Riemann Surfaces

This course on Riemann surfaces assumes that the reader is familiar with sheaf theory.

1 Definitions

A **ringed space** (X, O_X) is a topological space together with a sheaf of commutative, unitary rings O_X on it. One calls O_X a sheaf of functions if there exists a commutative, unitary ring R such that $O_X(U)$ is a subring of the ring of all functions of U to R and such that the restriction maps for O_X coincide with the restrictions of functions. The basic example for our purposes is (D, O_D) = the unit disc $D = \{z \in \mathbb{C} \mid \mid z \mid < 1\}$ provided with the sheaf of holomorphic functions on it. This is of course a sheaf of functions with values in $\mathbb{C}$.
A **morphism** of ringed spaces $(f, g) : (X, O_X) \to (Y, O_Y)$ is a continuous map $f : X \to Y$ together with a morphism of sheaves of unitary rings $g : f^*O_Y \to O_X$, i.e. for every open $V \subset Y$ there is given a ringhomomorphism $g(V) : O_Y(V) \to O_X(f^{-1}(V))$ and for any two open sets $V_1 \subset V_2 \subset Y$ and any element $h \in O_Y(V_2)$ one has $g(V_1)(h|_{V_1}) = g(V_2)(h)|_{f^{-1}(V_1)}$. If the sheaves involved are sheaves of functions with values in the same ring then we will always take g to be the composition with f.
A **Riemann surface** is a ringed space (X, O_X) such that:
(1) X is a connected Hausdorff space.
(2) (X, O_X) is locally isomorphic to (D, O_D).
A morphism of RS (or analytic map) is a morphism of the corresponding ringed spaces. The basic example of a RS is $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Further any open and connected subset of $\mathbb{P}^1$ is a RS. The ring $O_{X,x} \cong \mathbb{C}\{t\}$ (this is the ring of convergent power series in t), is a discrete valuation ring. Its field of quotients (the field of convergent Laurent series) is denoted by $\mathbb{C}(\{t\})$; the valuation is denoted by 'ord_x'(the order at x); t is called a local parameter at x. The completion of $O_{X,x}$ is isomorphic to the ring of formal power series $\mathbb{C}[[t]]$. The field of quotients of the last ring is denoted by $\mathbb{C}((t))$ (the field of formal Laurent series). A meromorphic function on the RS X is a function with values in $\mathbb{C} \cup \{\infty\}$ (or equivalently a morphism $X \to \mathbb{P}^1$) which has locally the form f/g where $f, g \neq 0$ are holomorphic functions .

The **sheaf of holomorphic differential forms** Ω^1 on X is defined by $\Omega^1(U) = O_X(U)dt$, where U is any open subset of X provided with an isomorphism $t : U \to D$. A **holomorphic differential form** ω on X is a section of the sheaf Ω^1. A **meromorphic differential form** ω on X is locally of the form $f\ dt$ where f is a meromorphic function on U as above. The order of a meromorphic differential form at a point $x \in X$ with local coordinate t is $ord_x(\omega) := ord_x(f)$ if locally $\omega = f\ dt$. The residue of a meromorphic differential form ω at a point $x \in X$ is $res_x(\omega) := a_{-1}$ where locally $\omega = \sum_n a_n t^n\ dt$. Contour integration around the point x shows that the definition does not depend on the choice of the local parameter t. For any closed Jordan curve (provided with an orientation) λ on a compact X and any meromorphic differential form ω on X, Cauchy's theorem remains valid:

$$\frac{1}{2\pi i}\int_{\lambda} \omega = \sum_{x \in \lambda^0} res_x(\omega)$$

where one has supposed that there are no poles on the Jordan curve and where λ^0 denotes the interior of the Jordan curve. We note that the interior of an oriented Jordan curve makes sense since any RS is an orientable surface. By cutting the RS into pieces isomorphic to D one easily finds a proof of Cauchy's theorem for RS.

Exercise 1: Let D denote the unit disc. Show that the ring $O(D)$ is not noetherian. Let $\bar{D}$ denote the closed disc. Give the definition of $O(\bar{D})$ and show that this ring is noetherian.
Exercise 2: Show that the groups of analytic automorphisms of the following spaces $\mathbb{P}^1, \mathbb{H} := \{z \in \mathbb{C} \mid \ im(z) > 0\}, \mathbb{C}$ are $PGl(2,\mathbb{C})$, $PSl(2,\mathbb{R})$ and $\{z \mapsto az + b \mid a \in \mathbb{C}^*,\ b \in \mathbb{C}\}$.
Exercise 3: Show that $D \cong \mathbb{H}$. What are the automorphisms of D?
In the near future we will use the following elementary result.

1.1 Lemma

Let X be a compact RS and let $f : X \to \mathbb{P}^1$ be a non constant meromorphic function. Then f has finitely many poles and finitely many zeros. The'divisor' of f is defined to be the formal expression $\sum_{x \in X} ord_x(f)x$. It *has the property* $\sum_{x\in X} ord_x(f) = 0$. *Moreover* f *is surjective. If* f *happens to have only one pole then* f *is an isomorphism.*

Proof. The poles and zeros of f are isolated. Hence the first statement follows from the compactness of X. By integrating the meromorphic differential form $\frac{df}{f} \neq 0$ over some Jordan curve on X and applying Cauchy's theorem one finds $\sum ord_x(f) = 0$. We note that the maximum principle implies that f has at least one pole and so also at least one zero. Let $a \in \mathbb{C}$, then $f - a$ has also a zero. It follows that f is surjective. Suppose that f has precisely one pole. Then reasoning as above yields that f is bijective. Similarly one shows that the derivative of f with respect to a local coordinate at any point can not be zero. This proves that the inverse of f is also an analytic map.

2 Construction of Riemann surfaces

X a RS and Γ a group of automorphism. Γ acts **discontinuously** on X if every point has a neighbourhood U such that $\#\{\gamma \in \Gamma \mid \gamma U \cap U \neq \emptyset\}$ is finite.

2.1 Example

A subgroup Γ *of* $PSl(2, \mathbb{R})$ *(with the topology induced by* $\mathbb{R}$*) acts discontinuously on* $\mathbb{H}$ *if and only if this group is discrete.*

Proof. If Γ is not discrete then there exists a non constant sequence γ_n in Γ with limit 1. Then Γ is not discontinuous since $\lim \gamma_n(i) = i$. Suppose that Γ is discrete. The stablizer of any $a \in \mathbb{H}$ in $PSl(2, \mathbb{R})$ is a compact group (for $a = i$ this group is $SO(2, \mathbb{R})$). Hence $Z := \{\gamma \in PSl(2, \mathbb{R}) \mid \mid \gamma(a) - a \mid \leq \epsilon\}$ is a compact set depending on a and ϵ. From $Z \cup \Gamma$ is finite one can easily deduce that Γ is discontinuous.

Exercise 4: Show that there are topological isomorphisms $PSl(2, \mathbb{R})/SO(2, \mathbb{R}) \cong \mathbb{H}$ and $\Gamma\backslash PSl(2, \mathbb{R})/SO(2, \mathbb{R}) \cong \Gamma\backslash\mathbb{H}$.

2.2 Theorem

Let Γ be a group acting discontinuously on the RS X. Then the quotient $Y := \Gamma\backslash X$ has a unique structure of RS such that the canonical map $\pi : X \to Y$ is analytic and such that every analytic and Γ-invariant map $X \to Z$ factors as an analytic map over Y.

Proof. The topology of Y is given by: $V \subset Y$ is open if and only if $\pi^{-1}(V)$ is open in X. The map π is open. Let $\pi(a) \neq \pi(b)$ hold for some points $a, b \in \mathbb{H}$. Let S denote the finite subgroup of Γ stabilizing a. There exists an S-invariant neighbourhood $U = U_a$ of a with such that $\gamma \in \Gamma, \gamma(U) \cap U \neq \emptyset$ implies $\gamma \in S$. One can take U_a small enough such that also $U_a \cap \Gamma b = \emptyset$. Around b one can make a similar neighbourhood U_b such that also $U_a \cap \Gamma U_b = \emptyset$. The open sets $\pi(U_a), \pi(U_b)$ separate the points $\pi(a), \pi(b)$. Hence Y is a connected Hausdorff space. The sheaf of rings O_Y is defined by $O_Y(V) = O_X(\pi^{-1}(V))^\Gamma$. The restriction of this sheaf to the neighbourhood $V := \pi(U_a)$ of the point $\pi(a)$ is equal to $W \subset V \mapsto O_X(\pi^{-1}(W) \cap U_a)^S$. For a suitable local parameter t at a the action of S has the form $t \mapsto \zeta t$, where ζ runs in a finite group of roots of unity. From this one easily sees that for a good choice of V one has $(V, O_Y|_V) \cong (D, O_D)$. This proves that the quotient has a structure of RS such that $\pi : X \to Y$ is analytic. Let now $f : X \to Z$ be a Γ-invariant analytic map to the RS Z. There is a unique map $g : Y \to Z$ with $f = g\pi$, this is a continuous map and for every open $W \subset Z$ and $h \in O_Z(W)$ the element $hf \in O_X(f^{-1}(W))^\Gamma = O_Y(g^{-1}(W))$. Hence g is an analytic map. This proves the last part of the theorem.

3 Lots of examples

3.1 Lattices in $\mathbb{C}$

A discrete subgroup $\Lambda \subset \mathbb{C}$ of rank 2 can be seen as discontinuous group of automorphism by $\{z \mapsto z + \lambda \mid \lambda \in \Lambda\}$. The quotient $E_\Lambda := \mathbb{C}/\Lambda$ is a compact Riemann surface. Let $\Gamma := \{z \mapsto \pm z + \lambda \mid \lambda \in \Lambda\}$. This group contains Λ as a subgroup of index two and is therefore also a discontinuous group. We claim that $Y \cong \mathbb{P}^1$.

Define the **Weierstrass function:**

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda,\lambda\neq 0}\left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}\right)$$

Elementary analysis shows that $\mathcal{P}$ is a meromorphic function on $\mathbb{C}$. Clearly $\mathcal{P}$ is invariant under the group Γ and thus provides an analytic map $f : Y \to \mathbb{P}^1$. Let y_0 denote the image of $0 \in \mathbb{C}$. Then f has a pole of order 1 at y_0 and no other poles. By (1.1) f is an isomorphism.
The analytic map $f : E_\Lambda \to \mathbb{P}^1$ induced by $\mathcal{P}$ has the properties: for general $a \in \mathbb{P}^1$ the set $f^{-1}(a)$ consist of two elements; for 4 special values $a_i, 1 \leq i \leq 4$ the set $f^{-1}(a_i)$ consist of one element. Suppose that $E_\Lambda \cong \mathbb{P}^1$ then f can be considered as a meromorphic function on $\mathbb{P}^1$. One can easily see that any meromorphic function on $\mathbb{P}^1$ is a rational function F/G. In our case the maximum of the degrees of F and G must be two. But then there are at most two special values a for which $f^{-1}(a)$ consist of one point. This contradiction shows the following:
E_Λ is not isomorphic to $\mathbb{P}^1$.

Exercise 5: Let Λ denote a discrete subgroup of rank 1. Find an easy representation for the quotient $\mathbb{C}/\Lambda$.
Exercise 6: Find all discontinuous groups acting on $\mathbb{C}$ and determine their quotients.
Exercise 7: Let Λ_1, Λ_2 denote two lattices in $\mathbb{C}$. Show that the RS $E_{\Lambda_1}, E_{\Lambda_2}$ are isomorphic if and only if there exists a complex number a with $a\Lambda_1 = \Lambda_2$.

3.2 Lattices in $\mathbb{C}^*$

A lattice Λ is a discrete subgroup of $\mathbb{C}^*$. The standard example is $\langle q\rangle := \{q^n \mid n \in \mathbb{Z}\}$ where $0 <\mid q \mid< 1$. The RS $E_q := \mathbb{C}^*/\langle q\rangle$ is again compact.

Exercise 8: Prove that there exists a lattice $\Lambda \subset \mathbb{C}$ with $E_\Lambda \cong E_q$. When does $E_{q_1} \cong E_{q_2}$ hold?
Exercise 9: Find all discontinuous groups operating on $\mathbb{C}^*$ and determine the corresponding RS.

3.3 The modular group

The group $\Gamma(1) := PSl(2,\mathbb{Z})$, called the modular group, acts, according to (2.1), discontinuously on $\mathbb{H}$. For any positive integer n one defines the 'congruence subgroup'

$$\Gamma(n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n\}$$

This is a group of finite index in $\Gamma(1)$. In general, a congruence subgroup is a subgroup (of finite index) containing $\Gamma(n)$ for some n. A special case is the group

$$\Gamma_0(n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid \ c \equiv 0 \bmod n\}$$

Exercise 10: Let p be a prime number. Calculate the indices of $\Gamma(p), \Gamma_0(p)$ in $\Gamma(1)$.

3.3.1 Proposition

Define $F := \{z \in \mathbb{H} \mid |z| \geq 1, \ |re(z)| \leq 1/2\}$ *and* $S, T \in \Gamma(1)$ *by* $S(z) = \frac{-1}{z}$, $T(z) = z + 1$. *Let* $\sim$ *be the equivalence relation on* F *given by the identification of* $\{z \in F \mid re(z) = -1/2\}$ *with* $\{z \in F \mid re(z) = 1/2\}$ *via the transformation* T *and the identification on* $\{z \in F \mid |z| = 1\}$ *under the transformation* S. *Then:*
(1) The map $F/_\sim \to \Gamma(1)\backslash\mathbb{H}$ *is a topological isomorphism.*
(2) The group $\Gamma(1)$ *is generated by* S *and* T.
(3) The only relations between the generators S, T *are* $S^2 = 1$ *and* $(ST)^3 = 1$.
(4) The quotient $\Gamma(1)\backslash\mathbb{H}$ *has a compactification by one point. This compact RS is topologically isomorphic to the sphere.*

Proof. Let G denote the subgroup of $\Gamma(1)$ generated by S and T. For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ one has $im(gz) = \frac{im(z)}{|cz+d|^2}$. Since c, d are integers there is a $g \in G$ such that $im(g)$ is maximal. Choose n such that $z' := T^n gz$ satisfies $|re(z')| \leq 1/2$. We claim that $z' \in F$. Indeed $z' \notin F$ implies $|z'| < 1$ and gives the contradiction $im(Sz') > im(z')$. Let $z, gz \in F$ with

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ and $g \neq 1$. After a possible interchange between z, g and gz, g^{-1} we may suppose $im(gz) \geq im(z)$. This gives $|cz + d| \leq 1$ and so c can only be $0, 1, -1$. Now one has to consider some cases:
$c = 0$, one finds $g = T$ or $g = T^{-1}$.
$c = 1$, one finds $d = 0$ or $z = \rho := e^{2\pi i/3}$($z = -\bar{\rho}$ resp.) with $d = 0, 1$ ($d = 0, -1$ resp.).
$c = 1$ and $d = 0$ leads to $g = S$.
$c = 1$ and $d = 1, z = \rho$ leads to $g = ST$.
$c = 1$ and $d = -1, z = -\bar{\rho}$ is similar.
$c = -1$ is the same as $c = 1$ since the matrix of g is only determined upto ± 1.
This proves (1) and (2). The group $\Gamma(1)$ is also generated by the two elements $D := ST$ and S. A reduced word w of length s in the two generators is an expression $w = \delta_1 \delta_2 ... \delta_s$ with $\delta_i \in \{D, D^2, S\}$ such that no succession of D by D^2 or D occurs and similar no succession of S by S occurs. One has to show that a reduced word is equal to 1 if and only if $s = 0$. This can be done by making a drawing of the images wF of the 'fundamental domain' F and induction on the length of w. This proves (3). Finally, it follows from (1) that the image of $A := \{z \in \mathbb{H} \mid im(z) > 1\}$ in $\Gamma(1)\backslash\mathbb{H}$ is $B := A/\{T^n \mid n \in \mathbb{Z}\}$. Using the map $z \mapsto e^{2\pi iz}$ one sees that $B \cong \{t \in \mathbb{C} \mid 0 < |t| < e^{-2\pi}\}$. Glueing $\hat{B} := \{t \in \mathbb{C} \mid |t| < e^{-2\pi}\}$ to $\Gamma(1)\backslash\mathbb{H}$ over B one finds the 'one point compactification' of (4). A drawing shows that this space is homeomorphic to the sphere.

Remark. There is a natural isomorphism $j : \Gamma(1)\backslash\mathbb{H} \to \mathbb{C}$. We will maybe return to this later. The RS $\Gamma_0(n)\backslash\mathbb{H}$ has also a compactification by finitely many points. The resulting compact RS is called the **modular curve.**

3.4 Kleinian and Fuchsian groups

Let G be a subgroup of $PGl(2, \mathbb{C})$. A point $p \in \mathbb{P}^1$ is called a limit point if there is a point $q \in \mathbb{P}^1$ and a sequence of distinct elements g_n of G such that $\lim g_n(q) = p$. The set of all limit points $\mathcal{L}$ is easily seen to be a compact set; the complement Ω of $\mathcal{L}$ is an open set. The group G is called a Kleinian group if $\Omega \neq \emptyset$. For such a group Ω is the largest open subset of $\mathbb{P}^1$ on which G acts discontinuously. In general Ω has many connected components and the quotient is a disjoint union of RS. A **Fuchsian group** is a Kleinian group which leaves a disc or a half plane invariant. Hence after conjugation in $PGl(2,\mathbb{C})$ one can suppose that the Fuchsian group is a subgroup of $PSl(2,\mathbb{R})$.

Exercise 11: Show that a Fuchsian subgroup of $PSl(2,\mathbb{R})$ is the same thing as a discrete subgroup and show that its set of limit points is contained in $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.
Exercise 12: Show that the group $PGl(2,\mathbb{Z}[i])$ is a discrete and not discontinuous subgroup of $PGl(2,\mathbb{C})$.

3.5 Schottky groups

A Schottky group is a special case of a Kleinian group. Let D be a Jordan curve in $\mathbb{P}^1$ and suppose that $\infty \notin D$. The complement of D in $\mathbb{P}^1$ has two connected components; the component containing ∞ will be called the outside D^+ of D; the other component will be called the inside D^- of D. Choose $2g$ disjoint Jordan curves $B_1, C_1, ..., B_g, C_g$ not containing ∞ in such a position that the sets $B_1^-, C_1^-, ..., C_g^-$ are disjoint. We suppose that there are $\gamma_1, ..., \gamma_g \in PGl(2,\mathbb{C})$ such that:

1. $\gamma_i(B_i) = C_i$
2. $\gamma_i(B_i^-) = C_i^+$ and hence also $\gamma_i(B_i^+) = C_i^-$

In case all B_i, C_i are circles (this is Schottky's original definition) it is clear that the γ_i exists. The group Γ generated by the elements γ_i is by definition a Schottky group. Let the compact set F denote the complement of all B_i^-, C_i^-. The interior F^o of F is easily seen (by making a drawing) to be $B_1^+ \cap C_i^+ \cap ... \cap B_g^+ \cap C_g^+$.

3.5.1 Proposition

(a) Γ *is a free group on* $\gamma_1, ..., \gamma_g$ *and* Γ *is discontinuous.*
(b) $\mathcal{L} := \mathbb{P}^1 - \cup_{\gamma\in\Gamma}\gamma F$ *is the collection of limit points of* Γ. *This set is compact and totally disconnected.*
(c) $\gamma F \cap F \neq \emptyset$ *if and only if* $\gamma \in \{1, \gamma_1, ..., \gamma_g, \gamma_1^{-1}, ..., \gamma_g^{-1}\}$.

(d) $\gamma F^o \cap F = \emptyset$ *if* $\gamma \neq 1$.
(e) Let Ω *be the set of ordinary points for* Γ. *The quotient* $X = \Gamma \backslash \Omega$ *is as a topological space isomorphic to* $F/_\sim$ *where the equivalence relation on* F *is given by the pairwise identification of the boundary components* B_i, C_i *via* γ_i.
(f) X *is a compact RS not isomorphic to* $\mathbb{P}^1$.

Proof. The proof is rather geometric and combinatorical. A **reduced word** w of length $l(w) = s$ in $\gamma_1, ..., \gamma_g$ is an expression $w = \delta_s ... \delta_1$ with $\delta_i \in \{\gamma_1, ..., \gamma_g, \gamma_1^{-1}, ..., \gamma_g^{-1}\}$ and such that no succession of γ_j, γ_j^{-1} occurs. Put $w = 1$ for the empty word with $s = 0$. Using induction on s one finds:

$$w(F^o) \subset C_i^- \text{ if } \delta_s = \gamma_i \text{ and } w(F^o) \subset B_i^- \text{ if } \delta_s = \gamma_i^{-1}$$

In particular $w(\infty) \neq \infty$ if $w \neq 1$. Hence Γ is a free group on $\{\gamma_1, ..., \gamma_g\}$.
For a point p we write $\mathcal{L}(p)$ for the set of limit points of the orbit $\Gamma(p)$. For $p \in F^o$ one has $\mathcal{L}(p) \subset \cup_i (B_i^- \cup B_i \cup C_i^- \cup C_i)$. Hence the points of F^o are not limit points for Γ and Γ is a discontinuous group. So (a) is proved.
We associate to a reduced word w of length $s > 0$ the Jordan curve D_w, given by: $D_w = \delta_s \delta_{s-1} ... \delta_2 (C_i)$ if $\delta_1 = \gamma_i$ and $D_w = \delta_s \delta_{s-1} ... \delta_2 (B_i)$ if $\delta_1 = \gamma_i^{-1}$. In particular $D_{\gamma_i} = C_i$ and $D_{\gamma_i^{-1}} = B_i$. Induction on the length of w shows that:

1. All D_w are disjoint.
2. $D_{w_1} \subset D_{w_2}^-$ if and only if $w_1 = w_2 z$ with $z \neq 1$ and $l(w_1) = l(w_2) + l(z)$.
3. The diameter of D_w is $\leq$ constant $.k^{l(w)}$ with $0 < k < 1$. This follows from the observation that γ_i(γ_i^{-1} resp.) is a contraction on any compact subset of B_i^+ (C_i^+ resp.).

Put $V_n := \mathbb{P}^1 - \cup_{l(\gamma) < n} \gamma F$ then clearly $\cap_{n>0} V_n = \mathbb{P}^1 - \cup_{\gamma \in \Gamma} \gamma F$. Further $V_n = \cup_{l(w)=n} D_w^-$ consists of $2g(2g-1)^{n-1}$ disjoint open sets. The statements (c) and (d) are now obvious. In order to prove (b) we introduce **infinite reduced words** $w = \delta_1 \delta_2 ....$ (i.e. each finite piece $w_s := \delta_1 ... \delta_s$ is a reduced word). We associate to $p \in \cap_n V_n$ an infinite word $[p]$ given by $[p]_s$ is the unique word w with length s such that $p \in D_w^-$. According to (2) one has $[p]_{s+1} = [p]_s \delta$ for some $\delta \in \{\gamma_1, ..., \gamma_g, \gamma_1^{-1}, ..., \gamma_g^{-1}\}$. Hence $[p]$ is well defined. Let $\hat{\Gamma}$ denote the set of infinite reduced words: this subset is a closed subset of $\{\gamma_1, ..., \gamma_g\, \gamma_1^{-1}, ..., \gamma_g^{-1}\}^{\mathbb{N}}$ with respect to the usual product topology. In fact $\hat{\Gamma}$ is a compact and totally disconnected set. The map $\hat{\Gamma} \to \cap V_n$ defined by $w \mapsto \lim\, w_s(\infty)$ is the inverse of the map above. It is easily seen that both maps are continuous. It follows that the set of limit points $\mathcal{L}$ of Γ is equal to $\cap V_n$ and that $\mathcal{L}$ is compact and totally disconnected. This proves (b). Further, the continuous map $F \to \Gamma\backslash\Omega$ is surjective and two points $p_1, p_2 \in F$ have the same image if and only if for some i one has $p_1 \in B_i$ (C_i resp.) and $p_2 \in C_i$ (B_i resp.) and $\gamma_i(p_1) = p_2$ ($\gamma_i(p_2) = p_1$ resp.). This proves statement (e). Finally, it is easily seen that X is not homeomorphic to the sphere. Using the last part of section 4 one can say more precisely that X is topologically a g-fold torus.

Exercise 13: Construct a Schottky group which is a subgroup of $PGl(2, \mathbb{R})$.

4 Coverings of Riemann surfaces

In this section we review quickly some topological properties related with RS. More details and proves can be found in [M]. A **covering** of topological spaces is a continuous map $f : Y \to X$ such that every point of X there is a neighbourhood U, a set I with the discrete topology and a homeomorphism $\phi : V \times I \to f^{-1}(V)$ such that $f\phi : V \times I \to V$ is the projection on the first factor. In the following we will suppose that the spaces X, Y are Hausdorff, connected and that they are locally homeomorphic to an open subset in some Euclidean space. The degree of a covering f is the number of points in a fibre $f^{-1}(a)$. This number does not depend on the choice of a since we have supposed that the spaces are connected. It turns out that there exists a **universal covering** $U \xrightarrow{u} X$, this means that for every covering $f : Y \to X$ there exists a covering $g : U \to Y$ with $u = fg$.

Fix a point $x_0 \in X$. The collection of all closed paths through x_0 is divided out by the equivalence relation 'homotopy'. Let $\lambda_1, \lambda_2 : [0,1] \to X$ denote two paths with $\lambda_i(0) = \lambda_i(1) = x_0$. A homotopy between the two paths is a continuous map $H : [0,1]^2 \to X$ satisfying $H(-,0) = \lambda_1$; $H(-,1) = \lambda_2$; $H(0,-) = H(1,-) = x_0$. The collection of equivalence classes $\pi = \pi_1(X, x_0)$ is called the **fundamental group** of X with base point x_0. The composition λ of two (equivalence classes of) paths λ_i is defined by the formula:

$$\lambda(t) = \lambda_1(2t) \text{ for } 0 \leq t \leq 1/2 \text{ and } \lambda(t) = \lambda_2(2t-1) \text{ for } 1/2 \leq t \leq 1$$

The group structure is derived form this formula. The space X is called **simply connected** if the fundamental group is trivial.
A simply connected space has only the trivial covering id:$X \to X$. In order to state more precisely the relation between fundamental group and coverings, we have to introduce pointed spaces. A pointed space is a space together with a chosen point of the space. A map between pointed spaces is a map which sends the chosen point to the other chosen point. In the following statements coverings will mean coverings of pointed topological spaces.
The fundamental group π acts as group of homeomorphisms on the universal covering U of X. The group acts without fixed points and the quotient $\pi\backslash U$ is topologically isomorphic to X. For every covering of X there is a unique subgroup $\Gamma \subset \pi$ such that the covering is isomorphic to $\Gamma\backslash U \to X$. The covering is called normal if Γ is a normal subgroup of π. For a normal covering Y the factor group $G = \pi/\Gamma$ acts as full group of automorphisms of the covering.
The importance of the above for RS is:
Let $f : Y \to X$ be a covering of the RS X. Then Y has a unique structure of RS such that f is analytic.
Indeed, for small open $V \subset X$ the set $f^{-1}(V)$ is a disjoint union of copies W of V. This induces the RS-structure on each W and on Y.

Exercise 14: Give all coverings of the circle.
Exercise 15: Find the universal covering of the RS $\mathbb{C}^*$.
Exercise 16: Give a plausible explanation for the fact that the fundamental group of $X := \mathbb{C}^* - \{1\}$ is a free (non commutative) group on two generators. What are the groups occuring for the finite normal coverings of X ?

Another aspect that we want to mention is the topology of a compact RS X. As a topological space X is a compact orientable surface. Such surfaces are classified by a number $g \geq 0$. The surface corresponding to $g = 0$ is the sphere; $g = 1$ corresponds with the torus; $g > 1$ corresponds to a g-fold torus. The fundamental group of the surface with number g is a group generated by elements $a_1, ..., a_g, b_1, ..., b_g$ and having only one relation

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}....a_gb_ga_g^{-1}b_g^{-1} = 1$$

Exercise 17: Find all coverings of the torus.

5 Divisors and line bundles

This section is rather formal in nature. Almost the same definitions and statements can be given for line bundle on an algebraic variety or on a scheme. A **divisor** on a RS X is a formal expression $D = \sum_{x\in X} d_x\ x$ with all $d_x \in \mathbb{Z}$ and such that the set $\{x \in X \mid d_x \neq 0\}$ is a discrete subset of X. The sum of divisors D and E is the divisor $\sum(d_x + e_x)\ x$. Further $D \geq E$ means $d_x \geq e_x$ for all $x \in X$. The divisor of a meromorphic function $f \neq 0$ is $\text{div}(f) = \sum_{x\in X} ord_x(f)\ x$. Such a divisor is called a **principal divisor**. The divisor of a non zero meromorphic differential form ω is $\sum_{x\in X} ord_x(\omega)\ x$. Such a divisor is called a **canonical divisor**. The set of divisors forms an additive group. For a divisor $D = \sum d_x\ x$ with only finitely many coefficients $\neq 0$ (this is always the case when X is a compact RS) one defines its degree by $\text{degree}(D) = \sum_{x\in X} d_x$. Lemma (1.1) states that the degree of a principal divisor on a compact RS is zero.
A **line bundle** L on X is a sheaf of O_X-modules locally isomorphic to O_X. This means that:
L is a sheaf of additive groups,
$L(V)$ has a structure of $O_X(V)$-module for every open $V \subset X$,
the restriction maps are compatible with the module structure,
every point of X has a neighbourhood U such that $L|_U \cong O_X|_U$, where the isomorphism respects the module structure.
Examples of line bundles are: O_X, the 'structure sheaf'; Ω^1, the sheaf of holomorphic differential forms (the 'canonical sheaf') and for every divisor D the sheaves $O(D), \Omega^1(D)$. The sheaf $O(D)$ is defined by:

$$O(D)(V) = \{f \text{ meromorphic on } V \mid \text{ the divisor of f on } V \text{ satisfies } \geq -D|_V\}$$

$\Omega^1(D)$ has a similar definition.
A line bundle L is called **trivial** on V if the restriction of L to V is isomorphic

to $O_X|_V$. It is clear how to define (iso)morphisms of line bundles. The collection of all isomorphism classes of line bundles on X is called the **Picard group** of X and is denoted by Pic(X). For two line bundles L, M one defines a third one $N = L \otimes M$ as the associated sheaf of the presheaf $V \mapsto L(V) \otimes_{O_X(V)} M(V)$. This makes Pic($X$) into an abelian group.

5.1 Lemma

There is a natural isomorphism between the groups $H^1(X, O_X^*)$ and Pic(X).

Proof. Let L denote a line bundle on X and let $\{U_i\}$ be an open covering of X such that there are isomorphisms $\phi_i : O_X|_{U_i} \to L|_{U_i}$. Then $(\phi_j)^{-1}\phi_i(1)$ is a collection of elements of $O_X(U_i \cap U_j)^*$ satisfying the 1-cocycle relation. This defines a group homomorphism Pic(X) $\to H^1(X, O_X^*)$. On the other hand, let an element of $H^1(X, O_X^*)$ be given by a 1-cocycle $\{\alpha_{i,j}\}$ for an open covering $\{U_i\}$ of X. On each U_i we take a trivial line bundle $L_i := O_X|_{U_i}e_i$. The line bundles L_i are glued together over the intersections $U_i \cap U_j$ by the isomorphism $e_i \mapsto \alpha_{i,j}e_j$. The glueing is valid because of the 1-cocycle relation and the result is a line bundle L on X. The two maps above are each others inverses.

For completeness we say a few words on **coherent sheaves** on a RS X. Coherent sheaves are sheaves of O_X-modules M on X having the property: For every point of X there is a neighbourhood V, and a morphism of O_X-modules $\alpha : O_V^m \to O_V^n$ such that $M|_V$ is isomorphic to the cokernel of α. (m, n, α depending on V).
Examples:
For a divisor $D > 0$ the quotient $O_X(D)/O_X$ is a coherent sheaf.
A vector bundle of rank n on X is a sheaf of O_X-modules which is locally isomorphic to O_X^n. In particular a vector bundle is a coherent sheaf.
A general coherent sheaf turns out to be a combination of the two examples above.

We end this section by some interesting exact sequences of sheaves.

$$0 \to \mathbb{Z} \to O_X \overset{exp}{\to} O_X^* \to 0$$

$$0 \to O_X \to M_X \to H_X \to 0$$

$$0 \to O_X^* \to M_X^* \to Div \to 0$$

The map in the first sequence is defined by $exp(f) = e^{2\pi i f}$. Since an invertible function is locally the exponent of a holomorphic function, the sequence is exact.
The sheaf M_X in the second sequence is the sheaf of meromorphic functions. The sheaf H_X is defined by the exactness of the second sequence. A section of H_X is supported on a discrete subset since the poles of a meromorphic function form a discrete set. Hence H_X is a skyscraper sheaf. The stalk of H_X at a point is equal to $\mathbb{C}(\{t\})/\mathbb{C}\{t\} = t^{-1}\mathbb{C}[t^{-1}]$ where t denotes a local parameter at the point. H_X is called the sheaf of **principal parts.** We note that M_X, H_X are not coherent sheaves. In the third sequence M_X^* is the sheaf of the invertible meromorphic functions and Div is the sheaf of divisors on X. This is again a skyscraper sheaf.

Exercise 18: Let $\alpha : O_X^* \to \Omega^1$ be the map $f \mapsto \frac{df}{f}$. Prove that the sequence:

$$0 \to \mathbb{C}^* \to O_X^* \xrightarrow{\alpha} \Omega^1 \to 0$$

is exact. Show that for $X = \{z \in \mathbb{C} \mid 0 < |z| < r\}$ the map $\alpha : O(X) \to \Omega^1(X)$ is not surjective. What is the cokernel?
Exercise 19: Let $Gl(n, O_X)$ denote the sheaf of invertible holomorphic matrices of rank n on X. Although $Gl(n, O_X)$ is not a sheaf of commutative groups it is possible to define its H^1 as a set.

Prove that $H^1(X, Gl(n, O_X))$ is isomorphic to the set of isomorphy classes of vector bundles of rank n on X.

6 Open subsets of the complex numbers

A domain $D \subset \mathbb{C}$ is by definition connected and open. The aim of this section is to give a proof of the statements:
$H^i(D, O) = H^i(D, O^*) = 0$ if $i \neq 0$.
The proofs in the literature are based on the $\bar{\partial}$ operator and harmonic functions or forms. One considers the sequence of sheaves on $\mathbb{C}$:

$$0 \to O \to C^\infty \xrightarrow{\bar{\partial}} C^\infty \to 0$$

where C^∞ denotes the sheaf of complex valued functions admitting partial derivatives of all orders with respect to the real coordinates x, y. Further $\bar{\partial} = \frac{\partial}{\partial \bar{z}} = 1/2(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.
Let g denote a C^∞ function on $\mathbb{C}$ with compact support. Then one can show that the integral

$$f(\zeta) := \frac{1}{2\pi i}\int\int_{\mathbb{C}} \frac{g(z)}{z-\zeta} dz \wedge d\bar{z}$$

is a C^∞-function on $\mathbb{C}$ and solves $\bar{\partial}(f) = g$. This shows that the sequence above is exact. The sheaf C^∞ is a fine sheaf and therefore has trivial cohomology. It follows at once that for any open subset D of $\mathbb{C}$ the groups $H^i(D, O)$ are 0 for $i > 1$ and that $H^1(D, O) =$ coker $\bar{\partial} : C^\infty(D) \to C^\infty(D)$. Approximation methods are then used to prove that this cokernel is in fact 0 for open subsets D of $\mathbb{C}$.

In this section we give a more geometric and combinatorical proof of the triviality of the various cohomology groups. By a Jordan curve in $\mathbb{C}$ we will mean an oriented Jordan curve which is a polygon. For such a curve k the space $\mathbb{C} - k$ has two connected components, k^+ will denote the inside (left hand side of k) and k^- denotes the outside of k. A **domain of finite type** D is a domain of the form $D = k_1^+ \cap k_2^+ ... \cap k_s^+$ where the k_i are disjoint Jordan curves. If one takes s minimal in this expression then the boundary of D is the union of the k_i. The **holes** of D are the connected components of $\mathbb{C} - D$, they are equal to the $\bar{k}_i^-$. For any domain E and a point $p \in E$ we write $O(E)_p := \{f \in O(E) \mid f(p) = 0\}$.

6.1 Proposition (Mittag-Leffler, Runge)

Let $D = k_1^+ \cap ... \cap k_s^+$ be a domain of finite type. Suppose that s is minimal, let p be a point of D and choose $a_i \in k_i^-$ for every i. Suppose for convenience that no a_i is ∞. Then:
(1) $O(D)_p = \oplus_{i=1}^s O(k_i^+)_p$
(2) Every $f \in O(D)$ can be approximated on compacta in D by rational functions having their poles outside D.
(3) Every invertible holomorphic function on D has the form

$$(z - a_1)^{n_1} ... (z - a_s)^{n_s} e^{2\pi i g}$$

where the integers n_i satisfy $\sum n_i = 0$ and where g is a holomorphic function on D. This representation is unique upto shifting g over an integer.

Proof. (1). Take $f \in O(D)$ and put

$$f_i(z) = \frac{1}{2\pi i} \int_{k_i} \frac{f(t)}{t-z} dt$$

This integral has the following meaning: approximate k_i by a Jordan curve $\tilde{k_i} \subset D$ such that there are no holes between k_i and $\tilde{k_i}$. Then $\tilde{f_i}(z) = \frac{1}{2\pi i} \int_{\tilde{k_i}} \frac{f(t)}{t-z} dt$ is well defined for the points in the interior of $\tilde{k_i}$. If one varies the curve $\tilde{k_i}$ one finds a holomorphic function f_i on k_i^+ and one has according to Cauchy's formula $f = \sum f_i$. For $f \in O(D)_p$ one can change the f_i such that $f_i \in O(k_i^+)_p$.
Suppose now that the $f_i \in O(k_i^+)_p$ are given and suppose that their sum is zero. Then $f_1 = -f_2 - ... - f_s$ extends to a holomorphic function on $\mathbb{P}^1$ and so f_1 is constant. Since $f_1(p) = 0$ it follows that $f_1 = 0$. As a consequence the decomposition of f is unique and we have proved (1).
(2) The integral for f can be approximated on any compact subset of D by a finite Riemann sum. This finite sum is a rational function with poles outside D.
(3) Let $f \in O(D)^*$. Define $n_i = \frac{1}{2\pi i} \int_{k_i} \frac{df}{f}$. As above one has to approximate k_i by a Jordan curve inside D to give the formula a sense. Cauchy tells us that $\sum n_i = 0$. After multiplying f with the inverse of $(z - a_1)^{n_1} ... (z - a_s)^{n_s}$ we may suppose that all n_i are zero. The differential form $\frac{df}{f}$ is exact on D since the integrals over all closed paths in D are zero. Hence there is a holomorphic g on D with $d(2\pi i g) = \frac{df}{f}$ and so after changing g with a constant one has $f = e^{2\pi i g}$. The uniqueness in (3) is easily verified.

By an **open subset of finite type** we will mean a finite union of domains of finite type E_i such that the closures of the E_i are disjoint. Mittag-Leffler can be generalized to such open sets. The intersection of any two open sets of finite type is again an open set of finite type.

6.2 Lemma

Let D_1, D_2 denote open sets of finite type then the map $d : O(D_1) \times O(D_2) \to O(D_1 \cap D_2)$ given by $d(f_1, f_2) = f_1 - f_2$ is surjective. The same statement *holds with O replaced by O^* and d replaced by d^* with $d^*(f_1, f_2) = f_1 f_2^{-1}$.*

Proof. We may suppose that $D_1 \cap D_2 \neq \emptyset$. The boundary of $D_1 \cap D_2$ consists of segments L either belonging to D_1 or D_2. Any holomorphic f on $D_1 \cap D_2$ is a sum $f = \sum_L f_L$ where $f_L(z) = \frac{1}{2\pi i} \int_L \frac{f(t)}{t-z} dt$. (Again one has to shrink the polygons a little to give the formula a meaning). Each term f_L belongs to either $O(D_1)$ or $O(D_2)$. This proves that d is surjective. The statement for O^* can be proved by using the holes of D_1, D_2 and $D_1 \cap D_2$ and part (3) of (6.1).

6.3 Lemma

Let D be a domain in $\mathbb{C}$. There exists a sequence of domains of finite type $D(n)$ such that $D = \cup_{n=1}^{n=\infty} D(n)$, $\bar{D(n)} \subset D(n+1)$ for every n. Define $d : \prod_{n=1}^{\infty} O(D(n)) \rightarrow \prod_{n=1}^{\infty} O(D(n))$ by $d(g_1, g_2, ...) = (g_2 - g_1, g_3 - g_2, ...)$. Then d is surjective. A similar statement in multiplicative form holds for O^*.

Proof. The existence of the $D(n)$ is rather obvious. For every n there is a number $s(n) \geq n$ such that every hole of $D(n)$ which is entirely contained in D is already contained in $D(s(n))$. Let the element $f = (f_1, f_2, f_3, ...)$ be given. We will construct $g = (g_1, g_2, g_3, ...)$ with $d(g) = f$. Define first $\tilde{g}$ by $\tilde{g_n} = -(f_n + ... + f_{s(n)})$. Then $\tilde{g}_{n+1} - \tilde{g_n} = f_n - \sum_{i=1+s(n)}^{s(n+1)} f_i$. The expression $R_n = \sum_{i=1+s(n)}^{s(n+1)} f_i$ exists on $D(s(n))$ and its restriction to $D(n)$ has a Mittag-Leffler decomposition using only the holes of $D(n)$ which are not entirely contained in D. Hence R_n can be approximated by $A_n \in O(D)$ such that $\|R_n - A_n\|_{D(n)} \leq \frac{1}{n^2}$ where the norm means the maximum norm on the set $D(n)$. Define g' by $g'_n = A_1 + A_2 + ... + A_n + \sum_{i \geq n}(A_i - R_i)$. The infinite sum converges uniformly and we conclude that $g'_n \in O(D(n))$. Further $g'_{n+1} - g'_n = R_n$. Hence $g := \tilde{g} + g'$ satisfies $dg = f$. This proves the first statement. The second statement has a similar prove and uses (6.1) part (3).

6.4 Lemma

Let D_1, D_2 denote two open subsets of $\mathbb{C}$. Then the maps

$$d : O(D_1) \times O(D_2) \to O(D_1 \cap D_2) \text{ and}$$

$$d^* : O(D_1)^* \times O(D_2)^* \to O(D_1 \cap D_2)^* \text{ are surjective.}$$

Proof. We suppose for simplicity that the D_i are domains and that $D_1 \cap D_2 \neq \emptyset$. Choose the domains of finite type $D_i(n)$ as in (6.3). We want to write $f \in O(D_1 \cap D_2)$ as $f_1 - f_2$ with $f_i \in O(D_i)$. According to (6.2) one has $f|_{D_1(n) \cap D_2(n)} = f_1(n) - f_2(n)$ with $f_i(n) \in O(D_i(n))$. The $f_1(n)$ do not glue to a function $f_1 \in O(D_1)$. We consider the holomorphic h_n on $D_1(n) \cup D_2(n)$ given by $h_n = f_1(n+1) - f_1(n)$ on $D_1(n)$ and $h_n = f_2(n+1) - f_2(n)$ on $D_2(n)$. We apply the lemma (6.3) to the sequence (h_n) for the domains of finite type $D_1(n) \cup D_2(n)$ filling up $D_1 \cup D_2$. Thus $h_n = g_{n+1} - g_n$ with $g_n \in O(D_1(n) \cup D_2(n))$. Define $f_i'(n) = f_i(n) - g_n$ for $i = 1, 2$. Then $f_1'(n) - f_2'(n) = f|_{D_1(n) \cap D_2(n)}$ and $f_i'(n) = f_i'(n+1)$ on $D_i(n)$. Hence the $f_i'(n)$ glue to holomorphic functions f_i on D_i with the required property $f = f_1 - f_2$.
The proof of the second statement is similar.

6.5 Lemma

Let $\{D_1, ..., D_n\}$ be a finite open covering of D. Then:

$$\prod_{i=1}^{n} O(D_i) \overset{d^0}{\to} \prod_{1 \le i < j \le n} O(D_i \cap D_j) \overset{d^1}{\to} \prod_{1 \le i < j < k \le n} O(D_i \cap D_j \cap D_k) \text{ is exact}$$

The analoguous statement for O^* holds as well.

Proof. The sequence in the lemma is part of the Cech-complex for the covering and the sheaf O. The case $n = 2$ is the last lemma. We will prove the case $n = 3$ and $n > 3$ is proved by induction in the same way. Take $f \in \ker d^1$. Choose $f_i \in O(D_i)$ for $i = 1, 2$ with $f_1 - f_2 = f(1, 2)$. Then $g = f - d^0(f_1, f_2, 0)$ satisfies $g(1, 2) = 0$ and $g(1, 3) = g(2, 3)$ on $D_1 \cap D_2 \cap D_3$. Let h denote the function on $(D_1 \cup D_2) \cap D_3$ given by $h = g(1, 3)$ on $D_1 \cap D_3$

and $h = g(2,3)$ on $D_2 \cap D_3$. The function h is holomorphic and can be written as $h = a - b$ with $a \in O(D_1 \cup D_2)$ and $b \in O(D_3)$. Then $g = d^0(a, a, b)$. This proves the exactness. The statement for O^* has an analoguous proof.

6.6 Theorem

For every open subset D of $\mathbb{C}$ one has $H^i(D, O) = H^i(D, O^*) = 0$ for $i \geq 1$. In particular every line bundle on D is trivial.

Proof. We may suppose that D is connected and we take a sequence of domains of finite type $D(n)$ in D as in (6.3). Let an open covering $\{D_i\}$ and a 1-cocycle $f = (f(i,j))_{i<j}$ for O be given. The restriction f_n of this 1-cocycle to $D(n)$ with the covering $\{D(n) \cap D_i\}$ has a finite subcovering since $D(n)$ is relative compact in D. Applying (6.5) one finds elements $f_n(i) \in O(D_i \cap D(n))$ with $f_n(i) - f_n(j) = f(i,j)|_{D(n) \cap D_i \cap D_j}$. The $f_n(i)$ do not glue to a function on D_i. We use (6.3) to change the $f_n(i)$.

$$f_{n+1}(i) - f_n(i) = f_{n+1}(j) - f_n(j) \text{ holds on } D(n) \cap D_i \cap D_j$$

for all i, j. Define now $A_n \in O(D(n))$ by $A_n(z) = (f_{n+1}(i) - f_n(i))(z)$ if $z \in D(n) \cap D_i$. According to the lemma one has $A_n = B_{n+1} - B_n$ for certain $B_n \in O(D(n))$. The elements $f'_n(i) := f_n(i) - B_n$ satisfy:
(1) $f'_n(i) = f'_{n+1}(i)$ on $D(n) \cap D_i$.
(2) $f'_n(i) - f'_n(j) = f(i,j)$ on $D(n) \cap D_i \cap D_j$.
Hence the $f'_n(i)$ glue to $f(i) \in O(D_i)$ and $f(i) - f(j) = f(i,j)$ holds on $D_i \cap D_j$. This proves that $H^1(D, O) = 0$. A similar proof gives $H^1(D, O^*) = 0$. The statements on H^i for $i > 1$ are left as an exercise.

Exercise 20: Prove that $H^i(\mathbb{P}^1, O) = 0$ for $i \geq 1$. Prove that $H^1(\mathbb{P}^1, O^*) = \mathbb{Z}$ and $H^i(\mathbb{P}^1, O^*) = 0$ for $i > 1$. Describe the line bundles on $\mathbb{P}^1$.

7 The finiteness theorem

7.1 Fréchet spaces

A function p on a complex vector space E with values in $\mathbb{R}_{\geq 0}$ is called a semi-norm if:
$p(a+b) \leq p(a) + p(b)$ for all $a, b \in E$ and
$p(\lambda a) = |\lambda| p(a)$ for $a \in E$ and $\lambda \in \mathbb{C}$.
The semi-norm p is called a norm if moreover $p(a) = 0$ implies that $a = 0$. A **Fréchet space** is a vector space E together with a sequence of semi-norms p_n such that the following is satisfied:
(1) $p_n(a) = 0$ for all n only if $a = 0$.
(2) If a sequence a_k in E is a Cauchy sequence for all semi-norms p_n then there is an $a \in E$ such that a_k converges to a for all semi-norms p_n.
One can make E into a metric space by defining the distance function

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(x-y)}{1 + p_n(x) + p_n(y)}$$

Condition (2) means that E is complete with respect to this metric.
Example. *Let X denote a RS (which is the union of at most countably many discs). Then $O(X)$ is made into a Fréchet space by choosing a sequence of compact subsets K_n with union X and by defining the semi-norms p_n by $p_n(f) = \|f\|_{K_n} =$ the supremum norm for f on K_n. The topology of $O(X)$ does not depend on the choice of the K_n.*
Proof: A sequence of holomorphic functions which is a Cauchy sequence with respect to the supremum (semi-) norm on all compact subsets of X converges with respect to the same (semi-) norms to a holomorphic function on X.

Remarks. Similarly for any line bundle on a RS (of countable type) X the complex vector space $L(X)$ has a unique structure as Fréchet space.

We note further that the countable product of Fréchet spaces is again a Fréchet space and that any closed linear subspace of a Fréchet space is also a Fréchet space.

A continuous linear map of Fréchet spaces $\phi : E \to F$ is called **compact** if there exists a neighbourhood U of $0 \in E$ such that the closure of its image

in F is compact.
Example. *Let Y be a relative compact open subset of the RS X (of countable type). This property is abbreviated by $Y \subset\subset X$. Then the restriction map $\phi : O(X) \to O(Y)$ is a compact map.*
Proof. Let K denote an open subset of X such that Y lies in the interior of K. Let the open neighbourhood of O be $V = \{f \in O(X) \mid \|f\|_K < 1\}$. The image of V is contained in the set W of holomorphic functions on Y with supremum norm ≤ 1 on Y.

We will use the following **theorem of L.Schwarz.**:
Let $\phi : E \to F$ be a continuous surjective map between Fréchet spaces, and let $\psi : E \to F$ be a compact linear map. Then im$(\phi + \psi)$ is a closed subspace of F with finite codimension.

7.2 Theorem

Let X be a Riemann surface and L a line bundle on X, then
(1) dim $H^1(X, L)$ is finite.
(2) If X is not compact then $H^1(X, L) = 0$.

Proof. (1)⇒(2). Choose an infinite discrete sequence $p_1, p_2, ..$ in X and put $D = \sum_{i=1}^{\infty}(-1)p_i$. Define the skyscraper sheaf $\mathcal{Q}$ by the exact sequence

$$0 \to O(D) \to O \to \mathcal{Q} \to 0$$

The exactness of $O(X) \to \mathcal{Q}(X) \to H^1(X, O(D))$ and dim $H^1(X, O(D)) < \infty$ implies that $O(X)$ is infinite dimensional. Let $f \in O(X)$ be a non constant function. Consider the exact sequence

$$0 \to L \xrightarrow{\cdot f} L \to \mathcal{R} \to 0$$

which defines the skyscraper sheaf $\mathcal{R}$. It follows that $H^1(f) : H^1(X, L) \to H^1(X, L)$ is surjective. If $H^1(X, L) \neq 0$ then $H^1(f)$ has some eigenvalue $\lambda \in \mathbb{C}$ and we find the contradiction that $H^1(f - \lambda)$ is not surjective. This proves $H^1(X, L) = 0$.
(1). Suppose that $Y_1 \subset\subset Y_2 \subset\subset X$. There are finitely many open sets $U_i, 1 \leq i \leq m$ in Y_2, all isomorphic to an open subset in $\mathbb{C}$ such that $Y_2' =$

$Y_1 \cup U_1 ... \cup U_m$ satisfies $Y_1 \subset\subset Y_2'$. We replace now Y_2 by Y_2'. Let L be a line bundle on X. Applying the exact Mayer-Vietoris sequence

$$H^1(Y \cup U, L) \to H^1(Y, L) \oplus H^1(U, L) \to H^1(Y \cap U, L)$$

for an U which is isomorphic to some open subset of $\mathbb{C}$ and using (6.6) one finds that $H^1(Y \cup U, L) \to H^1(Y, L)$ is surjective. This argument and induction on m shows that $H^1(Y_2, L) \to H^1(Y_1, L)$ is surjective.
Let $\{D_i\}$ denote a covering of Y_2 be open sets isomorphic to open subsets of $\mathbb{C}$. By Leray's theorem, the Cech-complex of L with respect to this covering calculates $H^1(Y_2, L)$. So we have an exact sequence

$$\prod_i L(D_i) \to Z \to H^1(Y_2, L) \to 0$$

where $Z \subset \prod_{i<j} L(D_i \cap D_j)$ is the space of 1-cocycles. Choose $D_i' \subset \subset D_i$ such that $Y_1 \subset \cup_i D_i'$, then the Cech- complex of L with respect to the covering $D_i' \cap Y_1$ calculates $H^1(Y_1, L)$ and we find an exact sequence

$$\prod_i L(D_i' \cap Y_1) \to Z' \to H^1(Y_1, L) \to 0$$

where Z' denotes the set of 1-cocycles. Using that $H^1(Y_2, L) \to H^1(Y_1, L)$ is surjective it follows that the natural map $\phi : \prod_i L(D_i' \cap Y_1) \oplus Z \to Z'$ (i.e. the sum of the given map $\prod_i L(D_i' \cap Y_1) \to Z'$ and the restriction map $Z \to Z'$) is surjective. The spaces $\prod_i L(D_i' \cap Y_1)$, Z and Z' are Fréchet spaces and ϕ is a continuous linear map. The map $\psi : \prod_i L(D_i' \cap Y_1) \oplus Z \to Z'$ is given by ψ is 0 on the first factor and equal to the restriction map on the second factor. This map is compact since $D_i' \cap D_j' \cap Y_1 \subset \subset D_i \cap D_j$. The cokernel of the map $\phi - \psi$ is equal to $H^1(Y_1, O)$ and by the theorem of L.Schwarz one finds dim $H^1(Y_1, L)$ is finite.
Suppose now that X is compact. Then we apply the above to $Y_1 = Y_2 = X$ and find that dim $H^1(X, L)$ is finite.
For a non compact X (of countable type) one makes a covering $X = \cup Y_k$ where $Y_k \subset \subset y_{k+1}$ for all k. We know already that dim $H^1(Y_k, L)$ is finite. The argument in (1)⇒(2) shows that in fact $H^1(Y_k, L) = 0$. An approximation argument like (6.3) shows that also $H^1(X, L) = 0$.

8 Riemann-Roch and Serre duality

The text of the next sections is rather brief and compact. There are many good sources available for more detailed information. Let X be a compact RS and let D be a divisor on X. One associates a line bundle $O_X(D)$ with D in the folowing way

$$O_X(D)(U) = \{f \text{ meromorphic on } U |\ (f) \geq -D \text{ on } U\}$$

where (f) denotes the divisor of f on U. If $D = \sum n_i x_i$ then the condition means that $\text{ord}_{x_i}(f) \geq -n_i$ for every point x_i belonging to U. For any coherent sheaf S on X (in particular for the line bundle $O_X(D)$) one defines the **Euler characteristic** $\chi(S) := dim\ H^0(X, S) - dim\ H^1(X, S)$. For the line bundle O_D one also writes $\chi(D) = \chi(O_X(D))$. The **genus** of X is $g := dim\ H^1(X, O_X)$. The first result is:

8.1 Proposition

For every divisor D on X one has $\chi(D) = 1 - g + deg(\ D)$.

Proof. For $D = 0$ the statement is just the definition of g and the easy assertion that the only holomorphic functions on X are the constant functions. For $D = \sum_{i=1}^{r} n_i x_i$ with all $n_i > 0$ the sheaf $O_X(D)$ contains O_X as a subsheaf and there is an exact sequence

$$0 \to O_X \to O_X(D) \to S \to 0$$

which defines S. One sees that the stalk S_x is zero if $x \notin \{x_1, ..., x_r\}$ and that S_{x_i} is a vector space over $\mathbf{C}$ with dimension n_i. Hence S is a skyscraper sheaf. The long exact sequence of cohomology is here a short one of finite dimensional vector spaces

$$0 \to H^0(X, O_X) \to H^0(X, O_X(D)) \to H^0(X, S) \to H^1(X, O_X) \to H^1(X, O_X(D)) \to 0$$

Then $\chi(D) = 1 - g + dim\ H^0(X, S)$ and further $dim\ H^0(X, S) = \sum n_i = deg(\ D)$. For an arbitrary divisor D one can find a positive divisor E with $D \leq E$. Then $O_X(D)$ can be seen as a subsheaf of $O_X(E)$. An exact sequence as above shows that $\chi(E) = \chi(D) + deg(\ E) - deg(\ D)$. This ends the proof.

Exercise 21. Prove that $\mathbf{P}^1$ has genus 0 by an explicit calculation with a Čech complex. Calculate the groups $H^i(\mathbf{P}^1, O_X(D))$ for every divisor D with positive degree. Prove that any divisor of degree 0 is the divisor of a meromorphic function.

Exercise 22. Let the complex number q satisfy $0 < |q| < 1$. Define the Riemann surface $E := E_q := \mathbf{C}^*/\{q^n|n \in \mathbf{Z}\}$. Let $\pi : \mathbf{C}^* \to E$ denote the canonical map. One considers a small positive ϵ and to open sets $U_1 = \{z \in \mathbf{C}|\ |q|(1-\epsilon) < |z| < |q|^{1/2}(1+\epsilon)\}$ and $U_2 = \{z \in \mathbf{C}|\ |q|^{1/2}(1-\epsilon) < |z| < (1+\epsilon)\}$. Prove that the genus of E is one by an explicit calculation with the Čech complex of O_E with respect to the covering $\{\pi(U_1), \pi(U_2)\}$ of E.
Exercise 23. Let $D = x$ denote the divisor on a genus 1 Riemann surface X where x is any point. Use an exact sequence to prove that $H^1(X, O_X(D)) = 0$. Use exact sequences to show that for any positive divisor $F \neq 0$ one has $H^1(X, O_X(F)) = 0$ and $dim\ H^0(X, O_X(F)) = deg(\ F)$.

Two divisors D and E are said to be **equivalent** if there is a meromorphic function f on X such that $(f) = D - E$. One easily sees that D and E are equivalent if and only if the sheaves $O_X(D)$ and $O_X(E)$ are isomorphic as sheaves of O_X modules. For any coherent sheaf S and any divisor D we write $S(D)$ for the coherent sheaf $O_X(D) \otimes_{O_X} S$. The last sheaf is defined as the sheaf associated to the presheaf $U \mapsto O_X(D)(U) \otimes_{O_X(U)} S(U)$ on X. We note that $O_X(D) \otimes_{O_X} O_X(E) \cong O_X(D+E)$. We will further show that

8.2 Lemma

For any line bundle L on X there is a divisor D with $L \cong O_X(D)$.

Proof.
Suppose first that $H^0(X, L) \neq 0$ and let $F \in H^0(X, L)$; $F \neq 0$. Define a morphism $\alpha : O_X \to L$ of O_X modules by giving for every open U a map $\alpha_U : O_X(U) \to L(U)$ by the formula $g \mapsto gF$. This gives rise to an exact sequence

$$0 \to O_X \to L \to S \to 0$$

and defines S. It can be seen that S is a skyscraper sheaf. Let $x_1, ..x_r$ denote the points of X where $S_x \neq 0$ and define $n_i := dim\ S_{x_i}$. Then $L \cong O_X(D)$ where $D := \sum n_i x_i$.
In general L has no sections (other than 0) and we have to change L into $L(E)$ where E is a positive divisor on X with high degree. If we succeed in finding an E such that $L(E)$ has a non trivial section then $L(E) \cong O_X(D)$ for some divisor D and $L \cong O_X(D - E)$.
For any choice of a positive divisor E one has again an exact sequence of sheaves

$$0 \to L \to L(E) \to S \to 0$$

where S is a skyscraper sheaf isomorphic to $O_X(E)/O_X$. The long exact sequence of cohomology reads

$$0 \to H^0(X, L) \to H^0(X, L(E)) \to H^0(X, S) \to H^1(X, L) \to H^1(X, L(E)) \to 0$$

If the degree of E, which is equal to the dimension of $H^0(X, S)$, is chosen greater than dim $H^1(X, L)$ then clearly $H^0(X, L(E)) \neq 0$ and this finishes the proof.

The degree of a line bundle L on X is now defined as degree (D) where D is any divisor such that $L \cong O_X(D)$. This does not depend on the choice of D because the degree of the divisor of any meromorphic function on X is 0. The *dual line bundle* $L^{-1} := Hom_{O_X}(L, O_X)$ turns out to be isomorphic to $O_X(-D)$. One is mainly interested in the dimension of $H^0(X, L)$ and (8.1) only gives an inequality dim $H^0(X, L) \geq 1 - g + deg(\ L)$. In the following we will establish a relation between H^0 and H^1 called Serre duality (and is in fact a very special case of a powerful theorem with the same name). From this one can start calculating dimensions and properties of meromorphic functions and differential forms.

8.3 The Serre duality

The cohomology group $H^1(X, \Omega)$ can be calculated with the Čech-cohomology with respect to any covering of X by proper open subsets according to (7.2). Choose any finite set $S = \{a_1, ..., a_n\}$ of points in X and consider the covering of X by two open sets U_0, U_1 where U_0 is the disjoint union of n disjoint

small disks around the points of S and where $U_1 = X - S$. One considers the map from $\Omega(U_{0,1})$ to $\mathbf{C}$, given by $\omega_{0,1} \mapsto \sum_{i=1}^{n} Res_{a_i}(\omega_{0,1})$.
Any $\omega_{0,1} = \omega_0 - \omega_1$ with $\omega_i \in \Omega(U_i)$ for $i = 0, 1$ has image 0 under this map. Therefore we find a $\mathbf{C}$-linear map $Res_S : H^1(X, \Omega) \to \mathbf{C}$ which depends a priori on S. But enlarging S does not change the map, hence we found a canonical linear map, called *the residue map*, $Res : H^1(X, \Omega) \to \mathbf{C}$ which is independent of the choice of S. Let D be any divisor on X, then one defines a pairing

$$H^0(X, \Omega(-D)) \times H^1(X, O(D)) \to H^1(X, \Omega) \stackrel{Res}{\to} \mathbf{C}$$

For $\omega \in H^0(X, \Omega(-D))$ and $\xi \in H^1(X, O(D))$ we write $< \omega, \xi >_D = Res(\xi\omega) \in \mathbf{C}$. This is a 'pairing', i.e. a bilinear form. The pairing induces a linear map $i_D : H^0(X, \Omega(-D)) \to H^1(X, O(D))^*$, where V^* is a notation for the dual of a vector space V.

The statement of the Serre duality theorem is:

$$i_D : H^0(X, \Omega(-D)) \to H^1(X, O(D))^* \text{ is an isomorphism.}$$

Proof.
(1) Let $\omega \neq 0$ then there is a ξ with $< \omega, \xi >_D = 1$. Indeed, choose $S = \{a\}$ and a small disk U_0 around a such that U_0 does not intersect the support of D and such that $\omega = f dz$ where z is a local parameter at a and f is an invertible function on U_0. Let as before U_1 denote $X - \{a\}$. Define $\xi = \frac{1}{zf} \in O(D)(U_{0,1})$. Then clearly $< \omega, \xi >_D = Res_a(\frac{dz}{z}) = 1$. The statement that we just proved implies that the map i_D is injective.
(2) The surjectivity of i_D is more difficult to see. The proof that we give here is somewhat artificial.
We have to use divisors $E \leq D$. For such a divisor E there is a natural injective map $\alpha : H^0(X, \Omega(-D)) \to H^0(X, \Omega(-E))$ and a natural surjective map $\beta : H^1(X, O(E)) \to H^1(X, O(D))$ and an injective dual map $\beta^* : H^1(X, O(D))^* \to H^1(X, O(E))^*$.
For $\omega \in H^0(X, \Omega(-D))$ and $\xi \in H^1(X, O(E))$ one has

$$< \alpha\omega, \xi >_E = < \omega, \beta\xi >_D$$

Indeed, one can represent ξ and $\beta\xi$ by the same element $f \in O(U_{0,1})$ for the Čech complexes calculating $H^1(X, O(D))$ and $H^1(X, O(E))$. The statement implies that the four injective maps satisfy

$$i_E\alpha = \beta^* i_D$$

The following statement is helpful

$$im\ i_E \cap \beta^* H^1(X, O(D))^* = \beta^* im\ i_D$$

Proof of the statement. Let $\omega \in H^0(X, \Omega(-E))$ such that $i_E\omega = \beta^*(\lambda)$ for some $\lambda \in H^1(X, O(D))^*$. If $\omega \notin H^0(X, \Omega(-D))$ then there is a point $a \in X$ such that $\text{ord}_a(\omega) < n_a$ where n_a is the value of the divisor D at a. Let z be a local coordinate at a; write $\omega = fdz$ at a; define the element $(\frac{1}{zf}, 0) \in O(D)(U_0) + O(D)(U_1)$ where U_0 is a small disk around a and $U_1 = X - \{a\}$. Let $\xi \in H^1(X, O(E))$ denote the image of $\frac{1}{zf} \in O(E)(U_{0,1})$. Then $< \omega, \xi >_E = \lambda\beta(\xi) = 0$ since $\beta(\xi)$ is a trivial 1-cocycle for $O(D)$. On the other hand, $< \omega, \xi > = Res_a(\frac{dz}{z}) = 1$. This contradiction shows that $\omega \in H^0(X, \Omega(-D))$.

Continuation of the proof.
Given a $\lambda \in H^1(X, O(D))^*$ it suffices to construct an $\omega \in H^0(X, \Omega(-E))$ for some $E \leq D$ such that $\lambda = i_E(\omega)$. Take a point $P \in X$ and some big positive integer n and a $f \in H_0(X, O(nP)), f \neq 0$. Then f induces an injective morphism of O_X-modules $f : O(D - nP) \to O(D)$, a surjective map $H^1(f) : H^1(X, O(D - nP)) \to H^1(X, O(D))$ and an injective map $H^1(f)^* : H^1(X, O(D))^* \to H^1(X, O(D - nP))^*$. In particular the vector space $\{H^1(f)^*\lambda \mid f \in H^0(X, O(nP))\}$ is isomorphic to $H^0(X, O(nP))$ and has a dimension $\geq n$+some constant. Also $im\ i_{D-nP}$ is a vector space of dimension $\geq n$+some constant. The two vector spaces are subspaces of $H^1(X, O(D - nP))^*$. Since the last vector space has dimension $g - 1 + n - deg(D)$ it follows that for big enough n there is a $\omega' \in H^0(X, \Omega(-D + nP))$ such that $i_{D-nP}(\omega') = H^1(f)\lambda$. Then $\omega := \frac{1}{f}\omega' \in H^0(X, \Omega(-E))$ for some $E \leq D$ and $i_E(\omega) = \lambda$. According to the statement above one has that ω belongs to $H^0(X, \Omega(-D))$. This finishes the proof of the Serre duality.

8.4 Some consequences

1. dim $H^0(X, \Omega) = g$, dim $H^1(X, \Omega) = 1$, $\deg(\Omega) = 2g - 2$.

2. Any meromorphic divisor $\omega \neq 0$ satisfies $\deg((\omega)) = 2g - 2$.

3. $dim\ H^0(X, O(D)) = 1 - g = deg(\ D) + dim\ H^0(X, \Omega(-D))$.

4. $H^0(X, O(D)) = 0$ if $deg(\ D) < 0$.

5. If $\deg(D) = 0$ then $H^0(X, O(D)) \neq 0$ if and only if D is equivalent to the zero divisor.

6. $H^1(X, O(D)) = 0$ if $\deg(D) > 2g - 2$.

The first statement follows at once from the duality and (8.1). The sheaf $O_X\omega$ is isomorphic to O_X and has degree 0. From this the second statement follows. The third statement follows from duality and (8.1). Let $f \in H^0(X, O(D)), f \neq 0$, then $(f) \geq -D$ and so $0 = deg((f)) \geq -deg(\ D)$. Hence $deg(\ D) \geq 0$ if $H^0(X, O(D)) \neq 0$. This proves (4). Statement (5) is easily seen to be true. For (6) we have to see that $H^0(X, \Omega(-D)) = 0$. But this follows from (4) and $deg(\ \Omega(-D)) = 2g - 2 - deg(\ D) < 0$.

8.5 Riemann-Hurwitz-Zeuthen

Let $f : X \to Y$ be a non constant morphism of degree d between compact RS of genera g_X, g_Y. Let for every point $x \in X$ the integer $e_x \geq 1$ denote the ramification index of x. Then one has the following formula:

$$2g_X - 2 = d(2g_Y - 2) + \sum_{x \in X}(e_x - 1)$$

Proof.
Let $\omega \neq 0$ be a meromorphic differential form on Y then $\deg((\omega)) = 2g_Y - 2$ and $\deg((f^*\omega)) = 2g_X - 2$. A local calculation at (the ramified)points of X gives the formula.

Exercise 24. Let X be a compact Riemann surface of genus g. Fix a point $P_0 \in X$. Show that every divisor of degree 0 is equivalent to a divisor of the form $P_1 + P_2 + ... + P_g - gP_0$. (Hint: Prove first that for any points

$Q_1, ..., Q_{g+1}$ there are points $R_1, ..., R_g$ such that $Q_1 + ... + Q_{g+1}$ is equivalent to $R_1 + ... + R_g + P_0$ by taking a suitable section of the line bundle $O_X(Q_1 + ... + Q_{g+1})$.

Exercise 25. Let the compact RS X have genus 1 and fix a point $P_0 \in X$. Show that the map: $P \mapsto$ the equivalence class of the divisor $P - P_0$, is a bijection of X with the group of equivalence classes of divisors on X of degree 0.

Exercise 26. A RS X is called hyperelliptc if there is a morphism $f : X \to \mathbf{P}^1$ of degree 2. Let n denote the number of ramified points of f. Show that $n = 2g + 2$ where g is the genus of X.

Exercise 27. Prove that the following sequence of sheaves on the compact RS X of genus g is exact

$$0 \to \mathbf{C} \to O_X \overset{f \mapsto df}{\to} \Omega_X \to 0$$

Use this to prove that dim $H^1(X, \mathbf{C}) = 2g$ and conclude that the topological genus of X is also g.(This means that X is homeomorphic to the g-fold torus. The classification of compact oriented surfaces and their cohomology for constant sheaves are assumed to be known for this exercise.)

9 Compact Riemann surfaces, algebraic curves

In section 3 we have seen that every (non singular, projective, connected) algebraic curve over $\mathbf{C}$ has the structure of a compact Riemann surface. In this section we will show that every compact RS is obtained in this way from a unique (non singular, projective, connected) algebraic curve. The method is the following. Let L denote a line bundle on the compact RS X with genus g and let $s_0, ..., s_n$ denote a basis of $H^0(X, L)$. One considers the "map" ϕ_L

$$x \in X \mapsto (s_0(x) : s_1(x) : ... : s_n(x)) \in \mathbf{CP}^n$$

Some explanation is needed. $\mathbf{CP}^n$ denote the projective space over $\mathbf{C}$ of dimension n. As a point set this space is $(\mathbf{C}^{n+1} - \{0\})/\sim$ where $\sim$ is the equivalence relation on non zero vectors v_1, v_2, given by $v_1 \sim v_2$ if there is a scalar λ with $v_1 = \lambda v_2$. The $s_0, ..., s_n$ are not functions on X. For any point $P \in X$ the line bundle is isomorphic to $O_X e$ in some neighbourhood of P and the elements $s_0, ..., s_n$ can be written as $s_i = f_i e$ for some holomorphic functions f_i. In this neighbourhood, the expression $(s_0(x) : s_1(x) : ... : s_n(x))$ is defined as $(f_0(x) : f_1(x) : ... : f_n(x))$. The last expression does not depend on the choice of e. Hence the expression is well defined except for the possibility that all $f_i(x)$ happen to be 0.

9.1 Lemma

If the line bundle L has degree $\geq 2g$ then ϕ_L is a well defined analytic map from X to $\mathbf{CP}^n$.
If the degree of the line bundle is $> 2g$ then ϕ_L is injective and the derivative of ϕ_L is nowhere zero.

Proof.
Let $P \in X$ and let the exact sequence

$$0 \to L(-P) \to L \to \mathcal{Q} \to 0$$

define the skyscraper sheaf $\mathcal{Q}$. Then $H^1(L(-P)) \cong H^0(\Omega(P) \otimes L^{-1})^* = 0$ since the degree of the sheaf $\Omega(P) \otimes L^{-1}$ is $2g - 2 + 1 - deg(L) < 0$. The cohomology sequence reads

$$0 \to H^0(X, L(-P)) \to H^0(X, L) \to \mathbf{C} \to 0$$

This implies the existence of a $s \in H^0(X, L)$ which generates L locally at P. From this the first statement follows.
One replaces in the argument above P by $P + Q$, for two points $P, Q \in X$. For $P \neq Q$ this gives that $\phi_L(P) \neq \phi_L(Q)$ and for $P = Q$ this gives that the derivative of the map ϕ_L is not zero at any point of X.

9.2 Proposition

If the line bundle L on X has degree $> 2g$ then the image Y of ϕ_L is a non singular connected projective algebraic curve. The map $\phi_L : X \to Y$ is an isomorphism of Riemann surfaces. The field of meromorphic functions on X

coincides with the function field of Y.

Proof.
The main difficulty here is to show that Y, the image of ϕ_L is an algebraic curve. This can be seen by using "big machinery" as follows. The "proper mapping theorem" shows that Y is an analytic subset of $\mathbf{CP}^n$ (i.e. locally the zero set of analytic equations.) Then "GAGA theorem" (so called after J-P. Serre's paper: Géométrie analytique et géométrie algébrique) states that every analytic subset of $\mathbf{CP}^n$ is an algebraic subset. Losely stated, "GAGA" says that all analytic objects on (an analytic subset of) $\mathbf{CP}^n$ are algebraic. In particular analytic line bundles are algebraic. Further the algebraic curve Y is uniquely determined by X (upto algebraic isomorphism). We will try to give a proof of (9.2) without these big machines.

One considers the complex vector space $R := \oplus_{n\geq 0} H^0(X, L^n)$, where $L^0 := O_X$ and so $H^0(X, L^0) = \mathbf{C}$. One uses the natural maps $H^0(X, L^n) \times H^0(X, L^m) \to H^0(X, L^{n+m})$ to define the multiplication in R. This gives R the structure of a graded $\mathbf{C}$- algebra. One can show, using that $\deg(L) > 2g$, that R is generated by $H^0(X, L)$ as a ring. This means that the homomorphism of graded $\mathbf{C}$- algebras $\alpha : \mathbf{C}[X_0, ..., X_n] \to R$, given by $X_i \mapsto s_i$ for all i, is surjective. It follows that $Z := V(ker(\ \alpha)$=the projective algebraic subspace defined by the homogeneous ideal $\ker(\alpha)$, contains Y. The homogeneous ring of Z is equal to R by definition. This ring is easily seen to have no zero divisors. Hence Z is an irreducible variety. The Hilbert polynomial $P \in \mathbf{Q}[t]$ of Z is defined as $P(n) = dim\ H^0(X, L^n) = ndeg(L) + 1 - g$ for $n >> 0$. Hence $P = tdeg(L) + 1 - g$. The degree of this polynomial is known to be the dimension of the variety. Hence Z is an irreducible algebraic curve. Let $s \in H^0(X, L), s \neq 0$. The set of zeros of s on X counted with multiplicity can be seen as a positive divisor on X. The exact sequence

$$0 \to O_X \stackrel{1 \mapsto s}{\to} L \to \mathcal{Q} \to 0$$

which defines the skyscraper sheaf $\mathcal{Q}$ gives an exact sequence

$$0 \to \mathbf{C} \to H^0(X, L) \to H^0(X, \mathcal{Q}) \to H^1(X, O_X) \to 0$$

With Riemann Roch one finds that the degree of the divisor above is equal to the degree of L. In the projective space one can intersect Z with the

hyperplane given by $s = 0$. According to the Hilbert polynomial of Z this intersection, seen as a divisor on Z has also degree equal to the degree of L. From this one concludes that $\phi_L : X \to Z$ is bijective. Since the derivative of ϕ_L is nowhere zero one concludes that $Z = Y$ is non singular. This finishes part of the proof.
The rational functions on Y are of course meromorphic functions on X. A counting of the dimensions of rational and meromorphic functions on Y and X with prescribed poles easily implies that every meromorphic function on X is a rational function on Y. This ends the proof.

9.3 Remarks

From the above it follows that there is no essential difference between compact Riemann surfaces and (non singular, projective, connected) curves over **C**. From this point on a course about compact Riemann surfaces could continue as a course on algebraic curves over some field. However the topological and the analytic structure of the compact Riemann surface gives information which one can not (or at least not without a lot of difficulties) obtain from the purely algebraic point of view. An example of this is the construction of the Jacobian variety of a curve. The analytic construction is transparant and the algebraic construction (valid over any field) is much more involved. Another example where the analytic point of view is valuable is the theory of uniformizations and Teichmüller theory.

Exercise 28. Let f be a homogeneous polynomial of degree n in three variables X_0, X_1, X_2 defining a compact Riemann surface X lying in $\mathbf{CP}^2$. Prove that the genus of X is $\frac{(n-1)(n-2)}{2}$ with the following method.
One may suppose that $(0 : 0 : 1) \not\in X$. The morphism $\phi : X \to \mathbf{CP}^1$, given by $(x_0 : x_1 : x_2) \in X \mapsto (x_0 : x_1)$ is well defined. We want to apply Riemann-Hurwitz-Zeuthen to this map. Prove that the degree of ϕ is n. Prove that the collection of ramified points for the map ϕ is given by the two equations $f = 0, \frac{df}{dX_2} = 0$. Use that the number of points of intersection (counted with multiplicity) of two curves in $\mathbf{CP}^2$ is equal to the product of the two degrees.(This statement is called after Bézout.) Finish the proof.

Exercise 29. Let X denote a Riemann surface of genus 1. Take a point

P on X and prove that the line bundle $O_X(2P)$ gives an isomorphism of X with an (non singular) algebraic curve in $\mathbf{CP}^2$ of degree 3 (i.e. given by a homogeneous equation of degree 3).

10 The Jacobian variety

10.1 Analytic construction of the Jacobian variety.

We will give some details about paths and integration of holomorphic forms on a Riemann surface X of genus $g \geq 1$. We start be describing the *singular homology* of X. Let $S_0, S_1, S_2, ...$ denote the standard simplices of dimension $0, 1, 2, ...$ (i.e. point, segment, triangle,...). Let S_iX denote the set of continuous maps from S_i to X and let C_iX denote the free $\mathbf{Z}$-module on S_iX (i.e. the elements of C_iX are formal finite sums $\sum n_j c_j$ with $n_j \in \mathbf{Z}$ and $c_j \in S_iX$.) One makes the following complex

$$... \to C_2X \stackrel{\partial_1}{\to} C_1X \stackrel{\partial_0}{\to} C_0X \to 0$$

where $\partial_0, \partial_1, ...$ are $\mathbf{Z}$-linear and are given on the generators as follows:
$\partial_0(c) = c(1) - c(0) \in C_0X$ for any continuous map $c : [0, 1] \to X$
Let $c : S_2 \to X$ be a continuous map. Each side of the triangle S_2 is identified with $[0, 1]$ in such a way that every vertex of S_2 is the beginning of only one segment $[0, 1]$. $\partial_1(c) =$ the sum of the restriction of c to the three sides of the triangle.
Similar definitions can be given for all ∂_n. Clearly $\partial_0\partial_1 = 0$ and the sequence above is a complex. The homology groups of the complex are denoted by $H_i(X, \mathbf{Z})$ and are called the singular homology groups of X.
Since X is connected one finds that $H_0(X, \mathbf{Z}) = \mathbf{Z}$. It is not difficult to show that $H_1(X, \mathbf{Z}) := ker(\partial_1)/im(\partial_0)$ is equal to the abelianization of the fundamental group of X. Another fact is that for any abelian group A one has $H^1(X, A) = Hom(H_1(X, \mathbf{Z}), A)$. For X with topological genus g one concludes that $H_1(X, \mathbf{Z}) \cong \pi_1(X, x)_{ab} \cong \mathbf{Z}^{2g}$. A further result is $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$ (this follows from the fact that X is orientable) and $H_i(X, \mathbf{Z}) = 0$ for $i > 2$ (this follows from the statement that the topological dimension of X is 2).
Continuous maps $c_1, c_2 : [0, 1] \to X$ are oriented paths on X. The intuitive way to count the number of points of intersection of c_1 with c_2 (including

signs) can be made into an exact definition. The result is a bilinear form $<,>$ on $H_1(X, \mathbf{Z})$(="closed paths on X modulo deformation") which is antisymmetric (or symplectic). There is a "symplectic basis" for this form, in standard notation $a_1, ...a_g, b_1, ..., b_g$, with the properties:

$$< a_i, b_i >= - < b_i, a_i >= 1 \text{ for all } i \text{ and all other } < .,. >= 0$$

Differential 1-forms and in particular holomorphic differential forms can be integrated over any continuous path. For any $\sum n_j c_j \in C_1 X$ the map

$$\omega \in H^0(X, \Omega) \mapsto \sum n_j \int_{c_j} \omega$$

induces a linear map, called the *period map*

$$Per : H_1(X, \mathbf{Z}) \to H^0(X, \Omega)^*$$

10.2 Theorem and Definition

The period map is injective and the image of the period map is a lattice in the g-dimensional vector space $H^0(X, \Omega)^*$. The "analytic torus" $H^0(X, \Omega)^*/im(Per)$ is called the Jacobian variety of X and denoted by Jac(X).

Proof.
We will need here some more analysis and topology of the RS X. From

Exercise 27, we know that the topological genus of X is equal to the algebraic genus g of X. The first *De Rham cohomology group* $H^1_{DR}(X, \mathbf{C})$ is the vector space of the closed (C^∞ and complex valued) 1-forms on X divided by the subspace of exact (C^∞ and complex valued) 1-forms. There is a natural identification of $H^1_{DR}(X, \mathbf{C}) \cong Hom(\ H_1(X, \mathbf{Z}), \mathbf{C})$, given by the map $\omega \mapsto (c \mapsto \int_c \omega)$. The group $H^1(X, \mathbf{C})$ which is defined by sheaf cohomology coincides with $Hom(\ H_1(X, \mathbf{Z}), \mathbf{C})$ as stated before. The resulting identification of $H^1_{DR}(X, \mathbf{C})$ with $H^1(X, \mathbf{C})$ is the same as the identification coming from the soft resolution of the constant sheaf $\mathbf{C}$ on X:

$$0 \to \mathbf{C} \to C^\infty\text{-functions} \ \to C^\infty\text{-1-forms} \ \to C^\infty\text{-2-forms} \ \to 0$$

In the statements above we have only used the topological and the C^∞-structure of the Riemann surface X. For the following decomposition of 1-forms on X we use the complex structure of X. Locally, every (C^∞-) 1-form ω can be written as $\omega = fdz + gd\bar{z}$ where z is a local parameter. One easily sees that the decomposition does not depend on the choice of the local parameter. This means that every 1-form ω on X has a unique global decomposition as $\omega = \omega_{1,0} + \omega_{0,1}$ satisfying: $\omega_{1,0}$ and $\omega_{0,1}$ have the forms fdz and $gd\bar{z}$ for every local parameter z. The complex conjugate $\bar{\omega}$ of a 1-form ω is given in local coordinates by $\omega = fdz + gd\bar{z} \mapsto \bar{\omega} = \bar{f}d\bar{z} + \bar{g}dz$. The $*$-operator on 1-forms is given in local coordinates by $*\omega = *(fdz + gd\bar{z}) = fdz - gd\bar{z}$. The decomposition of 1-forms above leads to the so called Hodge decomposition $H^1_{DR}(X, \mathbf{C}) = H^{1,0} \oplus H^{0,1}$. We will not continue this line of thought.

The g-dimensional space $H^0(X, \Omega)$ is a subspace of $H^1_{DR}(X, \mathbf{C})$ since every holomorphic 1-form is closed and an exact holomorphic 1-form has to be zero. Similarly $H^0(X, \bar{\Omega})$ is a g-dimensional subspace of $H^1_{DR}(X, \mathbf{C})$. We claim that

$$H^0(X, \Omega) \oplus H^0(X, \bar{\Omega}) = H^1(X, \mathbf{C})$$

It suffices to show that $H^0(X, \Omega) \cap H^0(X, \bar{\Omega}) = 0$. Let $\omega_1 - \bar{\omega}_2$ be an exact 1-form, where ω_1 and ω_2 are holomorphic. Then

$$0 = \frac{1}{i} \int_X (\omega_1 - \bar{\omega}_2) \wedge *\bar{\omega}_1 = \frac{1}{i} \int_X \omega_1 \wedge *\bar{\omega}_1$$

Using local coordinates, one sees that the last integral can only be 0 if $\omega_1 = 0$. This proves the claim.

Now we will show that the following map is bijective

$$H_1(X, \mathbf{R}) = H_1(X, \mathbf{Z}) \otimes \mathbf{R} \to H^0(X, \Omega)^*$$

The map is again given by $c \mapsto (\omega \mapsto \int_c \omega)$. Both spaces have real dimension $2g$. The map is injective, since a $c \in H_1(X, \mathbf{R})$ with $\int_c \omega = 0$ for all holomorphic ω, also satisfies $\int_c \bar{\omega} = 0$. Hence the image of c in $H^1(X, \mathbf{C})^*$ is zero and so $c = 0$.
Since $H^1(X, \mathbf{Z})$ is a lattice in $H^1(X, \mathbf{R})$, the period map is injective and has as image a lattice.

10.3 Theorem. The Riemann matrix.

Let $a_1, ..., a_g, b_1, ..., b_g$ denote a symplectic basis of $H_1(X, \mathbf{Z})$. There is a basis $\omega_1, ..., \omega_g$ of $H^0(X, \Omega)$ such that $\int_{a_i} \omega_j = \delta_{i,j}$. Further the "Riemann matrix" $Z := (\int_{b_i} \omega_j)$ has the properties: Z is symmetric and the imaginary part of the matrix $im(Z)$ is a positive definite matrix.

Remark.
A period matrix for X is a matrix $(\int_c \omega)$ where c runs in a basis of $H_1(X, \mathbf{Z})$ and where ω runs in a basis of $H^0(X, \Omega)$. For the choice of the theorem the period matrix reads (id_g, Z). The Jacobian variety $\text{Jac}(X) = H^0(X, \Omega)^*/\text{im}(\text{Per})$ is a compact complex analytic variety of dimension g since im(Per) is a lattice. From the properties of the Riemann matrix above one can conclude that Jac(X) has in fact a unique structure of algebraic variety compatible with the analytic structure. As an algebraic variety, Jac(X) is projective and the group structure makes it into an algebraic group. Such varieties are called Abelian varieties. The rather special form of the Riemann matrix implies more precisely that Jac(X) is a principally polarized Abelian variety.

Proof.
The symplectic basis $a_1, ..., b_g$ is represented by a set of closed curves on X through a chosen point $x_0 \in X$ such that the x_0 is the only point where the curves intersect. The complement Δ of the union of these curves is topologically a disk and it can be seen as the interior of a polygon in the plane with $4g$ sides; for each a_i and b_i there are two corresponding sides of

the polygon having opposite directions. If one follows the boundary of $\bar{\Delta}$ then the collection of sides reads $a_1, b_1, a_1^{-1}, b_1^{-1}, ..., a_g, b_g, a_g^{-1}, b_g^{-1}$.

The topological space X is obtained from $\bar{\Delta}$ by identifying the $4g$ sides pairwise according to their orientation. In particular, one sees that the relation in the fundamental group given in section 4 follows from this presentation.
Let ω be a holomorphic differential form on X, then $\omega = df$ for a certain holomorphic function on Δ which extends to an "analytic" function on the sides of the polygon.

Let P be a point lying on the curve a_i and let P^+, P^- denote the two points on the boundary $\partial\Delta$ of Δ such that P^+ lies on the side with positive orientation. Then $f(P^-) - f(P^+) = \int_{b_i} \omega$. For P on the curve b_i one has the similar result $f(P^+) - f(P^-) = \int_{a_i} \omega$. This will be used for the calculation of $\frac{1}{i} \int_X \omega \wedge *\bar{\omega}$. For non zero ω one has

$$0 < \frac{1}{i} \int_X \omega \wedge *\bar{\omega} = i \int_\Delta \omega \wedge \bar{\omega} = i \int_{\partial\Delta} f\bar{\omega}$$

The integral over the boundary is evaluated by combining the two sides corresponding to any of the $2g$ curves $a_1, ..., b_g$. The result is

$$i \sum_{k=1}^{g} (\int_{a_i} \omega \int_{b_i} \bar{\omega} - \int_{b_i} \omega \int_{a_i} \bar{\omega})$$

One conclusion is that for some i one has $\int_{a_i} \omega \neq 0$. Knowing this, one finds a unique basis $\omega_1, ..., \omega_g$ for $H^0(X, \Omega)$ such that $\int_{a_i} \omega_j = \delta_{i,j}$.

Suppose now that $0 \neq \omega = \sum_i \lambda_i \omega_i$ with all λ_i real numbers. Evaluation of the same integral gives

$$0 < \sum_{i,j} \lambda_i \lambda_j im(\int_{b_i} \omega_j)$$

This shows that the matrix $\mathrm{im}(Z)$ is positive definite, as required in the theorem.
The symmetry of Z follows from a similar evaluation of the integral

$$0 = \int_X \omega_i \wedge \omega_j = \int_\Delta \omega_i \wedge \omega_j = \int_{\partial\Delta} f_i \omega_j$$

where f_i denotes a holomorphic function on Δ with $df_i = \omega_i$. The result is

$$0 = \sum_{k=1}^{g} (\int_{a_k} \omega_i \int_{b_k} \omega_j - \int_{b_k} \omega_i \int_{a_k} \omega_j) = \int_{b_i} \omega_j - \int_{b_j} \omega_i$$

This proves the theorem.

Exercise 30. Let X be a Riemann surface with genus 1. Show that (10.3) is trivial in this case. Prove that X is isomorphic to $\mathrm{Jac}(X)$.

10.4 The Picard group.

The Picard group $\mathrm{Pic}(X)$ of a compact Riemann surface X of genus $g \geq 1$ has the following descriptions:
$H^1(X, O_X^*)$
The group of equivalence classes of line bundles on X.
The group of equivalence classes of divisors on X.
There is a degree map deg: $\mathrm{Pic}(X) \to \mathbf{Z}$ which maps every line bundle (or divisor) to its degree. The kernel of this map is denoted by $\mathrm{Pic}^0(X)$ and is equal to the group of divisors of degree 0 modulo equivalence. We can use the exact sequence of sheaves

$$0 \to \mathbf{Z} \to O_X \to O_X^* \to 0$$

in which the last non trivial map is $f \mapsto e^{2\pi i f}$. There results an exact sequence of cohomology

$$0 \to H^1(X, \mathbf{Z}) \to H^1(X, O_X) \to H^1(X, O_X^*) \xrightarrow{\phi} H^2(X, \mathbf{Z}) \to 0$$

We known that $H^1(X, \mathbf{Z}) \cong \mathbf{Z}^{2g}$ and that $H^2(X, \mathbf{Z}) \cong \mathbf{Z}$. The kernel of ϕ is a divisible group and so the degree map on $\ker(\phi)$ must be trivial. It follows from this that $\ker(\phi)$ is equal to $\mathrm{Pic}^0(X)$ and that the map $\phi : Pic(X) \to \mathbf{Z}$ is (possibly upto a sign) the degree map. Hence we found a natural exact sequence

$$0 \to H^1(X, \mathbf{Z}) \xrightarrow{\alpha} H^1(X, O_X) \to Pic^0(X) \to 0$$

The Serre duality gives an isomorphism $\beta : H^1(X, O_X) \to H^0(X, \Omega)^*$. We will prove that $\beta(H^1(X, \mathbf{Z})) = H_1(X, \mathbf{Z})$ where the latter group is seen as a subgroup of $H^0(X, \Omega)^*$. The result is:

10.5 Abel's theorem and Jacobi's inversion theorem.

The duality isomorphism β induces an isomorphism $Pic^0(X) \to$ Jac(X).

Proof.
We choose a point $a \in X$ and we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of X given by U_0 is a small disk containing a and $U_1 = X - \{a\}$. An element $r \in H^1(X, O_X)$ can be represented by a 1-cocycle $\rho \in O(U_{0,1})$. This 1-cocycle belongs to the subgroup $H^1(X, \mathbf{Z})$ if and only if the image of the 1-cocycle $e^{2\pi i \rho} \in O(U_{0,1})^*$ in $H^1(X, O^*)$ is trivial. One knows (the statement holds in fact for any sheaf of abelian groups and any open covering) that the map $\check{H}^1(\mathcal{U}, O^*) \to H^1(X, O^*)$ is injective. Under the assumption that $r \in H^1(X, \mathbf{Z})$ it follows that $e^{2\pi i \rho} = f_0 f_1^{-1}$ with $f_i \in O(U_i)^*$ for $i = 0, 1$. Since U_0 is a disk one has $f_0 = e^{2\pi i g_0}$ for some $g_0 \in O(U_0)$. One can change ρ into $\rho - g_0$ without changing r. This means that we may suppose that $f_0 = 1$.
The function f_1 on U_1 is in general not the exponential of a holomorphic function on U_1. From

$$0 \to \mathbf{Z} \to O_X \to O_X^* \to 0$$

one deduces the exactness of

$$0 \to \mathbf{Z} \to O(U_1) \to O(U_1)^* \to H^1(U_1, \mathbf{Z}) \to 0$$

We note that the canonical map $H^1(X, \mathbf{Z}) \to H^1(X - \{a\}, \mathbf{Z})$ is an isomorphism. One can use again the covering $\mathcal{U}$ to see this. The image of r under this canonical map coincides with the image of f_1^{-1} in $H^1(X - \{a\}, \mathbf{Z})$.
The image of r under β is the map $\omega \mapsto 2\pi i \ Res_a(\rho\omega)$. (Here we have introduced a harmless factor $2\pi i$.) We will calculate this residue by cutting the Riemann surface as in the proof of (10.3). There results a polygon Δ with $4g$ sides. As before we write $\omega = df$ for some holomorphic function f on $\bar{\Delta}$. Then

$$2\pi i \ Res_a(\rho\omega) = 2\pi i \ Res_a(\rho df) = -2\pi i \ Res_a(f d\rho) = Res_a(f \frac{df_1}{f_1})$$

The last expression is equal to $\frac{1}{2\pi i} \int_{\partial\Delta} f \frac{df_1}{f_1}$. This integral is evaluated by combining the pairs of sides corresponding to the $2g$ curves $a_1, ..., b_g$. The result is

$$\frac{1}{2\pi i} \sum_{k=1}^{g} (\int_{a_k} \omega \int_{b_k} \frac{df_1}{f_1} - \int_{b_k} \omega \int_{a_k} \frac{df_1}{f_1})$$

We note that for any closed path c the integral $\frac{1}{2\pi i} \int_c \frac{df_1}{f_1}$ has a value in $\mathbf{Z}$. This shows that

$$\beta(r) = \sum_{k=1}^{g} ((\frac{1}{2\pi i} \int_{b_k} \frac{df_1}{f_1}) a_k - (\frac{1}{2\pi i} \int_{a_k} \frac{df_1}{f_1}) b_k) \in H_1(X, \mathbf{Z})$$

The surjective map $O(U_1)^* \to H^1(U_1, \mathbf{Z}) \xrightarrow{\sim} H^1(X, \mathbf{Z}) = Hom(H_1(X, \mathbf{Z}), \mathbf{Z})$ implies that $\beta : Hom(H_1(X, \mathbf{Z}), \mathbf{Z}) \to H_1(X, \mathbf{Z})$ has the following form $\lambda \mapsto \sum_{k=1}^{g} (\lambda(b_k) a_k - \lambda(a_k) b_k)$. Using the symplectic form $< .,. >$ on $H_1(X, \mathbf{Z})$ this translates into: $\beta(\lambda)$ is the unique element satisfying $< \beta(\lambda), c >= \lambda(c)$ for all $c \in H_1(X, \mathbf{Z})$. In particular $\beta(H^1(X, \mathbf{Z})) = H_1(X, \mathbf{Z})$. This proves that β induces an isomorphism $\mathrm{Pic}^0(X) \to \mathrm{Jac}(X)$.

Remarks.
The map β can be given explicitly for divisors. Let $D = \sum (P_i - Q_i)$ denote a divisor on X of degree 0. Choose paths c_i on X from Q_i to P_i. Then the linear map $l_D : \omega \mapsto \sum_i \int_{c_i} \omega$ has the same image in $\mathrm{Jac}(X)$ as $\beta(D)$.
A classical form of Abel's theorem is:" l_D lies in the lattice if and only D is a principle divisor". Jacobi's inversion theorem states that every element of Jac(X) is the image of some l_D.

11 Uniformization

In this section we have collected some of the interesting results on uniformization. The basic result is the following theorem of Riemann:

11.1 Theorem (The Riemann mapping theorem)

Every simply connected Riemann surface is isomorphic to precisely one of the following Riemann surfaces:
The complex sphere $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$.
The complex plane $\mathbf{C}$.
The complex upper half plane $\mathbf{H} = \{z \in \mathbf{C} | \, im(z) > 0\}$.

We will give no proof of this. Let X be any Riemann surface. From section 4 we know that there is a universal covering $\Omega \to X$, with Ω a simply connected Riemann surface. Further the automorphism group Γ of the covering $\Omega \to X$ acts discontinuously, without fixed points on Ω and the quotient $\Omega/\Gamma \cong X$.

If Ω happens to be the complex sphere then $\Gamma = \{1\}$ since any automorphism $\neq$id on $\mathbf{P}^1$ has a fixed point. So $X = \mathbf{P}^1$.

If Ω equals to complex plane then Γ can only be a discrete subgroup of translations. Hence $\Gamma \cong \mathbf{Z}^s$ with $s = 0, 1, 2$. For $s = 0$, X is the complex plane. $X \cong \mathbf{C}^*$ for $s = 1$. Finally for $s = 2$ the quotient is an elliptic curve.

If Ω is the upper half plane. Then $\Gamma \subset PSl(2, \mathbf{R})$. If the quotient X is compact then it is a Riemann surface of genus ≥ 2. One can show that compactness of the quotient is equivalent to the compactness of the space $PSl(2, \mathbf{R})/\Gamma$.
The quotient $PSl(2, \mathbf{R})/\Gamma$ has finite volume (with respect to the Haar measure on $PSl(2, \mathbf{R})$) if and only if X can be identified with the complement of a finite subset of some compact Riemann surface.
If the discontinuous group Γ (acting without fixed points) has no finite "covolume" then there are many possibilities for the non compact RS X. There is an extensive theory of Fuchsian groups related to this. For the natural question:

For which discontinuous groups Γ *(acting without fixed points) can one embed the quotient* $X = \mathbf{H}/\Gamma$ *into a compact Riemann surface?*
I do not know the answer.

Some literature on Riemann surfaces

L.V.Ahlfors, L.Sario- Riemann surfaces. 1960.
A.F.Beardon-The Geometry of Discrete Groups. 1983.
H.M.Farkas, I.Kra-Riemann surfaces. 1979.
O.Forster-Riemann surfaces. 198?.
Ph.Griffiths, J.Harris-Principles of algebraic geometry. 1978
R.C.Gunning-Lectures on Riemann surfaces. 1966.
R.C.Gunning-Lectures on Riemann surfaces: Jacobian varieties. 1972.
B.Maskit-Kleinian Groups. 1988.
J-P.Serre-Groupes algébriques et corps de classes. 1959.
G.Springer-Introduction to Riemann surfaces. 1957.
H.Weyl-Die Idee der Riemannschen Fläche. 1955.

Aanvullende opgaven by Riemann Oppervlakken

(i) We bekijken Exercise 7 van het diktaat "Riemann Surfaces" op een wat andere manier. De situatie is als volgt:
$\Lambda_i \subset \mathbf{C}$ zijn twee roosters; we schrijven E_{Λ_i} voor $\mathbf{C}/\Lambda_i$ en $\pi_i : \mathbf{C} \to E_{\Lambda_i}$ voor de kanonieke afbeeldingen. Veronderstel dat er een (iso)morfisme $f : E_{\Lambda_1} \to E_{\Lambda_2}$ gegeven is. Een probleem is om aan te tonen dat er een morfisme $F : \mathbf{C} \to \mathbf{C}$ bestaat dat voldoet aan $f \circ \pi_1 = \pi_2 \circ F$.
Lokaal heeft dit een probleem een oplossing omdat de π_i lokaal isomorfismen zijn. We kunnen dus $\mathbf{C}$ overdekken met open verzamelingen U_i (met $i \in I$ en I voorzien van een totale ordening) zodat er voor iedere i een holomorfe $F_i : U_i \to \mathbf{C}$ bestaat zodat $f \circ \pi_1 = \pi_2 \circ F_i$ geldt op U_i. Als (toevallig) voor elke i_1, i_2 geldt dat de restricties van F_{i_1} en F_{i_2} tot $U_{i_1} \cap U_{i_2}$ gelijk zijn, dan plakken de F_i's aan elkaar tot een holomorfe $F : \mathbf{C} \to \mathbf{C}$ met de verlangde eigenschap. In het algemeen zal echter $F_{i_1} \neq F_{i_2}$ op $U_{i_1} \cap U_{i_2}$ zijn. Met de volgende stappen kunnen we de overdekking $\{U_i\}$ en de collectie $\{F_i\}$ wijzigen zodat de zaak aan elkaar plakt.
(a) Toon aan dat $\xi := (F_{i_1} - F_{i_2})_{i_1 < 1_2}$ een 1-cocykel is (d.w.z. in $ker(d^1)$ ligt) voor het Čech complex horend by $\{U_i\}$ en de constante schoof op $\mathbf{C}$ met groep Λ_2.
(b) Gebruik stelling 2.2 en toon aan dat $\{U_i\}$ een verfijning $\{V_j\}_{j \in J}$ heeft zodat het beeld van ξ een triviale 1-cocykel is (d.w.z. in $im(d^0)$ ligt).
(c) Het beeld van ξ voor de overdekking $\{V_j\}$ kan geschreven worden als $(G_{j_1} - G_{j_2})_{j_1 < j_2}$ waarbij de $G_j : V_j \to \mathbf{C}$ voldoen aan $f \circ \pi_1 = \pi_2 \circ G_j$ op iedere V_j. Verander nu de G_j's in G_j^* zodat voor elk tweetal j_1, j_2 geldt $G_{j_1}^* = G_{j_2}^*$ op $V_{j_1} \cap V_{j_2}$. Maak het bewijs van de existentie van F af.

(ii) Laat $A \subset \mathbf{C}$ een open samenhangende verzameling zijn. Een divisor op A is een formele uitdrukking $\sum_{a \in A} n_a a$, waarbij iedere $n_a \in \mathbf{Z}$ en waarbij $\{a \in A | \ n_a \neq 0\}$ een discrete deelverzameling is van A. Aan elke meromorfe funktie $f \neq 0$ op A kan men toevoegen $div(f) := \sum_{a \in A} ord_a(f)a$.
(a) Bewijs dat $div(f)$ een divisor is.
(b) Bewijs dat er bij elke divisor D op A een meromorfe funktie $f \neq 0$ is met $div(f) = D$. Aanwijzing:
Het lokaal construeren van f is geen probleem. De lokale oplossingen plakken a priori niet aan elkaar. Gebruik nu de methode van opgave (i), 1-cocykels voor de schoof O^* op A en theorem 6.6.

(c) Probeer het resultaat (b) in een boek over funktietheorie te vinden.

(iii) A is weer een samenhangende open deelverzameling van $\mathbf{C}$. Een (lokaal) hoofdeel in een punt $a \in A$ is gedefinieerd als een uitdrukking $\sum_{n=1}^{N} a_n(z-a)^{-n}$ (met $a_n \in \mathbf{C}$). Een globaal hoofddeel op A is een formele uitdrukking $\sum_{a\in A} h_a a$ waarbij elke h_a een (lokaal) hoofdeel in a is en de verzameling $\{a \in A|\ h_a \neq 0\}$ discreet is.
(a) Definieer voor elke meromorfe funktie f op A en punt $a \in A$ een lokaal hoofddeel $h_a(f)$. Laat zien dat $h(f) := \sum_{a\in A} h_a(f)a$ een globaal hoofddeel is.
(b) Zij een globaal hoofddeel H op A gegeven. Bewijs dat er een meromorfe funktie f op A bestaat met $h(f) = H$. Aanwijzing:
Gebruik de methode van (ii), produceer een 1-cocykel voor de schoof O op A, gebruik theorem 6.6.
(c) Probeer het resultaat (b) in een funktietheorieboek te vinden.