

## EXPLICIT ELLIPTIC $K3$ SURFACES WITH RANK 15

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ABSTRACT. This note presents a relatively straightforward proof of the fact that, under certain congruence conditions on  $a, b, c \in \mathbf{Q}$ , the group of rational points over  $\overline{\mathbf{Q}}(t)$  on the elliptic curve given by

$$y^2 = x^3 + t^3(t^2 + at + b)^2(t + c)x + t^5(t^2 + at + b)^3$$

is trivial. This is used to show that a related elliptic curve yields a free abelian group of rank 15 as its group of  $\overline{\mathbf{Q}}(t)$ -rational points.

**1. Introduction.** The theory of elliptic curves defined over the function field  $k(C)$  of a curve  $C/k$  is quite rich. To a large extent, this is due to the observation that any such elliptic curve  $E/k(C)$  corresponds to a minimal  $k$ -morphism  $\pi : \mathcal{E} \rightarrow C$  in which  $\mathcal{E}$  is a smooth surface over  $k$ , and the generic fiber of  $\pi$  is isomorphic to  $E$ . Rational points on  $E$  correspond to sections of  $\pi$ , and the geometry of  $\mathcal{E}$  gives a better understanding of the Mordell-Weil group  $E(k(C))$ . An exposition of this theory is given in [7].

For example, when  $\mathcal{E}$  is a rational surface and  $k$  is separably closed, one has the Shioda-Tate formula

$$\text{rank } E(k(C)) = 8 - \sum_{\nu} (m_{\nu} - 1),$$

where  $m_{\nu}$  is the number of irreducible components of the fiber  $\pi^{-1}(\nu)$  over  $\nu \in C$ . Using the formula, it is easy to construct explicit examples with a given rank  $r$  satisfying  $0 \leq r \leq 8$ . This is done in the table below, at least over  $\mathbf{C}(t)$ .

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rank	equation
0	$y^2 = x^3 + t^4x + t^5$
1	$y^2 = x^3 + t^3(t+1)x + t^5$
2	$y^2 = x^3 + t^4x + t^4$
3	$y^2 = x^3 + x^2 - t^4x + t^6$
4	$y^2 = x^3 + t^4x + t^3$
5	$y^2 = x^3 + x^2 - t^4x + t^6$
6	$y^2 = x^3 + t^4x + t^2$
7	$y^2 = x^3 + tx + t^6$
8	$y^2 = x^3 + t^4x + 1$

In fact, as may be verified using the result of [6], in all cases presented in the table above, the group of rational points is free of the given rank.

If the surface  $\mathcal{E}$  is not rational, then an explicit general formula for the rank cannot be expected. However, one has the general Shioda-Tate formula

$$\text{rank } E(k(C)) = \rho - 2 - \sum_{\nu} (m_{\nu} - 1),$$

with  $\rho$  the rank of the Néron-Severi group of  $\mathcal{E}$ . Only for special classes of surfaces are methods for determining  $\rho$  known. However, since a cycle class map [2] injects this Néron-Severi group into the  $l$ -adic cohomology group  $H^2(\mathcal{E}_{\bar{k}}, \mathbf{Q}_l(1))$ , the dimension of the latter space yields an upper bound.

The simplest nonrational surfaces  $\mathcal{E}$  are the so-called  $K3$  surfaces. Over  $\mathbf{C}$ , these are simply connected surfaces with a trivial canonical bundle. Over an algebraically closed field of characteristic 0, the corresponding elliptic curves  $E$  satisfy

$$\text{rank } E(k(C)) \leq 18 - \sum_{\nu} (m_{\nu} - 1).$$

David Cox in [1] proved that any integer  $r$  with  $0 \leq r \leq 18$  occurs as the rank over  $\mathbf{C}(t)$  of some elliptic curve corresponding to an elliptic  $K3$  surface  $\mathcal{E} \rightarrow \mathbf{P}^1$  defined over  $\mathbf{C}$ . Masato Kuwata [5] provided explicit examples for all  $r \neq 15$ . In his Ph.D. thesis [3], Kloosterman

showed that for a general  $c \in \mathbf{C}$  and a general quadratic polynomial  $f(t) \in \mathbf{C}[t]$ , the equation

$$y^2 = x^3 + t^3(t+c)f(t)^2x + t^5f(t)^3$$

yields an elliptic  $K3$  surface over  $\mathbf{P}^1$  with  $\rho = 17$ . Taking a carefully constructed base change  $\mathbf{P}^1 \rightarrow \mathbf{P}^1$  of degree 8 which is ramified only over  $t = 0$  and over the zeroes of  $f$ , Kloosterman obtains a new elliptic  $K3$  surface which also has  $\rho = 17$ , and which moreover has the property that all its fibers are irreducible. Hence, the relation between  $\rho$  and the rank here gives rank equal to 15. Nevertheless, this result is not regarded as the completion of the set of explicit examples which Kuwata constructed. Namely, the method of proof only shows that a general pair  $(c, f)$  (so,  $c$  and the coefficients of  $f$  outside an unspecified countable union of hypersurfaces in  $\mathbf{A}^1 \times \mathbf{P}^2$ ) meets the required condition.

In his later paper [4], Kloosterman was able to remove this objection and to provide an explicit pair  $(c, f)$  with  $c \in \mathbf{Q}$  and  $f \in \mathbf{Q}[t]$  such that the resulting elliptic curve indeed has rank 15 (over  $\overline{\mathbf{Q}}(t)$  and over  $\mathbf{C}(t)$ ). His proof uses the Artin-Tate conjecture, which in the case at hand (a  $K3$  surface) is a theorem.

Here we make a variation of this idea. This will allow us to avoid using the Artin-Tate conjecture, in fact to use calculations mod  $p$  for smaller primes than did Kloosterman and to provide infinitely many explicit examples.

**2. The results.** In this section the two main results of this note are presented.

**Theorem 1.** *Suppose that  $c \in \mathbf{Q}$  and  $f \in \mathbf{Q}[t]$  satisfy:*

- $c \equiv 1 \pmod{5 \cdot 7}$ ;
- $f$  has degree two, and  $f \equiv t^2 + 3t - 3 \pmod{5 \cdot 7}$ .

*Then  $E/\overline{\mathbf{Q}}(t) : y^2 = x^3 + t^3(t+c)f(t)^2x + t^5f(t)^3$  has rank 0 over  $\overline{\mathbf{Q}}(t)$ .*

This will be proven in Section 3.

Assume that the quadratic polynomial  $f \in \overline{\mathbf{Q}}[t]$  has two simple zeroes, and  $f(0) \neq 0$ . In particular, these conditions hold for the polynomials  $f$  appearing in Theorem 1. Define parameters  $s, u, v$  by

$$\mathbf{P}_v^1 \xrightarrow{2:1} \mathbf{P}_u^1 \xrightarrow{2:1} \mathbf{P}_s^1 \xrightarrow{2:1} \mathbf{P}_t^1,$$

in which the last  $2 : 1$ -map ramifies over the two zeroes of  $f$  and sends  $s = \infty$  and  $s = 0$  to  $t = 0$ . Furthermore, the first two maps ramify in  $v = 0, \infty$  which are mapped to  $u = 0, \infty$ , respectively, in  $u = 0, \infty$  which are mapped to  $s = 0, \infty$ . These morphisms correspond to a chain of function field extensions of degree 2

$$\overline{\mathbf{Q}}(t) \subset \overline{\mathbf{Q}}(t, \sqrt{f}) = \overline{\mathbf{Q}}(s) \subset \overline{\mathbf{Q}}(\sqrt{s}) = \overline{\mathbf{Q}}(u) \subset \overline{\mathbf{Q}}(\sqrt{u}) = \overline{\mathbf{Q}}(v).$$

To make this even more explicit, take a monic  $f(t) = t^2 + at + b$  with  $b \neq 0$  and  $a^2 - 4b \neq 0$ . The quadratic extension  $\mathbf{Q}(t) \subset \mathbf{Q}(s)$  given by  $t = 4bs/(s^2 - 2as + a^2 - 4b)$  only ramifies over  $f(t) = 0$ , and the map  $s \mapsto t$  clearly sends 0 and  $\infty$  to 0. Hence, the base change from  $t$  to  $v$  described above can in this case be given by  $t = 4bv^4/(v^8 - 2av^4 + a^2 - 4b)$ . Substituting this into the equation

$$y^2 = x^3 + t^3(t + c)f(t)^2x + t^5f(t)^3$$

for  $E/\mathbf{Q}(t)$ , one obtains an equation for  $E/\mathbf{Q}(v)$ . Clearing denominators and simplifying slightly more, one obtains

$$E/\mathbf{Q}(v) : y^2 = x^3 + A(v)x + B(v)$$

with  $A(v), B(v)$  given by  $A(v) = 4b(cv^8 + (4b - 2ac)v^4 + a^2c - 4bc)$  and  $B(v) = 16b^2v^2(v^8 - 2av^4 + a^2 - 4b)$ .

**Theorem 2.** *Assume the conditions stated in Theorem 1. Then  $E(\overline{\mathbf{Q}}(v))$  is torsion-free and has rank 15. It corresponds to an elliptic K3 surface  $\mathcal{E} \rightarrow \mathbf{P}_v^1$  having 24 fibers of type  $I_1$  (and no reducible fibers).*

As an example, the conditions in Theorem 1 hold for  $c = 1$  and  $f(t) = t^2 + 3t - 3$ . In this case,  $E/\mathbf{Q}(v)$  is given by

$$y^2 = x^3 - 12(v^8 - 18v^4 + 21)x + 144v^2(v^8 - 6v^4 + 21).$$

Theorem 2 implies that  $E(\overline{\mathbf{Q}}(v)) \cong \mathbf{Z}^{15}$ .

*Proof of Theorem 2.* Observe that the statements about the singular fibers follow from the fact that we know the fibers of the elliptic surface corresponding to  $E/\overline{\mathbf{Q}}(t)$ . Indeed, this surface has singular fibers over  $t = 0$ , over the zeroes of  $f$ , and over the solutions of  $4(t+c)^3 + 27t = 0$ . The condition  $c \equiv 1 \pmod{5}$  implies that the latter cubic polynomial is irreducible. Hence, its zeroes are simple, and we find an  $I_1$ -fiber over each of them. The conditions on  $f$  imply that  $f$  has two simple zeroes, both different from 0. It follows that, over these zeroes, we have an  $I_0^*$ -fiber and over  $t = 0$  a fiber of type  $III^*$ .

After the first quadratic extension, one obtains an  $I_1$ -fiber over six values of  $s$  and two  $III^*$ -fibers over  $s = 0, \infty$  (and smooth fibers over all other  $s$ ). The second extension results in 12 fibers of type  $I_1$  and two  $I_0^*$ -fibers (the latter over  $u = 0, \infty$ ). The last extension results in 24 values of  $v$  over which the fiber is of type  $I_1$ , and all other fibers are smooth. It is well known that this, and all intermediate, elliptic surface(s) are  $K3$ .

To prove the statement concerning the rank, one may repeatedly use the fact that for a quadratic extension  $L = K(\sqrt{d})/K$  and an elliptic curve  $E/K$ , one has  $E(L) \rightarrow E(K) \oplus E^{(d)}(K)$  with both the kernel and the cokernel killed by 2. Here  $E^{(d)}/K$  denotes the quadratic twist of  $E$  by  $d$ . In the present situation, this implies

$$\begin{aligned}
\text{rank } E(\overline{\mathbf{Q}}(v)) &= \text{rank } E(\overline{\mathbf{Q}}(u)) + \text{rank } E^{(u)}(\overline{\mathbf{Q}}(u)) \\
&= \text{rank } E(\overline{\mathbf{Q}}(u)) + 8 \\
&= \text{rank } E(\overline{\mathbf{Q}}(s)) + \text{rank } E^{(s)}(\overline{\mathbf{Q}}(s)) + 8 \\
&= \text{rank } E(\overline{\mathbf{Q}}(s)) + 6 + 8 \\
&= \text{rank } E(\overline{\mathbf{Q}}(t)) + \text{rank } E^{(f(t))}(\overline{\mathbf{Q}}(t)) + 6 + 8 \\
&= 0 + 1 + 6 + 8 = 15.
\end{aligned}$$

Here, in the second line it is used that  $E^{(u)}/\overline{\mathbf{Q}}(u)$  corresponds to a rational elliptic surface with no reducible fibers; hence, this gives rank 8 by the Shioda-Tate formula (see also [6]). In the fourth line one uses that  $E^{(s)}/\overline{\mathbf{Q}}(s)$  corresponds to a rational elliptic surface with two type  $III$  fibers and all other fibers irreducible, so this gives rank  $8 - 1 - 1 = 6$ . Finally, in the last line Theorem 1 is used, plus the fact that  $E^{(f(t) )}/\overline{\mathbf{Q}}(t)$  corresponds to an elliptic surface with as singular fibers one of type  $III^*$  and three of type  $I_1$ . Hence, this is a rational

elliptic surface giving rank  $8 - 7 = 1$  where the 7 comes from the fact that the  $III^*$ -fiber consists of eight irreducible components.

To show that  $E(\overline{\mathbf{Q}}(u))$  is torsion-free, note that  $E\overline{\mathbf{Q}}(u)$  corresponds to an elliptic surface with only irreducible fibres. The formula for the height presented in this case implies that any nonzero point has a positive integer as its canonical height. In particular, such a point cannot be torsion.  $\square$

So it suffices to prove Theorem 1, which will be done in the remaining section.

*Remark.* The fact that  $E(\overline{\mathbf{Q}}(u))$  is torsion-free could also be established by an argument more along the same line as the proof of the statement about the rank. Namely, note that it does not contain any torsion point of even order greater than one. Indeed, if it did, then it would contain a point of order two. It is easily checked that this is not the case. If  $E(\overline{\mathbf{Q}}(v))$  would contain a point of odd order, then so would one of  $E(\overline{\mathbf{Q}}(u))$  or  $E^{(u)}(\overline{\mathbf{Q}}(u))$ . However, both groups are torsion-free as follows from the fact that specialization is injective on torsion points, and the corresponding elliptic surfaces contain an additive fiber.

**3. Proof of the main result.** We will now study the implications which the conditions stated in Theorem 1 have for the resulting elliptic curves over  $\overline{\mathbf{F}}_5(t)$  and over  $\overline{\mathbf{F}}_7(t)$ .

**3.1.  $p = 5$ .** In characteristic 5, the elliptic curve with equation

$$y^2 = x^3 + t^3(t+1)(t^2+3t+4)^2x + t^5(t^2+3t+4)^3$$

is considered. Since we are only interested in this curve over  $\overline{\mathbf{F}}_5(t)$ , we will in fact study the equation

$$y^2 = x^3 + t^3(t+1)3^2(t^2+3t+4)^2x + t^53^3(t^2+3t+4)^3.$$

The latter defines an elliptic curve  $E_0$  which is isomorphic to our curve over  $\mathbf{F}_{25}(t)$ , but not over  $\mathbf{F}_5(t)$ .

The group  $E_0(\mathbf{F}_5(t))$  contains the point

$$P_0 := (3(t+1)(t+3)(t^2+3t+4), 2(t^2+3t+3)(t^2+3t+4)^2).$$

As a section of the corresponding elliptic surface over  $\mathbf{P}_t^1$ , this point meets the two  $I_0^*$ -fibers (at the zeroes of  $t^2 + 3t + 4$ ) in a component different from the identity component. And it meets the  $III^*$ -fiber (at  $t = 0$ ) in the identity component. It follows that the height of this point, as given in [7], equals

$$\langle P_0, P_0 \rangle = 4 - 1 - 1 = 2.$$

**Proposition 1.** *The group  $E_0(\overline{\mathbf{F}}_5(t))$  is free of rank 1, with generator  $P_0$  of height 2.*

*Proof.* The group is torsion-free since any torsion point  $Q$  in it would satisfy, using the formula for the height

$$0 = \langle Q, Q \rangle = 4 + 2(QO) - c_1 - c_2 - c_3,$$

with  $c_1, c_2 \in \{0, 1\}$  and  $c_3 \in \{0, 3/2\}$ . Since  $(QO) \geq 0$  for  $Q \neq O$ , this is only possible with  $Q = O$ .

Assume that the rank equals 1. The formula for the height of any point as given above shows that the height takes values in  $(1/2)\mathbf{Z}_{\geq 0}$ . So if  $P_0$  were not a generator, then  $P = 2R$  in which  $R$  has height  $1/2$ . But then  $(RO) = 0$ , which implies that the  $x$ -coordinate of  $R$  is a polynomial of degree at most 4, and moreover  $c_1 = c_2 = 1$  and  $c_3 = 3/2$ , which implies that this  $x$ -coordinate is divisible by  $t(t^2 + 3t + 4)$ . It is now easily checked that no such point  $R$  exists; hence, indeed  $P_0$  is a generator.

It remains to show that the rank  $r$  equals 1. This is done by considering the corresponding elliptic  $K3$  surface over  $\overline{\mathbf{F}}_5$ . Generators of the group of sections, together with the zero-section  $O$ , and the irreducible components of fibers yield, using a cycle class map, a subspace of dimension  $r + 2 + 7 + 4 + 4 = r + 17$  in  $H^2(\mathcal{E}_{0, \overline{\mathbf{F}}_5}, \mathbf{Q}_l)$ . The Frobenius automorphism acts semi-simply on  $H^2(\mathcal{E}_{0, \overline{\mathbf{F}}_5}, \mathbf{Q}_l)$ , and this subspace of dimension  $r + 17$  is the sum of eigenspaces with eigenvalues of the form  $5\zeta$  where  $\zeta$  is a root of unity. Since we already found one section of infinite order, we have a space of dimension 18 on which the action of Frobenius is known. Counting points on  $\mathcal{E}$  and using the Lefschetz trace formula, one can calculate the remaining four

eigenvalues on  $H^2$  (compare, e.g., [8, 9]). In the present case, one finds that they are the zeroes of

$$(x^2 - 2x + 25)(x^2 + 6x + 25).$$

Since none of these zeroes is of the form  $5\zeta$  with  $\zeta$  a root of unity, it follows that  $r + 17 \leq 18$ . Hence indeed the rank equals one.  $\square$

**3.2.**  $p = 7$ . The conditions modulo 7, assumed in Theorem 1, imply that we deal with  $E_1/\mathbf{F}_7(t)$  given by

$$y^2 = x^3 + t^3(t+1)^3(t+2)^2x + t^5(t+1)^3(t+2)^3.$$

This curve contains the rational point

$$P_1 := (3t^2(t+1)(t-1), 3t^4(t+1)^2).$$

The section corresponding to  $P_1$  meets the  $III^*$ -fiber over  $t = 0$  in a component different from the identity component. The same is true for the  $I_0^*$ -fiber over  $t = -1$ . However, the section meets the  $I_0^*$ -fiber over  $t = -2$  in the identity component. It follows that

$$\langle P_1, P_1 \rangle = 4 - \frac{3}{2} - 1 - 0 = 3/2.$$

**Proposition 2.** *The group  $E_1(\overline{\mathbf{F}_7}(t))$  is free of rank 1, with generator  $P_1$  of height  $3/2$ .*

*Proof.* The rank equals one by an argument similar to the one presented in the proof of Proposition 1. In the present case, the four remaining eigenvalues of Frobenius are zeroes of  $x^4 + 4x^3 + 14x^2 + 196x + 7^4$ . Since this polynomial is irreducible over  $\mathbf{Q}$ , none of these zeroes is of the form  $7\zeta$  for a root of unity  $\zeta$ .

The group of rational points is torsion-free by the same reasoning as before.  $P_1$  is a generator since the height takes values in  $(1/2)\mathbf{Z}_{\geq 0}$ , which immediately shows that  $P_1$  is not a multiple of a rational point with smaller height.  $\square$

**3.3. Conclusion.** To complete the proof of Theorem 1, observe that under the conditions given in Theorem 1, the group  $E(\overline{\mathbf{Q}}(t))$  is torsion-free. Indeed, specializing to a fiber where the reduction is additive shows that no points  $\neq O$  of finite order exist.

Now take  $p \in \{5, 7\}$ ,  $j := 0$  if  $p = 5$  and  $j := 1$  if  $p = 7$ , and consider a reduction homomorphism

$$E(\overline{\mathbf{Q}}(t)) \longrightarrow E_j(\overline{\mathbf{F}}_7(t)).$$

This map is injective (as follows from [9, Proposition 6.2]), and it preserves the height. Hence  $Q \in E(\overline{\mathbf{Q}}(t))$  has, by reducing to characteristic 5 and applying Proposition 1, height equal to  $n^2 \cdot 2$  for some integer  $n$ . Similarly, Proposition 2 and a reduction to characteristic 7 imply that this height is  $m^2 \cdot 3/2$  for an integer  $m$ . It follows that  $m = n = 0$ , hence  $Q = O$ , which completes the proof.  $\square$

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