}(\ell)$ is regarded as an element of the group A and $n\alpha$ denotes repeated addition in A.

Note that $\alpha \notin \mathbb{Q}$, since otherwise $a\alpha = b$ for some positive integers a, b, and therefore $\ell^a = 10^b$, which is absurd. So we can apply Theorem 3 to α and the interval

$$I := \{x \ : \ \log_{10}(k) \leq x < \log_{10}(k+1)\}$$

which has length $|I| = \log_{10}\left(\frac{k+1}{k}\right)$. This completes the proof of Part 1.

Part 2. To prove the second assertion, observe that in the group M the inverse of 2 is 5. As a consequence, the inverse of $\widetilde{2^n}$ is $\widetilde{5^n}$ for any integer n.

Now assume $n > 0$ and $\langle\!\langle 2^n \rangle\!\rangle = 2$ and $\langle\!\langle 5^n \rangle\!\rangle = 5$. By definition, this means

$$\widetilde{2^n} = 2 + \varepsilon_2 \quad \text{and} \quad \widetilde{5^n} = 5 + \varepsilon_5$$

for certain real numbers $\varepsilon_2, \varepsilon_5$ which satisfy $0 \leq \varepsilon_2, \varepsilon_5 < 1$. Multiplying in the group M shows

$$1 = \frac{(2+\varepsilon_2)(5+\varepsilon_5)}{10},$$

so $2\varepsilon_5 + 5\varepsilon_2 + \varepsilon_2\varepsilon_5 = 0$. Since $\varepsilon_2, \varepsilon_5$ are nonnegative, the latter equality implies

$$\varepsilon_2 = \varepsilon_5 = 0,$$

so $\widetilde{2^n} = 2$. This means $2^n = 2 \cdot 10^k$ for some integer k, which is clearly only possible for $k = 0$. In that case $2^n = 2$, so $n = 1$.

Since $\langle\!\langle \ell^1 \rangle\!\rangle = \ell$ for every $\ell \in \{2, 3, \ldots, 9\}$, this finishes the proof of Part 2.

Part 3. Take an integer $n > 0$ and assume $\langle\!\langle \ell^n \rangle\!\rangle = a$ for all $\ell \in \{2, 3, \ldots, 9\}$. Write

$$\widetilde{\ell^n} = a + \varepsilon_\ell$$

where, by assumption, $0 \le \varepsilon_\ell < 1$. In fact $\varepsilon_\ell \neq 0$ for every ℓ. Indeed, an equality $\widetilde{\ell^n} = a$ would imply that $\ell^n = a \cdot 10^k$ for some integer k. This is only possible with $k = 0$, implying that $\ell^n = a \le 9$. But then $n \le 3$, and in that case $\langle\!\langle 2^n \rangle\!\rangle \neq \langle\!\langle 3^n \rangle\!\rangle$.

So we conclude

$$0 < \varepsilon_\ell < 1$$

for every ℓ. Now multiply $\widetilde{2^n}$ and $\widetilde{5^n}$ in the group M. The result equals $1 \in M$, and as a consequence one has as real numbers

$$(a + \varepsilon_2) \cdot (a + \varepsilon_5) = 10.$$

The fact that every ε_ℓ is between 0 and 1 therefore shows

$$a^2 < 10 < (a + 1)^2,$$

hence $a = 3$ (this argument is also present in [**8**]).

Since $(2^n)^2 = 4^n$, it follows that in M the equality

$$(3 + \varepsilon_2)^2 = 3 + \varepsilon_4$$

holds. This is clearly absurd because the real number $(3 + \varepsilon_2)^2$ is between 9 and 16 while $3 + \varepsilon_4$ times any power of 10 is not.

So the assumption that $\langle\!\langle 2^n \rangle\!\rangle = \langle\!\langle 4^n \rangle\!\rangle = \langle\!\langle 5^n \rangle\!\rangle$ and $n > 0$ leads to a contradiction. This proves Part 3.

Part 4. The remaining assertion deals with the sequences

$$(\langle\!\langle 2^n \rangle\!\rangle, \langle\!\langle 3^n \rangle\!\rangle, \ldots, \langle\!\langle 9^n \rangle\!\rangle)$$

as n runs over the positive integers. These sequences are regarded as the decimal notation of numbers. We ask for the number of pairwise different sequences, and for the number of pairwise different primes represented by them.

The number of different sequences is bounded by $9^8 = 43,046,721$. We saw in the proof of Parts 2 and 3 that the actual number of sequences is smaller. As these proofs show, this is caused by multiplicative relations among the numbers $\{2, 3, \ldots, 8, 9\}$ in the group M.

Lemma 1. *The subgroup of M generated by $2,\ 3,\ 4, \ldots, 9$ is free of rank 3, with $2,\ 3,$ and 7 as generators.*

Proof. Indeed, $4 = 2^2$, $5 = 2^{-1}$, $6 = 2 \cdot 3$, $8 = 2^3$, and $9 = 3^2$ in M shows that the subgroup is generated by $2,\ 3,$ and 7.

A relation between these generators would mean that integers a, b, c exist, such that $2^a 3^b 7^c = 1$ in M. As rational numbers this is expressed as $2^a 3^b 7^c = 10^d$ for some integer d. Considering the contribution of the prime numbers 5, 2, 3, 7 to both sides of this equality (here unique factorization into prime numbers is used!) shows $d = a = b = c = 0$. This proves the lemma. ■

Translating the lemma into a statement about the additive group $A \cong M$ shows that $\log_{10}(2)$, $\log_{10}(3)$, and $\log_{10}(7)$ generate a free subgroup of A of rank 3. In particular, the three real numbers given here together with the number 1 are linearly independent over the rational numbers. Hence by Theorem 2 one concludes the following.

Corollary 1. *For every $a, b, c \in \{1, 2, \ldots, 8, 9\}$ there is an integer $n > 0$ such that*

$$\langle\langle 2^n \rangle\rangle = a \quad \textit{and} \quad \langle\langle 3^n \rangle\rangle = b \quad \textit{and} \quad \langle\langle 7^n \rangle\rangle = c.$$

Proof. By Theorem 2 and the remarks above, positive integers n exist (in fact, infinitely many) such that

$$n\left(\log_{10}(2), \log_{10}(3), \log_{10}(7)\right) \in I_a \times I_b \times I_c \subseteq A \times A \times A.$$

Here $I_a = \{x \in A \ : \ \log_{10}(a) \le x < \log_{10}(a+1)\}$ and I_b, I_c are defined analogously. If one translates this into a statement about the group M, it says

$$\langle\langle 2^n \rangle\rangle = a \quad \text{and} \quad \langle\langle 3^n \rangle\rangle = b \quad \text{and} \quad \langle\langle 7^n \rangle\rangle = c$$

which implies the result. ■

One finds a similar argument in King's paper [**8**]. In fact, from it one may conclude that the fraction of positive integers n such that $\langle\langle 2^n \rangle\rangle = a$ and $\langle\langle 3^n \rangle\rangle = b$ and $\langle\langle 7^n \rangle\rangle = c$ equals

$$\log_{10}\left(\frac{a+1}{a}\right) \cdot \log_{10}\left(\frac{b+1}{b}\right) \cdot \log_{10}\left(\frac{c+1}{c}\right).$$

Adopting the notation introduced in the proof of this corollary, one has

$$\langle\langle \ell^n \rangle\rangle = a_\ell \Leftrightarrow n\log_{10}(\ell) \in I_{a_\ell}.$$

Now $n\log_{10}(4) = 2n\log(2)$, and the condition $2n\log_{10}(2) \in I_{a_4}$ is equivalent to

$$n\log_{10}(2) \in \frac{1}{2}I_{a_4} \cup \left(\frac{1}{2}I_{a_4} + \frac{1}{2}\right).$$

Similarly the condition on $n\log_{10}(8)$ translates into

$$n\log_{10}(2) \in \frac{1}{3}I_{a_8} \cup \left(\frac{1}{3}I_{a_8} + \frac{1}{3}\right) \cup \left(\frac{1}{3}I_{a_8} + \frac{2}{3}\right).$$

In this way, all conditions can be rewritten as equivalent ones involving only $n\log_{10}(2)$, $n\log_{10}(3)$, and $n\log_{10}(7)$. The one for $\ell = 5$ becomes

$$n\log_{10}(2) \in \left(1 - I_{a_5}\right)$$

while $\ell = 9$ yields

$$n\log_{10}(3) \in \frac{1}{2} I_{a_9} \cup \left(\frac{1}{2} I_{a_9} + \frac{1}{2}\right).$$

The final one involves $\ell = 6$, which yields

$$n\log_{10}(2) + n\log_{10}(3) \in I_{a_6}.$$

So first ignoring the condition for $\ell = 6$, one obtains that $n\log_{10}(2)$ needs to satisfy four conditions which together demand that this element of the group A is in an intersection J_2 of the sets described above:

$$\begin{aligned} J_2 := \quad & \left(\tfrac{1}{3} I_{a_8} \cup \left(\tfrac{1}{3} I_{a_8} + \tfrac{1}{3}\right) \cup \left(\tfrac{1}{3} I_{a_8} + \tfrac{2}{3}\right)\right) \\ & \cap \\ & I_{a_2} \cap \left(\tfrac{1}{2} I_{a_4} \cup \left(\tfrac{1}{2} I_{a_4} + \tfrac{1}{2}\right)\right) \cap \left(1 - I_{a_5}\right). \end{aligned}$$

Similarly one needs $n\log_{10}(3) \in J_3$, with

$$J_3 := I_{a_3} \cap \left(\frac{1}{2} I_{a_9} \cup \left(\frac{1}{2} I_{a_9} + \frac{1}{2}\right)\right)$$

and of course $n\log_{10}(7) \in I_{a7}$.

It is not difficult to test for fixed $a_2, \ldots, a_9$ whether both sets J_2 and J_3 are nonempty and, if this is the case, whether $(J_2 + J_3) \cap I_{a_6} \neq \emptyset$. If these conditions hold and the nonempty sets involved in fact contain a nonempty open interval, then an application of Theorem 2 similar to how it is used in Corollary 1 shows that in fact infinitely many $n > 0$ exist such that $\langle\!\langle \ell^n \rangle\!\rangle = a_\ell$ for all $\ell \in \{2, 3, \ldots, 9\}$.

Running this process on a computer revealed $9 \cdot 1955 = 17\,595$ sequences for which the sets described above are nonempty, so all these sequences occur for infinitely many $n > 0$. The remaining one is the sequence for $n = 1$, bringing the total number of sequences to 17 596. A simple inspection shows that exactly 1127 of these are the decimal notations of prime numbers.

As an example of the computation, consider the sequence 48224987. Here the pairs $(a, b) = \left(n\log_{10}(2) \bmod \mathbb{Z}, n\log_{10}(3) \bmod \mathbb{Z}\right) \in A \times A$ such that the corresponding J_2, J_3 and $(J_2 + J_3) \cap I_4$ are nonempty are given by the three conditions $10^{3a} < 90$, $10^{2b} < 80$, and $10^{a+b} > 40$. The picture shows the solutions (a, b). By Weyl's equidistribution theorem, the area of the solution set equals the fraction of positive integers n giving rise to a sequence $48224 * 87$. Hence this fraction is quite small (roughly 4×10^{-7}). If also the condition $\langle\!\langle 7^n \rangle\!\rangle = 9$ is taken into account, then one obtains a ratio equal to the computed area times $\log_1 0\left(\frac{10}{9}\right)$. This shows that the fraction of positive n's leading to 48224987 is slightly less than 2×10^{-8}.

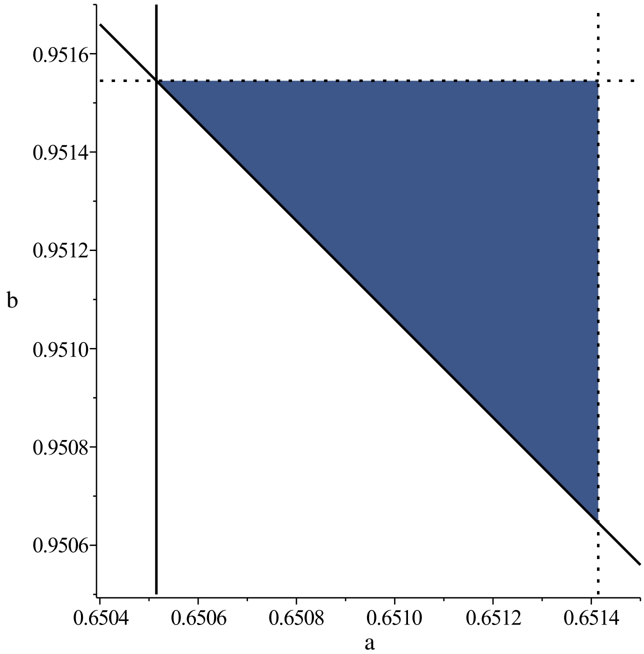

Figure 1. (a, b) such that $\langle\!\langle 10^a \rangle\!\rangle \langle\!\langle 10^b \rangle\!\rangle \langle\!\langle 10^{2a} \rangle\!\rangle \langle\!\langle 10^{1-a} \rangle\!\rangle \langle\!\langle 10^{a+b} \rangle\!\rangle \langle\!\langle 10^{3a} \rangle\!\rangle \langle\!\langle 10^{2b} \rangle\!\rangle = 4822487$

We are indebted to one of the referees of an earlier draft of this paper for the following argument, which in fact provides the total number of sequences without the help of a computer. All occurring conditions on a and b look like one of the following:

$$\begin{aligned}
&k \le 10^a < k+1,\\
&k \le 10^{2a} < k+1,\\
&10k \le 10^{2a} < 10k+10,\\
&k \le 10^{3a} < k+1,\\
&10k \le 10^{3a} < 10k+10,\\
&100k \le 10^{3a} < 100k+100,\\
&k \le 10^{1-a} < k+1,\\
&k \le 10^b < k+1,\\
&k \le 10^{2b} < k+1,\\
&10k \le 10^{2b} < 10k+10,\\
&k \le 10^{a+b} < k+1,\\
&10k \le 10^{a+b} < 10k+10
\end{aligned}$$

with a, b between 0 and 1 and $1 \le k \le 9$ an integer. The conditions involving only a or only b subdivide the unit square into $55 \times 24 = 1320$ rectangular pieces. The 24 horizontal strips, which involve the conditions on b only, reflect the fact that precisely 24 pairs $\langle\!\langle 3^n \rangle\!\rangle \langle\!\langle 9^n \rangle\!\rangle$ exist. Similarly, the 55 vertical strips correspond to the occurring sequences $\langle\!\langle 2^n \rangle\!\rangle \langle\!\langle 4^n \rangle\!\rangle \langle\!\langle 5^n \rangle\!\rangle \langle\!\langle 8^n \rangle\!\rangle$ with $n \ge 2$. The conditions on $a+b$ split some of the rectangles into smaller parts. Consider the lines where $a+b$ is constant which bound

Table 1. The Maple code which enumerates the sequences that occur infinitely often.

```
Digits := 20; verz := {};
for m from 10 by 50 to 1010
 do for n to m-1
       do x := n/m;
          if denom(m) > 3
          then tx := 10^x; v2 := floor(tx);
            v4 := floor(tx^2);
            if v4 > 9 then v4 := floor(tx^2/10) end if;
            v5 := floor(10/tx); v8 := floor(tx^3);
            if v8 > 9 then v8 := floor(tx^3/10) end if;
            if v8 > 9 then v8 := floor(tx^3/100) end if;
            v := 10*v8+10^4*v5+10^5*v4+10^7*v2+100;
            for k to m-1
              do y := k/m;
                 if denom(y)>2
                 then ty := 10^y; v3 := floor(ty);
                   v6 := floor(tx*ty);
                   if v6 > 9 then v6 := floor(tx*ty/10) end if;
                   v9 := floor(ty*ty);
                   if v9 > 9 then v9 := floor(ty*ty/10) end if;
                   w := v+v9+1000*v6+10^6*v3;
                   verz := 'union'(verz, {w})
                 end if
              end do
            end if
       end do; print(m, nops(verz))
 end do:
```

our conditions one by one. Each time such a line enters a new region, it will split the region into two parts. This happens precisely when the diagonal line crosses a vertical one, and when it crosses a horizontal one coming from the conditions. Every vertical and every horizontal strip is intersected by 9 of the diagonal lines. Since we know the total number of vertical and horizontal lines $(56 + 25)$ and also the number of occurrences where a diagonal line crosses precisely at an intersection of a horizontal and a vertical one (76 of them), the square is subdivided into $1320 + 9(55 + 24) - 76 = 1955$ regions. A picture illustrating these regions is presented below. Note that some are too small to be visible in the picture.

Note that the lines bounding our regions containing a point with both coordinates rational, satisfy at least one of the conditions $x \in \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}$ or $y \in \left\{0, \frac{1}{2}, 1\right\}$. Hence the Maple program in Table 1, which runs over rational values a, b, only uses points in the interior of the regions. With this code, indeed 1955 different sequences are found.

It is easy to add to each sequence the 9 possibilities for $\langle\langle 7^n \rangle\rangle$ and finally to add the sequence 123456789. It turns out that precisely 1127 of the resulting sequences represent a prime number.

Combining our results and ideas presented in [**8**], it is possible to settle the evident density problems related to Gelfand's question. In particular, for any given sequence $s = (a_2, \dots, a_9)$ the fraction of positive integers n such that $(\langle\langle 2^n \rangle\rangle, \dots, \langle\langle 9^n \rangle\rangle) = s$ equals the product of the area of the corresponding (a, b)-region as described above, and the number $\log_{10}\left(\frac{a_7+1}{a_7}\right)$. By adding 1127 such densities, one can even determine the fraction of integers $n > 0$ leading to prime numbers.

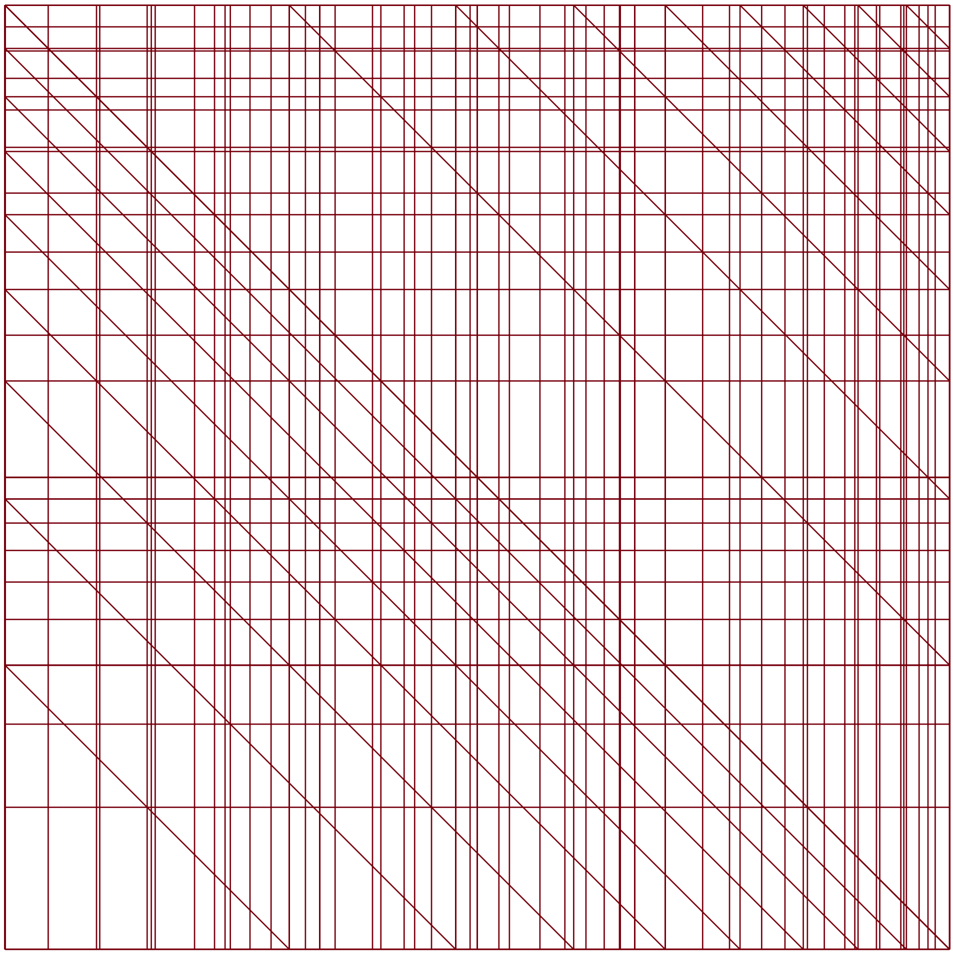

Figure 2. The 1955 regions

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